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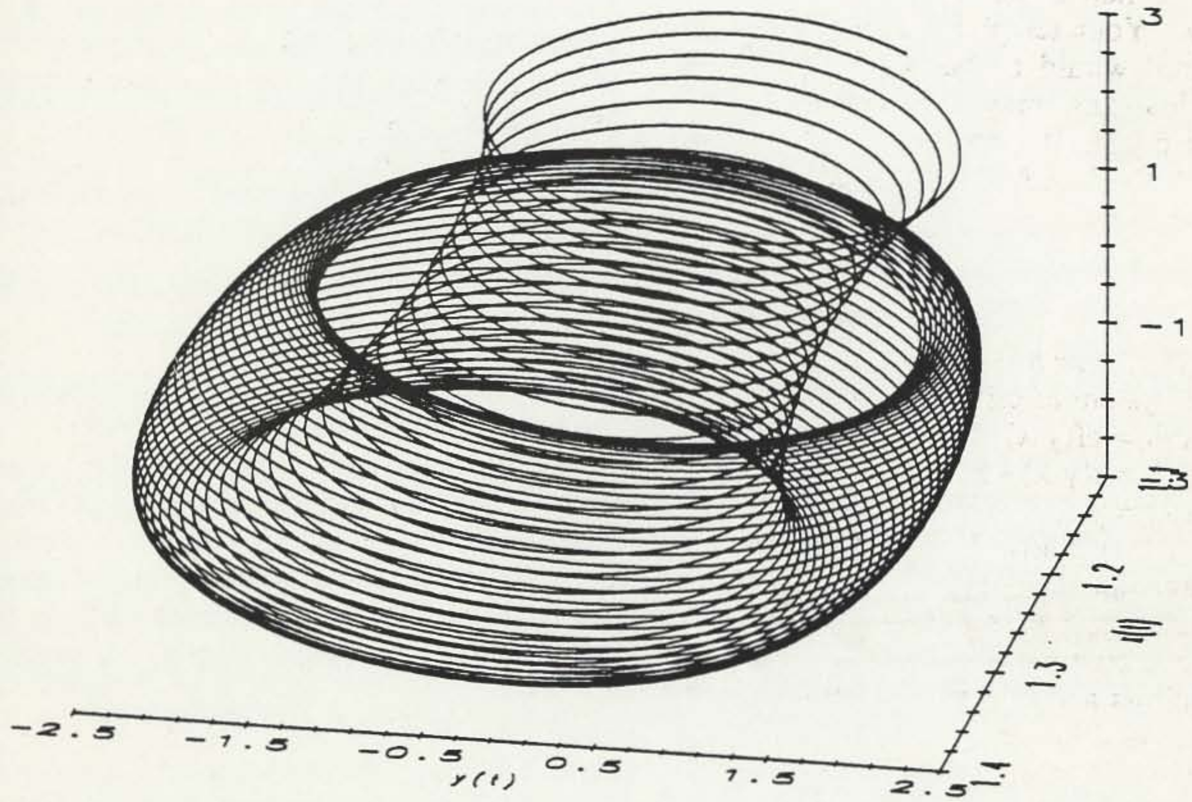

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Essays, book reviews, syllabi and letters are welcome. Two copies, double spaced, should be sent to Alvin White, HUM. MATH. NET., Harvey Mudd College, Claremont, CA 91711. If possible, avoid footnotes and put references and bibliography at the end using a consistent style. If you use a word processor please send a diskette in addition to the typed paper. *The Journal* is assembled using Microsoft Word 4.0 and PageMaker 4.0 on a Macintosh. It is possible, however, to convert from other word processing systems. Clean typed copy can be scanned (but not dot matrix). Your essay should have a title, you name and address, and a brief summary. Your telephone number (not for publication) would be helpful. Essays and communications may be transmitted by electronic mail to the editor at AWHITE@SIF.CLAREMONT.EDU. FAX (909) 621-8366.

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The Humanistic Math Network Journal #8, ISSN# 1065-8297 is the successor to the Humanistic Math Network Newsletter, ISSN# 1047-627X.

