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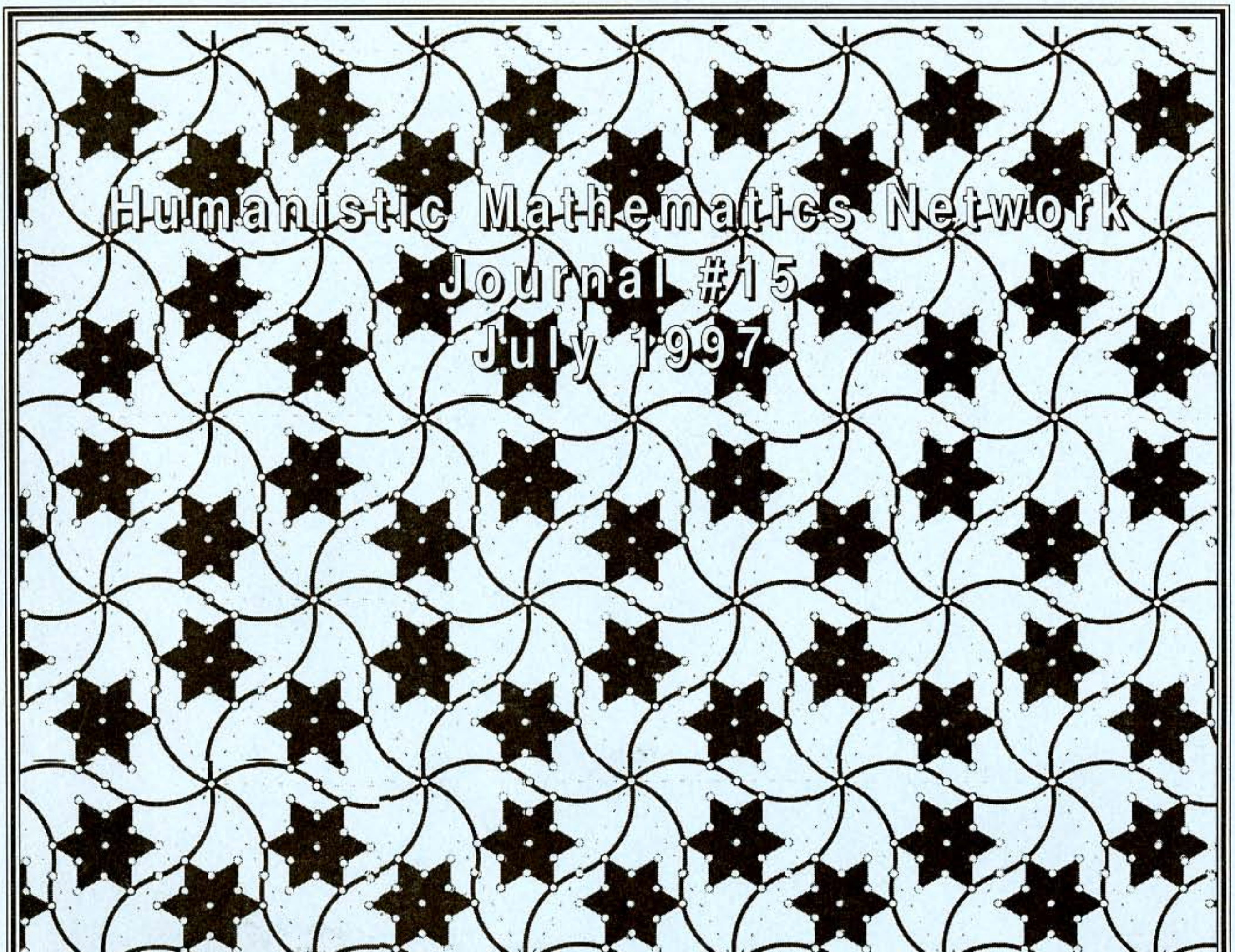

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Essays, book reviews, syllabi, poetry, and letters are welcomed. Your essay should have a title, your name and address, e-mail address, and a brief summary of content. In addition, your telephone number (not for publication) would be helpful.

If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text; these should be original photographs, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of your submission, double-spaced and preferably laser-printed, should be sent to:

Prof. Alvin White
Humanistic Mathematics Network Journal
Harvey Mudd College
Claremont, CA 91711

Essays and other communications may also be transmitted by electronic mail to the editor at AWHITE@HMC.EDU, or faxed to (909) 621-8366. The editor may be contacted at (909) 621-8867 if you have further questions.

The Journal uses the programs Microsoft Word and PageMaker for the Macintosh. If you used either of these programs to prepare your paper, it would be very helpful to us if you sent a copy of your file on diskette along with the printed copies of the paper.

Note to Librarians

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Subscriptions

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Cover

The pattern on the cover is an example of the symmetry and beauty of Islamic art, which is demonstrated to be a synthesis of mathematics and aestetics in Salma Marani's article, "Illumination and Geometry in Islamic Art." The specific pattern is called the Andalusian Variation.

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EXXON EDUCATIONAL FOUNDATION.

Humanistic Mathematics Network Journal #15
July 1997

From Newsletter #1

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other and how they might come to better understand mathematics as a meaningful rather than arbitrary discipline were among the ideas of the first theme.

The second theme focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, discovery to verification, mathematics to science, truth to utility, and in general, mathematics to the culture within which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:
a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."
b) An appreciation for the human dimensions that motivate discovery: competition, cooperation, the urge for holistic pictures.
c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, and why it is investigated.
d) A need for new teaching/learning formats that will help discourage our students from a view of knowledge as certain or to-be-received.
e) The opportunity for students to think like mathematicians, including chances to work on tasks of low definition, generating new problems and participating in controversy over mathematical issues.
f) Opportunities for faculty to do research on issues relating to teaching and be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures. . . the network might formally support writing, team-teaching, exchanges, conferences. . . .

Alvin White
August 3, 1987

From the Editor

The Exxon Education Foundation has been supporting the Humanistic Mathematics Network Journal for twelve years. EEF supports the production, publication, and mailing of HMNJ free of cost to all libraries and individuals who request copies.

The HMNJ circulation is close to 2000 readers all over the world. The diverse interest is reflected in the topics of the articles and book reviews in this issue.

The Production Manager, Matthew Fluet, had to take an unexpected leave in the middle of the semester. Fortunately, Justin Radick was able to step in to complete the production of this issue with skill and good ideas. Justin will work during the summer to produce the next issue, which will be mailed in September or October.

The Los Angeles Times of June 8 and 9, 1997 published two major articles that explored the effect of computers in school classrooms. The verdict is mixed: much excitement, little effect on traditional education such as reading, writing, and math.

An interesting newsletter that explores the theme of the L.A. Times articles is *Mathematics and Technology in the Classroom: A Moderating Voice*, edited by Kent Bessey. Copies and subscriptions are available from:

Mathematics and Technology
343 South 3rd East
Rexburg, ID 83440

Readers are invited to submit commentary to the Journal. All comments and letters should be sent to Harold Ness of the University of Wisconsin, Fond du Lac, WI 54935.

WHAT KIND OF A THING IS A NUMBER?

A Talk With Reuben Hersh

Interviewer: John Brockman
New York City

Reben Hersh
University of New Mexico
Albuquerque, NM 87131
rhersh@math.unm.edu

"What is mathematics? It's neither physical nor mental, it's social. It's part of culture, it's part of history. It's like law, like religion, like money, like all those other things which are very real, but only as part of collective human consciousness....That's what math is."

For mathematician Reuben Hersh, mathematics has existence or reality only as part of human culture. Despite its seeming timelessness and infallibility, it is a social-cultural-historic phenomenon. He takes the long view. He thinks a lot about the ancient problems. What are numbers? What are triangles, squares and circles? What are infinite sets? What is the fourth dimension? What is the meaning and nature of mathematics?

In so doing he explains and criticizes current and past theories of the nature of mathematics. His main purpose is to confront philosophical problems: In what sense do mathematical objects exist? How can we have knowledge of them? Why do mathematicians think mathematical entities exist forever, independent of human action and knowledge?

Reuben Hersh is professor emeritus at the University of New Mexico, Albuquerque. He is the recipient (with Martin Davis) of the Chauvenet Prize and (with Edgar Lorch) the Ford Prize. Hersh is the author (with Philip J. Davis) of *The Mathematical Experience*, winner of the National Book Award in 1983. His new book, *What is Mathematics, Really?* is forthcoming (Oxford).

JOHN BROCKMAN: Reuben, got an interesting question?

REUBEN HERSH: What is a number? Like, what is two? Or even three? This is sort of a kindergarten question, and of course a kindergarten kid would answer like this: (raising three fingers). Or two (raising two fingers). That's a good answer and a bad answer. It's good enough for most purposes, actually. But if you get way beyond kindergarten, far enough to risk asking really deep questions, it becomes: what kind of a thing is a number?

Now, when you ask "What kind of a thing is a number?" you can think of two basic answers—either it's out there some place, like a rock or a ghost; or it's inside, a thought in somebody's mind. Philosophers have defended one or the other of those two answers. It's really pathetic, because anybody who pays any attention can see right away that they're both completely wrong.

A number isn't a thing out there; there isn't any place that it is, or any thing that it is. Neither is it just a thought, because after all, two and two is four, whether you know it or not.

Then you realize that the question is not so easy, so trivial as it sounds at first. One of the great philosophers of mathematics, Gottlob Frege, made quite an issue of the fact that mathematicians didn't know the meaning of One. What is One? Nobody could answer coherently. Of course Frege answered, but his answer was no better, or even worse, than the previous ones. And so it has continued to this very day, strange and incredible as it is. We know all about so much mathematics, but we don't know what it really is.

Of course when I say, "What is a number?" it applies just as well to a triangle, or a circle, or a differentiable function, or a self-adjoint operator. You know a lot about it, but what is it? What kind of a thing is it? Anyhow, that's my question. A long answer to your short question.

JB: And what's the answer to your question?

HERSH: Oh, you want the answer so quick? You have to work for the answer! I'll approach the answer by gradual degrees.

When you say that a mathematical thing, object, entity, is either completely external, independent of human thought or action, or else internal, a thought in your mind—you're not just saying something about numbers, but about existence—that there are only two kinds of existence. Everything is either internal or external. And given that choice, that polarity or dichotomy, numbers don't fit—that's why it's a puzzle. The question is made difficult by a false presupposition, that there are only two kinds of things around.

But if you pretend you 're not being philosophical, just being real, and ask what there is around, well for instance there's the traffic ticket you have to pay, there's the news on the TV, there's a wedding you have to go to, there's a bill you have to pay—none of these things are just thoughts in your mind, and none of them is external to human thought or activity. They are a different kind of reality, that's the trouble. This kind of reality has been excluded from metaphysics and ontology, even though it's well-known—the sciences of anthropology and sociology deal with it. But when you become philosophical, somehow this third answer is overlooked or rejected.

Now that I've set it up for you, you know what the answer is. Mathematics is neither physical nor mental, it's social. It's part of culture, it's part of history, it's like law, like religion, like money, like all those very real things which are real only as part of collective human consciousness. Being part of society and culture, it's both internal and external. Internal to society and culture as a whole, external to the individual, who has to learn it from books and in school. That's what math is.

But for some Platonic mathematicians, that proposition is so outrageous that it takes a lot of effort even to begin to consider it.

JB: Reuben, sounds like you're about to push some political agenda here, and it's not the Republican platform.

HERSH: You're saying my philosophy may be biased by my politics. Well, it's true! This is one of the many novel things in my book—looking into the correlation between political belief and belief about the nature of mathematics.

JB: Do you have a name for this solution?

HERSH: I call it humanistic philosophy of mathematics. It's not really a school; no one else has jumped on the bandwagon with that name, but there are other people who think in a similar way, who gave it different names. I'm not completely a lone wolf here, I'm one of the mavericks, as we call them. The wolves baying outside the corral of philosophy.

Anyhow, back to your other question. The second half of my book is about the history of the philosophy of mathematics. I found that this was best explained by separating philosophers of mathematics into two groups. One group I call *mainstream* and the other I call *humanists and mavericks*. The humanists and mavericks see mathematics as a human activity, and the mainstream see it as inhuman or superhuman. By the way, there have been humanists way back; Aristotle was one. I wondered whether there was any connection with politics. So I tried to classify each of these guys as either right-wing or left-wing, in relation to their own times. Plato was far right; Aristotle was somewhat liberal. Spinoza was a revolutionary; Descartes was a royalist, and so on. These are well known facts. There are some guys that you can't classify. It came out just as you are intimating: the humanists are predominantly left-wing and the mainstream predominantly right wing. Any explanation would be speculative, but intuitively it makes sense. For instance, one main version of mainstream philosophy of mathematics is Platonism. It says that all mathematical objects, entities, or whatever, including the ones we haven't discovered yet and the ones we never will discover—all of them have always existed. There's no change in the realm of mathematics. We discover things, our knowledge increases, but the actual mathematical universe is completely static. Always was, always will be. Well, that's kind of conservative, you know. Fits in with someone who thinks that social institutions mustn't change.

So this parallel exists. But there are exceptions. For instance, Bertrand Russell was a Platonist and a socialist. One of my favorite philosophers, Imre Lakatos, was a right-winger politically, but very radical philosophically. These correlations are loose and statistical, not binding. You can't tell somebody's philosophy from his politics, or vice versa.

I searched for a suitable label for my ideas. There were several others that had been used for similar points of view—social constructivism, fallibilism, quasi-empiricism, naturalism. I didn't want to take anybody else's label, because I was blazing my own trail, and I didn't want to label myself with someone else's school. The name that would have been most accurate was social conceptualism. Mathematics consists of concepts, but not individually held concepts; socially held concepts. Maybe I thought of humanism because I belong to a group called the Humanistic Mathematics Network. Humanism is appropriate, because it's saying that math is something human. There's no math without people. Many people think that ellipses and numbers and so on are there whether or not any people know about them; I think that's a confusion.

JB: Sounds like we're talking about an anthropic principle of mathematics here.

HERSH: Maybe so; I never thought of that. I had a serious argument with a friend of mine at the University of New Mexico, a philosopher of science. She said: "There are nine planets; there were nine planets before there were any people. That means there was the number nine, before we had any people."

There is a difficulty that has to be clarified. We do see mathematical things, like small numbers, in physical reality. And that seems to contradict the idea that numbers are social entities. The way to straighten this out has been pointed out by others also. We use number words in two different ways: as nouns and adjectives. This is an important observation. We say nine apples, nine is an adjective. If it's an objective fact that there are nine apples on the table, that's just as objective as the fact that the apples are red, or that they're ripe, or anything else about them, that's a fact. And there's really no special difficulty about that. Things become difficult when we switch unconsciously, and carelessly, between this real-world adjective interpretation of math words like nine, and the pure abstraction that we talk about in math class.

That's not really the same nine, although there's of course a correlation and a connection. But the number nine as an abstract object, as part of a number system, is a human possession, a human creation, it doesn't exist without us. The possible existence of collections of nine objects is a physical thing, which certainly exists without us. The two kinds of nine are different.

Like I can say a plate is round, an objective fact, but the conception of roundness, mathematical roundness, is something else.

Sad to say, philosophy is definitely an optional activity; most people, including mathematicians, don't even know if they have a philosophy, or what their philosophy is. Certainly what they do would not be affected by a philosophical controversy. This is true in many other fields. To be a practitioner is one thing; to be a philosopher is another. To justify philosophical activity one must go to a deeper level, for instance as in Socrates' remark about the unexamined life. It's pathetic to be a mathematician all your life and never worry, or think, or care, what that means. Many people do it. I compare this to a salmon swimming upstream. He knows how to swim upstream, but he doesn't know what he's doing or why.

JB: How does having a philosophy of mathematics affect its teaching?

HERSH: The philosophy of mathematics is very pertinent to the teaching of mathematics. What's wrong with mathematics teaching is not particular to this country. People are very critical about math teaching in the United States nowadays, as if it was just an American problem. But even though some other countries get higher test scores, the fundamental mis-teaching and bad teaching of mathematics is international, it's standard. In some ways we're not as bad as some other countries. But I don't want to get into that right now.

Let me state three possible philosophical attitudes towards mathematics:

Platonism says mathematics is about some abstract entities which are independent of humanity.

Formalism says mathematics is nothing but calculations. There's no meaning to it at all. You just come out with the right answer by following the rules.

Humanism sees mathematics as part of human culture and human history.

It's hard to come to rigorous conclusions about this kind of thing, but I feel it's almost obvious that Platonism and formalism are anti-educational, and interfere with understanding, and humanism at least doesn't hurt and could be beneficial.

Formalism is connected with rote, the traditional method which is still common in many parts of the world. Here's an algorithm; practice it for a while; now here's another one. That's certainly what makes a lot of people hate mathematics. (I don't mean that mathematicians who are formalists advocate teaching by rote. But the formalist conception of mathematics fits naturally with the rote method of instruction.)

There are various kinds of Platonists. Some are good teachers, some are bad. But the Platonist idea, that, as my friend Phil Davis puts it, pi is in the sky, helps to make mathematics intimidating and remote. It can be an excuse for a pupil's failure to learn, or for a teacher's saying "some people just don't get it."

The humanistic philosophy brings mathematics down to earth, makes it accessible psychologically, and increases the likelihood that someone can learn it, because it's just one of the things that people do. This is a matter of opinion; there's no data, no tests. But I'm convinced it is the case.

JB: How do you teach humanistic math?

HERSH: I'm going to sidestep that slightly, I'll tell you my conception of good math teaching. How this connects with the philosophy may be more tenuous.

The essential thing is interaction, communication. Only in math do you have this typical figure who was supposedly exemplified by Norbert Wiener. He walks into the classroom, doesn't look at the class, starts writing on the board, keeps writing until the hour is over and then departs, still without looking at the class.

A good math teacher starts with examples. He first asks the question and then gives the answer, instead of giving the answer without mentioning what the question was. He is alert to the body language and eye movements of the class. If they start rolling their eyes or leaning back, he will stop his proof or his calculation and force them somehow to respond, even to say "I don't get it." No math class is totally bad if the students are speaking up. And no math lecture is really good, no matter how beautiful, if it lets the audience become simply passive. Some of this applies to any kind of teaching, but math unfortunately is conducive to bad teaching.

It's so strange. Mathematical theorems may really be very useful. But nobody knows it. The teacher doen't mention it, the students don't know it. All they know is it's part of the course. That's inhuman, isn't it?

Here is an anecdote. I teach a class, which I invented myself, called Problem Solving for High School and Junior High School Teachers and Future Teachers. The idea is to get them into problem solving, having fun at it, feeling confident at it, in the hope that when they become teachers they will impart some of that to their class. The students had assignments; they were supposed to work on something and then come talk about it in class. One day I called for volunteers. No volunteers. I waited. Waited. Then, feeling very brave, I went to the back of the room and sat down and said nothing. For a while. And another while. Then a student went to the blackboard, and then another one.

It turned out to be a very good class. The key was that I was willing to shut up. The easy thing, which I had done hundreds of times, would have been to say, "Okay, I'll show it to you." That's perhaps the biggest difficulty for most, nearly all, teachers—not to talk so much. Be quiet. Don't think the world's coming to an end if there's silence for two or three minutes.

JB: Earlier you mentioned the word beauty. What's with beauty?

HERSH: Fortunately, I have an answer to that. My friend, Gian-Carlo Rota, dealt with that issue in his new book, *Indiscrete Thoughts*. He said the desire to say "How beautiful!" is associated with an insight. When something unclear or confusing suddenly fits together, that's beautiful. Maybe there are other situations that you would say are beautiful besides that, but I felt when I read that that he really had something. Because we talk about beauty all the time without being clear what we mean by it; it's purely subjective. But Rota came very close to it. Order out of confusion, simplicity out of complexity, understanding out of misunderstanding—that's mathematical beauty.

Math Lingo vs. Plain English: Multiple Entendre

Stephen I. Brown
University at Buffalo
SUNY Buffalo, Amherst NY 14260
EUASTEVE@ubvms.cc.buffalo.edu

Once more, it is ordinary language with all its ambiguity that provides a clue that the concept of definition in mathematics might not be as monolithic as we are led to believe when the claim is made that definitions are arbitrary and we can define anything any way we wish.

"Beware the double entendre" would be a good slogan to summarize a recent article by Reuben Hersh—one that ends by enticing the reader to make up slogans with some words that have technical mathematical as well as ordinary language meanings.[1] The point of his creative exercise is to have the reader encounter and perhaps internalize what Hersh views as an important lesson that may account for difficulties students have in learning mathematics: that ordinary language is not only filled with ambiguous meanings, but that even when there is no ambiguity in ordinary language, there is generally either no connection or a tenuous one between that meaning and the mathematical one.

As an example of a tenuous connection, Hersh comments,

> If I say "I own a number of calculus books...," I don't mean *zero books*....I don't even mean *one* book....I mean two or more (p.48).

Hersh claims that he now understands that it was not mere ignorance that accounted for the comment many years ago by one of his students who asserted that zero was not a number.

Hersh offers a litany of other ordinary language expressions that are at odds with mathematical meaning: *adding* (which in ordinary language always leads to an increase in number), *difference* (signaling a comparison in ordinary language, but not necessarily subtraction), *multiplication* (repeatedly adding so that one arrives at something that is bigger than what was initially the case).

He points out that not only objects and operations but the logic of requests or demands is problematic as well. Thus when we ask someone to show that a number divisible by six is even, it is surely appropriate in ordinary usage to choose one example (like forty-two) to demonstrate the point rather than to come up with some general proof.

The connection between mathematics and ordinary language can be even more tenuous however in advanced mathematics, as Hersh points out. He comments:

> In advanced mathematics, there's more linguistic confusion. Surds (absurd), irrational and imaginary numbers, singular perturbations, degenerate kernels, strange attractors—all sound dangerous, undesirable, things to avoid (p.51).

It is true that the mismatch between mathematical and everyday meanings is significant enough to warrant our attention, and a disinclination to appreciate this observation may very well account for problems students have in appreciating mathematical meaning. There are, however, concomitant issues that are either ignored or distorted by Hersh's program to clear up the intended *entendre*—with the intention of minimizing ambiguity. They are issues that have deep consequences not only for students attempting to learn new bodies of knowledge, but for anyone attempting to appreciate the nature of mathematical thought as well as its intellectual history.

For this purpose, I would like to suggest the following complementary slogans:

1. "BE *AWARE OF* THE DOUBLE ENTENDRE."
2. "BE *AWARE OF* MULTIPLE ENTENDRE."

BE *AWARE OF* THE DOUBLE ENTENDRE

Precision of meaning is one thing. An appreciation for the evolution of ideas and the associated labor pains is another. The slogan "Be *Aware of* (rather than *Beware*) the Double Entendre" is intended to have an ameliorative rather than a dismissive quality with regard to the concept of double entendre. What do I have in mind? While Hersh has found out that some students have trouble understanding a concept like that of irrational or imaginary numbers because they seek association with such words which "sound dangerous, undesirable, things to avoid," I have discovered that many are frustrated by a disinclination to take seriously the ordinary language equivalent.

Take the case of "negative number" for example. While "negative" surely fits the bill of sounding dangerous and is something to avoid (unless of course it is associated with a biopsy), the Latin translation of that concept (which pre-dated the English translation) was just as foreboding and perhaps more revealing. These numbers were originally called *numeri ficti*—meaning *fictitious numbers*. The implication here is not only that these numbers are dangerous, but that they really do not exist—or if they do, their existence is shrouded in mystery.

