Reduction of cluster iteration maps

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Abstract

We study iteration maps of difference equations arising from mutation periodic quivers of arbitrary period. Combining tools from cluster algebra theory and presymplectic geometry, we show that these cluster iteration maps can be reduced to symplectic maps on a lower dimensional submanifold, provided the matrix representing the quiver is singular. The reduced iteration map is explicitly computed for several periodic quivers using either the presymplectic reduction or a Poisson reduction via log-canonical Poisson structures.

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1 Introduction

We study maps arising in the context of cluster algebras theory through the notion of mutation-periodic quivers ([FoMa]). These maps φ are defined as the composition of mutations at the nodes of a quiver and a permutation. The iterates $\varphi^{(n)}$ of such a map φ define a system of difference equations which is nonlinear since φ is a birational map. We call *cluster iteration maps* to this type of maps.

Since the introduction of cluster algebras by Sergey Fomin and Andrei Zelevinsky in [FoZe02], the theory of cluster algebras has grown in many research directions. Relevant to our work are the relations between cluster algebras and presymplectic/Poisson structures whose main achievements are surveyed in [GeShVa10]. The presymplectic and Poisson structures considered in this setting are known as log-canonical structures. A log-canonical Poisson structure

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is given by the simplest possible kind of homogeneous quadratic bracket. A log-canonical presymplectic form will be hereafter called *log presymplectic form*.

The interplay between cluster algebras theory and presymplectic/Poisson geometry opened new research directions in the study of the maps arising from mutation-periodic quivers. Notably, in [FoHo11], [FoHo14], [Ho07] and [HoIn], log presymplectic structures were used to show that cluster iteration maps associated to 1-periodic quivers can be reduced to symplectic maps. This reduction enabled the authors to show the integrability of the reduced Somos-4 and Somos-5 difference equations. More precisely, the respective iteration maps are instances of the widely studied family of QRT integrable maps (see [QRT] and the comprehensive study [Duist] by J. Duistermaat).

Although iteration maps associated to 1-periodic quivers have received considerable attention, the study of iteration maps associated to quivers of higher period is still in its infancy. Here we develop the theory of reduction of iteration maps associated to quivers of arbitrary period, with emphasis on the presymplectic approach.

Our main result, Theorem 2, shows that the iteration map arising from a quiver of arbitrary period can be reduced to a symplectic map on a $2k$-dimensional space with respect to a log symplectic form, where $2k$ is the rank of the matrix representing the quiver. The proof of this theorem does not rely on the classification of periodic quivers, which is only known for period 1 (see [FoMa]). The main ingredients of the proof are: (a) the invariance of the *standard log presymplectic form* under the iteration map, which is proved in Theorem 1; (b) a classical theorem of G. Darboux (or of E. Cartan for the linear version) for the reduction of an arbitrary presymplectic form to a symplectic form.

In the terminology of the theory of cluster algebras Theorem 2 can be stated as follows: the iteration map of a periodic quiver descends, as a symplectic map, to a secondary cluster manifold with its Weil-Petersson form (see [GeShVa03], [GeShVa05] and [GeShVa10]).

A reduced iteration map will be computed for several cluster iteration maps using either a presymplectic or a Poisson approach. This reduction will be done mainly for maps associated to 2-periodic quivers from the Physics literature and already present in [FoMa] as examples of periodic quivers.

The computation of the reduced iteration map via presymplectic forms illustrates the main result (Theorem 2). In all the examples the reduced map will be symplectic with respect to the *canonical log symplectic form*, i.e the reduction will be performed using Darboux coordinates.

Regarding the Poisson approach to reduction, we explain how to use log-canonical Poisson structures to reduce a cluster iteration map. In order to be able to carry out this reduction it is required that the Poisson structure leaves the iteration map invariant. Unlike the reduction via presymplectic forms, the Poisson approach is not always applicable since such structures may fail to exist. We also remark that, even when both approaches can be used, they do not produce necessarily a reduced map with the same properties, in particular the respective reduced maps may be defined on spaces of different dimensions.

Finally, let us remark that although the main focus of the paper is on the reduction of cluster iteration maps and not on its dynamics, we found two new examples of reduced iteration maps which are integrable. Moreover, these maps do not belong to the family of QRT maps since their first integrals (see (31) and (43)) are not the ratio of two biquadratic polynomials.

The organization of the paper is as follows. In Section 2 we introduce the basic notions of the theory of cluster algebras necessary to subsequent sections, in particular, the definition of mutation-periodic quiver and the construction of the associated iteration map. The next section is devoted to the proof of the main result, Theorem 2, on the reduction of cluster iteration maps to symplectic maps. Essential to this proof is Theorem 1 which shows that the standard log presymplectic form of a mutation-periodic quiver is invariant under its iteration map. In Section 4 we explain how to apply Theorem 2 and a theorem by E. Cartan to explicitly compute the reduced iteration map in Darboux coordinates. The last section is devoted to the Poisson approach to reduction. We exhibit log-canonical Poisson structures which are invariant under the iteration map of some periodic quivers with an even number of nodes, and use them to compute a reduced map.

2 Mutation-periodic quivers and iteration maps

In this section we introduce the notions of the theory of cluster algebras necessary to the following sections. We will work in the context of coefficient free cluster algebras $\mathcal{A}(B)$ where B is a skew-symmetric integer matrix.

An *initial seed* is a pair $(B, \mathbf{u})$ where B is an $N \times N$ integer skew-symmetric matrix and $\mathbf{u} = (u_1, \ldots, u_N)$ is an N-tuple of variables, called *cluster variables*, known as the *initial cluster*.

The basic operation in the theory of cluster algebras is called a *mutation*. A mutation μ_k in *the direction* of k (or at node k) acts on a seed $(B, \mathbf{u})$ with $B = [b_{ij}]$ and $\mathbf{u} = (u_1, \ldots, u_N)$, as follows:

- $\mu_k(B) = \left[b'_{ij}\right]$ with

$$b'_{ij} = \begin{cases} -b_{ij}, & (k-i)(j-k) = 0 \\ b_{ij} + \frac{1}{2}\left(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|\right), & \text{otherwise.} \end{cases} \tag{1}$$

- $\mu_k(u_1, \ldots, u_N) = (u'_1, \ldots, u'_N)$, with

$$u'_i = \begin{cases} u_i & i \neq k \\ \dfrac{\prod_{j:b_{kj}\geq 0} u_j^{b_{kj}} + \prod_{j:b_{kj}\leq 0} u_j^{-b_{kj}}}{u_k}, & i = k. \end{cases} \tag{2}$$

Formulae (1) and (2) are known as (cluster) *exchange relations*. It is easy to see that the following properties hold: (i) if B is skew-symmetric, then $\mu_k(B)$ is again skew-symmetric; (ii) μ_k is an involution, that is $\mu_k \circ \mu_k = Id$.

Remark 1. From (1) it follows that $b'_{ij} = b_{ij}$ whenever b_{ik} and b_{kj} have different signs (for all i, j different from k).

Given a matrix B and an initial cluster $\mathbf{u}$ we can apply a mutation μ_k to produce another cluster, and then apply another mutation μ_p to this cluster to produce another cluster and so on. The *cluster algebra* (of geometric type) $\mathcal{A}(B)$ is the subalgebra of the field of rational functions in the cluster variables generated by the union of all clusters.

To any integer skew-symmetric matrix B one associates a *quiver* Q in the way described below. By *quiver* we mean an oriented graph with N nodes and no loops nor 2-cycles, which will be represented by an N-sided polygon whose vertices are the nodes of the quiver and are labelled by $1, 2, \ldots, N$ in clockwise direction. Each oriented edge of the polygon is weighted by a positive integer which represents the number of arrows between the corresponding nodes of the quiver.

To a given $N \times N$ skew-symmetric matrix $B = [b_{ij}]$ one associates a quiver with N nodes by defining the number of arrows from node i to node j to be b_{ij} if this number is positive (no arrows otherwise). Conversely, a quiver Q with N nodes can be identified with an $N \times N$ skew-symmetric matrix $B = [b_{ij}]$ whose entry b_{ij} is equal to the number of arrows from node i to node j minus the number of arrows from node j to node i. Whenever necessary we will denote by B_Q the matrix representing the quiver Q.