COVER

The inverted mushroom of the figure is traced by a single orbit [x(t), y(t), z(t)] of the nonlinear system of differential equations

$$dx/dt = cf(y-x)$$
$$cy/dt = f(y-x) - z$$
$$dz/dt = y$$

$f(v) = 0.07v-0.085(|v+1| - |v-1|)$, c is a constant, and t increases from 0 to 400. The system models a simple two-looped electrical circuit built by L. O. Chua and his colleagues, a circuit that has two capacitors (c is the ratio of the capacitances), an inductor, and a nonlinear resistor with characteristic modeled by f. The functions x(t), y(t), and z(t) model the voltages across the capacitors and the current through the inductor. As the parameter c is increased from 0.1 (the value used in the figure) to 50, the circuit and the orbits of the system display an amazing sequence of bifurcations period doublings, and finally chaos.

References:

1. R.L. Borrelli, C. Coleman, W.E. Boyce, Differential Equations Laboratory Workbook, 249-252, 328-329; John Wiley and Sons, New York, 1992.
2. T. Matsumoto, L.O. Chua, R. Tokunaga, Chaos via torus breakdown; IEEE Trans. Circuits Syst., (CAS-34(3):240-253 (March 1987)).

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From the Editor

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An element that is absent in many education reform efforts is the interaction among students and between students and teachers. Martin Bonsangue's article describes how students working together in calculus workshop groups succeeded to a remarkable degree compared with students who did not participate in the workshop groups. His important study was reported to a group at the National Academy of Sciences. Bonsangue describes the workshops as a humanistic experience.

Kazem Mahdavi's discussion of Attracting Math Majors at Potsdam is a rehearsal of the study by Rick Luttmann on the Basis for the Success of the Potsdam Program and Clarence Stephens' description of a Humanistic Academic Environment for Learning Undergraduate Mathematics which are reprinted from earlier HMN Newsletters.

These essays and studies, I believe, describe examples of the suggestions found in *Everybody Counts* (National Research Council 1985, pp 58-59):

> Educational research offers compelling evidence that students learn mathematics well only when they *construct* their own mathematical understanding... This happens most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning.

Professional Standards for Teaching Mathematics (National Council of Teachers of Mathematics 1991, p 3) lists five major shifts that are needed to move from current practice to mathematics teaching for the empowerment of students. We need to shift—

* toward classrooms as mathematical communities—away from classrooms as simply a collection of individuals;
* toward logic and mathematical evidence as verification—away from the teacher as the sole authority for right answers;
* toward mathematical reasoning—away from merely memorizing procedures;
* toward conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding;
* toward connecting mathematics, its ideas and its applications—away from treating mathematics as a body of isolated concepts and procedures.

Every essay and every poem in the journal helps to describe and define humanistic mathematics. Bill Rosenthal takes a direct and whimsical approach in his essay. He reflects on the consequences of an unambiguous definition—that must be followed by theorem, proof. He then fantasizes that this would be followed by applied, pure, and homological humanistic mathematics, finally to be honored by a Bourbaki volume.

Humanistic Mathematics carries with it an awareness of and a sensitivity to what mathematics shares with the other humanities. Humanistic dimensions of mathematics include:

- An appreciation of the role of intuition in understanding and creating concepts that appear in their finished version, to be a result of a "merely technical" process.
- An understanding of the value judgements in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, or why it is investigated.

Many who have heard of humanistic mathematics quickly recognize this movement as a source of new vocabulary, and new colleagues who feel as they do about mathematics in their lives and in their students' lives. The movement, which began as the personal vision of a few, has now become a major part of mathematical culture. What was viewed with skepticism is now accepted and expected. In addition to the essays in this journal there are twenty-two essays in the forthcoming volume <u>Essays in Humanistic Mathematics</u>, in the MAA Notes series.

The MAA Notes volume is a broad introduction to the ideas of humanistic mathematics. After an introductory section, there are sections on Mathematics in the World, the Inner Life of Mathematics, Teaching and Learning Experiences, and Contemporary Views of Old Mathematics. The volume can be ordered from the Mathematical Association of America (800) 331-1622 in the fall of 1993.

