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Algebraic Properties of Generalized Fibonacci Sequence via Matrix Methods

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Abstract: Over the past centuries, the fascination over the Fibonacci sequences and their generalizations has been shown by mathematicians and the wider scientific community. While most of the known algebraic properties of these sequences were found based on the well-known Binet formula, new discoveries seemed to have been dwarfed by the nature of the complexity of its methodology. Recently, matrix method has become a popular tool among many researchers working on Fibonacci related sequences. In this study, we investigate the generalized Fibonacci sequence by employing two different matrix methods, namely, the method of diagonalization and the method of matrix collation, making use of several generating matrices. We obtained some new algebraic properties and the sum of the generalized fibonacci sequence with different indices.

Key words: Generalized fibonacci sequence, Binet formula, matrix methods, sequences, indices

INTRODUCTION

Over the past centuries, there has been a strong interest in the research related to Fibonacci sequence. The Fibonacci sequence is widely known and has much presence in our daily life, for example in nature where it is featured in the number of petals in the flowers and they are often linked to the golden ratio of the Fibonacci sequence (Hoggatt, 1969). There have been continuous research on Fibonacci and its related sequences. Some of the early interesting properties found include the "Pythagorean" property and the geometrical-paradox property (Horadam, 1967). From past research compilation shown in Table 1, there is a clear gravitation towards using the Binet formula approach but new discoveries seemed to have been dwarfed by its complexity. Hence, there is a strong motivation to search for new and improved methodologies and approaches. Recently, matrix method has become a popular and important tool among many researchers working on $W_{n+1} = pW_n + qW_{n-1}$ Fibonacci related sequences. A second order linear recurrence sequence is defined by the relation:

$$W_{n+1} = pW_n + qW_{n-1} \text{ and } W_0 = a,\ W_1 = b$$

Where:
a, b = Non-negative integers
p, q = Positive integers

Related to the above sequence (also known as the Horadam sequence (Horadam, 1965) are few special sequences shown below:

(p-Fibonacci): $F_{n+1} = pF_n + F_{n-1};\ F_0 = 0,\ F_1 = 1$
(p and-Lucas): $L_{n+1} = pL_n + L_{n-1};\ L_0 = 2,\ L_1 = p$
(Generalized Fibonacci): $U_{n+1} = pU_n + qU_{n-1};\ U_0 = 0,\ U_1 = 1$

Er (1984) and recently Kilic (2007) studied the Fibonacci sequence and p and-Fibonacci sequence, respectively by matrixmethods. In this study, we investigate the Generalized Fibonacci Sequence (GFS) by employing two different matrixmethods and derive some algebraic properties and obtain the sum of GFS with different indices by making use of several generating matrices.

MATERIALS AND METHODS

Binet formula approach: Table 1 summarizes some known algebraic properties obtained by researchers from the past (Singh *et al.*, 2014; Kalman, 1982; Bolat and Kose, 2010). Most of these results were mainly discovered using the Binet formula approach with the exception of few which made use of matrix algebra techniques. However, new discoveries seemed to have been restricted by the complex nature of the Binet Formula approach. Hence, there is a need to pay due attention to new methods or approaches and explore new frontier to attain further progress and knowledge advancement to research in the algebraic properties of Fibonacci and related sequence. we present two such approaches and obtained some algebraic properties and the sums of the GFS with different indices by making use of several generating matrices.

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Table 1: Summary of past research findings on p-Fibonacci sequence, p-Lucas sequence and generalized Fibonacci sequence

Name of sequense	Algebraic properties	Approach	References
p-fibonacci	Divisibility properties	Binet formula	Bolat and Kose (2010)
	Sums of odd and even terms	Binet formula	Panwar *et al.* (2014)
	Recursive application of geometrical transformation	Binet formula	Catarino (2014)
	identities	matrix methods	Falcon and Plaza (2007)
p-Lucas	Catalan's, gelin-cesaro's and d'ocagne identities	Binet formula	Falcon (2011)
	Convolution theorem		
	Binomial transform	Binet formula	Deepika
	Sum and alternating sum with arithmetic indices	Binet formula	Sengpanit *et al.* (2015)
	odd, even terms and sum of product	Binet formula	Cerin (1991)
Generalized Fibonacci	Matrix representation	Binet formula	Kalman (1982)
	sum of generalized fibonacci sequence	Matrix method	Kilic (2007)
	finite sums with lucas numbers	Binet formula	Kilic *et al.* (2011)
	divisibility properties	Matrix method	Yalciner (2013)
	odd and even sums	Matrix method	Ho and Chong (2014)

Two different approaches: In this study, we will consider two different matrix methods to find some algebraic properties and sums of the GFS without using the Binet Formula approach. They are the method of diagonalization and the method of matrix collation. First we derive some recursive formulas and identities for the GFS in. In we gave some generating matrices (Theorems to facilitate our proofs. Three different versions of the proofs of Theorem 4 will be illustrated.