What can students learn not by disassociating from an English translation, but by embracing such translations with an historical and multicultural perspective? Perhaps the deepest lesson to learn is that they are not fools if they do not immediately understand what the concept is all about. Not singly, but taken as a whole, words like "negative," "imaginary," "irrational," "complex" with regard to numbers signal something very important. That is, they suggest that these concepts evolved against considerable resistance. They may come to appreciate that in a quite deep sense, "ontogeny recapitulates phylogeny." If our students have trouble understanding how numbers are extended, then it would be a significant source of solace for them to appreciate that they are merely experiencing the labor pains of these ideas historically.

And why should these ideas have had such a labor intensive birth? Why were they not just accepted as reasonable extensions of existing knowledge? What does it mean to say, as Hersh points out, that mathematicians appreciate that zero may have meaning in the above context while ordinary language suggests the opposite? Who are these mathematicians that appreciate the meaning? Are we referring to those who gave birth to the ideas and found themselves walking on a tight-rope, or are we referring to a twentieth century embodiment of "mathematician?" Are there present day mathematicians who would have difficulty with the concept of zero defining a number of real world objects? Should there be?

One reason that each of the extensions of numbers (beyond natural numbers—those that Kronecker spoke of as God-given, but which Russell and Frege attempted to humanize by establishing them on a set-theoretic foundation) met with such resistance among professionals is that there was an important and healthy kind of confusion that had to be unraveled over time. It is a sort of confusion that is not easily conquered once and for all, but is perhaps built into the human mind, and reappears with each new discovery in all fields of inquiry. That is, in viewing an extension of already existing concepts, how do we connect with what exists? What do we expect of the newly emerging idea that is in common with the previous one?

Obviously a concept (of number, for example) which derives from an earlier one has *something* in common with the earlier one. Just as obviously, however, it differs from the original one. Each extension requires that we decide how much we want the emerging idea to deviate from the original. At what point is the deviation so significant that we can no longer speak of the two concepts in the same breath?

With each extension of number, mathematicians had to ask themselves what there was that was so fundamental about the concept from which it was to be derived that had to be held intact—such that letting it go would completely destroy the concept.

At early stages in the history of mathematics, extensions were characterized by mathematicians' search for a "visible" thread—something linked to the real world, or perhaps a model of some sort that might be a bit more abstract than what could be touched or seen. Just as mathematicians who were confronted with the search for some reality that linked the emerging concept of *numeri ficti* to the earthiness of the natural numbers, so our students experience discomfort when they cannot rely upon familiar models in a number system that is supposedly an extension of what is already comfortable.

We sometimes get the impression that an axiomatic formulation of mathematics was a watershed that enabled mathematicians to resolve this problem once and for all. We thus might conclude erroneously that it is our students' inability to appreciate an axiomatic perspective that accounts for their reluctance to accept some of these extensions. We might believe that the culprit then is an overly "concrete" hold on the prior number system, and furthermore that the concrete hold is rooted in an effort to connect each idea with ordinary language usage. Thus if natural number is associated with objects you can see or touch, then it surely is understandable that our students would have a problem that mathematicians do not have with zero or negative elements being numbers at all.

But the problem does not (and did not) disappear with the creation of an axiomatic perspective. If we think of the natural numbers as a system satisfying Peano's postulates, then we know that there are certain axioms that such a system must satisfy. But as we extend this system, we find out that some of the properties must be relinquished. It is not just that we cannot "touch" negative numbers that is problematic, but rather that the extended system loses some properties of number that are associated with the positive integers and such properties are cherished by different people in different ways. If the extension from positive integers to integers enables us to solve some new equations, it also raises some eyebrows. Thus, in the extended system we can no longer hold on to mathematical induction (a loss felt perhaps more dramatically in guise of the equivalent well-ordering property). Not every subset of the new system has a least element. Similarly, an awareness that is perhaps more intuitively understood (with machinery that may sound less technical than mathematical induction) is challenged to the hilt when an extension from positive to negative rationals leads us to reject the strongly held belief that a smaller number divided by a larger number cannot equal a larger number divided by a smaller one (as in $-1/1 = 1/-1$).

When do we reach a point of no return—such that we no longer think of the newly derived system as being a number system at all? We know that the deeply embedded property of commutativity had to be relinquished under matrix multiplication. Yet, we have come to think of matrices as being a number system of sorts.

As we depict the actual evolution of number systems, we can share with our students the historical debates that took place regarding the legitimacy of purported extensions. But we can do more. If we engage them in creating alternative extensions—ones that challenge some of their own cherished properties—at what point do our students get their backs up and say that the system being created no longer reflects what numbers are "really about"?

That's the sort of question that can engage our students, once we encourage them not to by-pass the ambiguity of ordinary language and to place mathematics on a different sort of pedestal, but rather to see how the presence of language in the evolution of ideas is a testimony to the most human problems of cognition and emotion as well: How badly do we want something that opens up totally new avenues to explore, and at what price will we buy it?

BE *AWARE OF* MULTIPLE ENTENDRE

So far, we have shown how attention to double entendre can be advantageous not from the point of view of making each new concept more easily understood, but rather as a tool in enabling us to better understand the problematic nature of an entire collection of concepts.

There is however another way in which attention to ordinary language can be enlightening. This has to do less with the translation (and mistranslations) of a family of words and grammatical uses in the domains of ordinary language vs. mathematics, and more with an awareness of certain concepts that are embedded in our culture in general.

It leads us to an issue alluded to in the above section, but it puts a totally new slant on the issue. I begin with the story of a classroom event of several years ago.

I was teaching a talented group (sic) of graduate students who had previously been exposed to a number of different strategies for extending number systems. Thus, they had postulated newly extended number systems; they had derived new systems from old ones making use of concepts such as ordered pairs of elements from the old ones; they had proved all sorts of things about the new systems in relation to the old ones; they knew what the concept of equivalence relation was all about and had seen the relevance of that concept to extensions; they had been exposed to the concept of new systems having a subset isomorphic to the old; they had been exposed to alternative historical development of the real numbers as in the case of Dedekind's cuts vs. Weierstrass' limits.

I then proposed the following (what I thought was) simple dilemma:

> The real numbers can be characterized in an axiomatic way (essentially an Archimedean ordered field, but I was careful to lay out the properties). I reviewed for them that within that system, it is possible to prove that there does not exist a number x so that $x^2 = -1$.

I then told them that one "popular" way of viewing the set of complex numbers is to define that set as a one that satisfies all the properties of the previous set, but in addition has the following property:

> There exists a number x so that $x^2 = -1$.

Question: How is such a contradiction possible?

I found their answers perplexing. Many of them claimed that the new set, the complex numbers, was a different set than the previous one—the real numbers—so that there was no implied contradiction.[2] Some people seemed to believe that the problem was resolved by *naming* the new system—as if such an act in and of itself had the power to dissolve a contradiction. Some claimed that it is not surprising to find out that what we previously held to be impossible was in fact possible since that is analogous to what growing up and being educated is all about.

Many other interesting comments were made, and in fact, encouraging students to analyze this sort of question in a non-threatening way served as a wonderful Rorschach test. By examining anomalies in a specific rather than in a global context, instructors may unearth some interesting student misconceptions. That is, if asked whether or not it would be acceptable to have a system that satisfies the two propositions X and not X simultaneously, they most likely would claim that such is not possible, and in fact is an important element in the arsenal of mathematical arguments.

Now there is a grain of truth in the students' reactions, and I perhaps misinterpreted their efforts to resolve the problem, but I still found it difficult to understand how they could not be bothered by what appeared to be an obvious contradiction. In fact, no one mentioned that the new system of complex numbers is not merely an add-on to the old system in the sense that everything that was assumed in the old system was also introduced into the new.

It is not that no one pointed out that in the new system, an important property of the old one must be relinquished (that of order), but rather that no one even entertained the possibility that *something* might be lost even if they could not name what it was.

Why is that? It took me a long time to come to appreciate what might have been going on, and I have finally come to an hypothesis that seems worth taking seriously. That is, I have come to believe that their disinclination to consider the possibility that something had to be relinquished is a function of one rather specific notion of *progress* in our culture. Adapting a phrase of Piaget's that has a slightly different connotation, I have dubbed this notion of progress *The American Phenomenon.* While there are multiple meanings of *progress* in ordinary language, a dominant one seems to assume that progress *involves getting more and more of what you find desirable* (like being able to get a solution to $x^2 = -1$ when it did not previously exist) *without ever losing anything that you previously held worthwhile.*

The fact that an extension of a number system provides you with something new and desirable but may at the same time deprive you of something you previously found desirable is not well understood. But why so? It may not be a result of the fact that the technical process of extension is poorly understood from a mathematical point of view, but rather because the concept of *progress* in general is filled with so many unexplored myths.

So, I am suggesting that it is not that we need to distinguish (and divorce) ordinary language from precise mathematical language in order to create a more accurate understanding of mathematical ideas. Rather it is worth doing some analysis of words and concepts in ordinary language that *do not at all* have mathematical counterparts, but that strongly influence the way in which our students think about mathematics and mathematical development in the first place. *Progress* is one such concept but there are others.

What is needed in order to fully appreciate that extension of systems may have a price to pay is not only an issue of mathematical logic. It requires simultaneously that we do some excavation on a concept of ordinary language that is popularly viewed as unambiguous: the concept of progress. Once more, what we need is to seek greater rather than lesser ambiguity in order to arrive finally at a view of the concept of progress that illuminates the interesting discomfort we feel when popularly held principles have to be relinquished.

I conclude with one other concept that is a meta-mathematical rather than a mathematical one. Sometimes it is our inability to appreciate fully the ambiguity of ordinary language that prevents us from understanding not only a particular mathematical concept or an array of concepts, but rather the nature of mathematical thought itself. Consider the concept of *definition.* Most of my students believe that definitions in mathematics are arbitrary. That is, they tell me that you can define things any way you want.

Holding on to a narrow and unambiguous notion of *definition,* they essentially see its application in mathematics as the replacement of one arbitrary English word with some mathematical formulation. Thus *the slope* of a line in a Cartesian co-ordinate system is meant to be a shorthand way of replacing the change in y values divided by the change in x values for any two points on a straight line.

What the concept of arbitrary definition neglects to appreciate is first of all that no one goes around just defining things arbitrarily and that considerable spade work is necessary in order to decide what is worth defining in the first place. That is, definitions single out objects with a *purpose* in mind, and frequently that purpose is arrived at as a culminating act of inquiry rather than as a first step (as most texts would have us believe). In addition, of course, there are logical criteria that need to be unearthed before definitions are accepted. For example, in most circumstances, we do not select definitions that we believe would lead to contradictions. Thus the concept of *the* slope of *a* straight line would make little sense if slope changed in value depending upon which points were selected along the line.

But there is something deeper about the concept of definition which does borrow from ordinary language use of definition. That is, there are occasions upon which definitions even in mathematics serve some function other than that of *stipulating* one expression for some other. That is, there are occasions upon which definitions are descriptive in nature.[3] Far from being arbitrary, these definitions are intended to convey with a degree of accuracy what it is that accords intuitively with our beliefs.

So, for example, there are many different ways of defining a circle in precise mathematical terms. Though, as Hersh would point out, common language usage might not distinguish carefully between points along the rim and interior points (for example), in no case would we expect that what we previously defined as slope would satisfy the definition of circle. Such a definition would not accord with our prior sense of what a circle "really is." To adopt the notion of definition in mathematics as *arbitrary* is to show a lack of appreciation for the interesting range of ways the concept of definition functions in ordinary language. It is to act as if the Socratic search for "justice" or "beauty" is a pointless venture on the grounds that any shorthand expression would do.

Once more, it is ordinary language with all its ambiguity that provides a clue that the concept of definition in mathematics might not be as monolithic as we are led to believe when the claim is made that definitions are arbitrary and we can define anything any way we wish.

CONCLUSION

So Hersh, in his delightful essay, reminds us that ordinary language can be misleading and can interfere with students' understanding of mathematical ideas. That lesson itself, however, is misleading if we do not also take into consideration that ambiguity of language can be an asset, especially when the goal is not necessarily to unearth the precise meaning of a relatively narrow mathematical concept (like negative integer), but rather to appreciate how it is that an array of related concepts (like number) has evolved.

It is by looking at the array of ordinary language meanings (and concomitant emotional baggage) associated with numbers that we can begin to imagine a state of mind that was behind Kronecker's reaction to Lindemann's demonstration of the transcendental nature of pi: Just a little over a century ago, he said:

> What good is your beautiful investigation regarding pi? Why study such problems, since irrational numbers do not exist?[4]

The pedagogical issues are complicated here and I have made no effort to spell this awareness out in terms of any teaching program. Furthermore, I have intentionally focused narrowly on the concept of number rather than upon the range of interesting specific concepts that Hersh has explored. I have also not explored in general the role that ordinary language plays in thinking, nor have I delved in particular into the role of metaphorical thinking in mathematics—a thinking that might account for the variety and richness of systems described by language such as "ring," "field," "ideal," and even "manifold" and "commutator."[5]

While what I have claimed does not negate Hersh's argument, I have attempted to point out that the ambiguity of ordinary language serves a number of interesting functions beyond the antiseptic one of identifying and delimiting (sic again) its potential in understanding mathematics.

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(revised March 24, 1997)

NOTES

[1] R. Hersh, "Math Lingo vs. Plain Language," *American Mathematical Monthly*, 104 (1), (1997), 48- 51.

[2] While having some surface validity, the problem with this way of resolving the contradiction is that I did not *construct* the new system from the old (in which case, this explanation could perhaps be justified). What I did was to postulate the new system so that I was not looking at the elements in the new system as having any existence other than what could be described by the axioms themselves. Though this strategy is frequently used in extending systems, it has a slippery enough quality so that Bertrand Russell was led to comment:

> The method of "postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil.

[3] I. Scheffler, *The Language of Education* (Springfield IL.: Charles C. Thomas, 1960) 11- 35.

[4] E.T. Bell. *Men of Mathematics*, (New York: Simon and Schuster, 1937).

[5] For a more general philosophical discussion of the issues raised by Hersh and myself with regard to different kinds of metaphors (extinct, dormant and active) and their uses/abuses, see

> M. Black, "More About Metaphor," *Metaphor and Thought*, Anthony Ortony, ed. (Cambridge: Cambridge University Press, 1979) 19 - 45.

For a discussion of the role in mathematics *per se*, see

> S.I. Brown, *Student Generations*, Consortium of Mathematics and Its Applications (COMAP) (Lexington, MA, 1988) 43 - 47.
>
> S.I. Brown and M.I. Walter, *The Art of Problem Posing*,, (Hillsdale, NJ: Lawrence Erlbaum Associates, 1990).
>
> D. Pimm, *Speaking Mathematically: Communication in Mathematics Classrooms* (New York: Routledge & Kegan Paul, 1987).
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> *Mathematical Reasoning: Analogies, Metaphors and Images*, L.D. English ed. (Mahwah, NJ: Lawrence Erlbaum Associates, 1997).

Abe Shenitzer at 75

Hardy Grant
539 Highland Avenue,
Ottawa ON R2A 2J8, Canada
hgrant@freenet.carlaton.ca

Widely recognized as a tireless crusader for "humanistic" approaches to mathematics, Abe Shenitzer earlier this year completed the third quarter of his first century. To mark the birthday, a celebratory conference in Abe's honor was held on October 5th at York University in Toronto, his home institution since 1969.

Abe was born in Warsaw but grew up in Sosnowiec, an industrial city in southwestern Poland. He says that in his school days he liked mathematics and was good at it but did not yet sense its cultural significance. He inclined at that time to prefer the study of languages, which indeed has remained one of his great loves. A deep sensitivity for linguistic nuance lies behind the success of his many translations of mathematical (and other) books and articles from Russian, German and Polish. (This activity he continues to pursue; his next major project is a translation of Detlef Laugwitz's recent intellectual biography of Riemann.)

Between 1943 and his liberation from Bergen-Belsen at the end of World War II Abe was in several labor and concentration camps. He continues to share his thoughts on the Holocaust by invitation with many groups, especially of high-school students. He came to the United States in 1946. He had by this time taught himself English, using the famous Langenscheidt method that based the study on a literary masterpiece (in this case *A Christmas Carol*). He took an undergraduate degree in mathematics at Brooklyn College, then went on to earn his Ph.D. at New York University. His supervisor was the late Wilhelm Magnus, whom Abe remembers as "a man of towering intellect and wonderful kindness." It was while at NYU that he met the wise and gracious lady who became his wife; he and Sarah now have two grown daughters and two grandsons.

A brief stint at a Bell Telephone research laboratory convinced him that his future lay in academia. He taught at Rutgers University in New Jersey for a year and a half, then at Adelphi University on Long Island until his move to Toronto. His classroom career was crowned by his winning of a prestigious Ontario-wide award for teaching excellence; the testimonials cited not only his command of his subject and his communicative skills but also his concern for his students as people. He retired officially from York University a few years ago, but now is busier than ever with scholarly pursuits. He has of course many interests and passions outside mathematics. Two of his recent translations from Polish are of books about literature. He is among other things a lover of good music (with a special reverence for Bach), an enthusiastic skier, and a skilled craftsman in wood.

The conference marking Abe's 75th birthday was superbly organized by two of his York colleagues and longtime friends, Israel Kleiner and Martin Muldoon. Five speakers graced the program, and the diversity of their themes mirrored the breadth of the guest of honor's mathematical interests. Ed Barbeau (University of Toronto) spoke on "Fourier Series"; Harold Edwards (New York University) on "The Fundamental Theorem of Algebra"; Peter Hilton (University of Central Florida and SUNY at Binghampton) on "From Geometry to Algebra: Reflections on the Birth of Homological Algebra"; Walter Littman (University of Minnesota) on "The Two-Way Street Between Control Theory and Partial Differential Equations"; and Helena Pycior (University of Wisconsin at Milwaukee) on "George Berkeley, Mathematics and Philosophy: Berkeleian Scholarship into the 1990s." Tributes to Abe at the ensuing banquet were glowing, but luckily his sense of humor and his sense of perspective, both of which are quite out of the ordinary, should ensure that he will be able to go on wearing the same hats as before.

Abe Shenitzer's work consistently champions what he calls the intellectual aspects of mathematics as opposed to the merely technical. His many talks and articles strive always to close the gap between these two facets of the subject. Specialization, he has written, "is the price we pay for creative achievements", but it entails that "the 'average' productive mathematician sometimes knows little about mathematical ideas outside his speciality and even less about their evolution and role." He once contrasted this narrowness of vision among mathematicians with the situation in a discipline such as English literature. "The term 'English major,'" he wrote, "implies some historical, philosophical and evaluative training and competence. It is sad but true that the term 'mathematician' does not imply corresponding training and competence."

These concerns underlie Abe's approach to the column, called "The Evolution of ...," which he has edited for the *American Mathematical Monthly* since January 1994. The column's articles are chosen for their ability to expand readers' mathematical horizons by paying special attention to (as Abe puts it) "ideas and issues that overlap different domains of mathematics, or overlap mathematics and other disciplines, such as physics, philosophy and so on." The articles have solid mathematical substance, with an emphasis on developments since 1700; but always the goal is to shed light on larger themes. This policy should make the column especially valuable to teachers, whose effectiveness can be much increased by awareness of their curriculum's wider mathematical and cultural context.

It is difficult for me to write dispassionately about Abe Shenitzer—so I hope that it is not necessary. For more than a quarter of a century he has figured in my life as colleague, collaborator, guru, travelling companion and much more; ours is a friendship with many dimensions. I can echo the several people at the conference in his honor who said, privately or publicly, that they count Abe among their greatest teachers though they never sat in one of his classrooms. I owe him debts that are not easy to express, let alone to repay—and I know that many others would say the same. It is a joy to report that at 75 he enjoys a mental and physical robustness scarcely if at all diminished by time. That is a lucky state of affairs for the cause of humanistic mathematics, which Abe has served so devotedly and so well.

A Brief Tribute to π

J.D. Phillips
Saint Mary's College
Morgana, CA 94575-3517
phillips@stmarys-ca.edu

So close, **yet so far away**

3.14159265358979323846264 . . .

Reminiscences of Paul Erdös (1913-1996)

Melvin Henriksen
Harvey Mudd College
Claremont CA 91711
Henriksen@hmc.edu

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http://www.maa.org/features/erdos.html

I met Paul Erdös shortly after his 40th birthday in April 1953 at Purdue University in West Lafayette, Indiana. Hewas already a living legend because of his substantial contributions to the theory of numbers, the theory of sets, what is now called discrete mathematics, as well as to many other areas of mathematics. (For example, although he had little interest in topology, his name appears in most topology texts as the first person to give an example of totally disconnected topological space that is not zero-dimensional.) I was a 26-year old instructor in my first year at Purdue. Many of my colleagues knew him well. He had been a visiting research associate at Purdue for a couple of years during World War II, and had visited so many universities and attended so many conferences that he was well known to most of the others. Those that were active in research admired his mathematical accomplishments, while others on the faculty were amused by his eccentricities. What I remember most clearly is his announcement to everyone that "death begins at 40".

I am not qualified to write a biography of Erdös, but some background seems necessary. There is an excellently written and accurate obituary of him by Gina Kolata in the Sept. 21, 1996 issue of the New York Times, beginning on page 1. An interview conducted in 1979 which reveals much of his personality appeared in the volume *Mathematical People* edited by D.J. Albers and G.L. Alexanderson (Birkhauser 1985). The Mathematical Association of America (MAA) sells two videos of Erdös, and Ronald Graham, a long time collaborator, has edited together with Jarik Nesetril two volumes on his mathematical work and life. (Both volumes have been published by Springer-Verlag and were available in January 1997. They include a detailed biographical article by Bella Bollobas.)

Erdös was born in Budapest in 1913 of parents who were Jewish intellectuals. His brilliance was evident by the time he was three years old. For this reason, and perhaps because two older sisters died of scarlet fever shortly before he was born, his parents shielded him almost completely from the everyday problems of life. For example, he never had to tie his own shoelaces until he was 14 years old, and never buttered his own toast until he was 21 years old in Cambridge, England. In return for the freedom to concentrate almost exclusively on intellectual pursuits, he paid the price of not learning the social skills that are expected of all of us and usually acquired in childhood.

He became internationally famous at the age of 20 when he got a *simple* proof of a theorem that was originally conjectured by Bertrand and later proved by Tchebychev: For every positive integer n, there is a prime between n and $2n$. Tchebychev's proof was quite hard! Erdös completed the requirements for the Ph.D. at the University of Budapest about a year later, but had no chance of getting a position in Hungary because he was a Jew living under a right wing dictatorship allied with Nazi Germany. He spent some time at Cambridge University in 1935. There, his life as a wandering mathematician began. In fact, he had visited Cambridge three times the year before. He liked traveling and had no trouble working while doing so. He liked people, and except for those who could not tolerate his ignorance of the social graces, they liked him. He tried his best to be pleasant to everyone and was generous in giving credit and respect to his collaborators.

I do not know when he first came to the United States, but he spent the years of World War II here, two of them at Purdue. Nor can I give a list of the many universities he visited for any substantial length of time. By the time I met him, he had written joint papers with many mathematicians most of whom had established research reputations before working with Erdös. The only Erdös collaborator who worked with him unwillingly was Atle Selberg. In the late 1940s, both of them, working independently, had obtained "elementary" proofs (meaning: proofs that did not use complex analysis) of the prime number theorem. The theorem states that the number of primes less than or equal to (a positive real number) x is asymptotically equal to $x/\log(x)$. This had been conjectured by Gauss and Legendre based on empirical data, but it had only been proved many years later, by two French mathematicians, Jacques Hadamard and Charles de la Vallée Poussin (also working independently). Both proofs depended heavily on complex analysis. What Selberg and Erdös did in their "elementary" proofs was to avoid using complex analysis (the proofs were in no sense "easy"). In those pre-email days, the fastest courier of mathematical news was Paul Erdös. He told anyone who would listen that Selberg and he had devised an elementary proof of the prime number theorem.