The importance of our Journal continues to grow. The Library of Congress was alerted to the change from Newsletter to Journal (and the need for a new ISSN number) by its Paris office. The librarian at the Sprague Library of Harvey Mudd College tells me of the many requests for interlibrary loans of articles from our publication. Other U.S. libraries that carry the Journal are:

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Calculus Workshop Groups as a Humanistic Experience

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The latest mathematics report card from the California State University is not encouraging. In 1989-90, 534 bachelors degrees in mathematics were awarded, about 1.1 % of all bachelors degrees at all 20 CSU campuses. Of these, 36 were earned by Latino students, 7 by African-Americans, and 4 by Native Americans. That same year, 98 masters degrees in mathematics were awarded in the CSU system. Exactly one of the 98 degrees went to an underrepresented minority student, in this case, a Latino male.

These statistics have probably caused some eyes to glaze over by now, perhaps causing an important readership group to stop reading. For the unglazed reader, though, these data are alarming, especially in light of the continued existence of so many programs that target minority achievement in mathematics and related fields. One of the critical elements in many programs is having role models in academia for students of color. I do not know if the lone 1990 MA graduate entered teaching, say, at the community college, but if he did, there are about half a million Latino students in the California Community College system's 107 campuses looking for a role model in their required algebra classes. He will be busy.

In The Challenge of Diversity, Daryl Smith discusses what she calls the perceived conflict between access and quality. "The continuing message that a fundamental conflict exists between issues of access to the institution and quality is perhaps the most disturbing indication that present institutional approaches to diversity are inadequate.

1. The concern about preparation of students, while affecting many minority students, is not a minority problem. Indeed, most poorly prepared students are white.

2. Much of the evidence concerning the tension between quality and diversity rests on lower standardized scores. Serious questions exist about the predictive validity and the power of these instruments for women, for many minorities, and for those with learning disabilities.

3. Early evidence focused attention on academic preparation as the most significant factor in achievement, leading many researchers to conclude that academic success is a function of preparation, not race. However, to the degree that issues of racism, sexism, homophobia, and the general presence of an alienating environment also affect performance, then lack of performance cannot be focused entirely on the student. All too often we have assumed the institution's perfection and students' incompetence." The mathematics degree completion data reported above seem to reflect the reality of this dilemma in the mathematics community.

The Workshop Experience

Consider the experience of the Academic Excellence Workshop Program in Calculus at California Polytechnic State University, Pomona. From 1986 to 1991, 130 African-American and Latino students have taken the traditional one-year calculus sequence while participating in twice-a-week workshop sessions outside of class. The AEW program, based on the Berkeley workshop program founded by Dr. Uri Treisman, targeted first-year freshmen students accepted in the College of science or the College of Engineering. Every African-American, Latino, or Native-American incoming student received a personal phone call from a faculty member or student leader in the program inviting the student to attend an introductory meeting about the program. Students paid no additional fees to be in the program, but received no additional credits either. Participation, though encouraged, remained the student's choice.

Lack of performance cannot be focused entirely on the student. All too often we have assumed the institution's perfection and students' incompetence.

Based on a 4-point scale, the average grade in first-quarter calculus was 2.67, compared to the department average of 1.91. More than one-fourth (26/130) of the workshop students earned grades of B+ or higher, while fewer than five percent (6/130) failed with a grade of D or lower. Second-quarter and third-quarter averages were likewise one-half to three-quarter grade points above the department mean.

This trend of success continued past the first-year courses, even though workshops were not offered with courses beyond first-year calculus. Between 1986 and 1989, seventy-eight students participated as freshmen in the workshop program. About four-fifths of these students (61/78) completed their second-year calculus sequence, this time with 40 % (31/78) earning grades of B+ or higher. Moreover, 75 of the 78 students were still enrolled in some major at Cal Poly after three years, with 85 % (66/78) still in mathematics, science, or engineering. The latest records show that as of Spring 1992, 60 of the 66 students had either graduated or were fifth-year seniors having completed all mathematics prerequisites for their degree.

Time-on-task may not be the key factor so much as time on the *right* task.

This trend of continued success past the first year was especially evident for minority women in the workshop program. Twenty-two of the 78 calculus workshop students from 1986-89 were Latinas. After three years, all 22 women were still enrolled at Cal Poly, with 19 remaining in their mathematics-based major (mostly engineering). By 1992, all 19 had either graduated in a math-based field or were within two quarters of graduation. By comparison, 23 Latina women were enrolled in the same Calculus I course as freshmen, but did not participate in the workshop sections. Within three years, only 12 were still enrolled at the university, with seven of these in mathematics, science, or engineering. By Spring, 1992, exactly four of the original 23 students had graduated or were within two quarters of graduation.