The mutation at a node k of a quiver Q is defined in such a way that the mutated quiver $\mu_k(Q)$ corresponds to the matrix $\mu_k(B_Q)$ defined by (1). This leads to the following set of rules for mutating the arrows (see [FoMa] and [Ke]):

1. Reverse all arrows which either originate or terminate at node k.
2. If Q has q arrows from node i to node k and r arrows from node k to node j, then add qr arrows to the existing arrows from node i to j.
3. If Q has q arrows from node j to node k and r arrows from node k to node i, then subtract qr arrows to the existing p arrows from node i to j (a negative result is understood as $qr - p$ arrows from j to i).

The notion of *mutation-periodicity* of a quiver introduced in [FoMa] enables to associate a system of m difference equations to an m-periodic quiver. Let us recall the definition of periodic quiver.

Consider the permutation

$$\sigma : (1, 2, \ldots, N) \longmapsto (2, 3, \ldots, N, 1) \tag{3}$$

and define σQ to be the quiver in which the number of arrows from node $\sigma(i)$ to node $\sigma(j)$ is the number of arrows in Q from node i to node j. Equivalently, the action of σ on the polygon representing Q leaves the weighted edges fixed and moves the vertices in the counterclockwise direction.

It is easy to check that the action $Q \mapsto \sigma Q$ corresponds to the conjugation $B_Q \mapsto \sigma^{-1} B_Q \, \sigma$, that is,

$$B_{\sigma Q} = \sigma^{-1} B_Q \, \sigma,$$

where, slightly abusing notation, σ also denotes the matrix

$$\sigma = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{4}$$

Definition 1. Let σ be the permutation (3) and Q a quiver with N nodes. The quiver Q is said to have *period* m if m is the smallest positive integer such that

$$\mu_m \circ \cdots \circ \mu_1(Q) = \sigma^m Q, \tag{5}$$

or equivalently if

$$\mu_m \circ \cdots \circ \mu_1(B_Q) = \sigma^{-m} B_Q \, \sigma^m. \tag{6}$$

In Figure 1 the notion of mutation-periodicity is illustrated with a 2-periodic quiver of 4 nodes (r, s, t are non-negative integers with r not equal to t; if r and t are equal the quiver is 1-periodic).

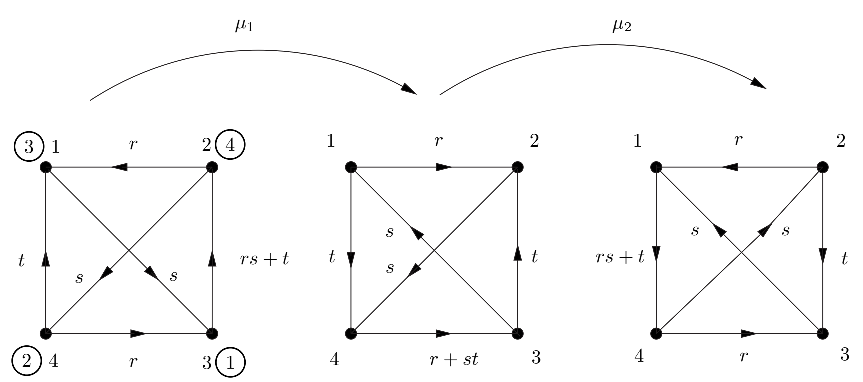

Figure 1: On the left the quivers Q and $\sigma^2 Q$ (with circled nodes) are superposed. The middle and right quivers are respectively $\mu_1(Q)$ and $\mu_2 \circ \mu_1(Q)$. The quiver Q is 2-periodic since $\sigma^2 Q = \mu_2 \circ \mu_1(Q)$.

The notion of mutation periodicity means that if a quiver is m-periodic, then after applying $\mu_m \circ \cdots \circ \mu_1$ we return to a quiver which is equivalent (up to a certain permutation) to the original quiver, and so mutating this quiver at node

$m+1$ produces an exchange relation identical in form to the exchange relation at node 1 but with a different labeling. The next mutation at node $m+2$ will produce an exchange relation whose form is identical to the exchange relation at node 2 with a different labeling, and so on. This process produces a list of exchange relations, which is interpreted as a system of difference equations.

We now use the quiver represented in Figure 1 to explain how to construct the difference equations (and the respective iteration map) corresponding to a periodic quiver. The matrix B representing the quiver in Figure 1 is

$$B = \begin{bmatrix} 0 & -r & s & -t \\ r & 0 & -rs-t & s \\ -s & rs+t & 0 & -r \\ t & -s & r & 0 \end{bmatrix}, \quad r \neq t. \tag{7}$$

Using the exchange relations (1) we have

$$\mu_1(B) = \begin{bmatrix} 0 & r & -s & t \\ -r & 0 & -t & s \\ s & t & 0 & -r-st \\ -t & -s & r+st & 0 \end{bmatrix} \Rightarrow \mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & -r & -s & rs+t \\ r & 0 & t & -s \\ s & -t & 0 & -r \\ -rs-t & s & r & 0 \end{bmatrix}$$

which satisfies (6) with $m=2$. Taking the initial cluster $\mathbf{u} = (u_1, u_2, u_3, u_4)$, the exchange relations (2) produce for the mutation at node 1 followed by the mutation at node 2 the clusters $\mu_1(\mathbf{u}) = (u_5, u_2, u_3, u_4)$ and $\mu_2 \circ \mu_1(\mathbf{u}) = (u_5, u_6, u_3, u_4)$ with

$$u_5 u_1 = u_2^r u_4^t + u_3^s \tag{8}$$

$$u_6 u_2 = u_3^t u_5^r + u_4^s, \tag{9}$$

where $u_5 = u_1'$ and $u_6 = u_2'$.

We note that relations (8) and (9) can be read directly from the first row of B and the second row of $\mu_1(B)$ respectively, and since $r \neq t$ the second equation is not a shift in the indices of the first. Moreover, as the quiver is 2-periodic, the third row of $\mu_2 \circ \mu_1(B)$ is just a permutation of the first row of B and so the equation

$$u_7 u_3 = u_4^r u_6^t + u_5^s,$$

defining the cluster $\mu_3 \circ \mu_2 \circ \mu_1(\mathbf{u}) = (u_5, u_6, u_7, u_4)$, is just a shift of 2 units in the indices of (8). Likewise, mutating next at node 4 one obtains an equation which is a 2-shift of (9). Therefore successive mutations of the 2-periodic quiver at consecutive nodes give rise to the following system of difference equations

$$\begin{cases} u_{2n+3} u_{2n-1} & = u_{2n}^r u_{2n+2}^t + u_{2n+1}^s \\ u_{2n+4} u_{2n} & = u_{2n+1}^t u_{2n+3}^r + u_{2n+2}^s \end{cases} \qquad n = 1, 2, \ldots$$

where (u_1, u_2, u_3, u_4) is the initial cluster. The iteration map corresponding to this system is

$$\varphi(u_1, u_2, u_3, u_4) = (u_3, u_4, \underbrace{\frac{u_2^r u_4^t + u_3^s}{u_1}}_{u_5}, \underbrace{\frac{u_3^t u_5^r + u_4^s}{u_2}}_{u_6}) \tag{10}$$

which, written in terms of the initial cluster $\mathbf{u}$, is

$$\varphi(\mathbf{u}) = \sigma^2 \circ \mu_2 \circ \mu_1(\mathbf{u}).$$

For future reference, let us remark that the iteration map φ associated to an m-periodic quiver is always given by

$$\varphi(\mathbf{u}) = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1(\mathbf{u}) \tag{11}$$

where $\mathbf{u}$ is the initial cluster. The associated system of difference equations is given by the iterates $\varphi^{(n)}$ of the map φ.

It was observed in [FoMa] that there are several quivers appearing in the context of quiver gauge theories which fall within the framework of mutation-periodic quivers. In particular, quivers associated to the complex cones over Hirzebruch zero and del Pezzo 1-3 surfaces are examples of periodic quivers (see [FoMa] and references therein). We list these quivers in figures 2 and 3 since they will be used, in subsequent sections, as a source of examples to illustrate the reduction procedure. For convenience we use letters for labeling the nodes of these quivers since the quiver's diagram is better pictured if the numerical labelling of the nodes does not follow the clockwise convention. We will denote by H0 the quiver associated to the Hirzebruch zero surface and by dPk the one associated to a del Pezzo k-surface.