Recursive formula and identities for the GFS: Before we start to discuss our main results, some important preparatory results on the recursive formula and identities will be developed. The collection of these results follows.

Proposition 1: For positive integers $n, L_n = U_{n+1} + qU_{n-1}$.

Proof: The proof is by induction. The result is true for n = 1. Assume it is also true for n = k. Now, for n = k+1:

$$\begin{aligned} L_{k+1} &= pL_k + qL_{k-1} \\ &= p[U_{k+1} + qU_{k-1}] + q[U_k + qU_{k-2}] \\ &= pU_{k+1} + qU_k + q[pU_{k-1} + qU_{k-2}] \\ &= U_{k+2} + qU_k \end{aligned}$$

Proposition 2 (Cassini's identity): For non-negative integers, $n, U_{n+2}U_n - U_{n+1}^2 = (-1)^{n+1} q^n$.

Proof: The identity is true for n = 0 because $U_{n+1} = pU_n + qU_{n-1}; U_0 = 0, U_1 = 1$ assume the identity holds true for n = k and. Then:

$$U_{n+2}U_n - U_{n+1}^2 = (-1)^{n+1} q^n$$

For n = k+1, we have:

$$\begin{aligned} & U_{k+3}U_{k+1} - U_{k+2}^2 \\ &= (pU_{k+2} + qU_{k+1})U_{k+1} - U_{k+2}^2 \\ &= U_{k+2}[pU_{k+1} - U_{k+2}] + qU_{k+1}^2 \\ &= U_{k+2}(-qU_k) + qU_{k+1}^2 \\ &= -q[U_{k+2}U_k - U_{k+1}^2] \\ &= -q[(-1)^{k+1}q^k] \\ &= (-1)^{k+2}q^{k+1} \end{aligned}$$

This completes the proof.

Proposition 3 (d'Ocagne Identity): For positive integers m, n where, $m \geq n\ U_{m+1}U_n - U_m U_{n+1} = (-1)^{n+1} q^n U_{m-n}$.

Proof: t is clear that the result is true for m = n (the fact that) $U_0 = 0$ and for m = n+1 (this follows from proposition 2). Suppose the result is also true for m = n+k-1 and m = n+k where k≥1. Now, for the case m = n+k +1, we have:

$$\begin{aligned} & U_{n+k+2}U_n - U_{n+k+1}U_{n+1} \\ &= (pU_{n+k+1} + qU_{n+k})U_n - (pU_{n+k} + qU_{n+k+1})U_{n+1} \\ &= p[(-1)^{n+1}q^n U_k] + q[(-1)^{n+1}q^n U_{k-1}] \\ &= (-1)^{n+1}q^n[pU_k + q^n U_{k-1}] \\ &= (-1)^{n+1}q^n U_{k+1} \end{aligned}$$

This completes the proof.

RESULTS AND DISCUSSION

Generating matrix: The following result (Theorem 1) can be easily verified and is written without proof. Theorem 1 For positive integers n:

$$\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_{n+1} & qU_n \\ U_n & qU_{n-1} \end{pmatrix}$$

Proof (omitted): Next, we verify the following Convolution Theorem (Proposition 4).

Proposition 4: For positive integers m and $n, U_{m+n} = U_m U_{n+1} + qU_{m-1} U_n$.

Proof: Using the fact that $A^m A^n = A^{m+n}$ for any matrix A and and then applying Theorem 1, we see that:

$$\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_{m+1} & qU_m \\ U_m & qU_{m-1} \end{pmatrix} \begin{pmatrix} U_{n+1} & qU_n \\ U_n & qU_{n-1} \end{pmatrix} = \begin{pmatrix} U_{m+1} U_{n+1} + qU_m U_n & q(U_{m+1} U_n + qU_m U_{n-1}) \\ U_m U_{n+1} + qU_{m-1} U_n & q(U_m U_n + qU_{m-1} U_{n-1}) \end{pmatrix}$$

Notice that by Theorem 1 again:

$$\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{m+n} = \begin{pmatrix} U_{m+n+1} & qU_{m+n} \\ U_{m+n} & qU_{m+n-1} \end{pmatrix}$$

By comparing the entries in the first column and the second row of both matrices, we get the desired result.