Self-Selection or Program Effects?

The data reported above raises many suspicions about the effects of the program versus those of self-selection. Plainly put, wouldn't these successful students have been successful anyway? The literature on this type of research identifies a host of variables that affect academic performance, including past achievement, motivation, socio-economic status, and parental factors (for a more exhaustive list, see Sandy Astin's work with the Higher Education Research Institute data). Of course, the data reported above is historical, so that perfect controls, such as a behavioral psychologist might use, are not possible. Did the workshop students have an edge to start with, and, if so, how much?

David Drew, my colleague in this inquiry, and I tried to triangulate the study, that is, look at the above results from three different ways and see if these all seem to say the same thing. First, we gathered all the past achievement data that was available: SATs, high school grades, qualify exam scores (given by the Cal Poly Department of Mathematics). Second, we surveyed the 1990-1991 cohort of calculus students about their study habits and involvement both inside and outside of school. And third, we interviewed juniors and seniors who had participated in the calculus workshops as freshmen (though some had dropped out of the workshop program), to listen to the effects of being in the program as the students had experienced them.

Academic Achievement

Past academic achievement showed workshop students to be at the same or slightly below the levels of their non-workshop minority and non-minority peers. SAT-math scores averaged around 540 for African-American and Latino students, and around 580 for white and Asian students. Likewise, SAT-verbal scores were lowest for Asians (around 370), medium for blacks and Latinos (430), and highest among whites (480). High school grade point average ranged between 3.3 and 3.4 for all groups except blacks, who were at 3.2. Finally, diagnostic test scores were about 29 (out of 40) for blacks and Latinos, and slightly higher (30-33) for Asians and whites. Moreover , none of the measures described here were different for workshop minority students and their ethnic peers not in the workshop. In summary, workshop students were indistinguishable on paper from students not in the workshop.

Study Patterns

The survey did reveal differences among students' study patterns. Thirty-six workshop students and 150 non-workshop students enrolled in the same lecture sections of first-year calculus were asked about their time spent studying individually, in groups, on-school activities, and off-campus commitments. Workshop students reported spending 8.25 total hours per week studying calculus (alone and in groups), while non-workshop students reported 6.25 hours. For workshop students, four of the 8.25 hours were spent studying in groups, while reports for non-workshop students varied: about 1.25 for Asian students and about 0.6 for white students. Interestingly, Latino and black students not in the workshop reported spending nearly 2 hours per week (out of 7 total) studying in groups, probably because most of these students study in the Minority Engineering Program tutorial center. There were no differences between workshop and non-workshop students' involvement off-campus, although workshop students reported spending more time involved in on-campus activities than did non-workshop students.

Although the data are based on self-report, if we assume that students uniformly exaggerate their study time, three patterns seem to emerge. First, students in the workshop spent 25 % more time studying for their calculus course than did non-workshop students. However, this difference was in time spent studying in highly structured groups, not in studying alone. Thus, while there is some evidence that the main difference in calculus performance may be explained by "time-on-task," there is little evidence to suggest that the workshop students were more highly motivated to work on their own than were their non-workshop peers. Second, minority students not in the workshop spent almost as much time per week as did workshop students, even though the average calculus workshop grade was a full grade point higher. This suggests that time-on-task may not be the key factor so much as time on the *right* task. Third, workshop students did not seem to have more "free" time than did other students.

Student Interviews

Twenty-three former workshop students were asked to describe how they felt their experience in the workshop contributed or did not contribute to their subsequent academic performance in their mathematics-based courses. The interviews ranged from twenty to sixty minutes in length, and were recorded and then transcribed with the student's written permission. Twenty-two of the interviewees were Latino students, including seven women, with one interviewee an African-American male. Rather than merely summarizing the interviews, students' comments are reported here in the context in which they were spoken. I think it is important here to let the students speak in their voices.

When you teach other people how to do it, you also get a better grasp of it. Sometimes you think you know how to do the problem, but when someone asks you how to do it, you kind of get stuck, and it shows you that you don't really know it as well as you thought you did.