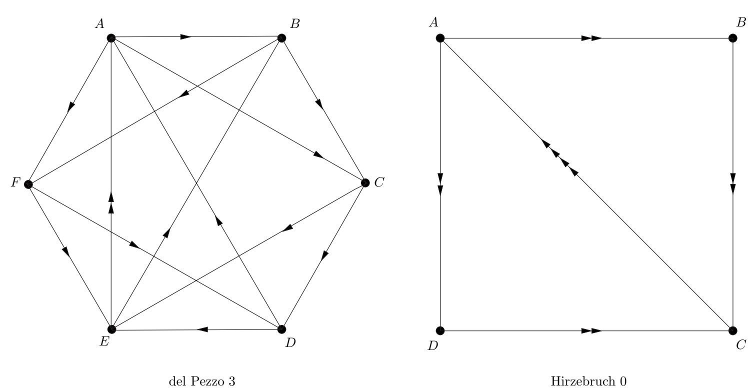

Figure 2: The quivers del Pezzo 3 (dP3) and Hirzebruch 0 (H0) are 2-periodic.

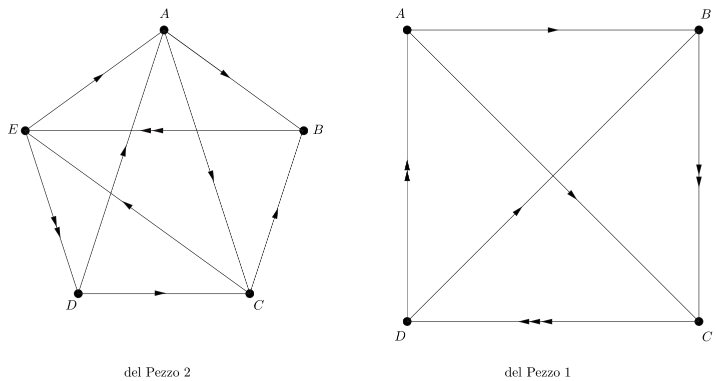

Figure 3: The quivers del Pezzo 2 (dP2) and del Pezzo 1 (dP1) are 1-periodic and associated to the Somos-5 and Somos-4 difference equations, respectively.

3 Symplectic reduction of iteration maps

Presymplectic structures associated to a cluster algebra $\mathcal{A}(B)$ were introduced in [GeShVa03] and [GeShVa05] and are of the type $\omega = \sum_{i<j} w_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}$. We call these forms *log presymplectic forms* and the coordinates u_i *log coordinates with respect to* ω.

The relevance of these log presymplectic forms in the study of iteration maps associated to 1-periodic quivers can be found for instance in [FoHo11], [FoHo14] and [HoIn], where they were used to reduce the associated iteration maps. Here we focus on the reduction of iteration maps arising from quivers of arbitrary period which, to the best of our knowledge, has not yet been done for quivers of period higher than 1.

In this section we prove that whenever the matrix representing an m-periodic quiver is singular, the corresponding iteration map can be reduced to a map on a lower dimensional space which is symplectic with respect to a *log symplectic form*. Our main result, Theorem 2, is a generalization to quivers of arbitrary period of the reduction result in [FoHo14] (see Theorem 2.6) and its proof is independent of the classification of m-periodic quivers, which is unknown for m greater than 1. The key property behind this symplectic reduction is precisely the invariance of the standard log presymplectic structure (12) under the iteration map, which is proved in Theorem 1.

If $(B, \mathbf{u})$ is the initial seed with $B = [b_{ij}]$ we call the 2-form

$$\omega = \sum_{1 \le i<j \le N} b_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}, \tag{12}$$

the *standard log presymplectic form* associated to the cluster algebra $\mathcal{A}(B)$ and the skew-symmetric matrix B will be called the *coefficient matrix of* ω.

We note that the 2-form ω, written in the coordinates $v_i = \log u_i$, has the form
$$\omega = \sum_{i<j} b_{ij} dv_i \wedge dv_j.$$
Although we will not use the notion of compatibility of a log presymplectic form with a cluster algebra, we remark that the form (12) is in fact compatible with the cluster algebra $\mathcal{A}(B)$ (see for instance Theorem 6.2 in [GeShVa10] which characterizes such compatible forms).

We now prove that the standard log presymplectic form is invariant under the iteration map of an m-periodic quiver.

Theorem 1. *Let $(B, \mathbf{u})$ be an initial seed and ω the associated standard log presymplectic form* (12)*. Then the following are equivalent:*

1. *The matrix B represents an m-periodic quiver, that is*
$$\mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1(B) = \sigma^{-m} B \sigma^m.$$
2. *$\varphi^* \omega = \omega$, where $\varphi = \sigma^m \circ \mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1$.*

The proof of the above theorem relies on the following two lemmas.

Lemma 1. *Let $(B, \mathbf{u})$ be an initial seed, ω the associated standard log presymplectic form* (12) *and σ the permutation* (4)*. Then the pullback of ω by σ is given by*
$$\sigma^* \omega = \sum_{i<j} \left(\sigma^{-1} B \sigma\right)_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}. \tag{13}$$

Proof. As $\sigma(u_1, u_2, \ldots, u_N) = (u_2, \ldots, u_N, u_1)$, the pullback of ω by σ is given by
$$\begin{aligned}\sigma^* \omega &= \sum_{1 \le i \le N-1} b_{iN} \frac{du_{i+1}}{u_{i+1}} \wedge \frac{du_1}{u_1} + \sum_{1 \le i<j \le N-1} b_{ij} \frac{du_{i+1}}{u_{i+1}} \wedge \frac{du_{j+1}}{u_{j+1}} \\ &= -\sum_{2 \le k \le N} b_{k-1,N} \frac{du_1}{u_1} \wedge \frac{du_k}{u_k} + \sum_{2 \le k<l \le N} b_{k-1,l-1} \frac{du_k}{u_k} \wedge \frac{du_l}{u_l}. \end{aligned} \tag{14}$$
Therefore the coefficient matrix of $\sigma^* \omega$ is
$$\hat{B} = \begin{bmatrix} 0 & -b_{1,N} & -b_{2,N} & \cdots & -b_{N-1,N} \\ b_{1,N} & 0 & b_{1,2} & \cdots & b_{1,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{N-2,N} & -b_{1,N-2} & -b_{2,N-2} & \cdots & b_{N-2,N-1} \\ b_{N-1,N} & -b_{1,N-1} & -b_{2,N-2} & \cdots & 0 \end{bmatrix}. \tag{15}$$
A straightforward computation shows that $\sigma^T B = \hat{B} \sigma^T$. As σ is an orthogonal matrix, this is equivalent to $\sigma^{-1} B \sigma = \hat{B}$, which concludes the proof. $\square$

Lemma 2. *Let $(B, \mathbf{u})$ be an initial seed, ω the associated standard log presymplectic form (12) and μ_k the mutation in the direction k given by (1) and (2). Then the pullback of ω by μ_k is given by*

$$\mu_k^*\omega = \sum_{i<j} (\mu_k(B))_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}. \tag{16}$$

Proof. Recall from (2) that

$$\mu_k(u_1, u_2, \dots, u_N) = \left(u_1, u_2, \dots, u_{k-1}, \underbrace{\frac{A^+ + A^-}{u_k}}_{u'_k}, u_{k+1}, \dots, u_N \right),$$

with

$$A^+ = \prod_{l: b_{kl} \geq 0} u_l^{b_{kl}}, \quad A^- = \prod_{l: b_{kl} \leq 0} u_l^{-b_{kl}}.$$

The pullback of ω by μ_k is then given by

$$\mu_k^*\omega = \sum_{k \neq i < j \neq k} b_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j} + \sum_{j>k} b_{kj} \frac{du'_k}{u'_k} \wedge \frac{du_j}{u_j} + \sum_{i<k} b_{ik} \frac{du_i}{u_i} \wedge \frac{du'_k}{u'_k}. \tag{17}$$