Proposition 5: For positive integers $n, U_{2n} = U_n L_n$.

Proof: Let m = n andin Proposition 4 and applying proposition 1, we get:

$$U_{2n} = U_n U_{n+1} + qU_{n-1}$$
$$U_n = U_n [U_{n+1} + qU_{n-1}] = U_n L_n$$

Theorem 2 for positive integers h and n:

$$\begin{pmatrix} L_h & (-1)^{h+1} q^h \\ 1 & 0 \end{pmatrix}^k = \frac{1}{U_h} \begin{pmatrix} U_{h(k+1)} & (-1)^{h+1} q^h U_{hk} \\ U_{hk} & (-1)^{h+1} q^h U_{h(k+1)} \end{pmatrix}$$

Proof: By proposition 5, the expression is true for n = 1. Assume the truth of the expression for n = k and. Then:

$$\begin{pmatrix} L_h & (-1)^{h+1} q^h \\ 1 & 0 \end{pmatrix}^k = \frac{1}{U_h} \begin{pmatrix} U_{h(k+1)} & (-1)^{h+1} q^h U_{hk} \\ U_{hk} & (-1)^{h+1} q^h U_{h(k+1)} \end{pmatrix}$$

Now, for n = k +1:

$$\begin{pmatrix} L_h & (-1)^{h+1} q^h \\ 1 & 0 \end{pmatrix}^{k+1} = \frac{1}{U_h} \begin{pmatrix} U_{h(k+1)} & (-1)^{h+1} q^h U_{hk} \\ U_{hk} & (-1)^{h+1} q^h U_{h(k-1)} \end{pmatrix} \begin{pmatrix} L_h & (-1)^{h+1} q^h \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{U_h} \begin{pmatrix} U_{h(k+1)} L_h + (-1)^{h+1} q^h U_{hk} & (-1)^{h+1} q^h U_{h(k+1)} \\ U_{hk} L_h + (-1)^{h+1} q^h U_{h(k-1)} & (-1)^{h+1} q^h U_{hk} \end{pmatrix}$$
$$= \frac{1}{U_h} \begin{pmatrix} U_{hk+h} (U_{h+1} + qU_{h-1}) + \\ (-1)^{h+1} q^h U_{hk} \, (-1)^{h+1} q^h U_{h(k+1)} \\ U_{hk} (U_{h+1} + qU_{h-1}) + (-1) \\ {}^{h+1} q^h U_{h(k-1)} \, (-1)^{h+1} q^h U_{hk} \end{pmatrix}$$

(by proposition):

$$= \frac{1}{U_h} \begin{pmatrix} U_{hk+h+1} U_h + qU_{hk+h} U_{h-1} & (-1)^{h+1} q^h U_{h(k+1)} \\ U_{hk+1} U_h + qU_{hk} U_{h-1} & (-1)^{h+1} q^h U_{hk} \end{pmatrix}$$

(by proposition):

$$= \frac{1}{U_h} \begin{pmatrix} U_{h(k+2)} & (-1)^{h+1} q^h U_{h(k+1)} \\ U_{h(k+1)} & (-1)^{h+1} q^h U_{hk} \end{pmatrix}$$

(by proposition 4). Hence, the expression is also true for n = k +1. By the mathematical induction, the proof is completed. We provide an extension to our 2×2 generating matrix in Theorem 2 to a 3×3 generating matrix in Theorem 3. Theorem 3 for positive integers h and n:

$$\begin{pmatrix} 1 & 0 & 0 \\ U_h & L_h & (-1)^{h+1} q^h \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ \sum_{i=0}^{n} U_{hi} & \frac{U_{h(n+1)}}{U_h} & \frac{(-1)^{h+1} q^h U_{hn}}{U_h} \\ \sum_{i=0}^{n-1} U_{hi} & \frac{U_{hn}}{U_h} & \frac{(-1)^{h+1} q^h U_{h(n-1)}}{U_h} \end{pmatrix}$$

Proof: Let:

$$S_n = \sum_{i=0}^{n} U_{hi}$$

The result is true for n = 1 and let's assume is also true for n = k and so that:

$$\begin{pmatrix} 1 & 0 & 0 \\ U_h & L_h & (-1)^{h+1}q^h \\ 0 & 1 & 0 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 & 0 \\ \sum_{i=0}^{n} U_{hi} & \frac{U_{h(n+1)}}{U_h} & \frac{(-1)^{h+1}q^h U_{hn}}{U_h} \\ \sum_{i=0}^{n-1} U_{hi} & \frac{U_{hn}}{U_h} & \frac{(-1)^{h+1}q^h U_{h(n-1)}}{U_h} \end{pmatrix}$$