Study Issues

Students were asked if they felt they would have done as well in the calculus course if they had not taken the workshop. Five of the sixteen men indicated that they would have done just as well, with one male student not sure, whereas only one of the seven women so indicated. There seemed to be two common factors in those who felt they would have done as well: first, that they studied hard, and would have learned the material anyway, and second, that the group was sometimes a "distraction" to learning. However, two of the five men felt that, in retrospect, they should have taken the workshop more seriously. One student felt he was "cocky" coming to college after a successful high school career, and failed his calculus course. R. R., an Electrical and Computer Engineering major, indicated that the workshop was a "disappointment," but that he still should have been more committed:

> I underestimated the level of work, particularly in math or science. Calculus is still a block to my other science courses.

The students indicating that they would not have done as well without the workshop echoed the common theme that the workshop sessions were a time of intensive study. More than half of the students mentioned the facilitator's importance in structuring the study time. One student said that the facilitators "pushed you" into being prepared for the exams. Several students indicated that those four hours per week were reserved for calculus, so that there was consistency in their own preparation for the course. Although not asked directly, five students discussed their amazement at being academically challenged by the other workshop students. M. J., a junior in Industrial Engineering, said that he was a

> . . . good student in high school, in the top 5 % of my class. But the workshop people were smarter than me or more disciplined. I saw that people were better than me. They challenged me to get to their level.

However, not all workshop participants desired this level of academic involvement. One interviewee who had dropped out of the workshop indicated that the program was good for "serious" students, but did not provide the social experience he needed.

Twelve of the 23 of the students indicated that their individual study time was affected by their workshop involvement, although the nature of that effect varied considerably. Most of these students indicated that the course material that they learned during the workshop sessions decreased the amount of time it took for them to learn the material on their own. Several students, however, felt that the workshop "cut into" their homework time. One of these students felt that the purpose of the workshop session was "like tutoring," and was frustrated that he could not work on the homework during the workshop time. Conversely, two students indicated that the level of difficulty of the worksheets in the workshop sessions made them realize that they needed to increase their individual work time in order to "get up to level that [other students] were at." One student felt being in the workshop "guaranteed" him four hours per week, especially when he was inclined to procrastinate his studying. R. D., a transfer student in Mechanical Engineering, articulated her realization of the amount of study time necessary to succeed in technical courses:

> You realize how much time you put into that one class, and that was, you know, a freshman-sophomore level calculus, so you get higher, you think 'Gosh, it took me that much time, plus the workshop, plus the tutoring,' it makes you really realize that you do need to put in time, it takes time to do your classes.

Hence, although workshop effects on individual study time were mixed, students indicating a change felt that their study time had changed both in terms of quality and quantity.

Just seeing that other women have made it, you feel that maybe you can make it.

All of the interviewees indicated that studying in groups was helpful in learning the material, although for the majority of students working collaboratively was a new experience. G. R., a senior majoring in Mechanical Engineering, summed up:

> When you teach other people how to do it, you also get a better grasp of it. Sometimes you think you know how to do the problem, but when someone asks you how to do it, you kind of get stuck, and it shows you that you don't really know it as well as you thought you did.

Several other students felt that the workshops "made me explain myself" in working with peers on calculus problems, particularly in solving applied problems. One student who now enjoys working in groups described himself in high school as a "loner," and that working with other students was "distracting." Further, he felt that in high school he could not depend on others for help: "I trusted myself and no one else." He felt that the combination of the social and academic aspects of the workshop were for him important in learning to adapt to group learning. Moreover, one student indicated that although he agreed in theory that group learning was helpful, he didn't perceive the need for personal involvement:

> I can do it on my own. I work 35 hours per week. But sometimes I didn't work; I would use that as an excuse. If I did have time, I would waste it, not work on school.

Thus, although his work schedule made workshop participation difficult, he admitted that even without work he probably would not have felt much differently.

The effect of the workshop experience on personal study habits was mixed. About three-fifths (14/23) of the students felt that the workshop

There is, however, strong evidence that the experience of women in the technological courses is qualitatively different than that of men.

had a definite impact on their study habits, particularly in working together in groups. Five students mentioned that they now seek out other students in their classes to study with, even though the students were not workshop classmates. Several students remarked that they had learned from their facilitators how to use different study tools, including writing their own worksheets as test reviews. Two of the students that did not indicate any changes in their own study habits still felt that group learning was "fun," and felt that the workshop experience had been beneficial. Two students indicated that they had gained very little benefit in learning how to improve their study effectiveness.