Almost every number theorist knew of Erdös, while few had heard of the young Norwegian Selberg. So when the news traveled back to Selberg, it appeared that Erdös had claimed all the credit for himself. The ensuing bitterness was not healed by the two of them writing a joint paper. Selberg later published another elementary proof on his own,and went on to a brilliant mathematical career, eventually becoming a permanent member of the Institute for Advanced Study in Princeton, the Valhalla for mathematicians. Erdös had been a visitor there earlier, but was not offered a membership. Exactly what happened is controversial to this day, and reading the article by Bollobas will shed more light on this matter than this short summary can.

Erdös spent the academic year 1953-54 at the University of Notre Dame in South Bend, Indiana. Arnold Ross, the chairman of the Mathematics Department, had arranged for him to teach only one (advanced) course, and supplied an assistant who could take over his class if he had the urge to travel to talk with a collaborator. Erdös had rejected organized religion as a young man, and had been persecuted in Roman Catholic Hungary. So we teased him about working at a Catholic institution. He said in all seriousness that he liked being there very much, and especially enjoyed discussions with the Dominicans. "The only thing that bothers me," he said, "there are too many plus signs." He came by bus to West Lafayette fairly often for short periods because he had so many friends there and because he liked the mathematical atmosphere.

At that time, Leonard Gillman and I were trying to study the structure of the residue class fields of rings of real-valued continuous functions on a topological space modulo maximal ideals. We had learned quite a bit about them, but had run into serious set-theoretic difficulties. Erdös had little interest in abstract algebra or topology, but was a master of set-theoretic constructions. Without bothering him with our motivation for asking them, we asked him a series of questions about set theory, which he managed to answer while we could not.

He was not terribly interested when we supplied him with the motivation, and I have often said that Erdös never understood our paper; all he did was the hard part. This paper by Erdös, Gillman and Henriksen was published in the *Annals of Mathematics* in 1955. Without any of us realizing it in advance, it became one of the pioneering papers in nonstandard analysis, and was often credited to Erdös, et al.

Erdös got an offer allowing him to stay indefinitely at Notre Dame on the same generous basis. His friends urged him to accept. "Paul", we said "how much longer can you keep up a life of being a traveling mathematician?" (Little did we suspect that the answer was going to turn out to be "more than 40 years.") Erdös thanked Ross, but turned him down. As it turned out, he would not have been at Notre Dame the next year whatever his answer had been.

The cold war was in full swing, the United States was in the grip of paranoia about communism, and many regarded unconventional behavior as evidence of disloyalty. Erdös had never applied for citizenship anywhere he lived, and had acquired Hungarian citizenship only by accident of birth. He belonged to no political party, but had a fierce belief in the freedom of individuals as long as they did no harm to anyone else. All countries who failed to follow this were classified as imperialist and given a name that began with a small letter. For example, the U.S. was **samland** and the Soviet Union was **joedom** (after Joseph Stalin). He talked of an organization called the f.b.u—a combination of the F.B.I and O.G.P.U (which later became the K.G.B) and conjectured that their agents were often interchanged.

In 1954, Erdös wanted to go the International Congress of Mathematicians (held every four years), which was to be in Amsterdam that August. As a non-citizen leaving the U.S. with plans to return, he had to apply for a re-entry permit. After being interviewed by an INS agent in South Bend in early 1954, he received a letter saying that re-entry would be denied if he left the U.S. He hired a lawyer and appealed only to be turned down again. No reason was ever given, but his lawyer was permitted to examine a portion of Erdös' file and found recorded the following facts:

- He corresponded with a Chinese number theorist named Hua who had left his position at the University of Illinois to return to (red) China in 1949. (A typical Erdös letter would have begun: Dear Hua, Let p be an odd prime...)

- He had blundered onto a radar installation on Long island in 1942 while discussing mathematics with two other non-citizens.

- His mother worked for the Hungarian Academy of Sciences, and had had to join the communist party to hold her position.

To Erdös, being denied the right to travel was like being denied the right to breathe, so he went to Amsterdam anyway. He was confident that he could easily obtain a Dutch and an English visa. The Dutch gave him a visa good for only a few months, and England would not let him come, likely because if they chose to deport him, the only country obligated to accept him was communist Hungary. By then, Erdös was a member of the Hungarian Academy of Sciences, but he would go to Hungary only if his friends could assure him that he would be permitted to leave. At this point, he swallowed his pride and obtained a passport from israel (note the punctuation) which served to give him freedom to travel anywhere in western Europe. He was permitted to return to the United States in the summer of 1959 on a temporary visa to attend a month long conference on number theory in Boulder, Colorado. He stopped at Purdue on his way back to Europe to give a colloquium talk. When I picked him up at the airport, what struck me first was that he had a suitcase! For many years, he traveled only with a small leather briefcase containing a change of socks and underwear in addition to a wash-and-wear shirt, together with some paper and a few reprints. About a year later, the United States government lost its fear of Erdös and gave him resident alien status once more. He never had trouble going in or out of the U.S. again. Erdös had lived from hand to mouth most of the time until the late 1950s. When the Russians sent Sputnik into orbit and the space race began, there was a vast increase in government support of research. This made it possible for his many friends and co-authors to give him research stipends. This had little effect on his lifestyle. His suitcase was rarely more than half full, and he gave away most of his money to help talented young mathematicians or to offer cash prizes for solving research problems of varying degrees of difficulty. (The cash prizes were not as costly as he had expected. The winners would often frame his checks without cashing them. Solving a $1000 problem would make you internationally famous, and being able to say that you solved any of his prize problems enhanced your reputation.) Around 1965, Casper Goffman concocted the idea of an Erdös number. If you had written a joint paper with him, your Erdös number was 1. If you had written a joint paper with someone with Erdös number 1, your Erdös number is 2, and so on inductively. There is now an Erdös Number Project home page on the web where you can see a list of all who have an Edos number of 1 (there are 462 of us) and 2 (all 4566 of them, including Albert Einstein). All in all, Erdös wrote about 1500 research papers, and 50 or so more will appear after his death.

While we did no more joint research, we often met at conferences or when we were both visiting the same university. Sometimes I could hardly talk to him because he was surrounded by mathematicians eager to ask him questions, but when I could, he inquired about mutual friends and asked about follow-up work on our paper and progress about solving the open problems we had posed. While he devoted his life to mathematics, he was widely read in many areas and I almost always learned a great deal talking to him about many non-mathematical ideas. I saw him last in Budapest last Sept. 4. He attended the first half of a talk I gave about separate vs. joint continuity. He apologized in advance about having to leave early because he had made an appointment he could not break before he knew I would be speaking. Even then, he made two helpful comments while present. Before I left the Academy of Sciences, I stopped to say goodbye and saw him going over a paper with a young Hungarian mathematician. He died in Warsaw of a heart attack on Sept. 20. He worked on what he loved to do to the last!

Erdös had a special vocabulary that he concocted and used consistently in his speech. Some samples are:

- **Children** are **Epsilons**
- **Women** are **Bosses**
- **Men** are **Slaves**
- **Married Men** have been **Captured**
- **Alcoholic Drinks** are **Poison**
- **God** is **The Supreme Fascist** or **SF**
- **Music** is **Noise**.

Examples:

I asked Barbara Piranian (President of the League of Women Voters in Ann Arbor, Michigan in the early 1950s) "When will you bosses take the vote away from the slaves?" Answer :"There is no need; we tell them how to vote anyway."

"Wine, women, and song" becomes "Poison, bosses, and noise".

Erdös said that the SF had a Book containing elegant proofs of all the important theorems, and when a mathematician worked very hard, the SF could be distracted long enough to allow her or him to take a brief peek. Particularly elegant proofs were described as fit to be placed in the Book.

There are many Erdös stories that were embellished over the years and made more delightful than the truth. For example, consider the story about blundering into a radar installation in 1942:

- **Embellished version:** Erdös, Hochschild (a German) and Kakutani (a Japanese) drove a car out onto Long Island and held an animated mathematical conversation in German. They walked onto a radar installation and were apprehended by a guard who was convinced that he had caught a group of foreign spies. They were questioned closely by military intelligence and released with a warning when they promised never to do such a thing again.

- **Actual version:** The car was driven by Arthur Stone (an Englishman). Hochschild was supposed to come,but did not because he had a date. They were speaking English because it was their only western language understood by Kakutani. The guard was satisfied as soon as they presented proper identification, and they were visited individually and briefly a few days later by military intelligence agents.

Erdös liked to tell many stories about himself. In particular, when he grew older, he claimed to be two billion years old because when he was in high school, he was taught that the earth was two and a half billion years old—but now we know it is four and a half billion years old.

Because he seemed to be in a state of Brownian motion, it was often hard to locate him at any given time. Erdös visited Claremont twice in the 1970s and could often be found at UCLA. For many years the way to contact him was to call Ron Graham of Bell Labs on the east coast, Paul Bateman of the University of Illinois, or Ernst Strauss at UCLA to find out where he was. Strauss died in 1983 and was replaced by Bruce Rothschild. Paul Bateman retired. Although Ron Graham himself traveled a great deal, until the end he was the person most likely to know of Erdös' whereabouts.

With Erdös' death we have lost one of the great mathematicians and free spirits of this century and it is hard to imagine that we will see anyone like him again. I feel fortunate to have had the privilege of knowing and working with him.

Sand Songs: The Formal Languages of Warlpiri Iconography

James V. Rauff
Millikin University
Decatur, IL 62522
jrauff@mail.millikin.edu

This essay is an investigation into the mathematical ideas implicit, if not explicit, in the iconographic designs (sand scenes, yawalyu, site-path designs, and guruwari) from the point of view of formal language theory and provides short formal grammars that generate the languages of the designs. The essay closes with a suggestion of the pedagogical value of viewing Walpiri iconography as mathematics and whether or not the iconographic system may be properly termed "mathematics."

1. THE WARLPIRI

Historically, the Warlpiri[1] were a group of seminomadic hunters and gatherers who wandered the desert of Central Australia. Today they live in outstations and towns in their ancestral homelands. Many have claimed their traditional land and established small homesteads near larger settlements. The Warlpiri treasure their traditions and still teach their children the techniques of survival in the desert. Their sand stories and iconographical designs play an important role in keeping their nomadic traditions alive. Because these iconographical designs, their composition, and the stories they tell are my primary concerns, my description of Warlpiri culture will be of their traditional culture.

The western desert region of central Australia is a harsh place. It is a place of limited diversity and resources. Sand ridges often stretch for miles separated by flat sand plains. The most prevalent plant-life are pale green spiny grasses. Occasionally this pattern is broken by red gravel cliffs upon which various species of wattle grow. Pervading this rather desolate landscape is a scarcity of water. Rainfall is unreliable and unevenly distributed; when it comes it averages less than twenty centimeters per year.[2] Drinking water collects in waterholes while the few lakes, usually dry, hold undrinkable saltwater after heavy rains. Fred Myers describes the situation succinctly for the Pintupi, another Australian aboriginal people.

> For hunters and gatherers, the unreliability of water supplies poses the fundamental subsistence challenge. It is important to understand the nature of this resource. Although there are no permanent surface waters in the area, the Pintupi have found it possible to exploit other types of water supply. They have used large, shallow, transient pools formed by heavy rain; claypans and rock reservoirs in the hills that might be filled from lighter rainfalls; soakage wells in sandy creek beds; and 'wells' in the sand or in the rock between the sand ridges.[3]

Living off of this land is challenging, but the Warlpiri, like the Pintupi, have adapted to it, obtaining a variety of food substances from grass seeds to kangaroos as they move from water hole to water hole. This environment of the western desert, although not a determiner of Warlpiri culture, has a profound influence on their cosmology and their sand stories.

The Warlpiri trace descent from totemic ancestors who are personified environmental entities like rain, the honey ant, fire, and the yam. The wanderings of these ancestors created the present day features of the desert landscape. All of the Warlpiri ancestors traveled routes that can still be located. Indeed, major features of the landscape are the result of ancestral footprints, imprints from an ancestor sitting or lying down, or transformations of parts of an ancestor's body into a geological or topographical feature. Rock formations are metamorphosed limbs and genitals, water holes were dug by ancestral beings looking for food, and dried river beds were cut when an ancestral being dragged his tail or some object along the ground.

These geology-altering and geology-creating events took place in the time the Warlpiri call the *Jukurrpa*, or

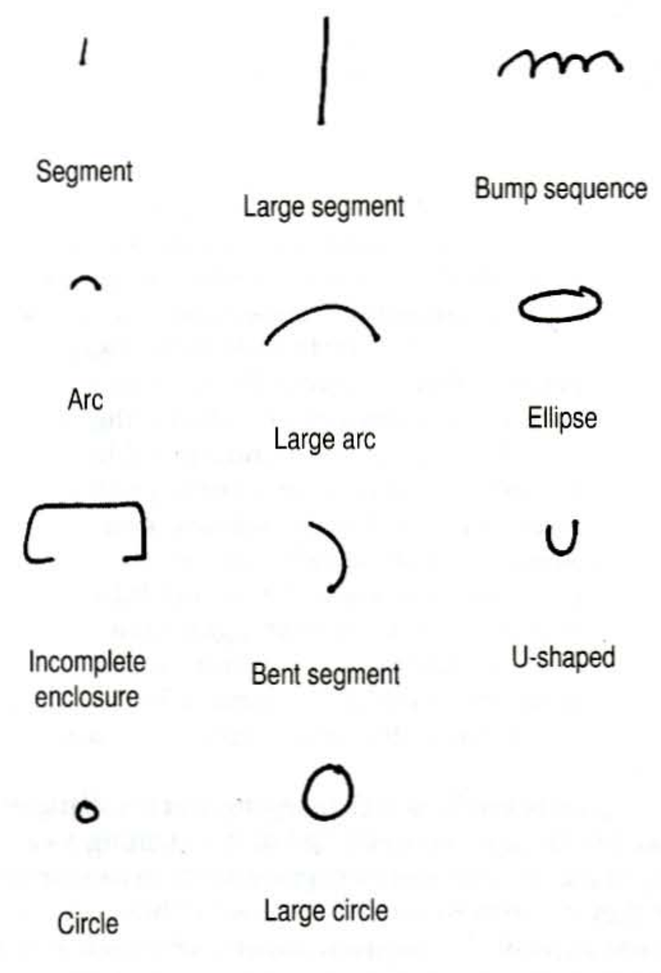


Figure 1
Sand story signs.

the Dreaming. The Dreaming, which is not unique to the Warlpiri, is arguably the most well-known and most enigmatic aspect of Australian aboriginal culture. The Dreaming refers to the time of the creation of people and of the world. Stories of the Dreaming are invariably tales of journeys. People and creatures and spiritual beings traverse the landscape, creating new landscapes, beginning traditions, defining classifications, and providing the reasons for why things in the natural world are the way they are. When a story about the Dreaming is told, it is considered a true story, a fact, a description of what is.

The Dreaming is a time of the creation of the world, but it is also the present. It is always there. The Dreaming is real, an accurate description of the world and a true history of what happened. The Dreaming is symbolic, a structure of ideas, values, and social norms.

2. FORMAL MODELS OF WARLPIRI ICONOGRAPHY

The iconographic designs of the Warlpiri reflect the Dreaming metaphysic, represent the ancestral actors of the Dreaming, and relate the Dreaming stories. Here I will examine four categories of Walpiri iconography from a formal algebraic point of view.[4] My analysis sees the iconographic systems as formal languages. Formal languages are algebraic systems consisting of two sets. One set is a finite set of symbols called the *alphabet* of the language, and the other is a set of *words*. The words of the language are created by combining symbols of the alphabet in certain specified ways. (The analogy with human languages is intentional!) It is usually the case that some combinations of alphabet symbols are words of the language and some are not. For example, if we consider English to be a formal language, then *elephant* is certainly a word of the language, whereas the string of English alphabet letters *glpoki* is not.

For formal languages, strings of alphabet symbols which are words are separated from those which are not by something called a *formal grammar*. Formal grammars provide the rules for constructing words and for excluding non-word combinations. The most familiar formal grammars are those of computer programming languages which provide the legal syntax from statements. The formal grammar of the computer language BASIC will allow us to write A = C + 10, but not = C A 10 +.

As an example of a formal language and its grammar, consider the infinite set of words {01, 0011, 000111, 00001111, ...}. This language has the set {0,1} as its alphabet. Its set of words includes only those strings that begin with one or more zeros and end with a like number of ones. Its grammar can be expressed with these rules:

1. S := 01
2. S := 0A1
3. A := 01
4. A := 0A1

The letter S is a special symbol called the *start symbol*. We may generate words in the language by beginning with any rule that has S on its left side (Rules 1 and 2 in this example). If we select Rule 1 we get the word 01. If we select Rule 2 then we get 0A1. Because A is not in the alphabet (symbols not in the alphabet are called *nonterminals*) we do not yet have a word. Thus, we must select a rule with A on its left side and replace A by the right side of that rule. For example, if we apply Rule 3, the string 0A1 becomes the word 0011. On the other hand, if we apply Rule 4, 0A1 becomes the string 00A11. We continue to apply rules until a string consisting only of alphabet elements (i.e. a word) results. It is easy to see that the four rules given above will produce only the words in the set {01, 0011, 000111, 00001111, ... }.

It is customary in formal language theory to combine rules that have the same left hand side into one rule with the options separated by pipes (|). Using this convention, our grammar becomes the two rule grammar:

1. S := 0A1 | 01
2. A := 0A1 | 01

In my discussion of Warlpiri iconography, I will suggest a formal grammar for the formation of iconographic complexes. To provide cultural context, I will also discuss the use of the iconography by the Warlpiri. It should be kept in mind that the algebraic analysis is intended as a model of the iconographic systems and is not necessarily the way the Warlpiri view their system. Whether or not the iconographic systems are actually formal algebraic systems is discussed in Section 3.

2.1 Sand Stories

Sand stories are usually told by women. A patch of ground is swept clean by hand and a story is related by drawing figures in the sand, singing corresponding songs, and providing minimal narration. The sand stories are about ancestral events and the Dreaming. Sand stories involve the patterns of daily life like gathering food, traveling, interpersonal relationships, birth, death, and ceremonies.

The nature of the sand story signs reflects the medium in which they appear. The signs are drawn in the sand with the fingers and thus consist of simple lines, and curves. Figure 1 shows the basic sand story signs.

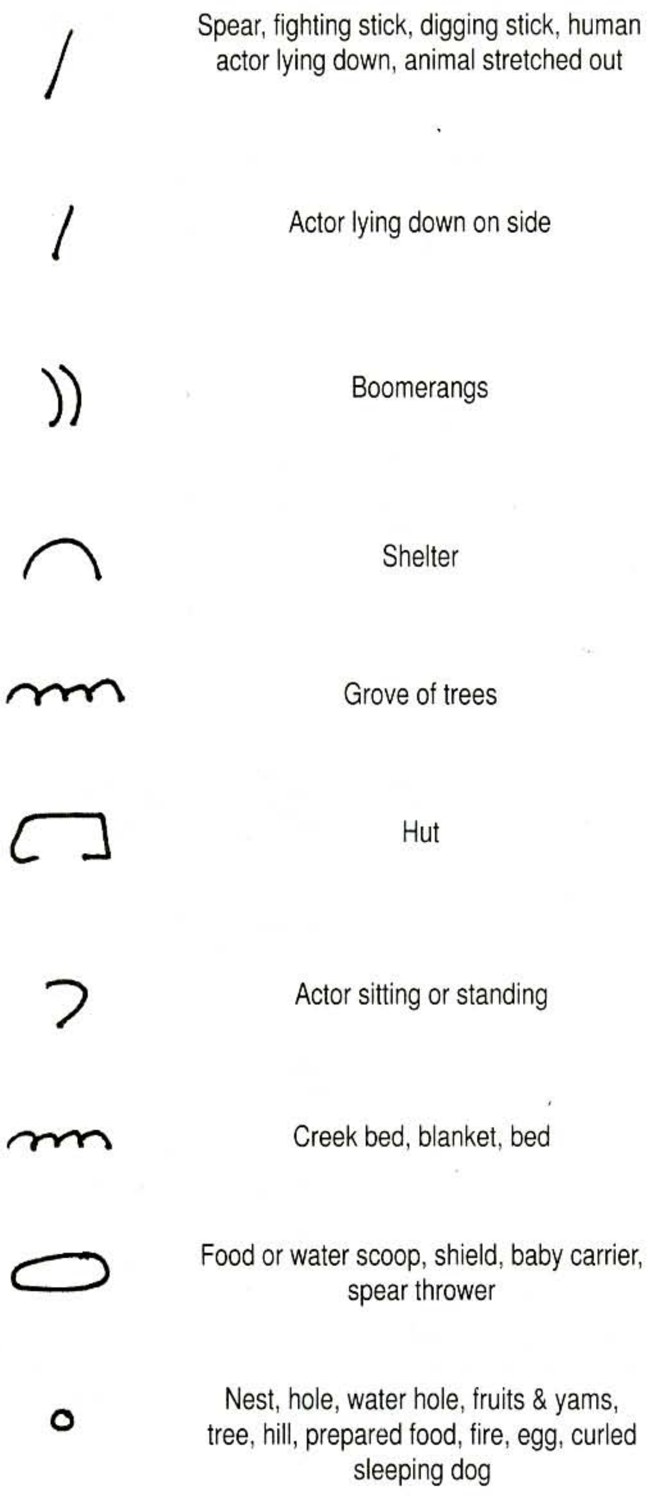
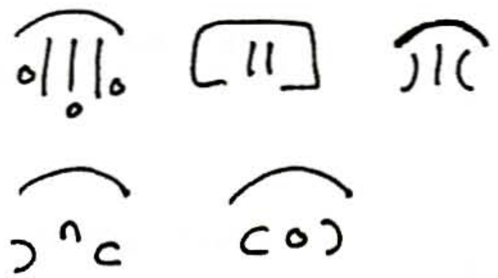
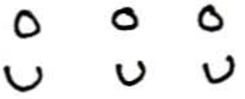

Spear, fighting stick, digging stick, human actor lying down, animal stretched out
Actor lying down on side
Boomerangs
Shelter
Grove of trees
Hut
Actor sitting or standing
Creek bed, blanket, bed
Food or water scoop, shield, baby carrier, spear thrower
Nest, hole, water hole, fruits & yams, tree, hill, prepared food, fire, egg, curled sleeping dog

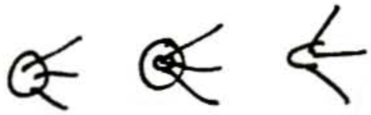
Figure 2
Range of meaning of single signs.



Camps



Foraging



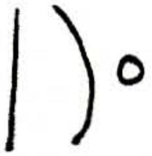
Going into the ground

Figure 3
Sand story scenes.

These signs are combined in a finite number of ways to create sand story scenes. The scenes are then sequenced to form the sand story. The ranges of meanings of the single signs are given in Figure 2. A formal language can be described that takes the basic story signs as its alphabet and produces sand story scenes as words.[5] Some typical sand story scenes are shown in Figure 3.[6]

A formal language of the sand stories, which I will call **SAND**, may be defined as follows. I use the descriptive titles from Figure 1.