As

$$\frac{du'_k}{u'_k} = -\frac{du_k}{u_k} + \frac{A^+}{A^+ + A^-} \sum_{l: b_{kl} > 0} b_{kl} \frac{du_l}{u_l} - \frac{A^-}{A^+ + A^-} \sum_{l: b_{kl} < 0} b_{kl} \frac{du_l}{u_l},$$

substituting into (17) and re-arranging all the terms, we obtain

$$\mu_k^*\omega = \sum_{i<j} \hat{b}_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j} \tag{18}$$

with $\hat{B} = [\hat{b}_{ij}]$ given by:

a) for $i < k$ (resp. for $j > k$): $\hat{b}_{ik} = -b_{ik}$ (resp. $\hat{b}_{kj} = -b_{kj}$);

b) for $i < j < k$:

$$\hat{b}_{ij} = b_{ij} + \begin{cases} \frac{A^+(b_{ik}b_{kj} - b_{jk}b_{ki})}{A^+ + A^-} = 0, & \text{if } b_{ki} > 0,\ b_{kj} > 0 \\ -\frac{A^-(b_{ik}b_{kj} - b_{jk}b_{ki})}{A^+ + A^-} = 0, & \text{if } b_{ki} < 0,\ b_{kj} < 0 \\ \frac{A^+ b_{ik}b_{kj} + A^- b_{jk}b_{ki}}{A^+ + A^-} = b_{ik}b_{jk}, & \text{if } b_{ki} < 0,\ b_{kj} > 0 \\ -\frac{A^- b_{ik}b_{kj} + A^+ b_{jk}b_{ki}}{A^+ + A^-} = -b_{ik}b_{jk}, & \text{if } b_{ki} > 0,\ b_{kj} < 0 \end{cases}$$

where the equalities inside the above bracket follow from the skew-symmetry of $B = [b_{ij}]$.

Using similar arguments, the remaining entries $\hat{b}_{ij}$ are as follows.

c) for $i < k < j$:

$$\hat{b}_{ij} = b_{ij} + \begin{cases} \frac{A^+(b_{ik}b_{kj}+b_{kj}b_{ki})}{A^++A^-} = 0, & \text{if } b_{ki} > 0,\ b_{kj} > 0 \\ -\frac{A^-(b_{ik}b_{kj}+b_{kj}b_{ki})}{A^++A^-} = 0, & \text{if } b_{ki} < 0,\ b_{kj} < 0 \\ \frac{A^+b_{ik}b_{kj}-A^-b_{kj}b_{ki}}{A^++A^-} = b_{ik}b_{kj}, & \text{if } b_{ki} < 0,\ b_{kj} > 0 \\ -\frac{A^-b_{ik}b_{kj}-A^+b_{kj}b_{ki}}{A^++A^-} = -b_{ik}b_{kj}, & \text{if } b_{ki} > 0,\ b_{kj} < 0 \end{cases}$$

d) for $k < i < j$:

$$\hat{b}_{ij} = b_{ij} + \begin{cases} \frac{A^+(-b_{ki}b_{kj}+b_{kj}b_{ki})}{A^++A^-} = 0, & \text{if } b_{ki} > 0,\ b_{kj} > 0 \\ \frac{A^-(b_{ki}b_{kj}-b_{kj}b_{ki})}{A^++A^-} = 0, & \text{if } b_{ki} < 0,\ b_{kj} < 0 \\ -\frac{A^+b_{ki}b_{kj}+A^-b_{kj}b_{ki}}{A^++A^-} = b_{ik}b_{kj}, & \text{if } b_{ki} < 0,\ b_{kj} > 0 \\ \frac{A^-b_{ki}b_{kj}+A^+b_{kj}b_{ki}}{A^++A^-} = -b_{ik}b_{kj}, & \text{if } b_{ki} > 0,\ b_{kj} < 0 \end{cases}$$

Summing up, if $i = k$ or $j = k$ then $\hat{b}_{ij} = -b_{ij}$, and in any other case

$$\hat{b}_{ij} = b_{ij} + \begin{cases} 0, & \text{if } b_{ik}b_{kj} < 0 \\ b_{ik}b_{kj}, & \text{if } b_{ik} > 0,\ b_{kj} > 0 \\ -b_{ik}b_{kj}, & \text{if } b_{ik} < 0,\ b_{kj} < 0 \end{cases} \tag{19}$$

It is now easy to check that each $\hat{b}_{ij}$ coincides with b'_{ij} given in (1), which shows that $\hat{B} = [\hat{b}_{ij}]$ is precisely $\mu_k(B)$. □

Remark 2. Although we do not require compatibility of ω with the cluster algebra $\mathcal{A}(B)$, the proof of the above lemma could follow from the proof of Theorem 6.2 in [GeShVa10] which characterizes log presymplectic structures compatible with a cluster algebra. However, to the best of our knowledge, the expression (16) in Lemma 2 is new and not obvious from the setup used in the theory of cluster algebras.

Proof of Theorem 1. Using properties of the pullback operation and the identities (13) and (16) from lemmas 1 and 2, we have

$$\begin{aligned} \varphi^*\omega &= (\sigma^m \circ \mu_m \circ \ldots \circ \mu_1)^* \omega = \mu_1^* \circ \ldots \circ \mu_m^* \circ (\sigma^m)^*\omega \\ &= \sum_{i<j} \big(\mu_1(\cdots(\mu_m(\sigma^{-m}B\sigma^m)))\big)_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}. \end{aligned}$$

Therefore $\varphi^*\omega = \omega$ if and only if

$$\mu_1 \circ \cdots \circ \mu_m(\sigma^{-m} B \sigma^m) = B.$$

As any mutation is an involution we have the following equivalence

$$\varphi^*\omega = \omega \quad \Longleftrightarrow \quad \sigma^{-m} B \sigma^m = \mu_m \circ \cdots \circ \mu_1(B),$$

which concludes the proof. $\square$

We now prove that whenever the matrix B represents an m-periodic quiver, the corresponding iteration map can be reduced to a symplectic map with respect to a *log symplectic form*. Our strategy to prove this theorem relies on a classical theorem of G. Darboux for presymplectic forms and on the invariance of the standard presymplectic form under the iteration map (Theorem 1).

Theorem 2. *Let Q be an m-periodic quiver with N nodes, $(B_Q, \mathbf{u})$ the initial seed, φ the iteration map* (11) *and $2k = rank(B_Q)$. Then there exist*

i) a submersion $\pi : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+^{2k}$,

ii) a map $\hat{\varphi} : \mathbb{R}_+^{2k} \longrightarrow \mathbb{R}_+^{2k}$

such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{R}_+^N \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{R}_+^N \arrow[d, "\pi"] \\ \mathbb{R}_+^{2k} \arrow[r, "\hat{\varphi}"'] & \mathbb{R}_+^{2k} \end{tikzcd}$$

Furthermore $\hat{\varphi}$ is symplectic with respect to the canonical log symplectic form

$$\omega_0 = \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} + \cdots + \frac{df_{2k-1}}{f_{2k-1}} \wedge \frac{df_{2k}}{f_{2k}}, \tag{20}$$

that is $\hat{\varphi}^*\omega_0 = \omega_0$.

Remark 3. If B is singular then $2k < N$ and so the above theorem guarantees that the iteration map can be reduced to a lower dimensional manifold.

The proof of this theorem relies on the next proposition which in turn relies on Darboux's theorem for closed 2-forms (or presymplectic forms) of constant rank.

Proposition 1. *Let ω be a closed 2-form of constant rank $2k$ on a manifold M and $x_0 \in M$. Then there exists a set $\{g_1, g_2, \ldots, g_{2k}\}$ of $2k$ locally independent functions on M such that*

$$\omega = dg_1 \wedge dg_2 + \cdots + dg_{2k-1} \wedge dg_{2k}. \tag{21}$$

Moreover, if φ is a local diffeomorphism preserving ω, that is $\varphi^*\omega = \omega$, then each of the functions

$$\psi_1 = g_1 \circ \varphi, \quad \psi_2 = g_2 \circ \varphi, \quad \dots \quad , \psi_{2k} = g_{2k} \circ \varphi$$

depends only on $\{g_1, g_2, \dots, g_{2k}\}$.

Proof. The first statement is just Darboux's theorem for closed 2-forms of constant rank (see for instance [AbMa] or [SS]).