Students were asked if participation in the workshop had affected their decision to continue in their MSE major. More than half (4/7) of the women reported a direct effect, compared to less than a third (5/16) of the men. The common theme among the nine students indicating a positive effect was that of encouragement. Although the interview sample is small for general inference, the importance of encouragement from peers and from the facilitator seemed to be especially important to the women. This encouragement sometimes came in the form of discipline and accountability. O. S., a senior majoring in Computer Science, reflected about her first year:

> I never thought I could do it. I had to take remedial courses, couldn't pass the stupid diagnostic test. I got a B+ in physics. Before it was, like, C's. It brought me up one to two levels from F to C, or D to B. [The facilitators] push you to a limit that you think you're gonna die, that you can't do it any more, and they say, 'It's not that hard.' I had many facilitators stay with me until I got that problem, 'cause they knew I wouldn't do it if I left.

S. B., a senior in Electrical and Computer Engineering, mentioned the importance for her of the facilitator as a role model:

> I remember one specific math facilitator. She was Tau Beta Pi, the top 20 % for engineers. You look up to them.

However, several of the men indicated the importance of encouragement of their peers. M. J., a junior in Industrial Engineering, summarized the importance of encouragement after a test:

> It's the whole thing about the workshop, you know, to get you up there to another level. It's discouraging sometimes, you do bad on a test, you go to the workshop, it's like, 'Man, how you doing?' 'Not too good.' 'Let's talk about it. What was the problem?' [Your peers] encouraged you a lot.

Conversely, about two-thirds of the students felt that the workshop had not guided their decision to stay within a technical field. However, several students stated that working in an environment where excellence was expected confirmed their sense of being potential engineers or scientists.

<u>Ethnic Issues</u>

Interviewees were asked if the workshop experience had affected their perceptions of minority students in the mathematical sciences. Exactly half of the 22 Latino students said that it had, while the other half said that it had not. There were strong feelings on the part of some students on both sides. Almost all of the students answering no indicated that their high schools had been ethnically diverse, with minority students enrolled in the top mathematics and science courses. Conversely, most of the students answering yes came from backgrounds where minority participation in academic pursuits was the exception. One student indicated that in high school there were few Hispanics in the advanced mathematics courses; here, he felt like "part of the crowd." C.A., a senior, shared a strong personal experience:

> In high school, I never considered myself Hispanic. Sure, my name was Antonio, but everyone called me Tony. I felt white,

> I didn't feel Hispanic. I felt myself being prejudice against Hispanics. I denied my heritage. Coming to Cal Poly, I was shocked. I'd never seen so many blacks and Hispanics. But I was signed up for the workshop. Plus I found out that this place was OK. People knew my name now, this is pretty good, y'know? Wow, suddenly these guys are OK, and I started perceiving them differently, like, 'Yeah, I'm Hispanic. What's wrong with saying my name, 'Antonio,' or saying I'm Hispanic?' I started feeling Hispanic, and admitting it, and not feeling ashamed, like, 'Yes, I'm part of this group.' With minorities I used to look down upon, I feel now that I'm one of them. It changed my entire perception of them.

Conversely, A. B., a junior transfer student majoring in Chemistry, felt resentful that the workshop program was aimed only at minority groups:

> Why is it just for us? Do we really need that much more help? In [previous institution], the workshop was open to all groups. I was uncomfortable that the workshop was for minorities only. It felt like cultural singling out.

A. B.'s was the only strong negative reaction to the question. However, it does suggest that while many students enjoyed the fellowship of other minority students, there were some who were not comfortable with the ethnic selectivity of the workshop.

The interview group included one black student, H. A., a senior majoring in Computer Science. When asked if the workshop experience had affected his perception of Blacks in the MSE fields, he responded candidly:

> I always tried not to think about it [being one of the few Black students], 'cause it's what I've grown up with. I try not to put any undo pressure to say, 'You're the only one here, you gotta booster the culture, the society. I just try to say, 'Let's do what we can...and do it as well as we can.' Every now and then I'll look around again and say, 'Yep, [I'm the] only one again.' I think in this quarter I have four classes, two of them are CS, and I have 2 Blacks in those four classes."