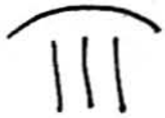
SAND: The alphabet is the set {small **s**egment, long **S**egment, ben**t** segment, **b**umps, small **c**ircle, large **C**ircle, small **a**rc, large **A**rc, **U**-shaped, **I**ncomplete enclosure, **e**llipse, [ ], { }, < >}. The **bold** letters will be used as alphabet symbols in the grammar. Thus, **stc** signifies a segment followed by a bent segment followed by a circle. Like this:



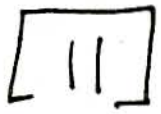
It is also necessary to introduce special symbols (like accent or punctuation marks) to specify how the combinations are to be constructed. So, I've added the following symbols to my alphabet[7]:

[ ] indicates that the enclosed string is below the previous symbol
{ } indicates that the enclosed string is inset into the previous symbol
< > indicates that the enclosed string converges on the previous symbol

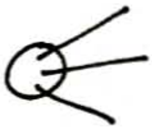
For example, **A[SSS]** represents the scene:



I{ss} represents:



and **c<S[S[S]]>** represents:

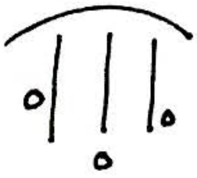


The formal grammar for **SAND** is given by the following rules:

1. Start := Camp | Forage | Finale
2. Forage := **c[U]** | **c[U]** Forage
3. Camp := Regular | Ceremonial

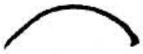
4. Regular := **A** [P]
5. P := P1 | P2 | P3 | **b**[**b**]
6. P3 := P1 P2
7. P2 := P1 P1
8. P1 := Sleep | Sit
9. Sleep := **S** | **Sc** | **cS** | **Sc** | **S**[**c**] | **t**
10. Sit := **U** | **U**{**c**} | **U**{**Uc**}
11. Ceremonial := **A**[**b**[**b**]] | **(cccc)**[**C**{**c**}] | dance
12. dance := dancers [singers]
13. dancers := **b** | **b**[dancers]
14. singers := **U**[**U**[**U**]] | singers[**U**]
15. Finale := F<**S**[**S**[**S**]]>
16. F := **c** | **C**{**c**} | **U** | **C**{**U**}

The **SAND** grammar can be seen to produce each of the sand story scenes in Figure 3 as well as many others. For example, the camp scene:

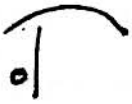


may be generated by the following sequence of grammar rules:

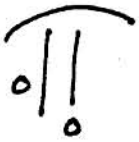
Rule 1: Start := Camp
Rule 3: Camp := Regular
Rule 4: Regular := **A**[P] **A**



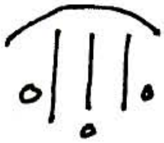
Rule 5: P := P3
Rule 6: P3 := P1 P2
Rule 8: P1 := Sleep
Rule 9: Sleep := **cS** **A[cS]**



Rule 10: P2 := P1 P1
Rule 8: P1 := Sleep
Rule 9: Sleep := **S[c]** **A[cSS[c]]**



Rule 8: P1 := Sleep
Rule 9: Sleep := **Sc** **A[cSS[c]Sc]**



Considered as a formal language, **SAND** has several interesting features that reveal something about the iconography it models. For example, the rules pertaining to the ceremonial dances (Rules 11 -14) are infinitely recursive allowing any number of singers and dancers. Actual sand stories known to me go no larger than three rows of dancers (bump sequences) and five singers (U-shapes), but although there are certain practical limits, there appears to be no potential limit to the size of this scene.

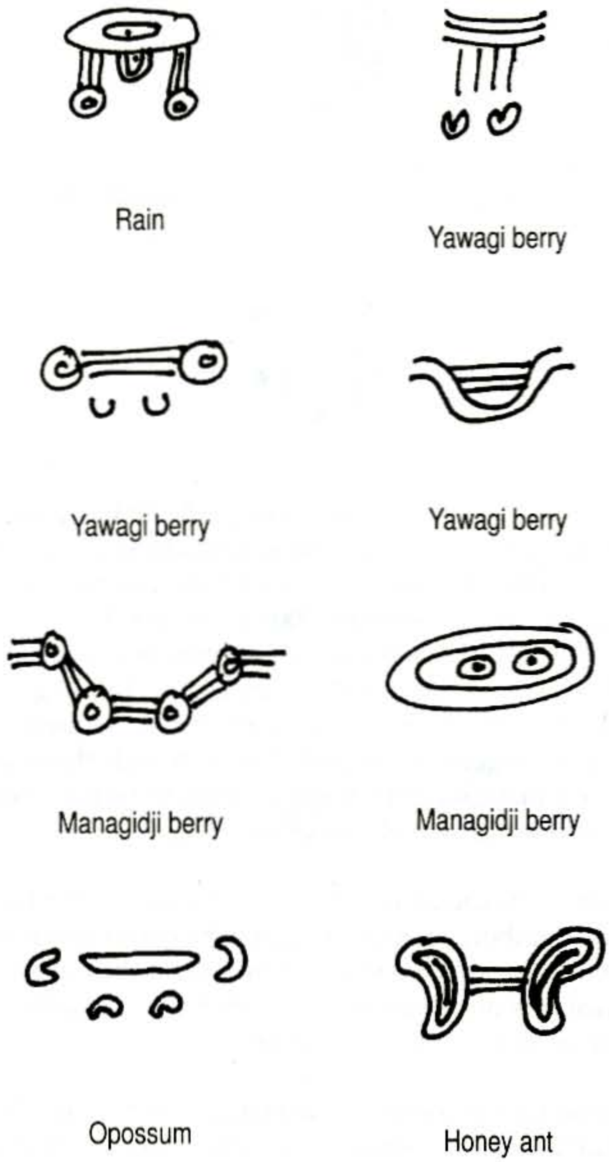
In contrast to the dances, there is a definite upper limit on the number of people occupying a campsite enclosure. This is reflected in the grammar by the bounded derivations possible through Rule 5 which limits enclosures to three adult inhabitants.

We now turn to the female designs known as *yawalyu*. Nancy Munn[8] has shown the iconographic relationships between sand stories and yawalyu. Here we will investigate their formal properties.

2.2 Yawalyu

Yawalyu designs are generally revealed to women in dreams. A ceremony is usually performed to reveal a new design and the presentation of the design is accompanied by songs. The yawalyu dream is a story about the Dreaming. How the Warlpiri view these dreams is summed up by Nancy Munn:

> The Warlpiri view is certainly not that "life is like a dream" but more nearly the opposite: that whenever event sequences are cut off from the world of everyday life so that they seem to con-



Rain

Yawagi berry

Yawagi berry

Yawagi berry

Managidji berry

Managidji berry

Opossum

Honey ant

Figure 4
Yawalyu designs.

> stitute a closed totality of their own and can be "talked about" but not "lived through" in the day-to-day involvement of social life, then such events are *djugurba*—stories, dreams, the ancestral past.[9]

Yawalyu designs represent specific totemic species that refer to ancestors. Some typical yawalyu designs are shown in Figure 4.

Notice that the number of signs in yawalyu designs is much less that that of the sand story scenes. (Figure 5 shows the basic yawalyu design elements.[5]) Consequently, my formal language for yawalyu designs, which I will call **YAW**, has a very small alphabet. Specifically, the formal language for yawalyu designs may be given as follows.

> **YAW**: The alphabet is {**U**-shaped, **R**ing, **C**ircle, **S**tick, **P**arallel line segments, **H**ooked segment, [ ], { }}.

As was the case with the language **SAND**, it will be necessary to introduce special symbols (like accent or punctuation marks) to specify how the combinations are to be constructed. I use the same conventional symbols as I did with the sand story language. Thus,

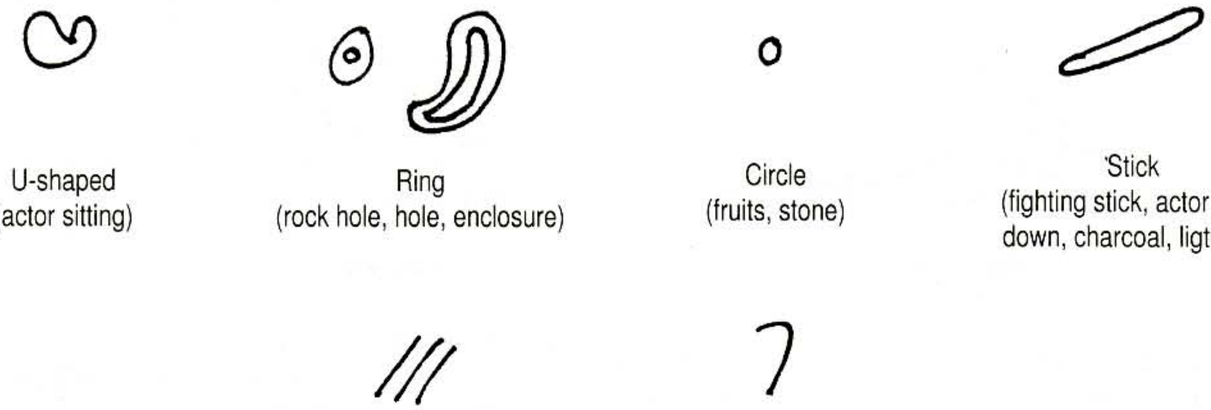
[ ] indicates that the enclosed string is below the previous symbol
{ } indicates that the enclosed string is inset into the previous symbol

The grammar for **YAW** has the following rules:

1. Start := Arch | Locus | Path | Triad | **R**{**R**{EE}}
2. Arch := Cover[Attached]
3. Cover := **P** | E | Bar
4. Attached := ABA | **P**[**UU**] | **UCU**
5. A := **P**[E]
6. E := **R**{**C**}
7. Bar := E Pseg
8. B := **U**{**C**}
9. Locus := **U**[**U** Core **U**][**U**]
10. Core := **P** | **S**
11. Path := Pseg Path | **P** | Pseg Pseg **P**
12. Pseg := **P**E
13. Triad := **R**{**H**} Center **R**{**H**}
14. Center := **R** | **S** | **P**

These rules will generate the yawalyu designs shown in Figure 4 as well as many more, including some that have not been reported in the literature. An interesting test of the model would be to see if the grammar can predict yawalyu designs not yet dreamt.

A rule of particular interest is Rule 11 which generates site path designs, a fundamental and popular Warlpiri design that appears in contexts other that yawalyu designs. I turn my attention to these designs



U-shaped
(actor sitting)

Ring
(rock hole, hole, enclosure)

Circle
(fruits, stone)

'Stick
(fighting stick, actor lying down, charcoal, ligtning)

Parallel line segments
(falling rain, paths, headbands, teeth, yawalyu designs)

Hooked segment
(lightning)

Figure 5
Basic yawalyu elements with range of meanings.

now.

2.3 Site-Path Designs

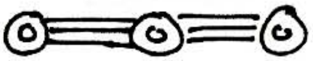
Site-path designs, usually drawn by men but sometimes occurring in yawalyu designs, depict the various routes taken by ancestral beings in the Dreaming. Consisting almost entirely of parallel lines and concentric circles, site path designs appear in Australian aborigine rock art, body painting, on shields and other artifacts, and in contemporary art for sale.[11] A typical example of a site-path design appears in Figure 6.[12]

Site-path designs are clearly reminiscent of formal graphs with the concentric circles serving as vertices and the parallel line segments as edges. One significant difference between mathematical graphs and Warlpiri site-path designs is that the Warlpiri designs allow for free edges connected to only a single vertex. My formal language for site-path designs, which I will call **SITE**, is as follows.

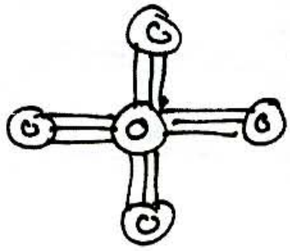
> **SITE**: The alphabet for **SITE** has only two Warlpiri elements, but also contains some special symbols and the natural numbers. The alphabet is {**P**arallel lines, **C**oncentric circles, :, ( ), $, 1, 2, 3, 4, 5, ...}.

I'll use the numerals, parentheses, and colons to express complex linkages. A **PC** sequence preceded by a numeral *k* and a colon and enclosed in parentheses is attached to the *k*th concentric circle of the preceding or following sequence. Multiple numerals indicate multiple attachments. Thus, we have the following notations:

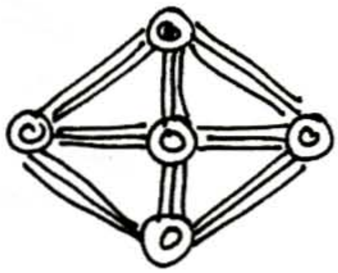
CPCPC for:



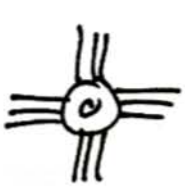
(2:**CP**)**CPCPC**(2:**PC**) for:



and (1:2:3:**CP**)**CPCPC**(1:2:3:**CP**) for:

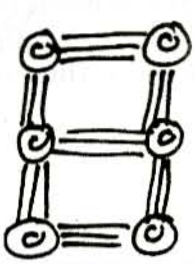


Free edges attached to the same concentric circle are denoted by repeating the **P** symbol. Thus, we have the following notation:
PPPPC for:

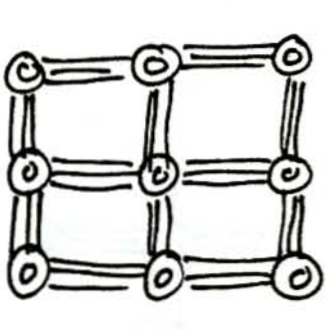


The dollar sign, $, applied to a site path sequence enclosed in parentheses attaches a copy of that sequence via corresponding concentric circles. Thus, we have the following notations:

(**CPCPC**)$ for:



and (**CPCPC**)$$ for:



We may then give the grammar rules for SITE as follows.

1. Start := Road | Star | Poly | Net | Five
2. Road := **CPCPC** | Road **PC**
3. Star := Free | Jack
4. Free := **PPC** | **P** Free
5. Jack := (2:**CP**)Road (2:**PC**)
6. Poly := (1:2:**PC**)**CPC** | (1:3:**PC**)**CPCPC**
7. Net := (Road) $ | Net$
8. Five := (1:2:3:**CP**) **CPCPC** (1:2:3:**CP**) | (1:2:3**CP**) ((1:2:**PC**)**CPC**)

Rules 6 and 8 are potentially families of rules. Larger polygons and complete graphs may be specified by using longer "Roads" and changing the numerals in the parenthetical attachments. The alternative to the family of rules is a context sensitive grammar or some sort of regulated rewriting system.[13] However, the site path designs known to me are small in size and only the "Roads" and "Nets" seem to be potentially infinitely extendable.

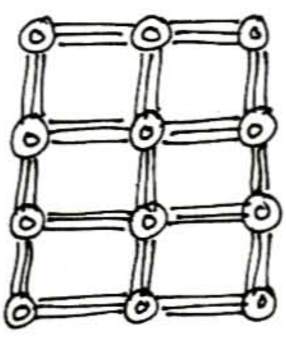


Figure 6
A Site-Path Design

2.4 Guruwari

Guruwari are men's ancestral designs. Guruwari are painted on ceremonial regalia, boards, stones, the ground, and on bodies. They tell the stories of the ancestors, their travels, the founding of the clans, history, ecology, geography, and geology of Warlpiri country. Guruwari designs are powerfully charged with dream value. They originate in dreams and tell about the Dreaming.

Figure 7 presents a sampling of men's ancestral designs. Notice that the designs, like the sand story scenes and the yawalyu, are complexes composed of a small set of basic signs. The basic signs are variables, taking on a variety of semantic values. The undulating line can represent both "snake" and "lightning" and the dots may be "eggs" or "ants."

Also, notice that the undulating line and straight line serve as base symbols which are flanked on both sides by the satellite symbols (dots, small circles, short pairs of parallel lines, etc.). These patterns make the formal language of men's ancestral designs, I'll call it **GURU**, relatively straightforward.

> **GURU**: The alphabet for the formal language of guruwari designs is {**S**nake line, **D**ots, **C**ircle, d**A**shes, se**G**ment, **F**ootprints, &, ( ), { }, [ ]}. Figure 8 shows each of these alphabetic elements.

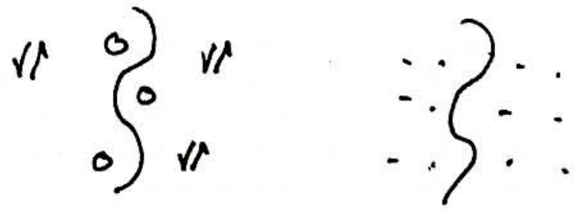
I will also use the braces to denote inset symbols and the square brackets to denote positioning under a symbol as I did in the **SAND** grammar. In addition, notice that the guruwari designs often include a scattering of an arbitrary number of basic elements as in Figure 7c (honey ants). To denote this scattering, **GURU** includes the symbol & as a prefix to denote scattering. Thus, Figure 7c may be represented as **&DG&D**.

The grammar of **GURU** consists of the following rules:

1. Start := Adjunct Core B
2. Core := **S** | G | **c[p[c[p[c]]]]**
3. c := **C**{**C**}
4. p := **GG**
5. Adjunct := **&D** | **&C** | **&F** | **&A**
6. **&D** Core B := **&D** Core **&D**
7. **&C** Core B := **&C** Core **&C**
8. **&A** Core B := **&A** Core **&A**
9. **&F** Core B := **&F** Core **&F**

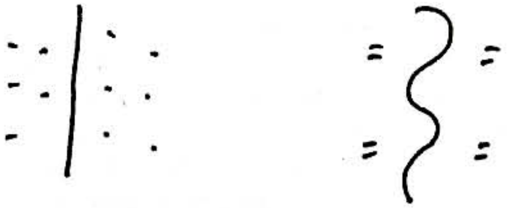
Rules 6-9 are context-sensitive rules that are necessary to insure that the left adjunct is the same symbol as the right adjunct.

We have seen how various subsets of Warlpiri iconography may be modeled as formal languages with context-free or context-sensitive rules. It is natural to ask at this point if this is but an empty exercise or is the Warlpiri iconography a mathematical system of some sort. A beginning of an answer to that question is the topic of the next section.



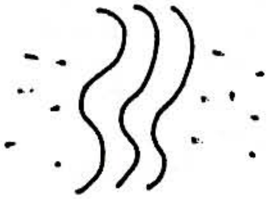
Snake with ribs and footprints

Snake with eggs



Central passage with honey ants

Lightning with rain clouds



Rain

Figure 7
Guruwari.

3. IS WARLPIRI ICONOGRAPHY MATHEMATICS?

Originally, I began to think of Warlpiri iconography as a formal language in an attempt to provide an interesting, but manageable, formal language modeling examples for my computer science and mathematics students. The Warlpiri designs are appealing in their own right, just unusual enough to engage students, much more interesting than arbitrary sequences of letters, and smaller in scope than natural languages or computer languages. Additionally, Warlpiri designs are a real set of objects with an uncoverable formal algebraic structure. As such they provide an entry to the mathematical modeling of human artifacts.

My students were successful in producing formal languages with grammars for selected collections of Warlpiri designs.[14] However, the question of whether or not the iconographic system of the Warlpiri is a Warlpiri mathematics arose frequently in our discussions. That is, can we say that the iconographic system of the Warlpiri is a mathematical system?
A short answer to the question, in my opinion, is a tentative yes, the Warlpiri iconographic system is mathematics. The Warlpiri iconographic system has the components that we expect of an algebra interacting in a way that is only slightly different from the

abstract algebra we learned.

The visual algebraic components of Warlpiri iconography are obvious. We have a finite set of symbols and the symbols may be combined to make more complex structures according to a finite set of rules. Compare this to the formation of equations in college algebra where the letters x and y, the integers 2 and 3, and the symbols =, + ,*, and ^ combine to make the complex notion of a quadratic function: $y = x^\wedge 2 + 3x$.

Beyond what we see in the iconography is its power to model the real world of the Warlpiri. The iconographic designs are, to a large extent, models of the Dreaming, the fundamental reality of the Warlpiri. The Warlpiri iconographic system can, in this way, be seen as a mathematical model of aspects of reality parallel in form and function to the mathematical models of trajectories we study in college algebra.

The Warlpiri recognize the components and rules of their iconography and they recognize its modeling functions. What they don't seem to have is a "theory of iconography" that abstracts general patterns from the sand scenes, yawalyu, and guruwari. Here we may question whether or not the iconography is mathematics. Perhaps we are safer to say, as Marcia Ascher has suggested, that the Warlpiri iconography be called "mathematical ideas" rather than mathematics.[15]

I prefer to leave the question open at this time. Obviously, that we can model the iconography with mathematics does not imply that the iconography is mathematics, but neither does it imply that it isn't. The quandary may be resolved by researchers working in the field of ethnomathematics.

Ethnomathematics includes "all practices of a mathematical nature, such as sorting, classifying, counting, and measuring, which are performed in different cultural settings, through the use of practices acquired, developed, and transmitted through generations."[16]

The Austrian mathematician Roland Fischer provides a way of viewing mathematics that is helpful in understanding "ethnomathematics." Fischer writes,

> Mathematics provides a *means* for individuals to explain and control complex situations of the natural and of the artificial environment and to communicate about those situations. On the other hand, mathematics is a *system* of concepts, algorithms and rules, *embodied in us*, in our thinking and doing; we are subject to this system, it determines parts of our identity.[17]

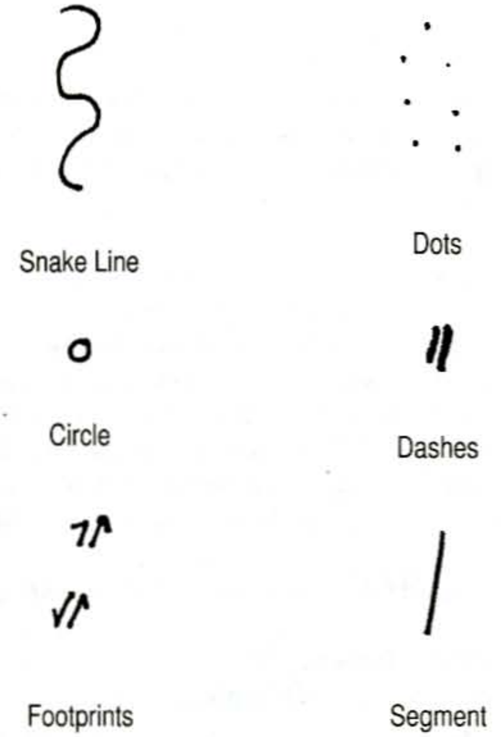


Figure 8
Guruwari alphabet.

When mathematics is viewed as a means and as a system embedded within a culture, our understanding of what mathematics enlarges to encompass much more than formal school mathematics. Instead, mathematics includes a multitude of practices that are characterized by algorithms, formal processes, and abstraction. In this context, the Warlpiri iconography emerges as an ethnomathematical system.

Whatever our position on the mathematical nature of Warlpiri iconography, one thing, however, is clear. The Warlpiri have developed a sophisticated symbolic system for describing their world.

A pervasive myth in the history of mathematics is that the Australian aborigines are one of the least competent mathematical thinkers in the world. These arguments arise from early anthropological reports of the simplicity and lack of power of aboriginal counting systems. These reports were misguided at best, reflections of a cultural superiority complex at worst. It has been shown that the aboriginal people can count perfectly well if they want to. However, traditional aboriginal culture had no need for counting because it did not value possessions. If they counted at all, aborigines counted for purposes of sharing or sorting.[18] The case of Warlpiri iconography suggests that the simplicity may have been on the part of the European anthropologists. They were looking for counting and arithmetic in Aboriginal culture, but they missed abstract algebra!

NOTES

[1] Ethnographic information on the Warlpiri is from Nancy Munn's *Walbiri Iconography* (Chicago: University of Chicago Press, 1986).