To prove that the functions $\psi_i = g_i \circ \varphi$ just depend on $\{g_1, g_2, \dots, g_{2k}\}$, let $\omega^{(k)}$ be the k^{th} exterior power of ω. As $\varphi^*\omega = \omega$ then

$$\varphi^*\omega^{(k)} = \omega^{(k)}. \tag{22}$$

Using the expression (21) for ω, it turns out that

$$\omega^{(k)} = k!\, dg_1 \wedge dg_2 \wedge \dots \wedge dg_{2k-1} \wedge dg_{2k},$$

and so (22) is equivalent to

$$d\psi_1 \wedge d\psi_2 \wedge \dots \wedge d\psi_{2k-1} \wedge d\psi_{2k} = dg_1 \wedge dg_2 \wedge \dots \wedge dg_{2k-1} \wedge dg_{2k}. \tag{23}$$

Now complete the set $\{g_1, g_2, \dots, g_{2k}\}$ to a full set of coordinates $\{g_1, g_2, \dots, g_N\}$ on M, and consider the Jacobian matrix of $\psi = (\psi_1, \psi_2, \dots, \psi_{2k})$,

$$J = \begin{bmatrix} \frac{\partial \psi_1}{\partial g_1} & \cdots & \frac{\partial \psi_1}{\partial g_{2k}} & \frac{\partial \psi_1}{\partial g_{2k+1}} & \cdots & \frac{\partial \psi_1}{\partial g_N} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \psi_{2k}}{\partial g_1} & \cdots & \frac{\partial \psi_{2k}}{\partial g_{2k}} & \frac{\partial \psi_{2k}}{\partial g_{2k+1}} & \cdots & \frac{\partial \psi_{2k}}{\partial g_N} \end{bmatrix}.$$

Condition (23) implies that the determinant of the leftmost $(2k) \times (2k)$ submatrix of J is equal to 1 and all the other $(2k) \times (2k)$ determinants of J are equal to 0.

Linear algebra arguments assure that the rightmost $(2k)\times(N-2k)$ submatrix of J is the zero matrix, which concludes the proof. □

Proof of Theorem 2. Since the standard log presymplectic form

$$\omega = \sum_{1 \le i < j \le N} b_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}.$$

has rank equal to $2k$ on $\mathbb{R}^N_+$, Darboux's theorem (see Proposition 1) implies the existence of functions $g_1, \dots, g_{2k}$ such that ω is given by (21). Considering

$$\begin{array}{rcl} \pi : \mathbb{R}^N_+ & \longrightarrow & \mathbb{R}^{2k}_+ \\ \mathbf{u} & \longmapsto & (\exp(g_1(\mathbf{u})), \dots, \exp(g_{2k}(\mathbf{u}))) \, , \end{array}$$

and ω_0 given by (20), we have

$$\pi^*\omega_0 = \omega. \tag{24}$$

By Theorem 1 the iteration map φ preserves ω, that is $\varphi^*\omega = \omega$. Then, again by Proposition 1, $\pi\circ\varphi(\mathbf{u})$ depends only on $\pi(\mathbf{u})$, since each $g_i\circ\varphi$ depends only on $\{g_1,\ldots,g_{2k}\}$. This is equivalent to say that $\hat{\varphi}$ exists and makes the diagram commutative.

It remains to prove that $\hat{\varphi}$ is symplectic. For this purpose, we note that the commutativity of the diagram is equivalent to

$$(\pi\circ\varphi)^*\omega_0 = (\hat{\varphi}\circ\pi)^*\omega_0 \quad \Longleftrightarrow \quad \varphi^*(\pi^*\omega_0) = \pi^*(\hat{\varphi}^*\omega_0).$$

Using (24) and the fact that φ preserves ω, we have

$$\varphi^*(\pi^*\omega_0) = \pi^*(\hat{\varphi}^*\omega_0) \quad \Longleftrightarrow \quad \pi^*\left(\omega_0 - \hat{\varphi}^*\omega_0\right) = 0.$$

As π is a submersion, this implies $\hat{\varphi}^*\omega_0 = \omega_0$. $\square$

It was shown in [GeShVa05] that any cluster algebra carries a compatible log presymplectic structure (of which the standard log presymplectic form (12) is an instance). Moreover, the same reference shows that when the coefficient matrix W of a compatible structure has not full rank there exits a rational (symplectic) manifold of dimension $2k = \operatorname{rank} W$, named *secondary cluster manifold*, whose symplectic form is called the *Weil-Petersson form* associated to the cluster algebra $\mathcal{A}(B)$.

Theorem 2 shows that, if the quiver Q is m-periodic, then its iteration map descends to a symplectic map on a secondary manifold of $\mathcal{A}(B_Q)$ with its Weil-Petersson form.

Remark 4. In the conditions of Theorem 2, the reduced map $\hat{\varphi}$ preserves the volume form

$$\Omega = \frac{1}{f_1 f_2 \ldots f_{2k}} df_1 \wedge df_2 \wedge \ldots \wedge df_{2k}.$$

This volume-preserving property is relevant to the dynamics of continuous dynamical system and its consequences for discrete systems deserve further studies.

4 Computation of the reduced iteration map

In this section we compute the reduced iteration map $\hat{\varphi}$ as well as the reduced variables $\pi(\mathbf{u})$ for several periodic quivers.

Our approach to the computation of the reduced symplectic iteration map $\hat{\varphi}$ is based on the construction of the functions $g_1,\ldots,g_{2k}$ appearing in Proposition 1. We note that these functions are given by Darboux's theorem whose proof is not constructive. However, due to the particular form of a log presymplectic form, it is possible to use a theorem of Cartan (Theorem 2.3 in [LibMa]) whose proof is constructive, in order to obtain explicit Darboux coordinates.

Consider the standard log presymplectic form ω written in coordinates $v_i = \log u_i$,

$$\omega = \sum_{1\leq i<j\leq N} b_{ij} dv_i \wedge dv_j,$$

and let $2k$ be its rank. Cartan's Theorem says that there exist $2k$ functions g_i depending linearly on the v_i variables such that

$$\omega = \sum_{1\leq i<j\leq 2k} dg_i \wedge dg_j.$$

We recall the main steps of the proof of Cartan's Theorem, which explicitly produces the functions g_i.

Reordering if necessary the v_i-coordinates, we can assume that $b_{12} \neq 0$. Let

$$g_1 = \frac{1}{b_{12}} \sum_{k=1}^{N} b_{1k} v_k \quad \textit{and} \quad g_2 = \sum_{k=1}^{N} b_{2k} v_k, \tag{25}$$

so that

$$dg_1 \wedge dg_2 = b_{12} dv_1 \wedge dv_2 + \sum_{i=3}^{N} b_{1i} dv_1 \wedge dv_i + \sum_{j=3}^{N} b_{2j} dv_2 \wedge dv_j + \alpha$$

where α depends only on $\{v_3, \dots, v_N\}$. Then the 2-form

$$\tilde{\omega} = \omega - dg_1 \wedge dg_2$$

is a closed 2-form on the $(N-2)$-dimensional space with coordinates $\{v_3, \dots, v_N\}$, and rank$(\tilde{\omega}) = 2k-2$.

If rank$(\omega) = 2$ then $\tilde{\omega} = 0$ and $\omega = dg_1 \wedge dg_2$. Otherwise, the previous procedure is repeated, replacing ω by $\tilde{\omega}$. After k steps all the functions $g_1, \dots, g_{2k}$ will be obtained.

As each function g_i is a linear function of the variables $v_i = \log u_i$, it can be written in the form

$$g_i(u_1, \dots, u_N) = \log(f_i(u_1, \dots, u_N)), \quad i = 1, 2, \dots, N.$$

The submersion π in Theorem 2 is then given by

$$\pi(u_1, \dots, u_N) = (f_1(\mathbf{u}), \dots, f_{2k}(\mathbf{u})),$$

and the reduced iteration map is

$$\hat{\varphi}(f_1, \dots, f_{2k}) = (f_1 \circ \varphi, \dots, f_{2k} \circ \varphi),$$

which, by Theorem 2, is symplectic with respect to the canonical log symplectic form:

$$\omega_0 = \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} + \cdots + \frac{df_{2k-1}}{f_{2k-1}} \wedge \frac{df_{2k}}{f_{2k}}.$$

We remark that although it is possible to take for reduced variables the ones obtained from a basis of the range of B, the reduced map in these variables is not necessarily symplectic with respect to the canonical log symplectic form.