Although H. A. was otherwise very enthusiastic about his experience in the workshop, his response indicated a mix of frustration and determination in representing the very small community of Black students succeeding in the mathematics, science, and engineering majors.

The effect of the workshop upon perception of minority students as a competitive group in technical fields seemed to depend largely on the students' previous experience. Students attending institutions with diverse populations felt far less of a change than did students who attended schools where minority students were not greatly involved

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In working difficult mathematics problems designed by the facilitator during workshop sessions, interview students recalled not only developing a deeper understanding of the material, but becoming acutely aware of the level of comprehension that their professors would expect of them.

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in advanced mathematics or science courses. Some interview students indicated finding a true sense of community with fellow ethnic students, while others felt resentment or embarrassment at being singled out. The one Black student interviewed articulated his frustration at being one of very few Blacks who have managed to get through the mathematics pipeline into higher level courses.

Gender Issues

The seven women who were interviewed were asked if their workshop experience had affected their perception of women in technical fields. Each of the students indicated that it had, with several of the students having a strong reaction to the question. The common theme was that of isolation in a male-dominated discipline. C. T., a senior in Electrical and Computer Engineering, summarized the problem:

> There are very few women in ECE classes. You look around, see half Asian, half white, and you're the only one. It's kind of lonely.

O.S., a Computer Science major, implied that for her, the issue of gender was far more important than ethnicity or race:

> There are very few women in Computer Science and in technical fields. Being a minority, Hispanic, and being female, when you're walking to class, you're the only girl, and it was very awkward. It doesn't matter if you're Hispanic, or orange, or purple, or whatever. It's like the guys are, 'What is she doing here?', that kind of attitude. They get over it, 'cause...you're doing good in the class. When you are doing better [than the men], it's a shocker!

Moreover, C. L., an above-average student majoring in Mechanical Engineering, indicated the importance of having met "a lot" of female engineers. Although she was glad to have had two workshop facilitators that were women, she discussed the barriers to women in engineering:

> You hear about how few women actually graduate in engineering, even the ones that start in it, and I seem to think it might be harder for women to actually graduate in it. Whether they mean to or not, you see a lot of, you know, prejudice against the women...within the teachers or fellow students as if they don't seem to think you can actually make it sometimes.

Although not prompted to do so, she went on to discuss more specifically the issue of sexism in her engineering courses.

> You also hear about the teachers that are sexist, or the ones maybe you shouldn't take. Of course, there's no way to prove anything about that, but the teachers themselves say that sometimes it causes a difference in their grading, even though they don't mean to.

She concluded with the importance of women role models who have completed the engineering degree:

> Just seeing that other women have made it, you feel that maybe you can make it.

Still, success seemed to be something that was not entirely in C. L.'s control, in spite of earning grades that were higher than those of most of her male colleagues.

The concerns that these women raise are reflected in the low proportion of women graduates in engineering discussed at the beginning of this chapter. From 1985 through 1990 at Cal Poly, 12.3 % (382/3112) of the graduates in the College of Engineering were women, with 11.5 % (44/382) of this group Black or Latino. Stated differently, African-American and Latino women accounted for less than 1.5 % of the engineering degrees at Cal Poly over a five-year period. The extent to which academic or social barriers in technical courses affected the women in the sample

The strongest reaction to interview questions regarding effects of group learning came from the women, with each student reporting that she had experienced some degree of isolation enrolling and competing quarter after quarter in classes dominated by men.

is difficult to determine without other instruments of precise measurement. There is, however, strong evidence that the experience of women in the technological courses is qualitatively different than that of men. The extent to which ethnicity of the student or gender of the instructor play a role was not measured in this analysis. However, the women interviewed here articulated a strong need for women role models in engineering with whom they had direct contact, as well as a peer community of women sharing the same or similar experiences in college. R. D., a transfer student in Mechanical Engineering, succinctly described the importance of the community for women generated during the hours spent together in workshop sessions:

> Your minority women, it gets them together, and it's like, Let's hang together and get through this.'