Now, for n = k+1:

$$\begin{pmatrix} 1 & 0 & 0 \\ U_h & L_h & (-1)^{h+1}q^h \\ 0 & 1 & 0 \end{pmatrix}^{k+1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ S_k & \frac{U_{h(k+1)}}{U_h} & \frac{(-1)^{h+1}q^h U_{hk}}{U_h} \\ S_{k-1} & \frac{U_{hk}}{U_h} & \frac{(-1)^{h+1}q^h U_{h(k-1)}}{U_h} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ U_h & L_h & (-1)^{h+1}q^h \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ S_k + U_{h(k+1)} & \frac{L_h U_{h(k+1)} + (-1)^{h+1}q^h U_{hk}}{U_h} & \frac{(-1)^{h+1}q^h U_{h(k+1)}}{U_h} \\ S_{k-1} + U_{hk} & \frac{L_h U_{hk} + (-1)^{h+1}q^h U_{h(k-1)}}{U_h} & \frac{(-1)^{h+1}q^h U_{hk}}{U_h} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ S_{k+1} & \frac{U_{h(k+2)}}{U_h} & \frac{(-1)^{h+1}q^h U_{h(k+1)}}{U_h} \\ S_k & \frac{U_{h(k+1)}}{U_h} & \frac{(-1)^{h+1}q^h U_{hk}}{U_h} \end{pmatrix}$$

This completes the proof. Following the compilation of the above results we are now ready to prove the sums $\sum_{i=0}^{n} U_{hi}$ for any positive integer h (Theorem 4). Three different proofs will be given, one using the method of diagnalization and the other two using the method of matrix collation.

Method of diagonalization: Theorem 4 for positive integers h and n:

$$\sum_{i=0}^{n} U_{hi} = \frac{U_h - U_{h(n+1)} + (-1)^h q^h U_{hn}}{1 - L_h + (-1)^h q^h} \quad |\lambda I - M| = 0$$

Proof: First, we find the eigenvalues of the matrix:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ U_h & L_h & (-1)^{h+1}q^h \\ 0 & 1 & 0 \end{pmatrix}$$

by solving $|\lambda I - M| = 0$. Their eigenvalues are 1 and $\frac{L_h \pm \sqrt{L_h^2 + 4(-1)^{h+1}q^h}}{2}$ the eigenspace can be found as:

$$X = \begin{pmatrix} 1 - L_h + (-1)^h q^h & 0 & 0 \\ U_h & 1 & 1 \\ U_h & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \end{pmatrix}$$

Since, M is diagonalizable, we have $M^n X = XD^n$ where D is a diagonal matrix. By Theorem 3, the left hand side of the equation $M^n X = XD^n$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ \sum_{i=0}^{n} U_{hi} & \frac{U_{h(n+1)}}{U_h} & \frac{(-1)^{h+1}q^h U_{hn}}{U_h} \\ \sum_{i=0}^{n-1} U_{hi} & \frac{U_{hn}}{U_h} & \frac{(-1)^{h+1}q^h U_{h(n-1)}}{U_h} \end{pmatrix} \begin{pmatrix} 1 - L_h + (-1)^h q^h & 0 & 0 \\ U_h & 1 & 1 \\ U_h & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \end{pmatrix}$$

The right hand side of the equation is:

$$\begin{pmatrix} 1 - L_h + (-1)^h q^h & 0 & 0 \\ U_h & \lambda_2^n & \lambda_3^n \\ U_h & \lambda_2^{n-1} & \lambda_3^{n-1} \end{pmatrix}$$

Comparing the entry in the second row and first column and with the corresponding entry in the resulting matrix multiplication of the left hand side of the equation $M^n X = XD^n$ we get the desired result. In the next subsection, two more proofs of Theorem 4 will be shown.

Method of matrix collation: This method is quite similar to the approach adopted by Falcon and Plaza (2007) in their study. Let:

$$T_1 = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

and:

$$T_2 = \begin{pmatrix} L_h & (-1)^{h+1}q^h \\ 1 & 0 \end{pmatrix}$$

Suppose:

$$S = T_1^h + T_1^{2h} + \ldots + T_1^{nh}$$

Then:

$$T_1^h S = T_1^{2h} + T_1^{nh} + \ldots + T_1^{(n+1)h}$$

Consequently:

$$S = \left(T_1^{(n+1)h} - T_1^h\right)\left(T_1^h - I_2\right)^{-1}$$

By Theorem 1, we see that the sum of the entries in the second row and first column of the matrices $T_1^h, T_1^{2h}, \ldots, T_1^{nh}$ is equal to the corresponding entry in the matrix and its value is equal to $\sum_{i=0}^{n} U_{hi}$. Suppose:

$$S^* = T_2 + T_2^2 + \ldots + T_2^n$$

Then:

$$T_2 S^* = T_2^2 + T_2^n + \ldots + T_2^{(n+1)}$$

Consequently:

$$S^* = \left(T_2^{n+1} - T_2\right)\left(T_2 - I_2\right)^{-1}$$

By Theorem 2, we see that the sum of the entries in the second row and first column of the matrices $T_2, T_2^2, \ldots, T_2^n$ is equal to the corresponding entry in the matrix S^* and its value is equal to:

$$\frac{1}{U_h}\sum_{i=0}^{n} U_{hi}$$

We shall now produce two alternative proofs of Theorem 4 as a verification of the result obtained by the method of diagonalization.

Theorem 4 (second proof): Consider the generating matrix, we have:

$$S = \left(T_1^{(n+1)h} - T_1^h\right)\left(T_1^h - I_2\right)^{-1}$$

By applying Theorem 1, the term simplifies to:

$$\begin{pmatrix} U_{h(n+1)+1} - U_{h+1} & qU_{h(n+1)} - qU_h \\ U_{h(n+1)} - U_h & qU_{h(n+1)-1} - qU_{h-1} \end{pmatrix}$$

while the term $\left(T_1^h - I_2\right)^{-1}$ reduces to:

$$\frac{1}{qU_{h+1}U_{h-1} - U_{h+1} - qU_{h-1} + 1 - qU_h^2}\begin{pmatrix} qU_{h-1} - 1 & -qU_h \\ -U_h & U_{h+1} - 1 \end{pmatrix}$$

and by propositions 1 and 2, further simplified to:

$$\frac{1}{1 - L_h + (-1)^h q^h}\begin{pmatrix} qU_{h-1} - 1 & -qU_h \\ -U_h & U_{h+1} - 1 \end{pmatrix}$$

Finally, by comparing the entry in the first column and the second row of the matrix S and with the corresponding entry in the matrix multiplication $\left(T_1^{(n+1)h} - T_1^h\right)\left(T_1^h - I_2\right)^{-1}$, we get:

$$\sum_{i=0}^{n} U_{hi} = \frac{U_h - U_{h(n+1)} + (-1)^h q^h U_{hn}}{1 - L_h + (-1)^h q^h}$$

Theorem 4 (third proof): Consider the generating matrix, T^2 we have:

$$S^* = \left(T_2^{n+1} - T_2\right)\left(T_2 - I_2\right)^{-1}$$

By applying Theorem 1, the term simplifies to:

$$\begin{pmatrix} \frac{U_{h(n+2)}}{U_h} - L_h & \frac{(-1)^{h+1} q^h U_{h(n+1)}}{U_h} - (-1)^{h+1} q^h \\ \frac{U_{h(n+1)}}{U_h} - 1 & \frac{(-1)^{h+1} q^h U_{hn}}{U_h} \end{pmatrix}$$

while the $\left(T_2 - I_2\right)^{-1}$ term reduces to:

$$\frac{1}{1 - L_h + (-1)^h q^h}\begin{pmatrix} -1 & (-1)^h q^h \\ -1 & -1 \end{pmatrix}$$

Finally, by comparing the entry in the second row and first column of the matrix S* and with the corresponding entry in the matrix multiplication $\left(T_2^{(n+1)} - T_2\right)\left(T_2 - I_2\right)^{-1}$, we get:

$$\sum_{i=0}^{n} U_{hi} = \frac{U_h - U_{h(n+1)} + (-1)^h q^h U_{hn}}{1 - L_h + (-1)^h q^h}$$

CONCLUSION

We briefly discussed past research on Fibonacci and related sequences and argued some facts about the presumptive limitation provided by the Binet Formula approach in the search for their algebraic properties. Hence, there is a strong motivation to search for new and improved methodologies and approaches.

We discovered new algebraic properties and several important generating matrices. Using these, we employed two different matrix methods namely, the Method of Diagonalization and the Method of Matrix Collation to obtain the sum of the GFS, $\sum_{i=0}^{n} U_{hi}$ for any positive integer h and, in which three different proofs. Our discoveries contribute to the literature and shed new lights into the research on Fibonacci and related sequences. It is also noted that our methods could be used to analyze Diphantine Matrix Equation (Shang, 2014). This aspect will be further explored in our upcoming researches.

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