In the next examples we will consider several periodic quivers and reduce their iteration maps using the Darboux coordinates provided by the application of the aforementioned proof of Cartan's theorem.

Example 1. A 6-node quiver

Consider a quiver with 6 nodes represented by the matrix

$$B = \begin{bmatrix} 0 & -r & s & -p & s & -t \\ r & 0 & -t-rs & s & -p-rs & s \\ -s & t+rs & 0 & -r-s(t-p) & s & -p \\ p & -s & r+s(t-p) & 0 & -t-rs & s \\ -s & p+rs & -s & t+rs & 0 & -r \\ t & -s & p & -s & r & 0 \end{bmatrix} \tag{26}$$

where r,s,t,p are non-negative integers. Using (1) and (4) it is easy to check that one has $\sigma^{-1}B\sigma = \mu_1(B)$ if r and t are equal and $\sigma^{-2}B\sigma^2 = \mu_2 \circ \mu_1(B)$ otherwise. That is, the quiver is 1-periodic if $r = t$ and 2-periodic otherwise.

Taking $r \neq t$ and $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6)$, as the second row of $\mu_1(B)$ is $(-r, 0, -t, s, -p, s)$, the quiver gives rise to the following system of difference equations

$$\begin{cases} u_{2n+5}u_{2n-1} & = u_{2n}^r u_{2n+2}^p u_{2n+4}^t + u_{2n+1}^s u_{2n+3}^s \\ & \\ u_{2n+6}u_{2n} & = u_{2n+1}^t u_{2n+3}^p u_{2n+5}^r + u_{2n+2}^s u_{2n+4}^s, \end{cases} \quad n = 1, 2, \ldots \tag{27}$$

The respective iteration map is:

$$\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_3, u_4, u_5, u_6, \underbrace{\frac{u_2^r u_4^p u_6^t + u_3^s u_5^s}{u_1}}_{u_7}, \underbrace{\frac{u_3^t u_5^p u_7^r + u_4^s u_6^s}{u_2}}_{u_8}). \tag{28}$$

del Pezzo 3 quiver: The quiver dP3 in Figure 2 is 2-periodic and is represented by the matrix (26) with $r = s = p = 1, t = 0$. In fact, taking $(A, B, C, D, E, F) = (2, 4, 6, 3, 5, 1)$ for the nodes, dP3 is represented by the matrix

$$B = \begin{bmatrix} 0 & -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & -2 & 1 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 2 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}. \tag{29}$$

This matrix has rank 2 and the corresponding iteration map (28) is

$$\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_3, u_4, u_5, u_6, \underbrace{\frac{u_2u_4 + u_3u_5}{u_1}}_{u_7}, \underbrace{\frac{u_5u_7 + u_4u_6}{u_2}}_{u_8}).$$

Taking into account the first and second rows of B, it follows from (25) that:

$$\omega = d\left(v_2 - v_3 + v_4 - v_5\right) \wedge d\left(v_1 - v_3 + v_4 - 2v_5 + v_6\right) = dg_1 \wedge dg_2,$$

where $v_i = \log u_i$. As

$$g_1 = v_2 - v_3 + v_4 - v_5 = \log\left(\frac{u_2u_4}{u_3u_5}\right)$$
$$g_2 = v_1 - v_3 + v_4 - 2v_5 + v_6 = \log\left(\frac{u_1u_4u_6}{u_3u_5^2}\right),$$

the projection π in Theorem 2 is

$$\pi(\mathbf{u}) = (f_1, f_2) = \left(\frac{u_2u_4}{u_3u_5}, \frac{u_1u_4u_6}{u_3u_5^2}\right)$$

and the reduced log symplectic form is the canonical one: $\omega_0 = \frac{df_1}{f_1} \wedge \frac{df_2}{f_2}$.

The reduced iteration map is just $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2)$ with $\hat{\varphi}_i = f_i \circ \varphi$. Computing these compositions as functions of f_1 and f_2, we obtain

$$(\hat{\varphi}_1, \hat{\varphi}_2) = \left(\frac{f_2}{1+f_1}, \frac{f_2(1+f_1+f_2)}{f_1(1+f_1)^2}\right). \tag{30}$$

We note that this reduced map $\hat{\varphi}$ admits a first integral, that is a function I such that $I \circ \hat{\varphi} = I$, namely

$$I(f_1, f_2) = \frac{f_2\left(1+f_1^2+f_2\right) + f_1\left(1+f_1\right)^2}{f_1f_2}. \tag{31}$$

Consequently each orbit of the reduced map stays on a level set of I, which in the 2-dimensional case implies integrability.

Example 2. A 5-node quiver

We now consider a quiver of 5 nodes represented by the following matrix

$$B = \begin{bmatrix} 0 & -r & 1 & 1 & -s \\ r & 0 & -r-s & 1-r & 1 \\ -1 & r+s & 0 & -r-s & 1 \\ -1 & -r-1 & r+s & 0 & -r \\ s & -1 & -1 & r & 0 \end{bmatrix} \tag{32}$$

where r and s are non-negative integers. This quiver is 1-periodic if $r = s$ and 2-periodic otherwise. In the latter case, as the second row of $\mu_1(B)$ is $(-r, 0, -s, 1, 1)$, the iteration map becomes

$$\varphi(u_1, u_2, u_3, u_4, u_5) = (u_3, u_4, u_5, \underbrace{\frac{u_2^ru_5^s + u_3u_4}{u_1}}_{u_6}, \underbrace{\frac{u_3^su_6^r + u_4u_5}{u_2}}_{u_7}). \tag{33}$$

We remark that this quiver was considered in [FoMa] where the corresponding system of difference equations was described as a difference equation of order 2 in the plane subject to a boundary condition. Our interpretation does not require any boundary condition.

The rank of B is equal to 4 unless $(r, s) = (1, 1)$, in which case the quiver would be 1-periodic. To keep the expressions simpler we consider $r = 1$ and $s \neq 1$ (the general case is done in an entirely analogous way). In this case we have

$$u_6 = \frac{u_2 u_5^s + u_3 u_4}{u_1}, \qquad \frac{u_3^s u_6 + u_4 u_5}{u_2}.$$

Considering the first row of B divided by -1 and its second row, respectively the vectors $(0, 1, -1, -1, s)$ and $(1, 0, -(1 + s), 0, 1)$, we construct the form

$$\tilde{\omega} = d(v_2 - v_3 - v_4 + s v_5) \wedge d(v_1 - (1 + s)v_3 + v_5),$$

where $v_i = \log u_i$. The standard log presymplectic form ω has rank 4 and $\omega - \tilde{\omega}$ is a form of rank 2. More precisely

$$\omega - \tilde{\omega} = -(s^2 + s - 2) dv_3 \wedge dv_5,$$

and hence

$$\omega = d(v_2 - v_3 - v_4 + s v_5) \wedge d(v_1 - (1 + s)v_3 + v_5) - (s^2 + s - 2) dv_3 \wedge dv_5. \quad (34)$$

Taking

$$f_1 = \frac{u_2 u_5^s}{u_3 u_4}, \quad f_2 = \frac{u_1 u_5}{u_3^{1+s}}, \quad f_3 = \frac{1}{u_3^{s^2+s-2}}, \quad f_4 = u_5,$$

then (34) reduces to the canonical log symplectic form ω_0. Computing the reduced symplectic map $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4)$, with $\hat{\varphi}_i = f_i \circ \varphi$, we obtain

$$\begin{aligned}
\hat{\varphi}_1 &= \frac{(1 + f_1 + f_2)^s}{f_1^s f_2^{s-1}(1 + f_1)} f_4^{s^2+s-2} \\
\hat{\varphi}_2 &= \frac{1 + f_1 + f_2}{f_1 f_2} \\
\hat{\varphi}_3 &= \frac{1}{f_4^{s^2+s-2}} \\
\hat{\varphi}_4 &= \frac{1 + f_1 + f_2}{f_1 f_2} f_3^{\frac{1}{s^2+s-2}} f_4^{1+s}.
\end{aligned}$$

del Pezzo 2 quiver: The quiver dP2 in Figure 2 is a 1-periodic quiver, represented by the matrix B in (32) if we take $r = s = 1$ and number the nodes $(A, B, C, D, E) = (5, 4, 1, 2, 3)$. The corresponding iteration map is

$$\varphi(u_1, u_2, u_3, u_4, u_5) = (u_2, u_3, u_4, u_5, \frac{u_2 u_5 + u_3 u_4}{u_1}).$$

This map corresponds to the so-called Somos-5 difference equation. The reduction of this map has been done in [FoHo11] and in [CrSD12] where it was shown that the reduced map (in Darboux coordinates) is

$$\hat{\varphi}(f_1, f_2) = \left(f_2, \frac{1 + f_2}{f_1 f_2} \right).$$

This map belongs to the widely studied QRT family of integrable maps (see for instance [QRT], [Duist], [IaRo] and references therein).