[2] See *Aboriginal Man in Australia*, edited by D. Mulvaney and J. Golson (Canberra: Australian National University Press, 1971).

[3] See p.26 of Fred Myers ethnography of the Pintupi, *Pintupi Country, Pintupi Self* (Berkeley: University of California Press, 1991).

[4] The categories of Warlpiri iconographic designs and the typical examples are from Nancy Munn's *Warlpiri Iconography* (Chicago: University of Chicago Press, 1986).

[5] The sand stories themselves may be considered a subset of the nth order Cartesian product on the set of sand story scenes or as a formal language in their own right.

[6] Adapted from *Warlpiri Iconography* by Nancy Munn (Chicago: University of Chicago Press, 1986), pp.70-71.

[7] The sand stories are supplemented by finger movements showing direction of action. These non-pictorial signs, although important aspects of the sand story are not dealt with in my grammar which focuses on the static aspects of Warlpiri iconography.

[8] Op. Cit. pp. 89-118.

[9] Ibid. p.117.

[10] Nancy Munn (Ibid. p.104) sees five basic elements, but in my opinion her data clearly show six.

[11] See Rockman, Peggy and Napaljarri Cataldi, *Warlpiri Dreamings and Histories* (San Francisco: HarperCollins, 1994); Layton, Robert, *Australian Rock Art: A New Synthesis* (Cambridge: Cambridge University Press, 1992); and Morphy, Howard, *Ancestral Connections* (Chicago: University of Chicago Press, 1991).

[12] Adapted from Rockman and Cataldi, Op. Cit., Plate 4.

[13] See Dassow, J. and G. Paun, *Regulated Rewriting in Formal Language Theory* (Berlin: Springer-Verlag, 1989).

[14] Also successful in this endeavor were a group of junior high students faced with the same task. The results of this little experiment leads me to believe that similar formal language writing tasks may offer an earlier entry into modeling with abstract algebras.

[15] Personal communication, February 1996.

[16] D'Ambrosio, Ubiratan "Ethnomathematics: A Research Program on the History and Philosophy of Mathematics with Pedagogical Implications," *Notices of the American Mathematical Society*, Volume 39, No. 10, pp.1183-1185 (1992).

[17] Fischer, Roland. "Mathematics as a means and as a system". In Restivo, Sal; van Bendegem, Jean Paul; and Roland Fischer, Eds. *Math Worlds: Philosophical and Social Studies of Mathematics and Mathematics Education* (Albany: State University of New York Press, 1993), pp.113-133.

[18] Several detailed discussions of Australian Aboriginal counting systems and practices may be found in the *Work Papers of SIL-AAB, Series B, Volume 8, Language and Culture*, edited by S. Hargrave (Darwin: Summer Institute of Linguistics, Australian aborigine Branch, 1982).

Book Review: *The Crest of the Peacock: Non-European Roots of Mathematics*, by George Gheverghese Joseph

Bernadette A. Berken
St. Norbert College
De Pere, Wisconsin 54115

"It is a pioneering book that celebrates the magnificent heritage of non-Western mathematics and challenges the reader to cast off limiting European bias and see mathematics and its development as the product of civilizations from every corner of the globe."

The Crest of the Peacock: Non-European Roots of Mathematics. George Gheverghese Joseph. Penguin Books: London, 1990. 371 pp, ISBN 0-14-012529-9.

An exciting global mathematical journey awaits the reader of George Gheverghese Joseph's *The Crest of the Peacock-Non-European Roots of Mathematics*. Joseph is an apt mathematical tour guide and leads his readers on an intellectual journey to the four corners of the earth in search of an accurate understanding of the historical development of mathematics. What makes Joseph particularly suited for this challenging task of weaving a solid tapestry of mathematical history is the rich and diverse background that he possesses. He is the product of four different heritages: He was born in Kerala, Southern India, and spent the first nine years of his life there, steeped in the music, customs, and the rich diversity of Indian culture. Coming from a family of Syrian Orthodox Christians brings a second perspective to his background. Living and growing up in Mombasa, Kenya with a rich mixture of African and Arab influences adds a third aspect to his background while his studies in Britain at the University of Leicester and the University of Manchester furnishes his final Western heritage. In addition to these four significant and diverse heritages, Joseph's many travels and job experiences abroad contribute to his inclusive perspective of the global development and history of mathematics.

This inclusive perspective compels Joseph to clearly state that the capacity to 'make' science and technology (and mathematics) is not the prerogative of one culture alone. His book diverges sharply from the typical treatment of the history and development of mathematics that tends toward an extreme bias in favor of the early contributions of the Greeks and the subsequent domination of mathematical development by Europe and her cultural dependencies. Instead, Joseph substantiates the development of mathematics before the Greeks and celebrates the contributions of peoples from many diverse cultures around the world. Additionally, based on sound evidence, he proposes alternative perspectives for the development of mathematics and the diverse transmission of mathematical knowledge across cultures emphasizing the global nature of mathematical pursuits and suggesting the possibility of independent mathematical development within each culture.

After a short chapter introducing the reader to the global perspective of mathematical development, Joseph begins his global mathematical journey with a brief chapter that explores proto-mathematics, the mathematics that existed when no written records were available. Here he includes an examination of and conjectures about some very early bone artifacts that may well exhibit some of the earliest evidence of numerical recording. Inca *quipus* and the Inca abacus compose a majority of this chapter where Joseph explains the logic and usefulness of both. Counting systems and Mayan numeration and calendrics round out the chapter. Although some people may argue that these considerations should not be included in an examination of mathematical development, Joseph soundly refutes objections to their inclusion.

Throughout his book, Joseph emphasizes the global nature of mathematical pursuits. Nevertheless, he is unable to include every culture within the book. It would be unrealistic to expect anything else. Joseph does not include the mathematical experiences of native North America, Korea, Japan, or most of Africa. Nor does he elaborate on Hellenistic mathematics since Greek mathematics is the usual fare of most other books of this type. Instead, he chooses to focus on the development of mathematics in Egypt, Babylonia, China, India, and the Arab world.

In each of these cultures, Joseph explores mathematical development chronologically yet within the social, historical, and religious context of the particular culture. Further, he makes numerous connections among the various cultures so that the reader easily perceives the interactions that occurred between cultures and the process by which mathematical knowledge was transmitted and grew. Using available primary sources, Joseph examines each culture's counting system, including bases and numerals, as well as the algebraic, geometric, and trigonometric pursuits of each. In addition, he includes the significant or unique contributions of the culture. Frequently Joseph poses questions that challenge familiar and commonly held opinions that stem from a narrow Euro-centric bias.

Numerous maps, charts, tables, photos, and sketches contribute important detail to the text. Throughout the book, Joseph copiously sprinkles in examples taken from the original sources to illustrate important mathematical ideas. Although many scholars of the history of mathematics tend to label all mathematics before the Greeks merely as utilitarian and pre-scientific, Joseph dispels this view often in his exposition where numerous contributions by non-Hellenistic ancients around the globe are shown to be quite remarkable; what we today might call "awesome."

Because Joseph so competently incorporates a great variety of convincing evidence from a number of historical sources, the reader easily sees the unity of what we call mathematics. The strong historical profile that Joseph provides for each culture allows the reader to more fully understand why a specific culture focused its efforts on particular mathematical pursuits.

This superb book is a clearly written treatise that is an outstanding contribution to a true and more complete understanding of what comprises mathematics and the process by which mathematical knowledge came to be. It is a pioneering book that celebrates the magnificent heritage of non-Western mathematics and challenges the reader to cast off limiting European bias and see mathematics and its development as the product of civilizations from every corner of the globe. This literary work of art offers the reader both truth and beauty. Don't miss out on reading it!

Poetry

Sascha Cohen, sixth grade
Hale Middle School, Los Angeles, CA
Submitted by Margaret Schaffer, teacher

Red and blue
bumpy grass
sharp
an angle
measuring a wide 140 degrees
is close up by two thin acute corners
they make up the pointed yellow
obtuse triangles
that look like Swiss cheese
scattered in this design
and there is
a little green hexagon.

Framing each of the polygon's
six sides
are
deep purple
rectangles
all with
four straight parallel
lines
that form
90 degree angles.
Their lines are side
by side
connected
only to shape
a glorious
decagon
and around that
is an outer ring of
diamonds and
rhombuses.
And then the squares!
Each congruent square
was more beautiful than the last.
It grew more confusing
and less symmetrical
with each set
of patterns
and geometric figures
little green hexagon in the middle
sitting still
my mind now twisted
my eyes stretched as I
stand back and look at this
immense stained glass window.

Al-Khawarizmi's Algebra: The First Paradigm in Algebra

Murad Jurdak
American University of Beirut
Beirut, Lebanon

The rationalist historians of mathematics consider the history of mathematics as the history of homogeneous growth of mathematical knowledge using essentially the unchanged axiomatic method. The fallibilists regard the growth of mathematics as a result of a dialectical process in which counter-examples to conjectures (theorems) lead to restructuring knowledge in mathematics or in its sub-fields.

INTRODUCTION

This paper addresses the question of the first paradigm in algebra, an achievement universally recognized to be that of the Arab mathematician Mohammed Ibn Musa Al-Khawarizmi (first half of the ninth century AD). First, the paper will discuss the usefulness of Kuhn's concept of paradigm in describing major developments in mathematics and education. Second, it will describe the pre-Al-Khawarizmi paradigm in algebra (henceforth, referred to as the prehistoric paradigm). Third, Al-Khawarizmi's algebra will be described. Then fourth, Al-khawarzmi's algebra will be analyzed as a paradigm, and finally some pedagogical implications will be discussed.

KUHN'S PARADIGM

Kuhn[1] contends that the development of science proceeds in paradigm shifts which involve revolutionary transitions. For Kuhn, "a paradigm is what the members of a scientific community share, and, conversely, a scientific community consists of men who share a paradigm" (p. 176). In the first sense (i.e. what a scientific community shares) a paradigm may be viewed as a "disciplinary matrix" whose components include:

> **1.** Symbolic generalizations (expressions universally accepted by members of a scientific community).
>
> **2.** Shared commitments to certain beliefs among members.
>
> **3.** Values which are widely shared among different communities belonging to the same discipline.
>
> **4.** Shared examples of concrete problem-solutions found in laboratories, textbooks, examinations, and journals (p. 182).

In the second sense, a paradigm is viewed as the shared examples themselves which provide the basis for acquired similarity relations that enable the members of a scientific community to regard similar situations as subjects for applying the same scientific law. According to Kuhn, a revolution is a special "sort of change involving a certain sort of reconstruction of group commitments" (p. 181). Kuhn argues that in normative science, this change is triggered by a crisis generated by incompatible ways of practicing the discipline by the particular scientific community.

To what extent do the constructs of *paradigm* and *paradigm shift* describe mathematics and its historical development? The paradigm construct seems to be applicable to mathematics in both meanings of *paradigm*. Mathematicians have always constituted a well-defined group that has shared a sophisticated system of symbolic generalizations, commitments to accepted beliefs and values, and a distinct body of examples with highly structured relations.

When it comes to the description of the historical development of mathematics, sharp disagreements arise. One can recognize two schools of thought in this regard. The rationalist historians of mathematics consider the history of mathematics as the history of homogeneous growth of mathematical knowledge using essentially the unchanged axiomatic method. The fallibilists regard the growth of mathematics as a result of a dialectical process in which counter-examples to conjectures (theorems) lead to restructuring knowledge in mathematics or in its sub-fields. To Lakatos[2], who has best articulated the fallibilists' position, the inconsistencies and their refutations have led to changes in the dominant theory resulting in the reorganization of our knowledge. Thus, for instance, "the paradoxicality, and, indeed, seeming inconsistency of arithmetic induced the Greeks to abandon arithmetic as the dominant theory and replace it by geometry" (p. 125). Lakatos's interpretation of the history of mathematics seems to be consistent with Kuhn's concept of paradigm shift in science.

In a sense, these revolutions in education may be looked at as paradigm shifts not so much in the research concepts and methodology as in the conception and practices of education.

Education, however, is a different matter. It is very difficult to argue for a paradigm in education in Kuhn's sense. Historically, educators have neither formed a distinct group with shared symbolic generalizations, beliefs, values and exemplars, nor have the latter constituted a well-defined basis for theory-building. Nevertheless, there have been throughout history basic changes in education resulting in revolutions in educational concepts and practices. Examples of such revolutions are: the introduction of the alphabet, the shift of responsibility for teaching from home to school, and the introduction of printing. A fourth revolution is predicted as a result of computer technology. In a sense, these revolutions in education may be looked at as paradigm shifts not so much in the research concepts and methodology as in the conception and practices of education.

THE PRE-HISTORIC PARADIGM

Whatever algebra existed before Al-Khawarizmi had neither a specific form nor a specific name to distinguish it from other fields of knowledge. What actually existed was rudimentary knowledge of some concepts and techniques involved in quadratic equations not so much as independent and distinct techniques but rather as incidental solutions of specific and isolated problems. In the paragraph that follows, we present a brief historical account of algebraic analysis before Al-Khawarizmi. More details are given in Karpinski[3].

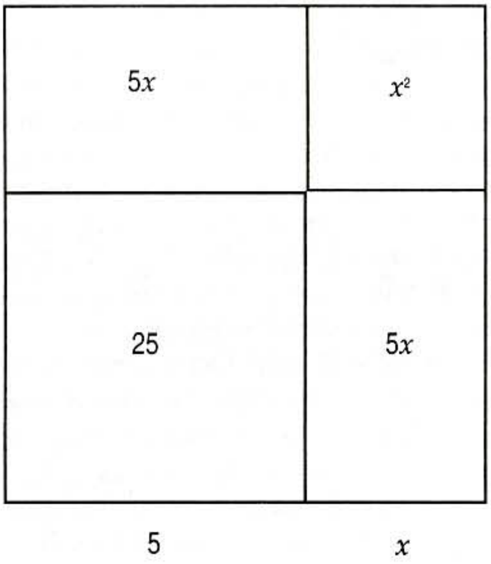
Simple equations of the first degree in one unknown are found in the oldest mathematical textbook, the Ahmes Papyrus of about 1700 B.C. Later quadratic equations appeared in Egypt in the context of area measurement. Square numbers such as $3^2 + 4^2 = 5^2$ were also used by Egyptians to construct right angles. Contemporary with the Egyptians, the ancient Babylonians also constructed tables of squares and cubes. Greek mathematicians were familiar with geometrical solutions of quadratic equations as early as the fifth century B.C. as it appears in the solutions of specific problems in the writings of Pythagoras, Hippocrates, and Euclid. Analytical solutions of quadratic equations appear around the beginning of the Christian Era in the works of Heron of Alexandria. Diophantus, the great Greek mathematician, solved analytically (about A.D. 250) the three types of quadratic equations ($ax^2 + bx = c$, $ax^2 + c = bx$, and $ax^2 = bx + c$, with positive coefficients and roots) in the context of solving other problems. In the fifth century B.C., Hindu mathematicians gave rules for the numerical solution of some quadratic equations using the method of completing the square. Though most of the algebraic ideas and techniques were known before Al-Khawarizmi, it is difficult to trace the algebra of Al-Khawarizmi to any of his predecessors. As Karpinski[3] comments:

> Yet we need to notice that we are dealing with the independent appearances of algebraic ideas and that the mathematics of Babylon, China, Greece, and India were developing from within (p. 11).

THE ALGEBRA OF AL-KHAWARIZMI

The algebra of Al-Khawarizmi will be briefly described using a photocopy (available at the Jafet Library of the American University of Beirut) of the English translation by Rosen[4] of the "Algebra of Mohammad Ben Musa" (Al-Khawarizmi). Rosen also included in his translation a printed version of the Arabic manuscript preserved in the Bodleian collection at Oxford.

Al-Khawarizmi starts his mathematical treatise by definitions of his basic mathematical terms: root or unknown, thing (variable), square (called 'money'), arithmetical operations, equality, equation. He then proceeds to define his mathematical concepts: first degree equation in one unknown, second degree equation in one unknown, binomial, trinomial, solution of an equation, proof. He ends this section by proving his first corollary that all six forms of quadratic equa-



$$(x+5)^2 = x^2 + 10x + 25$$
$$= 39 + 25$$
$$= 64$$

$$x+5 = 8$$

(negative roots were not recognized)

$$x = 3$$

Figure 1
Geometric proof for the case $x^2 + 10x = 39$

tions admissible under the conditions of positive solutions can be reduced to the three standard forms:

$$x^2 + px = q$$
$$x^2 + q = px$$
$$x^2 = px + q.$$

Al-Khawarizmi proceeds next to solve systematically each of the standard forms of quadratic equations using the method of completing the square, essentially in the form one would find in any high school algebra textbook three decades ago.

Geometric proofs are presented for each of the three cases. Figure 1 shows a geometric proof for the case of $x^2 + 10x = 39$.

Al-Khawarizmi, then, deals with binomials as numbers and introduces the four operations on them. Having developed the theory of quadratic equations, Al-Khawarizmi proceeds to apply his theory in four areas: numbers, mercantile transactions, measurement, and inheritance. The section on inheritance entitled "legacies" is by far the largest in the Al-Khawarizmi's book (more than half) and probably the most difficult to understand from a non-Islamic perspective. Inheritance laws are part of the Koran and hence are part of Islamic jurisdiction even in modern times. These laws—very sophisticated, detailed, and comprehensive—remain applicable even today. Because of the vastness of the Arab empire (from Morocco to China), many different algorithms existed in different parts of the empire. Developing standard techniques for dealing with inheritance problems was a priority for the judicial system of the state in order to work out inheritance deals and settle disputes. Al-Khawarizmi included in this section a comprehensive set of inheritance problems which model the various situations which may arise in Islamic inheritance laws. In many cases, the problems led to quadratic equations.

THE FIRST PARADIGM IN ALGEBRA

Historians of mathematics agree with the statement made by Karpinski[3] in the introduction to his translation to English of Chester's Latin manuscript:

> The activity of the great Arabic mathematicians Abu Abdallah Mohammed Ibn Musa Al-Khawarizmi marks the beginning of that period of mathematical history in which analysis assumed a place on a level with geometry; and his algebra gave a definite form to the ideas which we have been setting forth (p.13).

The work of Al-Khawarizmi in algebra constituted a core shared by a new scientific community of algebraists.

The prehistoric paradigm (prior to Al-Khawarizmi) was not a paradigm in Kuhn's sense. For instance, it is hard to establish that whatever algebraic knowledge had existed in the pre-historic period constituted a disciplinary matrix. None of the components of a disciplinary matrix as defined by Kuhn[1] is identifiable in the pre-historic algebraic paradigm. For one, no symbolic generalizations belonging to algebra had developed to be universally accepted by members of an identifiable scientific community. Not even a name existed to describe the rudimentary and isolated algebraic knowledge that Greeks, Hindus, and Arabs had possessed in the pre-historic period. Since there was no distinct and independent body of knowledge in algebra in the pre-historic paradigm, one can safely say that there was no distinct scientific community (algebraists) identifiable within each of the cultures in which algebraic ideas were independently emerging. We present and provide support to the hypothesis that Al-Khawarizmi's algebra marked the first algebraic paradigm in Kuhn's sense. First, the technical language that Al-Khawarizmi used or developed constituted a well-defined set of symbolic generalizations which had been universally accepted by a scientific community of algebraists through the centuries. One example of the lasting impact of Al-Khawarizmi's symbolic generalizations is best represented by the name he gave to the new fledgling field of knowledge. Al-Jabr (Arabic root is "jabara" meaning either to "compel" or to "reduce a fracture") has been used to refer to the field all through the centuries and has been universally adopted by almost all languages. It is not conceivable to assume that the name algebra would have been universally adopted were it not for the significance of the referent field it denotes. The name algebra is unmistakably of Arabic origin and its etymology has been the subject of many investigations.[5] The controversy involves whether the meaning of word "jabr" in ordinary Arabic refers to a specific mathematical operation or the field of science itself. The work 'algorithm' itself derives from "Al-Khawarizmi".

The work of Al-Khawarizmi in algebra constituted a core shared by a new scientific community of algebraists. On the Arab side, some algebraists of whom the best known is Omar Khayyam (about 1045-1123 AD), who tried to extend Al-Khawarizmi's work to solutions of higher degree equations. Other algebraists, of whom Al-Karkhi (died about 1029) is best known, tried to surpass Al-Khawarizmi's work by attempting to "arithmetize" algebra; i.e., to apply arithmetical operations on algebraic expressions. For both trends, the work of Al-Khawarizmi was the starting point to the extent that many of the equations used by Al-Khawarizmi (for example, $x^2 + 10x = 39$) appeared in the algebras of Arab as well as European mathematicians[3]; "When towards the beginning of the twelfth century European scholars turned to Islam for light, the works of Mohammad Ibn Musa came to occupy a prominent place in their studies" (p. 23). Mohammad Ibn Musa, of course, refers to Al-Khawarizmi. The many translations of Al-Khawarizmi's treatise on algebra attest to its role as the core of shared algebraic knowledge of a growing scientific community of algebraists. The best known Latin translation of Al-Khawarizmi's treatise was done by Robert of Chester. Translations to other European languages were based on Chester's translation except for that of Rosen.[4] The translated manuscripts constituted the basic text for the study of algebra and a reference for scholars in the field.

The scientific community of algebraists came to share common beliefs and values regarding their field of study. One such basic belief was that of an existence of a mathematical system distinct from known mathematical systems at the time. The core of the value system is the appreciation of algebra as an applied field of mathematics besides being of value to mathematics itself. Such belief and value systems could not have developed in isolation from the work of Al-Khawarizmi.

Next we turn our attention to the body of knowledge (shared examples in Kuhn's sense) produced by Al-Khawarizmi to establish the extent to which this knowledge represented points of departure from the then existing mathematical knowledge.

A New Mathematical System

In retrospect, Al-Khawarizmi's algebra seems to be the first mathematical theory in algebra; i.e., the theory of the first and second degree equations in one unknown. An examination of the structure and development of Al-Khawarizmi's algebra (refer to section on "The Algebra of Al-Khawarizmi") will reveal a conscious effort to construct a coherent mathematical theory as we know it now:

1. Al-Khawarizmi's theory starts with clear definitions of technical terms, the most important of which for algebra are the concepts of "thing" (variable), "unknown", and "root." The basic concepts are then defined in terms of the technical terms and are independent of any application. The most significant concepts are those of first and second degree equations together with the related binomial and trinomial algebraic expressions.

2. Having laid the ground, Al-Khawarizmi charac-

terizes systematically the types of quadratic equations whose coefficients are positive rational numbers ($ax^2 = bx$, $ax^2 = c$, $bx = c$,$ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$). This development differs from pre-historic algebraic practice in that the purpose of quadratic equations was not to solve a series of problems but rather to characterize all types of quadratic equations as mathematical objects to be investigated independently of any application.

3. The development progresses to a higher level of abstraction by transforming the six types of quadratic equations to the three canonical forms ($x^2 + px = q$, $x^2 = px + q$, $x^2 + q = px$).
4. A general algorithm (completing the square) for solving the three canonical forms is then presented. The solution (in the set of positive rational numbers) is complete and covers all cases including the case of $x^2 + q = px$ where Al-Khawarizmi indicates that "the instance is impossible"[4] (p.12) when $q > (p/2)^2$.
5. Next, Al-Khawarizmi provides geometric proofs for each type of quadratic equation. Geometric proofs were very well known to the Greeks (see the section "Algebra before Al-Khawarizmi"). The value of Al-Khawarizmi's contribution in this regard is the systematic utilization of proof to validate statements (algorithms) that followed from a mathematical system.
6. The last step in the theory is the extension of the four operations, including the square root, to the binomial algebraic expressions of the form $ax + b$.

Algebraic expressions in their full generality are dealt with as numbers. In other words, we have a new mathematical system on a new set of mathematical objects.

A New Representation System

According to Kaput[6], a referential extension provides meaning to a notation system. For example, the meaning of second degree polynomial expressions in one unknown (a notation system) may be provided by their graphical referents ($x^2 + 2x + 3$ represents its graphical referent; i.e., the parabola). When there is a well-defined correspondence between the syntax of the notation system A and the syntax of referential extension B, "then we often refer to A and B and the correspondence between them as comprising a representation system" (p.169).

What then are the contributions of Al-Khawarizmi's algebra to representation systems? One can cite three specific contributions in this regard. First, Al-Khawarizmi extended the notation system of positive rational numbers to include the "unknown" or "root" and the "square" as new notations (though in words and not letters) in such a way that the syntax rules of positive rational numbers apply to the new notations. It is clear from the opening paragraphs of his book that such a coherent extension of the notation system was his first task. His extension of the notation system of positive rational numbers to include the "variable" is a milestone in the history of representation systems. Second, Al-Khawarizmi used natural language as a referential extension for his algebra. The correspondence between the syntax of the notational system and that of natural language was well-defined and consistent. The representation system thus developed was of such power that it dominated algebra for four centuries and was used not only in the original language of the text but also in the languages to which Al-Khawarizmi's book was translated (Latin and other European Languages). Third, Al-Khawarizmi used geometric figures to represent the extended positive rational numbers (a notation system). A number (positive rational or "unknown") was represented by a line segment and a product by a rectangle. The correspondence between the syntactic structures of the two systems is natural and well-defined. Al-Khawarizmi was not by any means the inventor of this kind of representation because it was very well known to the Greeks and others. The contribution of Al-Khawarizmi in this regard is that he pushed the idea of representing numbers by figures to the level of a well-developed representation system.

A Model of a Mathematical System

Up to Al-Khawarizmi's time, the thrust of the development of mathematics had been either to develop a purely mathematical system (Euclidean geometry, for example) or to find mathematical techniques to deal with specific types of situations (Egyptian and Babylonian mathematics). Al-Khawarizmi's work marks an early and rare example in which the full cycle of a model was achieved. The algebra of Al-

Khawarizmi moved from identifying situations to modeling such situations by a mathematical system and validating the mathematical system by applying the latter to a variety of situations much broader than the one with which he started. That Al-Khawarizmi had exactly that frame of mind is clear from his manuscript.[4] In the preface to his book, he mentions that his intention was to provide:

> What is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned-relying on the goodness of my intention therein... (p.3).

Next, Al-Khawarizmi proceeds to develop a mathematical system which, though motivated by the contexts mentioned earlier, transcends these contexts and is independent of any of them. When the mathematical system (quadratic equations) is fully developed it is again applied in a variety of contexts including those which had motivated the development of the mathematical system itself.

PEDAGOGICAL IMPLICATIONS

The pedagogical obstacles that students experience in learning algebra derive, to a large part, from two sources: prior arithmetical experience and translation from natural language. Research has indicated many pedagogical problems arising from prior arithmetical experience. One example of such difficulties is the process-product dilemma; i.e., the failure to accept that the process "$x + 3$" is itself the final answer (product).[7,8] Another example of such difficulties is concatenation, which denotes implicit addition in arithmetic (for example, 3 1/2) and multiplication in algebra (for example $6x$). Among the many difficulties that students encounter due to the translation of natural language is the misconception that a letter that represents a word ("b" for blue) represents a set in natural language but represents a number in algebra.[9]

Clement, Lochhead, and Soloway[10] have identified another difficulty which arises from syntactic translation from natural language to variable symbols (for example, "six times as many students as professors" is very often mistakenly translated into $6S = P$, where S is the number of students and P the number of professors).

Algebra and Prior Arithmetical Experience

What lessons can we learn from the early stages of the evolution of algebra regarding the nature of such pedagogical obstacles? If one attempts to analyze the nature of the two pedagogical obstacles arising from prior arithmetical experience or translation from natural language, one is likely to observe that both obstacles relate to the representation of algebra. The pedagogical obstacle associated with prior arithmetical experience may probably be traced to the misconception that the symbol system of the alphabet with its morphology is a representation of algebra. Obviously, this is not the case because there is no well-defined correspondence between the morphology of the alphabet and that of algebra. The early algebra of Al-Khawarizmi provides a helpful insight into the relationship between arithmetic and algebra. Al-Khawarizmi views the relationship as that of extension rather than representation. In other words, the arithmetical system is extended by adding new symbols for new numbers; i.e., the root and the square. Consequently the syntax of arithmetic applies to the extended set in a natural way. In my judgment, this view of the relationship between arithmetic and algebra is critical in removing many misconceptions in the early stages of learning and teaching algebra.

Algebra and Natural Language

The relationship of algebra to natural language is another problematic area which may benefit from studying the historical development of algebra. The typical algebra curriculum has been characterized by an early introduction of variable and operation symbols. In the last three decades, a trend has been observed in which intermediate representations of place-holders are used as variable symbols in elementary classes in mathematics to prepare students for the introduction of the formal standard algebraic symbols in post-elementary grades. The early introduction of a symbol system does not seem to correspond to the historical development of algebra which was represented by natural language by Al-Khawarizmi and subsequently by other algebraists for four centuries. There is a natural correspondence between the syntax of the natural language and that of the generalized arithmetic of algebra. According to Kaput[6], one major source of mathematical meaning is "via translations between mathematical representations and non-mathematical systems" (p.168). It would perhaps be worth considering the payoff of using natural language to represent introductory algebraic concepts. Natural language is normally fully developed at the time students start their study of algebra. It is plausible to assume, therefore, that representation via a familiar and well-developed system is more effective in providing meaning than via an unfamiliar system (like the alphabet). This possibility ought to be looked into seriously, particularly because it had served the science of algebra for more than four centuries.

Geometric Representation of Quadratic Equations

A promising system for representing algebra is the sub-system of geometry. Al-Khawarizmi adapted this representation from Greek mathematicians and used it extensively in providing "proofs" for his algorithms for solving quadratic equations. The power of this representation resides in its visual concrete form, a feature which renders it valuable for beginning algebra students. The correspondence between the two is simple and natural. Segments represent numbers and "unknowns" by matching the length of a segment to the number or "unknown"; addition corresponds to joining, and multiplication corresponds to the area of the rectangle whose sides are the factors. Dienes Blocks provide a concrete representation system of this kind. By using a flat square of side x (x square), a flat rectangle (one x) and a (1×1) small square, trinomials can be concretely represented.[11] This linkage between algebra and metric measures of geometric figures is a powerful pedagogical tool which merits more attention. It was an efficient tool in the hands of the shaper of algebra, and there should be no reason why it should not be as powerful in the hands of a beginning learner of algebra.

Algebra and Applications

One last lesson which may benefit the pedagogy of algebra is the fact that the genesis of algebra was that of a science and not of a symbol system. Al-Khawarizmi's algebra was grounded in the needs of the society at the time, such as mercantile transactions, measurement, and inheritance. Although Al-Khawarizmi elevated his algebra to a theory which transcended the applications that initially motivated it, he proceeded to show the power of his theory in the extensive real-life applications which constituted more than half his book. Perhaps there is merit in patterning the learning of algebra after that of its historical development. If this is the case, we might as well de-emphasize algebra as a symbol system whose syntax is to be mastered in favor of structuring algebra as a science which is grounded in real life applications.

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Illumination and Geometry in Islamic Art

Salma Marani
University of Texas, MD Anderson Cancer Center
Houston, TX 77054
smarani@notes.mdacc.tmc.edu

A form of art was created in the Islamic world between the 10th and the 13th centuries that integrated rhythmic geometric patterns, calligraphy, and illumination. The dynamic geometric forms, infused with light, were created by artists working in collaboration with mathematicians. All Islamic monuments, spreading from Egypt to Spain, were decorated with these patterns. The mathematical basis of infinitely repeating geometric patterns includes the use of symmetry transformations like rotations, reflections, translations and glide reflections. Tessellations were created from these geometric transformations and an infinite variety of patterns resulted from the use of such elegant and simple mathematical laws. Oleg Grabar, a scholar of Islamic Art, describes this art as follows:

> After the 10th century a second type of ornament appears alongside the earlier one, emphasizing polygons and stars. It makes geometric pattern almost the only pattern of decoration.... From evidence which is only now being discovered, it seems that this art was made possible by a conscious attempt on the part of professional mathematicians to explain and to guide the work of the artisans.[1]

The Alhambra is a walled fortress and one of the masterpieces of Islamic architecture built during the 13th century in the city of Granada in Spain. Scholars of Islamic art have referred to the Alhambra as a "geometer's odyssey" due to the rich variety of illuminated geometric designs that decorate the walls and ceilings of the building.[2] Islamic art is for the most part abstract and geometric; illuminated geometry became an important feature of Islamic art and architecture and resulted in an art form that has been characterized as "transcendental."[3] Techniques of illumination evolved over time and included the use of special colors embedded into the building material, luster painting, marble inlays and the use of colored and stained glass. In some architecture no medium was used except abstract geometric forms that allowed the penetration of sunlight. The stars and polygons dispersed the incoming light into geometrical shapes and projected the illuminated forms into the space enclosed by the building. The rays of the sun created an interplay of light and geometry. This subtle design feature allowed one to experience a sublime interior light within the enclosures of sacred buildings. Such buildings as the Alhambra were made radiant with light in this manner.[4] Figure 1 is a scene from the interior of the tomb of Itimad al-Dawa built in 1628 in Agra, India.[5] It illustrates the use of illuminated geometric patterns and the interplay of light and geometry in Islamic art.

Exploring visual images of geometric art from the Islamic world is now possible through the Internet. The Rotch Visual Collections at the MIT library (**http://nimrod.mit.edu/rvc**) has a World Wide Web page on Islamic art and architecture subtitled "The Aga Khan Visual Archives" which contains a selection of 167 images representing historic and contemporary art and architecture from 26 countries. The images can be indexed by geographic location and architectural components. A tour through this archive provides an extensive visual experience of various art forms developed in the Islamic world. Another impressive web site is Jan Abas' Islamic Patterns page (**http://www.bangor.ac.uk/~mas009/islampat.htm**) which contains a mathematical treatment of the subject.

The Dutch artist M.C. Escher studied Islamic art when he visited the Alhambra palace in 1935. He declared that the Moorish majolica mosaics in the palace made a profound impression on him and realized that the Moors had used exclusively abstract mathematical motifs for their decorations.[6] Escher then decided to study the mathematical laws that formed the basis of these patterns. It was pointed out to him by the mathematical community that the regular division of the plane into mathematical congruent figures is part of the study of geometric crystallography including symmetry transformations and the theory of plane symmetry groups. He began to do his artistic compositions based on these mathematical laws. What the Moorish artists had done with abstract geometric figures, Escher began to do with figures from nature including bird, animal, human and architectural motifs. In his book *Escher on Escher: Exploring the Infinite*, Escher declares that the Moorish artists as well as the mathematicians, and in particular the crystallographers, have had a considerable influence on his work of the last twenty years. With this new style Escher tried to "approach infinity as purely and as closely as possible" and expressed by his compositions "the idea of endlessness, represented on a piece of paper: where our eyes reach the border of the print, the imagination must assume the task of the eye."[6]

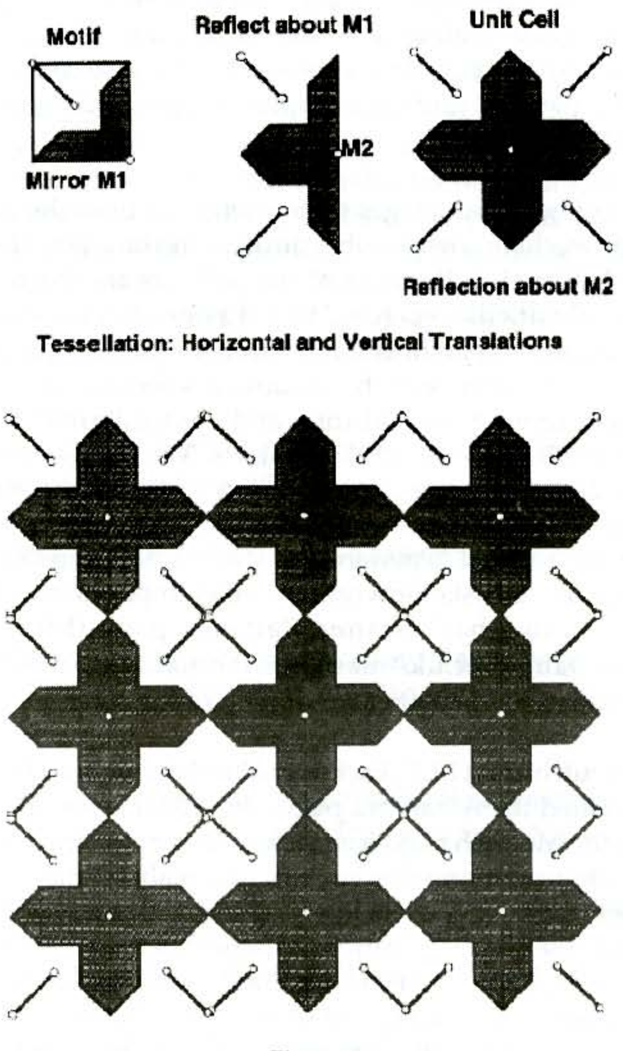

Figure 1
Star-Cross Pattern

The use of two dimensional transformation geometry in creating infinitely repeating geometric patterns was most pronounced in Islamic art. Its application forms the basis for the creation of rhythmic crystallographic patterns. This feature of Islamic art makes it a valuable tool for teaching mathematical topics such as symmetry, transformation and tessellation. It also allows the possibility of learning and integrating multiple disciplines including mathematics, art, computer graphics and Islamic culture in a secondary school curriculum.

The mathematical process involved in creating these patterns can be realized by applying symmetry transformations on a two dimensional motif that is sometimes referred to as the generating motif for the pattern. Symmetry transformations include rotations, reflections, glide reflections and translations together with their compositions. Doris Schattschneider introduces the idea of a plane symmetry group into which infinitely repeating geometric patterns can be classified.[7] A plane symmetry group is a group of symmetry transformations including rotations, reflections, glide reflections, translations and their compositions. Mathematically it has been proven that such patterns can be classified into seventeen distinct plane symmetry groups. All seventeen pattern types were utilized by the Moors in their decoration of the Alhambra palace although there is no documented evidence that the artists and mathematicians during that time were actually familiar with the theory of symmetry groups.[2]

The practical steps involved in creating these patterns include: drawing a 2-dimensional generating motif, applying a symmetry transformation or a combination of symmetry transformations to create a unit cell, and then applying further symmetry transformations like translations in the plane to create a tessellation. All these operations can be performed by a compass and straightedge. In recent years the advances in computer technology and geometry software have made possible the implementation of symmetry transformations and tessellations in fascinating ways. One such geometry program is *The Geometer's Sketchpad*.[9] It allows the exploration of ideas in transformation geometry interactively and with ease. The program allows the student to apply the concept of transformation for creating tessellations and to learn the corresponding mathematical terminology in the process. Since the program performs the necessary repetitive

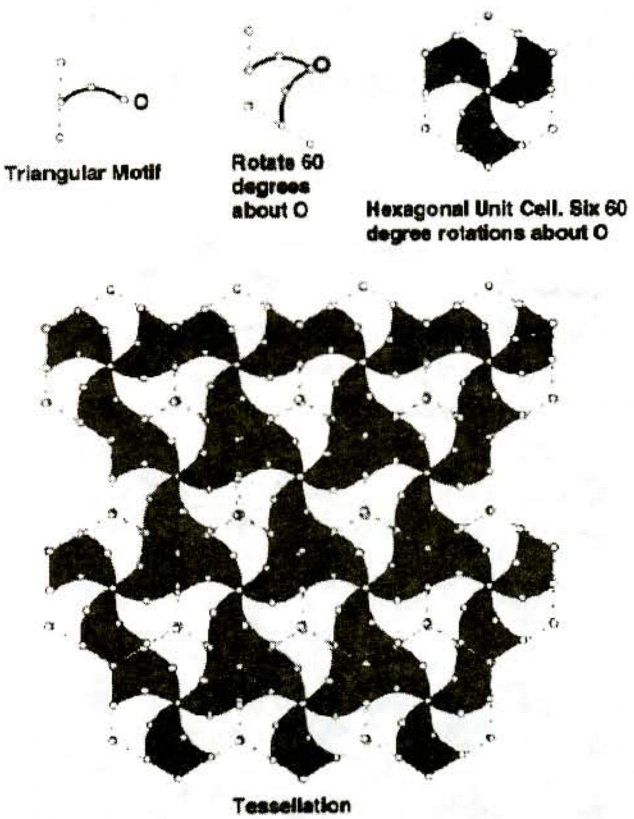
Triangular Motif
Rotate 60 degrees about O
Hexagonal Unit Cell. Six 60 degree rotations about O
Tessellation

Figure 2
Andalusian Pattern

steps involved in creating the unit cells and tessellations the students are able to explore more geometrically complex tessellations and investigate the necessary mathematical ideas that form the basis of Islamic geometric art and the art of Escher. The following illustrations describe some creations that are possible with *Sketchpad*.

The Star-Cross Pattern:

The star cross pattern is a frequently used pattern in Islamic art. It is found in many areas of the Middle East as well as in Spain. The pattern exhibits four-fold symmetry with four-fold rotations and reflections. The generating motif is a simple polygon and the unit cell is a square. The tessellation is obtained by translating the unit cell horizontally and vertically. The stars are often decorated with poetry and verses from the *Quran* and illuminated with luster paint. Islamic artists often integrated symbols like the cross from other religious traditions into their art. The technique of creating these patterns with *Sketchpad* is illustrated in Figure 1.

The Andalusian Pattern

This pattern is commonly found in Islamic monuments in Spain. It displays hexagonal symmetry and the generating motif is composed of a single arc in a triangular shape. The unit cell is hexagonal and is formed by performing a six-fold rotation of the generating motif. The tessellation is obtained by translating the unit cell horizontally and vertically or by rotation and translations. The technique of creating this pattern with the program is illustrated in Figure 2.

The Star-Hexagon Pattern

This is another repeating pattern that uses stars. The star is a popular design element in Islamic art and plays an important role in illuminating the art. The stars signify heaven; many artists used stars to decorate the ceilings and domes of Islamic monuments. In this way they created a vision of heaven through their art. The star hexagon pattern has a six-fold double hexagonal symmetry with rotations and reflections performed on a simple generating motif. The steps involved in creating this pattern with the program are illustrated in Figure 3.

Sketchpad also allows one to decorate these patterns with the use of color. Most Islamic patterns are filled

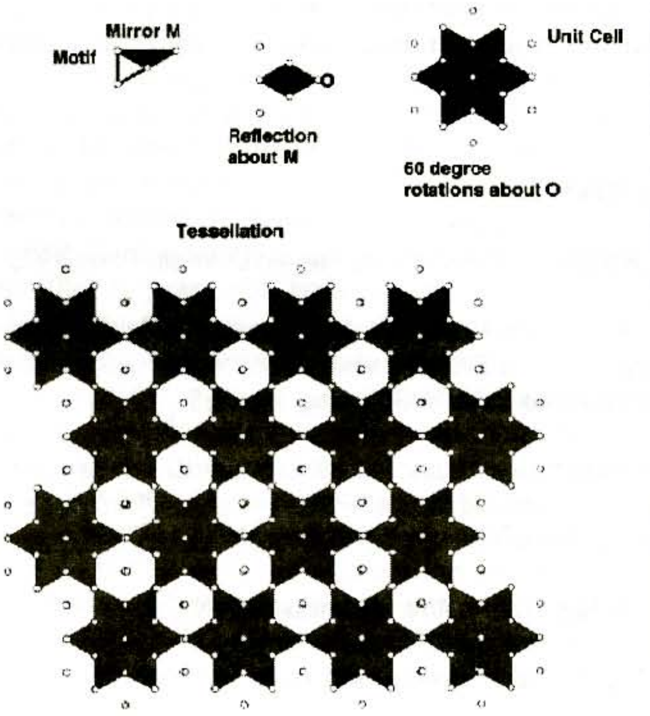

Figure 3
Star Hexagon Pattern

with brilliant colors dominated by blue and gold. Blue is a symbol of the infinite and gold symbolizes the glory of the creator. The patterns were often glazed with turquoise and cobalt, a common technique of illumination, and decorated with divine inscriptions and poetry that encompassed divine compassion.

Variations of Patterns and Original Designs

Simple variations of the above patterns can be implemented easily by changing the original motif and creating the unit cell and tessellation with this new motif. For example, the Andalusian pattern can be given a different expression by changing its motif. The resulting pattern is a tonal version of the pattern and shows how altering the generating motif and applying the same symmetry transformations gives a different expression to the pattern. The result is shown in Figure 4.

New designs can also be created by generating an original motif to create the tessellations. This process of creating unit cells and tessellations allows the student to explore ideas in transformation geometry with relative ease and since the results are interactively and visually generated a deeper understanding of the mathematical concepts becomes possible. Learning about transformations is an important part of mathematical understanding since transformations connect ideas from geometry and algebra as well as probability theory and statistics. This approach also allows the possibility of learning multiple disciplines including mathematics, art, computer graphics, and Islamic culture.

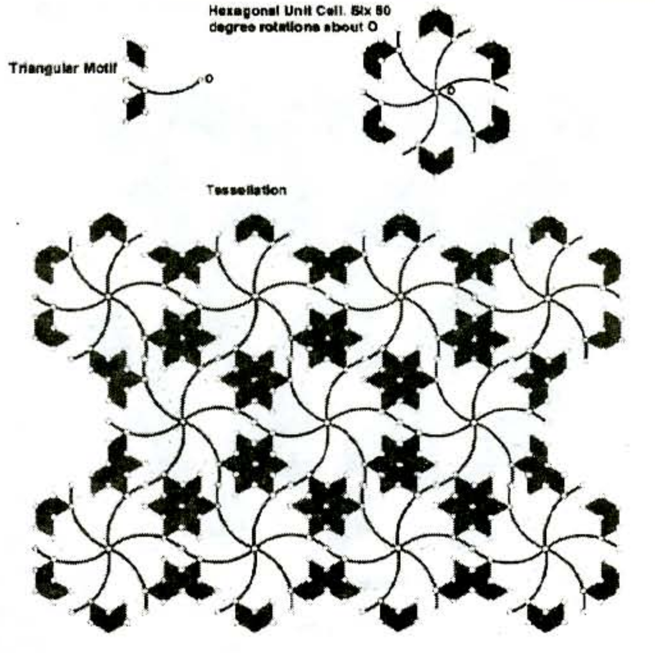

Figure 4
Andalusian Variation

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SEE-DUCTION
How Scientists & Artists Are Creating A Third Way Of Knowing

Howard Levine
California College of Arts and Crafts
Oakland, CA 94618
howard_levine@ccac-art.edu

> ***"If we trace out what we behold and what we experience through the language of logic we are doing science; if we show it in terms whose interrelationships are not accessible to our conscious thought but are intuitively recognized as meaningful, we are doing art."***
>
> **Albert Einstein**

In his 1959 Rede Lectures, C. P. Snow coined a now famous phrase—The Two Cultures—that has acted as a cautionary note for much of our modern life: "I believe the intellectual life of the whole of western society is increasingly being split into two polar groups. Intellectuals at one pole—at the other scientists. Between the two a gulf of mutual incomprehension—sometimes hostility and dislike, but most of all lack of understanding. They have a curious distorted image of each other. Their attitudes are so different that, even on the level of emotion, they can't find much common ground." Maybe so, but Lord Snow never met Brent Collins[1] or John Conway:

> As a boy John Conway was fascinated by knots. So much so that he spent weeks whittling complex knots out of solid blocks of wood so that he could study their form and shape from every conceivable angle. Today, Conway is still interested in visualizing knots which he often does by inviting friends to "dance" while holding different colored ropes. Brent Collins is also interested in visual representation, but for Collins the objects have esoteric names such as 'one-sided surface with opposed chiralities' and 'Haken surfaces of figure eight knots.' Even his explanation of his work is arcane, "The linear patterns are never arbitrary but issue as abstractions of the logical motifs constellated in a particular composition."