Example 3. A 3-periodc quiver with 3 nodes

Our last example is a 3-periodic quiver with 3 nodes. Using Remark 1 and the fact that $\sigma^3 = Id$, it is easy to construct such a quiver. Indeed if r, s and t are positive integers which are not all equal, the matrix

$$B = \begin{bmatrix} 0 & r & s \\ -r & 0 & t \\ -s & -t & 0 \end{bmatrix} \tag{35}$$

represents a 3-periodic quiver since $B = \mu_3 \circ \mu_2 \circ \mu_1(B)$. As the second row of $\mu_1(B)$ is $(r, 0, t)$ and the third row of $\mu_2 \circ \mu_1(B)$ is $(s, t, 0)$, the associated system is given by

$$\begin{cases} u_{3n+1}u_{3n-2} & = u_{3n-1}^r u_{3n}^s + 1 \\ u_{3n+2}u_{3n-1} & = u_{3n}^t \, u_{3n+1}^r + 1 \\ u_{3n+3}u_{3n} & = u_{3n+1}^s u_{3n+2}^t + 1 \end{cases} \qquad n = 1, 2, \ldots \tag{36}$$

The iteration map $\varphi(\mathbf{u}) = \mu_3 \circ \mu_2 \circ \mu_1(\mathbf{u})$ is

$$\varphi(u_1, u_2, u_3) = \left(\frac{u_2^r u_3^s + 1}{u_1}, \frac{u_3^t u_4^r + 1}{u_2}, \frac{u_4^s u_5^t + 1}{u_3} \right) = (u_4, u_5, u_6),$$

and the log presymplectic form (12) can be written as

$$\omega = d(v_2 + s/rv_3) \wedge d(-rv_1 + tv_3) = d \log \left(u_2 u_3^{s/r} \right) \wedge d \log \left(\frac{u_3^t}{u_1^r} \right),$$

where $v_i = \log u_i, i = 1, 2, 3$. Taking

$$f_1 = u_2 u_3^{s/r}, \quad f_2 = \frac{u_3^t}{u_1^r},$$

the form ω reduces to the canonical log symplectic form ω_0 in (20), and the map π in Theorem 2 is

$$\pi(u_1, u_2, u_3) = \left(u_2 u_3^{s/r}, \frac{u_3^t}{u_1^r} \right).$$

The system (36) associated to the quiver represented by (35) was obtained under the standard framework of the theory of cluster algebras. However, the proof of Lemma 2 shows that Theorem 2 still applies to the following iteration map

$$\begin{aligned} \varphi(u_1, u_2, u_3) &= \left(\frac{u_2^r u_3^s}{u_1}, \frac{u_3^t u_4^r}{u_2}, \frac{u_4^s u_5^t}{u_3} \right) \\ &= \left(\frac{u_2^r u_3^s}{u_1}, \frac{u_2^{r^2-1} u_3^{t+rs}}{u_1^r}, \frac{u_2^{tr^2-t+rs} u_3^{t^2+trs+s^2-1}}{u_1^{s+tr}} \right). \end{aligned} \tag{37}$$

In fact, after some algebraic manipulations we obtain the reduced map of (37)

$$(\hat{\varphi}_1, \hat{\varphi}_2) = \left(f_1^{\frac{r^3-r+rs^2+r^2st-st}{r}} f_2^{\frac{s^2+trs+r}{r^2}}, f_1^{trs+r^2t^2-t^2-r^2} f_2^{\frac{t^2r+st-r}{r}} \right),$$

which depends only on (f_1, f_2) as it should.

5 Reduction via Poisson Structures

Poisson structures which are compatible with a cluster algebra were introduced in [GeShVa03] and have been applied, for instance, to Grassmannians, to directed networks and even to the theory of integrable systems such as Toda flows in GL_n (see for instance [GeShVa10]). The Poisson structures considered in this framework are given by the simplest possible kind of homogeneous quadratic brackets.

In the context of the reduction of iteration maps arising from periodic quivers, the relevant notion is not the compatibility of the Poisson structure with the cluster algebra but that of a Poisson structure for which the iteration map φ is a Poisson map as we will show in Theorem 3 below.

We consider Poisson brackets of the form

$$\{u_i, u_j\} = c_{ij} u_i u_j, \quad i, j \in \{1, 2, \ldots, N\} \tag{38}$$

where $C = [c_{ij}]$ is a skew-symmetric matrix and $\mathbf{u} = (u_1, \ldots, u_N)$ is the initial cluster. Recall that a map $\varphi : M \to M$ is said to be *a Poisson map with respect to a Poisson bracket* $\{\ ,\ \}$, or the *bracket* $\{\ ,\ \}$ *is said to be invariant under* φ, if

$$\{f \circ \varphi, g \circ \varphi\} = \{f, g\} \circ \varphi, \quad \forall f, g \in C^\infty(M).$$

Given a Poisson structure for which the iteration map φ is a Poisson map, the functions f_i (known as *Casimir functions*) such that

$$\{f_i, f\} = 0, \quad \forall f \in C^\infty(M)$$

will provide "reduced" variables. This statement, and the reduction of cluster-iteration maps via Poisson structures, are fully justified with the next theorem concerning general Poisson structures and the lemma following it.

Theorem 3. *Let $\varphi : M \to M$ be a differentiable map with differentiable inverse, and $\{\ ,\ \}$ a Poisson structure on M, invariant under φ, of non maximal constant rank. If $\{f_1, \ldots, f_k\}$ is a maximal independent set of Casimir functions for $\{\ ,\ \}$, then there are functions $\hat{\varphi}_1, \ldots, \hat{\varphi}_k$ such that*

$$f_i \circ \varphi = \hat{\varphi}_i(f_1, \ldots, f_k), \quad i = 1, \ldots, k.$$

Proof. As φ is a Poisson diffeomorphism each function $f_i \circ \varphi$ is again a Casimir:

$$\{f_i \circ \varphi, f\} = \{f_i, f \circ \varphi^{-1}\} \circ \varphi = 0.$$

As the set of Casimir functions is the center of the Lie algebra $(C^\infty(M), \{\,,\})$, the hypothesis on the set $\{f_1, \ldots, f_k\}$ implies that each Casimir is a function of $f_1, \ldots, f_k$. $\square$

In the particular case where $\{\,,\}$ is the homogeneous quadratic Poisson structure (38) we can show that the bracket $\{\,,\}$ has constant rank on $M = \mathbb{R}_+^N$, equal to the rank of C, and so the theorem can be applied whenever $\ker C \neq \{0\}$. Moreover the next lemma provides a very simple way of computing an independent set of Casimirs.

Lemma 3. *Let* $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^N$, $\mathbf{u} = (u_1, \ldots, u_N)$ *and* $\mathbf{u^k} = u_1^{k_1} u_2^{k_2} \cdots u_N^{k_N}$. *Then* $\mathbf{u^k}$ *is a Casimir for the Poisson bracket*

$$\{u_i, u_j\} = c_{ij} u_i u_j$$

if and only if $\mathbf{k} \in \ker C$, *where* $C = [c_{ij}]$.

Proof. For any $i \in \{1, \ldots, N\}$ we have

$$\{u_i, \mathbf{u^k}\} = \sum_{j=1}^{N} c_{ij} k_j \mathbf{u^k} u_i = \mathbf{u^k} u_i \sum_{j=1}^{N} c_{ij} k_j = \mathbf{u^k} u_i \, (C\mathbf{k})_i \, .$$

Thus $\mathbf{u^k}$ is a Casimir if and only if $C\mathbf{k} = \mathbf{0}$. $\square$

We remark that a Poisson structure of the form (38) for which φ is a Poisson map might not exist. For instance, for the quiver of 5 nodes in Example 2 we can show that such a Poisson structure exists if and only if $r = s = 1$, which corresponds to the 1-periodic dP2 quiver (Somos-5 difference equation). A Poisson bracket for the dP2 quiver is given by

$$\{u_i, u_j\} = (j - i) u_i u_j \tag{39}$$

with $i, j \in \{1, 2, 3, 4, 5\}$ (see [FoHo11] and [CrSD12]).