Who's the artist and who's the scientist? Does it really matter what we choose to call them if they are both engaged in the same fundamental activity? Not according to Collins: "Scientists' forms are elaborated through first a collection of data looking for underlying relationships, quantifying them, and then seeing how they may be visually represented. I go direct to the visual representation. But clearly the whole modeling process is internalized in the human brain." (In case you haven't guessed, Conway is a world renowned Princeton mathematician; Collins is a sculptor whose works have been exhibited at Fermi Labs, the National Center for Supercomputing Applications, and AAAS.)

What Collins and Conway understand, and what Snow overlooked, is that not only are scientists and artists engaged in the same basic task—interpreting the fundamental nature of both the universe and our place within it—but they do so by employing the same essential artistic and scientific skill: seeing and interpreting. Furthermore, and Snow could not have foreseen this 35 years ago, both of these disciplines are using computers to discover and experiment with new observational opportunities, to give form and shape to dry mathematical equations, and to search for meaning among seemingly random, chaotic data. In using the computer as a tool to help us see and make sense of what we see, artists and scientists are creating a new and important third way of knowing: *see-duction*—the visualization, simulation, and modeling of real world phenomena using computers. In so doing, *see-duction* is helping to break down the artificial barriers between the two cultures.

FROM SCIENCE TO ART

What is the greatest scientific discovery of all time? Twentieth century denizens might choose the Theory of Special Relativity which unifies matter and energy or the discovery of DNA, the information code for all life forms. Those with a longer view might select the Theory of Natural Selection or the Laws of Motion. Still others might argue that since all science is based on mathematics, the greatest scientific discoveries have been mathematical: the invention of zero or the insight that all geometrical shapes can be numerically represented. But each of these great intellectual achievements pales in significance to the correct answer, the discovery that allows all other scientific achievement to occur—the invention of the scientific method.

Twenty-five hundred years ago the ancient Greeks invented *deduction*—a logical system of reasoning that started with indubitable axioms and employed precise rules to generate theorems (new knowledge); this was the birth of mathematics, the first great scientific way of knowing. Five hundred years ago the early Renaissance thinkers invented *induction*—a formal system of rules governing observation and experimentation designed to give us knowledge of the natural world; this was the birth of science, the second great way of knowing. Today, an interdisciplinary group of revolutionary scientists and mathematicians are inventing the third great way of knowing, *see-duction*:

Bill Thurston is one of the world's best mathematicians. A Fields Medal (the Nobel Prize for mathematics) winner and Director of Berkeley's Mathematical Sciences Research Institute, he is best known for his work establishing a deep connection between topology and geometry. As one might expect, his papers (i.e. "Three-dimensional manifolds," "Kleinian groups," and "Hyperbolic geometry") are not easy bedtime reading. The pleasant surprise is that one need not read the paper in order to understand the concepts. The Geometry Center at the University of Minnesota (Thurston is also a director there) has produced an award winning video, Not Knot[2]*, that uses animation to show and explain the concepts and reasoning behind Thurston's ideas. In fact, since he has not yet provided a complete paper-and-pencil proof of his theorem, the video stands as the proof. Although it is certainly not a technique without controversy, computer-aided visualization is allowing mathematicians to embrace a long cherished dictum of empirical science: Seeing is believing (and understanding).*

Although it is certainly not a technique without controversy, computer-aided visualization is allowing mathematicians to embrace a long cherished dictum of empirical science: Seeing is believing (and understanding).

In 1963 Edward Lorenz sowed the seeds for a scientific revolution when he published a dull-sounding paper ("Deterministic Nonperiodic Flow") in a somewhat obscure journal (Journal of Atmospheric Sciences). *Today, we recognize Lorenz's work as the foundation for chaos theory—the study of systems governed by nonlinear rules and equations which can be so sensitive to minor fluctuations that their behavior seems chaotic. The classic statement of such a system is Lorenz's, "The flapping of a butterfly's wings in China today may lead to a tornado in the Midwest next month." Thirty years later, a new generation of climate modelers is still struggling with chaos, but now they are aided by a staggering and ever-growing amount of computational power. The best current model is the National Center for Atmospheric Research's (NCAR) Community Climate Model, but competitors with names such as MOM (Modular Ocean Model) and POP (Parallel Ocean Program) are also seeking to develop a coupled atmospheric-ocean climate model. If the possibility of accurate, long term weather forecasts is still in question, the utility of visualizing the output from reams of arcane equations is not. As scientists continue to simulate increasingly complex phenomena (i.e. ozone depletion, economics), the knowledge gained from seeing these simulations on a computer screen will be the truest test of their worth and validity.*

The flapping of a butterfly's wings in China today may lead to a tornado in the Midwest next month."

There is one image that we never tire of seeing—the image of the human body. Whether it is Galen's anatomical sketches, or early x-ray images, or a CAT scan of our own head, the human form seems endlessly fascinating. But the body is decidedly three dimensional while each of these rendering techniques yields a two dimensional image. How much information is lost? You don't have to be an anatomist or computer scientist to realize that the answer must be "a whole lot." Researchers at Sandia National Laboratory and the Baylor University Medical Center have used massive parallel supercomputers to turn two dimensional MRI images into three dimensional views and the results are startling—the detection of breast tumors that were "invisible" to x-ray mammography. But why stop with the human breast? The Visible Human project seeks nothing less than a four trillion byte image library that will provide three dimensional numerical coordinates from which *both internal and external structures can be depicted, rotated, viewed from any angle and reversibly "dissected." Early* *scientists built physical models. Later scientists employed conceptual models. Today, scientists in fields as diverse as psychology, crystallography and medicine are employing computer models to help them better understand the natural world*[3].

Modern day neo-Luddites scoff at the idea that *see-duction* is a new way of knowing. "After all," they argue, "scientists have always used the processes of visualization, simulation, and modeling. The computer is just a tool." The trouble with this "argument" is that it totally fails to understand the power of revolutionary tools. Thirty years ago, Marshall McLuhan observed that we shape our tools and thereafter our tools shape us. The computer, the first meta-tool—or tool with no specified, overt purpose—and its human masters are engaged in an endless bootstrapping cycle of shaping both us and our machines. Truly revolutionary tools pass through three stages: First, they simply enable us to perform the same old tasks with greater efficiency (quantitative phase). Second, with enough speed and efficiency, the old task mutates into something inventive and unexpected (qualitative phase). Finally, we find ourselves using the tool to perform totally new and unforeseen tasks. In effect, the tool has shaped us so that we think in terms that would have been impossible without it (revolutionary phase). No one who looks at the work of Bill Thurston or Edward Lorenz or any of the hundreds of other scientists using the computer to help themselves see, can argue that it's simply business as usual. Today, *see-duction* is in its infancy, somewhere between the quantitative and qualitative phases; tomorrow, it will enable us to think in new ways and usher in a third scientific revolution.

FROM ART TO SCIENCE

Who is the greatest scientist of all time? Twentieth century denizens might choose Albert Einstein or Watson and Crick. Those with a longer view might select Charles Darwin or Isaac Newton. Still others might argue that since all science is based on mathematics, the greatest scientist has to be mathematician. They might choose Muhammad al-Khwarizmi or Rene Descartes. But each of these great scientists, as Newton so aptly pointed out, was only able to proceed because he already stood on the shoulders of giants—the shoulders of the inventors of the scientific method. Pythagoras, Plato, and Aristotle (and later, Euclid) who invented deduction; Brunelleschi, Alberti, and Leonardo (and later, Galileo and Bacon) the inventors of induction. But notice that those individuals we recognize as scientists were already building on the work of philosophers and artists. Revolution in scientific method has always required a synthesis of Snow's two cultures. Breaking the scientific paradigm (as Kuhn so ably documents) has always required forces outside the scientific community. The same is true today. *See-duction* is the work of artists as much as it is the work of scientists:

Tony Robbin[4] *is an artist with a simple, if incomprehensible, mission—to see and paint the fourth dimension. In 1975, Englebert Shucking, a physicist at NYU, told Robbin that he had seen the fourth dimension. Shucking said little else, but it was enough to send Robbin on his mission. Four years later, Robbin visited Tom Banchoff, a professor of mathematics at Brown University, and saw his first computer-generated graphics of a hypercube rotating in space. Today, Robbin has programmed his own computer to allow him to see the fourth dimension. He has sold his large, 4-D paintings to private collectors and corporations such as General Electric and AT&T. What's the attraction? Isn't a fourth spatial dimension some kind of conjurer's trick? Not according to Robbin: "Physics has confirmed what we really knew all along: three dimensional space is an arbitrary convention. In the future there will be many works by many artists based on visual experience of the fourth dimension. With new works of art and new computers, the tools are already available to us for learning to see the fourth spatial dimension that is all around us and hidden from our view for only a moment. When the fourth dimension becomes part of our intuition, our understanding will soar." For Robbin, visualizing the fourth dimension is analogous to the work of the Renaissance masters—it is the portal to knowledge.*

Donna Cox is an artist with an unusual institutional home—the National Center for Supercomputing Applications at the University of Illinois. Her job, to steal a title from Ed Tufte's[5] *classic book, is envisioning information. Whether it's the "Motion Analysis of Kink Instabilities in Supersonic Flow," "Plastic Injection Molding," or "Numerical Relativity: Black Hole Space Times," her task is making sure that the graphic displays of the supercomputers (with artist's names like Klimt, Courbet, and Mondrian) convey the maximum amount of information possible. But what rules are to be followed? How can dry equations be turned into meaningful pictures? Tufte closes* Envisioning Information *with a lament: "The essential dilemma of* *narrative designs is how to reduce the magnificent four-dimensional reality of time and three-space into little marks on paper flatlands. Perhaps one day high-resolution computer visualizations, which combine slightly abstracted representations along with a dynamic and animated flatland, will lighten the laborious complexity of encodings -- and yet still capture some worthwhile part of the subtlety of the human itinerary." Cox, and the scores of other artists who work at the National Computing Centers and proprietary computing firms around the world, have already taken the first step in that human itinerary. If a picture is worth a thousand words, how much information can be contained in the six minute computer simulation of a thunderstorm? The answer may just be the hundreds of lives that can be saved if such simulations enable us to better forecast the weather.*

Aaron is sui generis*—the world's first artist-computer (not an artist using a computer [a computer-artist], but a computer that is programmed to be an artist). Aaron is also the alter-ego of Harold Cohen*[6]*, a renowned abstract painter who gave up painting twenty years ago to enter into a strange, symbiotic relationship with a computer. What's the connection between art and computers? Between Harold and Aaron? For Cohen, art has always been about the representation of human knowledge; computer languages are also a form of representation—a set of rules, algorithms, and heuristics that encompass knowledge and might just lead to new knowledge. But could a computer program lead to the kinds of knowledge that an artist requires in order to create art? Harold has spent the last twenty years imbuing Aaron with all his painterly knowledge; Aaron's artwork speaks for itself. Cohen is emphatic that Aaron's work is not computer art: "The fact is that art is not, and never has been, concerned primarily with the making of beautiful or interesting patterns. The real power, the real magic, which remains still in the hands of the elite, rests not in the making of images, but in the conjuring of meaning." By creating a computer model of himself, Cohen has created a totally new method for cognitive scientists to study the ultimate question of knowledge: How do we mentally represent the world in order to create meaning?*

"The fact is that art is not, and never has been, concerned primarily with the making of beautiful or interesting patterns. The real power, the real magic, which remains still in the hands of the elite, rests not in the making of images, but in the conjuring of meaning."

This time, it's more than just the neo-Luddites who are scoffing. "How can math, and science, and computers have anything to do with artistic creation?" they complain. The essence of this plaint was anticipated almost fifty years ago by the Swiss sculptor Max Bill. After asserting his belief that "it is possible to evolve a new form of art in which the artist's work could be founded to quite a substantial degree on a mathematical line of approach to its content," Bill set forth what he believed would be the skeptical response to his manifesto: "It is objected that art has nothing to do with mathematics; that mathematics, besides being by its very nature as dry as dust and as unemotional, is a branch of speculative thought and as such in direct antithesis to those emotive values inherent in aesthetics; and finally that anything approaching ratiocination is repugnant, indeed positively injurious to art, which is purely a matter of feeling." The trouble with this "argument" is that it totally fails to understand art, science, and the longstanding, important relationship between them[7].

Far from being independent, these disciplines have always shared a five stage relationship as they engage in the same, vital, enterprise—observing and interpreting the universe and our place within it:

Shared tools Artists rely on scientific and mathematical tools to count, measure, design buildings, anneal glass and much more; scientists rely on artistic tools to model non-Euclidean spaces, create topological surfaces, enhance photos from space, and much more.

Mathematical foundations Neither art nor science could exist without a reliance on fundamental mathematical concepts. Perspective, proportion, and symmetry are just three mathematical ideas that are crucial to the practice of both art and science.

Mathematical inspiration There are no limits to what an artist may choose to depict, so it should not be surprising to discover that many artists have found inspiration in mathematical concepts and ideas: Phidias, Leonardo, Durer, Kandinsky, and Escher not only created works inspired by mathematics, they also wrote treatises explaining the role of science and mathematics to the arts. Today, the CyberArts movement, with its interest in chaos theory and fractals, is sometimes hardly distinguishable from the scientists working on those very subjects.

Epistemology Scientists and artists are seekers after the same thing: beautiful, elegant solutions. The famous British mathematician G.H. Hardy wrote that "the mathematician's patterns, like the painter's or poet's, must be beautiful." In his Messenger Lectures about the character of physical laws, Richard Feynman says, "[they] are simple, and therefore they are beautiful." Perhaps without realizing it, artists and scientists may be uniquely suited to judge the quality of each other's work.

Metaphysics Do science and mathematics tell us more about the inner workings of our own minds or the outer workings of the universe? Should artists be credited for inventing totally new ways of seeing (i.e. Cubism, 4D) or only with discovering preexisting modalities? Are the scientists' quarks and space-time wormholes really descriptions of our universe or simply current fictions that we use to explain our universe?

Such questions may ultimately have no answers, but this much is clear: artists, scientists, and mathematicians are engaged in the ultimate creative activity—creating something out of nothing. Today, and increasingly in the future, *see-duction* will contribute much to this creative quest.

See-duction is the second of a two part argument I have made regarding the relationship between art and mathematics. The first article, "The Art of Mathematics, The Mathematics of Art" appeared in Leonardo, *vol. 27, no. 1, 1994.*

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[1] Brent Collins has published a series of papers in *Leonardo* describing his mathematically based sculptures. Accepted for future publication in that journal is an article explaining his collaboration with Carlo Sequin, a computer scientist ant UC Berkeley.

[2] The *Not Knot* Video and booklet is available from Jones & Bartlett Publishers. There is also a wealth of information available on the University of Minnesota Geometry Center web site.

[3] In general, much of the most exciting *see-duction* work is being communicated through cyberspace. Two of the best sites are the University of Illinois' National Center for Supercomputing Applications (see especially the Renaissance Experimental Laboratory) and UC San Diego's Supercomputer Center.

[4] Tony Robbin explains his work in his book *Fourfield*. The book also comes with a computer program allowing the user to manipulate a hypercube in 4-space.

[5] Ed Tufte has self-published three classic books exploring the relationship between visualization and information. See *The Visual Display of Ouantitative Information*, *Envisioning Information*, and *The Brand New Visual Explanations*.

[6] Harold Cohen's story is told by Pamela McCorduck in *Aaron's Tale*.

[7] *The Visual Mind*, edited by Michele Emmer (MIT Press) is a first class collection of articles exploring the relationship between art and mathematics.

Curriculum Development via Literary and Musical Forms

Joel K. Haack
Department of Mathematics
University of Northern Iowa
Cedar Falls, IA 50614-0506
haack@uni.edu

In addition to the inclusion of humanistic concepts in the classroom presentation of mathematics and the assignments/projects that students are asked to complete, I have found it valuable to structure the courses I've developed in terms of literary and musical forms.

Mathematical knowledge, as that of many disciplines, is appropriately regarded as a web of ideas, having a great many interconnections. This can make it very difficult to imagine presenting material in a linear fashion—the richness of the discipline can be lost.

Upon reflection, it is clear that structural considerations are not and should not be limited to the development of new courses. Whenever we consider a course that we will teach in the next term, we have the opportunity of seeking an overarching structure for the subject.

One way to develop a path through the web of knowledge is to look to other arts for suggestions of ways to organize material. Two that I have found to be particularly helpful are literary and musical forms.

As a first example (and one which encouraged me to think of curriculum this way) a geometry course, based for example on Martin Jay Greenberg's *Euclidean and Non-Euclidean Geometries*[1], could have the form of a novel, with its plot the independence of the Euclidean parallel postulate. To this end, one would first introduce the setting—namely, the origins of geometry, the axiomatic method, Euclid's first four postulates, and the parallel postulate. This would be followed by ideas from logic that establish the context in which independence can be demonstrated or defeated: theorems, proofs, and models. The non-human characters, namely Hilbert's Axioms, would then appear and their consequences and interactions would be explored, constructing a portion of neutral (absolute) geometry. Demonstrating that a number of statements are equivalent to the parallel postulate would then show the students what is at stake in its proof or refutation. Tension would increase through a chapter on the history of the parallel postulate, introducing additional human characters and leading to the discovery of non-Euclidean geometry. Finally the independence of the parallel postulate would be established through models of hyperbolic geometry; by this time, the students welcome this as the culmination of the semester's story! After this denouement, the philosophical implications of the independence of the parallel postulate could be discussed, providing an additional reward for the completion of the plot of this novel.

Compare this description of a course based on Greenberg's text with the typical section-by-section, chapter-by-chapter presentation in most courses. Highlighting the difference: in a course with the plot sketched above, it would be unconscionable to run out of time before finishing the story!

It is worthwhile to let the students know what role the day's topic plays in the story of the independence of the parallel postulate—this helps establish a context for them, again exhibiting a significant difference from a course without a plot. In fact, when I asked students on the in-class final to sketch the plot of the course, all but one were quite successful.

A traditional first-semester calculus course adapts readily to this idea of structure via a novel. Exactly what the plot line is and who the major characters are will depend on the text adopted for the course. In a traditional calculus course, very often the Fundamental Theorem of Calculus is the goal of the plot. Almost every topic preceding this can either be related to the development of this theorem, or presented as subplots and asides. In the CCH reformed approach to calculus based on modeling[2], the fundamental theorem may have become an obvious consequence of the emphasis on derivatives as rates of change and integrals as total change. Now instead, the introductory chapter introduces the various families of functions as the characters in a logically developed sequence of mystery tales and rags-to-riches stories with unlikely heroes. Which function is responsible for modelling some situation, and what evidence can one present for this? How well can an unlikely hero, a linear function, perform as an adequate substitute for a more complicated function in some particular situation?

Another example of the use I've made of a form from the arts is in developing a portion of a history of mathematics course offered for middle grades education majors through mathematics masters candidates, and since adapted for inclusion in other survey courses. The approach taken was that of a novella or tone poem; within an overall story line, themes reappear. This story, based on number and numeration systems, traces ideas from pre-history through the second half of the twentieth century. Ideas from Eudoxus recur with Dedekind; prehistoric counting reappears as the basis of Cantor's cardinality of sets. The discomfort caused earlier mathematicians by irrational, negative, and complex numbers reappears in my students with an introduction to the infinitesimals of the hyperreal numbers of Abraham Robinson.

In contrast to a structure modeled after a novella/ musical tone poem, a unit on "Shape" based on the chapter in *On the Shoulders of Giants*[3] took the form of a theme and variations with some fugal entrances. Ideas of similarity, dimension, symmetry, dissections, and combinatorial geometry all appeared and were interwoven.

A book explicitly constructed in sonata-allegro form is John McCleary's *Geometry from a Differentiable Viewpoint*[4]; a summary of his comments in the introduction indicate the structure of his text (p.*ix-xi*). His first section of five chapters opens with a prelude of spherical geometry, then introduces some of the main themes, including Euclid's parallel postulate. The eight chapters in the development section establish what will be required to provide a rigorous model of non-Euclidean geometry, and introduce a new theme of an intrinsic feature of a surface, the Gaussian curvature. In the last three chapters, the recapitulation and coda, McCleary finishes the development, provides the climax with the construction of models of non-Euclidean geometry, and then provides a coda on the theme of the intrinsic.

In addition to providing an over-arching structure, there are additional advantages to thinking about courses in terms of literary or musical forms. Making the structure apparent to students, providing a conceptual narrative, and reminding them from time to time where they are in the course helps address the needs of students who are not in Sheila Tobias's[5] "first tier" (pp.31, 38, 46, 89). Such students often feel discomfort with an unmotivated section-by-section presentation.

Thinking of the course in these terms also can suggest potential projects and assignments for students that take a more humanistic approach. For example, one could ask students to select the most important theorem from a chapter containing major theorems of calculus leading up to the fundamental theorem, and explaining the reason for the choice. As another example, one could encourage students to write a poem after the students have seen hyperreal numbers, reacting as a Pythagorean might to the discovery of incommensurable magnitudes; a graduate student of education felt this was the best activity of a two-semester survey course.

Finally, considering over-arching themes for a course also encourages thought and discussion among faculty regarding strands in the mathematics curriculum. As institutional (or external) pressures grow to shorten (or at least not lengthen) majors, this will become increasingly important to the integrity of an undergraduate major in mathematics.

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Algebra Anyone?

Leslie Jones
Cape Cod Community College
West Barnstable, MA 02668

"Solving life's problems is rather like the process of doing algebra. Time, skill, wisdom, and determination are necessary keys for success in both endeavors."

INTRODUCTION BY TED PANITZ

Every once in a while a student does something that makes teaching exhilarating. Those moments underscore the whole purpose of teaching. I would like to share one of those with you that I had recently.

Leslie started college last September as a returning student having raised a family and worked at part time jobs. She had all the symptoms of a person unsure about the world of college, a completely new experience for her: high anxiety, low self-esteem, and lots of self doubt. She started in my elementary algebra class and is now completing intermediate algebra and is ready to take on college level math with a great deal of confidence.

Leslie responded to my precourse letter of introduction, where I ask students to write their math autobiography before coming to the first class, by writing a three page essay. Most students write one page double spaced, if that. She indicated that she appreciated the opportunity to write to me since she understood that form of expression while she couldn't comprehend math concepts. She responded to all my writing assignments with enthusaism and often mentioned how they helped her through the course. Throughout the course she, along with many of her peers, kept asking the eternal question, "What do I need this for???" My response was always that it would help her in life. That usually brought guffaws and giggles from everyone.

This week, Leslie approached me right before class and said she had something special for me. "Uh oh! What could that be??!" I wondered. She wrote an article for our school's literary book, "The Write Stuff," which is refereed by faculty, and her article was accepted. Let me tell you this is one proud student and one proud teacher. You will see why in a moment.

Leslie has captured the essence of why anyone should take classes at college. Her inspiration happens to have come through her experiences with algebra. She has made a few good friends through our classes and has learned a lot about herself. I have had the privilege to watch her grow and develop into a mature college student. I attribute her response in large part to her adapting to and using cooperative learning and a mastery approach to testing in a supportive, nonthreatening environment which was created by collaboration instead of competition.

Leslie has given me permission to duplicate her article for my other classes and to send it out over the Internet. If you feel it might have an impact on some of your students, feel free to use it.

This student's response will keep me humming and smiling down the corridors for some time.

ALGEBRA ANYONE?

When an elementary algebra class is in progress, repeated choruses of "Why do we have to take algebra anyway?" can be heard echoing down the college corridors. I was an avid member of this chorus in the early weeks of DE060 (Elementary Algebra) while struggling to get my aging brain to grasp the basic concepts. But, as my ability to handle algebraic complexities increased, I gradually became aware of the benefits of algebra. They have little to do with the usefulness of any particular math skills and formulas in the future and more to do with the reality that solving life's problems is rather like the process of doing algebra. Time, skill, wisdom, and determination are necessary keys for success in both endeavors.

The effort of gaining skills and solving problems, whether in life or algebra, often takes a sizable investment of time and energy. Adequate time does not magically appear in the crowded schedules of the late 20th century lifestyles; it has to be carved out, sometimes ruthlessly. New priorities have to be established, if only temporarily, and ways of using time more efficiently will have to be discovered in order to accomplish everyday chores more quickly. But first and foremost, finding time to learn skills and solve problems depends on a willingness to invest the required time and determination to give whatever it takes to accomplish the goal at hand. Some of life's problems hardly seem worth this investment, just like algebra. Doing so anyway increases discipline, focus, and understanding for those times that are critical.

Not only is it necessary to gain knowledge and skill to effectively solve problems, but wisdom is also needed to decide which option to choose and when. I have often thought about this when confronted with setting up and solving an equation or simplifying an algebraic expression, especially one that contains a complex fraction. Invariably, I forget to keep track of the signs or fail to remember the rules that govern them, and so I arrive at an incorect solution. There are so many questions to ask of myself and many that I forget to ask. Have I reduced or factored enough? Is it even factorable? Sometimes I forget to factor altogether or forget that eliminating fraction denominators anywhere but in an equation is not allowed. I love to get rid of those irritating fractions. When working on a personal problem, I have often pictured a complicated polynomial expression in my mind's eye and the confusion of all the rules, terms and variables involved in simplifying it. When I think of an equation, I am reminded of balance as I remember that what is done to one side of the equation must be done to the other. Balance is a good thing to keep in mind when solving problems. I don't always get my solutions right in life or in algebra, but the more I practice and understand my mistakes, the more my wisdom and skill improve. This is most definitely an advantage.

I can forsee that pushing through the dark chapters in algebra will help when the dark chapters of life occur, problems for which past experience has not been adequate preparation.

Finally, algebra is a wonderful opportunity for strengthening determination and self-confidence, valuable character traits when faced with a problem. It is the only academic subject I have taken where it is possible to gain some degree of understanding and confidence only to turn the page to the next chapter and not have the slightest idea about what the text is trying to explain. This is disequilibrium . . . "BIG time." At its worst, disequilibrium involves fear of the unknown, of not knowing the right way. At best, it is simply confusing and frustrating. Either way, the temptation is to avoid it rather than gather up the courage and patience to stand up on one's potential and trust that eventually the light will dawn. I can forsee that pushing through the dark chapters in algebra will help when the dark chapters of life occur, problems for which past experience has not been adequate preparation.

After satisfying the math and science requirements for a degree, it may be true that I will never again use the particular math skills I have learned along the way. I will not, however, consider the investment of my energy a waste of time. The reward of improved discipline, skill, wisdom, and determination will be useful keys of success for the rest of my life, especially when confronted with the "impossible."

RESPONSE BY WALTER BURLAGE

Thank you for passing along that student's essay regarding the importance of algebra. I have felt this way about algebra for a long time myself, but I find it difficult to communicate those feelings to my students. Usually they do not want to believe that doing well in algebra is going to have some future benefit. It is taken (by them) a little bit like the promise of organized religions (i.e., adhere to these beliefs, live according to these principles, and you shall be rewarded somehow). But when the message comes from a fellow student, it may have some added credibility.

I would like to share my story with you and I hope that you will pass it along to Leslie. When I completed the requirements for a B.S. degree in 1970, the Vietnam confict was going full tilt. A friend told me that joining the U.S. Marine Corps Reserve was a way to fulfill my military obligation and probably avoid going to Vietnam. Just one small hitch, however, was that I had to undergo six months of active duty training (boot camp and beyond) with regular Marines. Little did I know that this was about to become one of the greatest challenges of my life.

When I arrived at Marine Corps Recruit Depot, San Diego, CA, I was treated like all of the other poor slobs who had chosen the Marine Corps as their way to fullfil their military obligation. I felt very intimidated at first because I was not sure whether I could handle the physical challenges (running all day, carrying a 30 lb. pack and a 14 lb. M-14 rifle). I knew that most of the other recruits were younger than I and in much better physical condition.

Imagine my surprise when I discovered after a few short weeks that not only could I keep up with the physical challenges, but I was surpassing most of the younger recruits. It took me a while to figure out why this was so. I could not immediately see any logical reason why this was happening. Later, when it finally dawned on me, I had difficulty believing the truth that I had discovered.

The truth that I discovered is that Marine Corps Training, while it is extremely demanding physically, is also demanding both emotionally and psychologically. The Marine Corps, after all, is attempting to train its people to go into the worst of situations (a battlefield where an adversary is trying to kill you, where you may be outnumbered and out-gunned, where there is seemingly no hope . . .) and function as a soldier to the best of your ability. When this emotional/psychological element is added to the physical challenges, the training can rapidly become more than many young men are able to endure and they break down. First, they break emotionally and then they break physically. If your head is not in the right space, all of the physical strength in the world will not pull you through.

The Marine Corps drill instructors are trained to break recruits emotionally first. They know that they have succeeded when they begin to see the physical breakdown. Once this occurs, they then begin to rebuild the recruit emotionally to prepare him to survive the reality of warfare. Once the emotional component is back in place, most recruits quickly regain their physical powers.

I discovered through this experience that all of the mathematics I had studied had actually prepared me to face the "impossible." I already had acquired the emotional discipline that carried me through those harried few weeks of boot camp. It sustained me and carried me through that terrible, nightmarish experience. As I look back on my life, I can recount other times when the discipline that I learned in mathematics truly came to my aid when I was faced with a difficult life challenge, but few events will compare with my experience in the Marines.

Mathematical Rebuses

Arthur V. Johnson II
Nashua Senior High School
36 Riverside Drive
Nashua, NH 03062

decimaldecimaldecimal	**(repeating decimal)**
nomial **nomial** **nomial**	**(trinomial)**
$\frac{\textbf{vanilla}}{\pi}$	**(pie a la mode)**
skπ**y**	**(pie in the sky)**

Book Reviews: *Uncommon Sense* by Alan Cromer and *The Physics of Immortality* by Frank J. Tipler

Harald M. Ness
University of Wisconsin, Centers-Fond du Lac
Fond du Lac, WI 54935
hness@uwcmail.uwc.edu

George Polya, when asked why he became a mathematician, said that he was too good to be a physicist, and not good enough to be a philosopher.

Uncommon Sense: The Heretical Nature of Science. Alan Cromer. Oxford University Press: New York,1995. 256p. ISBN 0-19-509636-3.

The Pysics of Immortality: Modern Cosmology, God, and Resurrection of the Dead. Frank J. Tipler. Doubleday: Anchor NY, 1995. ISBN 0-385-46799-0.

This is a pair of interesting books written by physicists. Why, you might ask, are books written by physicists being reviewed in a mathematics journal, in particular, in a journal dedicated to mathematics as a humanistic discipline? Well, in the first place, they are of a mathematical nature. The first is about rational thought, which we believe we use in mathematics; the latter has a detailed mathematical development of theorems leading to the major conclusion. Furthermore, mathematicians and physicists are of the same ilk. I'm not sure, but I think it was George Polya who, when asked why he became a mathematician, said that he was too good to be a physicist, and not good enough to be a philosopher. I hope these reviews will make clear why they are appropriate for a humanistic journal.

Alan Cromer is a theoretical nuclear physicist at Northeastern University who is actively involved in school science education. When teaching elementary college physics, he was always troubled by the inability of students to follow the rational analytical thought which, he believed, was necessary for the understanding of basic physics. Well, welcome to the club. Anyone who has taught high school or beginning college mathematics or physical science has encountered and has been troubled by this. Cromer applied more rational thought than most of us to this problem and came up with the primary premise of this book. The reason for the difficulty for most people, he argues, is that the analytic, rational, deductive thought process so necessary, most of us in the business believe, for success in understanding mathematics, science and, hence, the universe, is unnatural. He argues that if it is natural, it would have evolved in most, if not all, cultures. The only culture where it did evolve was, according to Cromer, the Greek culture. It is part of our culture (so-called "Western" culture) because it was nurtured in Islam and came to Europe in the European Renaissance. Cromer lists seven cultural factors that stimulated the development of objective thinking in the Greek culture: (1) the assembly, where men first learned to persuade one another by means of rational debate, (2) the maritime economy, that prevented isolation and parochialism, (3) a widespread Greek speaking world, (4) an independent merchant class that could hire its own teachers, (5) the *Iliad* and the *Odyssey*, the epitome of rational thinking, (6) a literary religion not dominated by priests, and (7) the persistence of these factors for one thousand years. His presentation is convincing.

Unlike the egocentricity of other cultures, which Cromer says is natural, the Greeks were able to separate internal thought from external objectivity. In addition to objective thinking as unnatural, Comer cites monogamy, honesty, and democratic government. He says that in the Old and New Testaments, knowledge is belief. Regarding his beliefs, Comer states, "I believe that rational civilization, with its science, arts, and human rights, is humankind's greatest hope for nobility. But like Jericho, it's but an oasis in the midst of a vast desert of human confusion and irrationality." For elucidation of the last sentence in that quote, I invite you to peruse the preface of the book.

So, what is to be done about it? Cromer submits that since our higher rational abilities do not develop spontaneously, they must be cultivated by the formal educational system. He says that since many intelligent students are unable to grasp mathematical logic, the normal sequence does not lead to this ability. He concludes that while physical development, given adequate nutrition, is pretty well programmed in the genetic make-up, mental growth depends strongly on the cultural and social environment. We should nurture objective, rational thought in our culture, I would imagine, through our educational system. Is there something wrong with this picture? I grew up in a fairly stable environment. There was very little change in the student population and in the teaching staff. Yet, when we got to geometry (traditionally, the first chance at deductive thinking) some of us caught on early, while every week or so a couple more would catch on, and, perhaps, a few never did. Of course, those in the "other track" probably never had the chance. One of my earliest teaching positions (and a great experience it was) was in a small village where there was even less change in the student body and faculty. These students, as we, were subjected to essentially the same learning environment. Although with the "new math," I started with deductive processes in Algebra I, I still experienced the same thing with the rate and extent of student development in deductive abilities. Again, the students in the "other track" didn't have the opportunity. Could there be a gene for relatively quick development of the ability, one for a slow development of the ability, and one for no development of the ability? Or could it be testosterone, as some have concluded? At any rate, I think Cromer's suggestion of an educational environment that attempts to develop rational thought is a good one. I fear, however, the trend is in the opposite direction. One culprit, I believe, is the egalitarian movement which pervades current education; everyone should get the same education, they demand. Of course, there were some flaws in the old tracking system, but might there not be some middle ground? Another culprit, I believe, is the extensive use of the computer. It is not that the use of the computer is bad; it is very good and absolutely necessary. What is bad is the substitution of learning by observation for learning by thinking, and I think there is too much of that. Cromer presents a broad sweep of criticism of the schools in the United States. These broad generalizations are dangerous; there are many excellent schools in the U.S. I won't debate the ideas presented there (I could write pages about that. In fact, I did, but decided to zap them), but I do agree that in many cases teachers and parents are not demanding enough and there is a great need for improved methods of developing objective, rational thought in students.

It is not that the use of the computer is bad; it is very good and absolutely necessary. What is bad is the substitution of learning by observation for learning by thinking, and I think there is too much of that.

Cromer does a nice job of presenting historical and cultural information pertinent to his case. This is familiar stuff, I think, to most of us, but I think it is good to be reminded and to get it from a different perspective.

My major criticism of this book is the author's attack on religion. He is as irrational in his criticism of religion as he accuses religion to be. I really think this detracts from his presentation and should have been left out. Belief in God, Mr. Cromer, is not ego-centric. God is not an extension of self, but rather, self is an extension of God. I quote from Albert Einstein, who was, himself, a fair to middlin' physicist, "Science without religion is lame, religion without science is blind." All of which provides us with a neat segue to the other book to be reviewed.

Frank Tipler, also a theoretical physicist, has written a book, albeit a very formidable book, that provides us with the science that Einstein suggested is needed for religion. The author uses 339 pages of exposition, 35 pages of notes, and 123 pages of Appendix For Scientists (well, maybe for some scientists) where he provides the deductive development to prove the immortality of all. The concepts he uses in the exposition and the mathematical model he uses in the deductive development are quantum field theory. Now, we all know that for any deductive development, there must be definitions and postulates. In order to apply physics to the question, Tipler defines all life forms (including humans) as machines, the brain as an information processing device, and the soul as a program being run on a computer (brain); the basic postulate is that the universe is such that life can continue until

Tipler defines all life forms (including humans) as machines, the brain as an information processing device, and the soul as a program being run on a computer (brain).

the end of time. This definition may be somewhat troubling to some. We must not, however, consider it as a denigration of human life, but rather as necessary for the mathematical model in order to apply the deductive process to the question. Assuming that humans are machines allows for the proof of free will and life after death in a place that resembles the Heaven of major religions. Tipler explains that while we are machines, we differ from the machines we build in that we have "true free will." He further explains that the postulate that life can continue until the end of time is necessary because the Einstein field equations are maximally chaotic and it is impossible to make predictions regarding the universe in the near future, cosmologically speaking. The postulate, which chaos theory makes plausible, solves the prediction problem along with other puzzles of physics such as what boundary conditions to put on the wave function and why the universe exists, and leads to the conclusion of immortality.

I quote from the preface:

> When I began my career as a cosmologist some twenty years ago, I was a convinced atheist. I never in my wildest dreams imagined that one day I would be writing a book purporting to show that the central claims of Judeo-Christian theology are in fact true, that these claims are straight-forward deductions of the laws of physics as we now understand them. I have been forced into these conclusions by the inexorable logic of my own special branch of physics.

Tipler does a fine job of motivating and explaining the technical concepts needed for the deductive development. One is tempted to try to convey the essentials of this, but soon finds the ideas needed to do this expanding exponentially. I will, however, attempt to pass on some of the ideas without adhering to sequence or continuity. The postulate that life can continue until the end of time is made feasible by defining a living being as any entity which codes information. By developing self-replicating computers, it is possible to accomplish this. Tipler states, "From the Berkstein Bound it follows that, using computer memory capacity of the amount indicated by the Berkstein Bound, a computer simulation of a person...will not merely be very good, it will be perfect. It will be an emulation....an emulation of an entity *is* the entity. An emulated human will be made of emulated human cells, made of emulated molecules, quarks, and gluons." Since information processed (life) must diverge to infinity in finite proper time, we had better get crackin'. Well, you folks had better; I'm retired. Come to think of it, maybe you won't have to. Maybe someone out there is already well on the way, and we are merely simulations (remember New Mexico). On the other hand, we do have free will. Don't we? The devil made me say that. The theory requires that information, the available energy, the temperature and density of the universe all diverge to infinity as the universe converges to a single point (the Omega point) in finite proper time. Tipler distinguishes "proper" time from "subjective" time and relates these to the "tempus" and "aevum" respectively as described by Thomas Aquinas. He states that the mathematics of quantum mechanics forces us to accept the Many Worlds Interpretation. After an hour in the steel chamber, Schodinger's cat is in the quantum state—both dead and alive, and we, too, split into two worlds, observing both the cat dead and the cat alive. This Omega Point theory results in the existence of God as creator of the universe and immortality for all life with God at the Omega Point. The theory leads to a model of "God who is evolving in His/Her immanent aspect (the events in space time) and yet is eternally complete in His/Her transcendent aspect (the Omega Point, which is neither space nor time nor matter, but is beyond all of these). According to the author, the properties of the universal wave function constrained by the Omega Point Boundary Condition are those of the biblical Holy Spirit. This all sounds far out, but I have one caution. Don't scoff at this or reject it out of hand without studying this book.

Tipler admits that there are few physicists who understand quantum field theory. Prior to this book, belief in everlasting life had to depend on faith. Now, with Tipler's proof of the Omega Point Theory, at least most of us can base our beliefs on, well, faith. I'll see you all at the Omega Point. If you get there first, draw a blue line; if I get there first, I'll erase it.

The First CAMS Project: A Humanistic Endeavor

Barbara A. Wainwright & Homer W. Austin
Department of Mathematics and Computer Science
Salisbury State University
Salisbury, MD 21801

I. INTRODUCTION

One of the current themes in mathematical circles is humanistic mathematics. Educators want to make mathematics seem humanistic particularly to non-mathematics majors. There are various ways in which faculty at Salisbury State University attempt to do this in the liberal arts mathematics courses; however, this paper will describe something a little different. During the 1993-94 academic year Salisbury State University created a Center for Applied Mathematical Sciences (CAMS). The center attempts to connect a client (from industry) who has a project or problem to be solved with a team of students and faculty members who serve as advisors. Although most of the contracted projects are scientific in nature (physics, mathematics, computer science, etc.) and require a team of students knowledgeable in these sciences, there are many humanistic aspects involved in such projects. We share our experience as directors of the first CAMS project.

II. THE FIRST CAMS PROJECT: A SATISFACTION SURVEY

With much emphasis today being placed on accountability and how well faculty perform their jobs, the Director of CAMS thought a survey of the "mature graduates" of Salisbury State would be a good first project for a CAMS team. He thought also that such a survey would be of interest to the president of our University.

We agreed to co-direct such a project and proceeded to recruit some of our mathematics majors who were working toward a concentration in statistics. With these students we approached the president of the university and easily convinced him to fund the project. The president was enthusiastic about the project because assessment was one of his priorities for the year.

The first semester we had five students enrolled in "Directed Consulting," the course name given to a CAMS project. Four of the five were mathematics majors with concentrations in statistics and the other was a liberal studies major with a concentration in computer science. We decided that "mature graduates" would be those alumni who had been out of college from five to twelve years. We assumed that such graduates had been out long enough to have either gone to graduate school or be settled into a job or career. Part of the agreement with the provost and the president was that we interview the department chairs and deans to find out what types of information they would like to get from the survey of these "mature graduates."

It should be mentioned that the important aspects of a successful project are teamwork and good communication skills. This team worked well together and partitioned the work load fairly. The first task the team tackled was deciding who would speak to whom and then they began their interviews with the department chairs. These students soon discovered that good communication skills were essential. Many department chairs did not understand what the "math department" survey had to do with them. The students decided that since there is often a small return rate on mail surveys, they would offer an incentive to the early respondents. As a team, they agreed that a university mug would be a nice gift for the first 100 people to respond. They shopped and compared prices. They also considered various art designs and talked to administrators on campus about funding for these mugs.

Next came the development of the survey. Clarity and political correctness were absolutely necessary. Again, a great need for good written communication skills became apparent to the students. The authors each gave the survey to an upper level class that we were teaching that semester to serve as a pilot. After receiving feedback, the students edited and revised the instrument. These students were also very concerned about aesthetic appeal of the survey instrument. They wanted the appropriate color of paper and arrangement of questions on the page.

Students who participate in a CAMS project must present the results/conclusions of their project to the Department of Mathematics and Computer Science and to the client who contracted the project. At the end of the first semester, this first team gave such a presentation to the department, the president, and other administrators on campus. This presentation consisted of their discussing the various tasks they had performed, sharing of what value this experience had been for them, and presenting the final project, the survey instrument. With approval from the president, we planned for a January mailing.

III. MAILING THE SURVEY AND INCENTIVE.

We all agreed that with the semester ending in December, we should wait until after the holidays to mail our survey. During winter term we all met for three days to prepare for bulk mailing. The students were very concerned that the survey would be put aside with other papers and overlooked. They believed that if our survey were colorful it would not be easily misplaced. Their choice was to have the survey on gold paper. The cover letter was pink and the return sheet (for sending the mugs) was blue. We spent three days stuffing envelopes, sampling from a sampling frame of address labels generated by the alumni office and sorting by zip code for the bulk mail. We really began to know the students quite well and learned to work very well as a group.

IV. ANALYSIS AND RESULTS OF SURVEY

As the surveys started coming in we kept track of the first one hundred received. The blue slips were pulled from these so as to not have name and/or address associated with response. Once the first one-hundred responses were obtained, the team packaged and sent the mugs.

The second semester two additional members joined our team. One was a graduate student majoring in mathematics/secondary education. The other was a part-time student who was a mathematics major with a concentration in statistics.

We spent several meetings beginning to code the data from the surveys. A data file had to be created, and if each student was to enter data, we had to be particularly careful that everyone used the same format. In short, the data had to be coded. We had to agree on the proper numeric codes to represent each possible response. The students at this point behaved much like poets and writers. Just as a poet or author wants the exact word to communicate a thought, the students were very choosy in their selection of codes. With data entry, there had to be a division of labor. Each student entered approximately 100 surveys. Two of the students wrote an SPSSx program to analyze the data. These students were very conscientious about sticking to the objectives of the survey. There were many comparisons that could be made but some were not consistent with the original objectives. Only those comparisons that were pertinent were included in the analysis.

The results were written in a report and were given in a presentation. Two other students used Harvard Graphics to prepare bar graphs and pie charts for the responses for various questions. A beautiful color report was prepared and presented to the president of the University. Again they emphasized the artistic quality of the report. The colors and style made this easy to read and understand. These students decided that the final presentation should be a celebration. We reserved the great hall on campus. We invited the president and other administrators as before. However, notices were sent around campus notifying everyone in all departments of the event. CAMS board of trustees and other local business people who may have an interest for future CAMS projects were invited. Family members of the seven team members were also invited. Refreshments were served afterward.

V. REACTION TO THE PRESENTATION AND RESULTS

The presentation was a huge success. Of course the results were very favorable and certainly what we all liked to hear. The quality of the presentation was superb. A local businessman told us that he had attended many conferences and presentations and heard some very well known people present talks, but none seemed any more professional than these students.

VI. SUMMARY

Many people feel that mathematics is not humanistic in nature. They feel that it is different from the arts, music, literature, and communication. Most mathematicians know that mathematics is an art and that it is beautiful; however, for those who still believe that mathematics is only for "solving problems," we hope we can convince them that even when scientific and/or mathematical approaches are used to solve problems, one still needs the arts and communication. This first CAMS project would not have been successful without these interconnections between the arts and mathematics. The success of this project was in large part due to the humanistic aspects of mathematics.

Coming in Future Issues

Seymour Kass
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Ein-ya Gura
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Emamuddin Hoosain
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Julian F. Floren
Book Review: A Tour of the Calculus by David Berlinski

Alvin White
Book Review: Garbage Pizza, Patchwork Quilts, and Math Magic by Susan Ohanian

Bernard A. Fleishman
John Dewey: The Math and Science Standards and the Workplace

Josephina Alvarez
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Martin Bonsangue
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HARVEY MUDD
COLLEGE

301 East Twelfth Street
Claremont, CA 91711–5990