We now reduce the iteration map φ of some 2-periodic quiver by providing a Poisson structure which is invariant under φ.

Example 4. A 2-periodic quiver with 4 nodes

For the 2-periodic quiver of 4 nodes represented in Figure 1 and given by the matrix B in (7) there exists a Poisson bracket of the form (38) for which the iteration map φ is a Poisson map. In fact, consider the Poisson bracket (38) with coefficient matrix $C = [c_{ij}]$ given by

$$C = \begin{bmatrix} 0 & r & s & rs+t \\ -r & 0 & t & s \\ -s & -t & 0 & r \\ -rs-t & -s & -r & 0 \end{bmatrix}. \tag{40}$$

One can check that the iteration map φ in (10) is a Poisson map with respect to this bracket, since it satisfies the relations $\{u_i \circ \varphi\,,\, u_j \circ \varphi\} = \{u_i, u_j\} \circ \varphi$, for all $i, j \in \{1, 2, 3, 4\}$.

As the matrices B and C have the same determinant

$$\det C = \det B = (r^2 - s^2 + rst + t^2)^2,$$

this Poisson structure can be used to reduce φ to a map in two variables when $\det B = 0$.

Hirzebruch 0 quiver: If we number the nodes of the quiver H0 in Figure 2 as $(A, B, C, D) = (2, 1, 3, 4)$ the matrix B representing it is given by (7) with $r = s = 2$ and $t = 0$ and so the matrix B and the corresponding matrix C for the Poisson tensor are:

$$B = \begin{bmatrix} 0 & -2 & 2 & 0 \\ 2 & 0 & -4 & 2 \\ -2 & 4 & 0 & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 2 & 2 & 4 \\ -2 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ -4 & -2 & -2 & 0 \end{bmatrix}. \tag{41}$$

The iteration map (10) is $\varphi(u_1, u_2, u_3, u_4) = (u_3, u_4, u_5, u_6)$ with

$$u_5 = \frac{u_2^2 + u_3^2}{u_1}, \qquad u_6 = \frac{u_5^2 + u_4^2}{u_2}. \tag{42}$$

Both matrices B and C have rank equal to 2, and a basis for the kernel of C is formed by the vectors $(1, -2, 0, 1)$ and $(0, -1, 1, 0)$. By Lemma 3 we have the following Casimirs

$$f_1 = \frac{u_1 u_4}{u_2^2}, \qquad f_2 = \frac{u_3}{u_2}.$$

From Theorem 3 the reduced iteration map $\hat{\varphi}$ is obtained by computing the compositions $f_i \circ \varphi$ as functions of f_1 and f_2, which in this case gives

$$\hat{\varphi}(f_1, f_2) = \left(f_2 \left(1 + \frac{(1 + f_2^2)^2}{f_1^2} \right), \frac{1 + f_2^2}{f_1} \right).$$

We note that this 2-dimensional reduced map is also integrable since it admits the following first integral:

$$I(f_1, f_2) = \frac{(f_1 + f_2)(1 + f_2^2)^2 + f_1^2 f_2}{f_1 f_2 (1 + f_2^2)}. \tag{43}$$

del Pezzo 1 quiver: If we number the nodes of the quiver dP1 in Figure 2 as $(A, B, C, D) = (4, 1, 3, 2)$ the quiver is represented by the matrix B in (7) with $r = t = 1$ and $s = 2$. This is a 1-periodic quiver whose iteration map

$$\varphi(u_1, u_2, u_3, u_4) = \left(u_2, u_3, u_4, \frac{u_2 u_4 + u_3^2}{u_1} \right)$$

represents the Somos-4 difference equation:

$$u_{n+4} u_n = u_{n+3} u_{n+1} + u_{n+2}^2.$$

The coefficient matrix (40) is

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix},$$

which is a 4-dimensional instance of the Poisson bracket (39).

If one chooses the basis $\{(1,-2,1,0),(0,1,-2,1)\}$ for the kernel of C, the Casimirs are

$$f_1 = \frac{u_1 u_3}{u_2^2}, \qquad f_2 = \frac{u_2 u_4}{u_3^2}$$

and the iteration map reduces to the following map

$$\hat{\varphi}(f_1, f_2) = \left(f_2, \frac{1+f_2}{f_1 f_2^2} \right).$$

This map was obtained in [Ho07] using this Poisson structure, and like the Somos-5 reduced map is another example of an integrable (QRT) map.

Example 5. The 2-periodic quiver with 6 nodes revisited

Let us consider the 2-periodic quiver with 6 nodes represented by the matrix B in (26) with $r \neq t$ and iteration map

$$\varphi(u_1,u_2,u_3,u_4,u_6) = (u_3,u_4,u_5,u_6, \underbrace{\frac{u_3^s u_5^s + u_2^r u_4^p u_6^t}{u_1}}_{u_7}, \underbrace{\frac{u_4^s u_6^s + u_3^t u_5^p u_7^r}{u_2}}_{u_7}).$$

The rank of B is not maximal when $p = r + t$ and in this case there exists a Poisson bracket of the form (38) for which the iteration map φ is a Poisson map, namely the Poisson tensor with coefficient matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

This matrix C has rank 2 and a basis for its kernel is

$$\{(1,0,1,0,0,0),(0,1,0,1,0,0),(0,0,1,0,1,0),(0,0,0,1,0,1)\}.$$

Therefore, we can take for reduced variables

$$f_1 = u_1 u_3 \qquad f_2 = u_2 u_4 \qquad f_3 = u_3 u_5 \qquad f_4 = u_4 u_6.$$

The computation of the reduced iteration map gives

$$\hat{\varphi}(f_1, f_2, f_3, f_4) = (f_3, f_4, \frac{f_3\,(f_2^r f_4^t + f_3^s)}{f_1}, \frac{f_4\,\left(f_3^{r+t}(f_2^r f_4^t + f_3^s)^r + f_1^r f_4^s\right)}{f_1^r f_2}). \quad (44)$$

We note that, due to the particular form of $\hat{\varphi}$, this map is the iteration map of the following system of difference equations

$$\begin{cases} f_{2n+3} &= \frac{f_{2n+1}(f_{2n}^r f_{2n+2}^t + f_{2n+1}^s)}{f_{2n-1}} \\ f_{2n+4} &= \frac{f_{2n+2}(f_{2n+1}^t f_{2n+3}^r + f_{2n+2}^s)}{f_{2n}} \end{cases} \qquad n = 1, 2, \ldots$$

We remark that in this case ($p = r+t$) the matrix B is singular and therefore Theorem 2 also reduces φ to a map on a lower dimensional manifold. However, if B has rank 2 the presymplectic approach will produce a map defined on a lower dimensional manifold than the Poisson approach. This is the case of the dP3 quiver presented in Example 1 for which the reduced map is the 2-dimensional map (30).

5.1 Concluding remarks

Using the presymplectic approach provided by Theorem 2, the cluster iteration map associated to an m-periodic quiver Q can be reduced to a symplectic map with respect to the canonical log symplectic form. If the matrix $B = B_Q$ has rank $2k$, then this reduced iteration map is defined on a $2k$-dimensional space.

A cluster iteration map φ for which there exists a Poisson structure with non maximal rank of the form (38) leaving φ invariant can also be reduced by using such a Poisson structure. In this case the reduced map is defined on a space whose dimension is equal to the dimension of the kernel of the coefficient matrix C of the Poisson structure.

As $\dim\ker(C)$ and $\operatorname{rank}(B)$ are not necessarily equal, the two approaches can lead not only to different reduced maps but even to reduced maps defined on spaces of different dimensions. For instance, besides the dP3 case referred above, the presymplectic approach for the Somos-5 iteration map leads to a reduced (symplectic) map in two variables, while the Poisson approach via the bracket (39) leads to a reduced map in three variables.

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