<u>Summary</u>

Interviews with former workshop students who have persisted in their Mathematics, Science, or Engineering majors suggested that the quality of study may be at least as important as the quantity. Active involvement in the two-hour workshop sessions was a key component in the learning process, with students having to explain their own and challenge each other's solutions to mathematics problems, particularly with applications. About one-half of the students thought the workshop actually decreased or made more effective their individual study time, although several students felt they had to increase their time outside the workshop in order to keep up with their workshop

Science magazine recently reported that currently at the University of Texas at Austin, 23 % of the approximately 500 undergraduate mathematics majors are minorities, largely due to their participation in the Emerging Scholars Program, a workshop program for entering calculus students.

peers. All of the students found the collaborative model to be effective to varying degrees, although some did not indicate that the workshop had any lasting effect on their personal study methods. About three-fifths of the former workshop students indicated that they had since studied in groups regularly, often forming their own groups with non-minority students in their major classes.

The most prevalent interview theme centered on academic involvement and awareness. In working difficult mathematics problems designed by the facilitator during workshop sessions, interview students recalled not only developing a deeper understanding of the material, but becoming acutely aware of the level of comprehension that their professors would expect of them. Especially helpful was the interactive nature of the sessions, where students verbalized their ideas to one another. Community-building and developing interpersonal relationships occurred within the context of academic rather than social activities, although a number of students stated that the best workshops were ones in which the facilitator used games and other activities in some sessions.

Working in peer groups seemed to be a new approach for most of the workshop students. About three-fifths of the group indicated that this technique of group learning had carried over beyond the calculus classes into junior and senior-level courses within their major. However, the extent to which this carry-over was directly associated specifically with the calculus workshop was difficult to determine, since some of the interviewees had taken workshop sections of other courses as well, including Mechanical Engineering, Physics, and Chemistry. Fewer than half of the workshop students indicated that their workshop involvement had affected their decision to stay in a technical major, although the workshop may have had a greater impact on MSE persistence for women. The strongest reaction to interview questions regarding effects of group learning came from the women, with each student reporting that she had experienced some degree of isolation enrolling and competing quarter after quarter in classes dominated by men. The workshop seemed to provide a bi-weekly network for the women that was not possible in a tutorial drop-in setting. Several women described feelings of self-doubt in their ability to succeed in engineering, even though their in-major grade point average was higher than those of most of their male peers. In summary, the experience described by women of successfully getting through a mathematics, science, or engineering major was qualitatively different than that described by men.

Conclusion

In the interviews students describe a rich tapestry of experiences that were associated with the calculus workshops. Few, if any, of the students interviewed had ever studied in peer groups before their calculus courses. However, most of the students attribute their success in varying degrees to their workshop involvement. Of course, it is still not resolved whether these particular students would have done as well anyway, just as one's headache might have gone away even without the aspirin. However, almost every student reported that the workshop affected their academic lives, many in permanent ways. At any rate, the Cal Poly Workshop experience shows that there exists a nontrivial number of students who will participate, will succeed, and will form the nucleus of a successful group of minority students in mathematics-based disciplines that currently have little minority representation.

Dr. Treisman talks about workshop classes being "inroads to the major," where students spend a lot of time in the department or study room where the sessions are held. In this sense, students feel that they are part of the academic life of the department, and feel a connection to the department and some faculty. Thus, workshop-type programs may be less about enhancing academic performance

Achievement among underrepresented minority students in mathematics, science, and engineering disciplines is less associated with pre-college ability than with in-college academic experiences and expectations.

as about creating structures by which students can become assimilated in a meaningful way into the department. Science magazine recently reported that currently at the University of Texas at Austin, 23 % of the approximately 500 undergraduate mathematics majors are minorities, largely due to their participation in the Emerging Scholars Program, a workshop program for entering calculus students.

Julian Weissglass of the Mathematical Sciences Education Board proposes that "changing the system means changing ourselves" as a mathematics community. He poses the following beliefs and values that describe the practices of many mathematics classrooms.

Competition is necessary to motivate learning.
Noise is distracting.
Telling is teaching.
Paper-and-pencil assessment is adequate.
It is cheating to get help from another person.
Feelings are not part of the academic environment.
The system is O.K. (after all, I succeeded).

At the very least, the Cal Poly Workshop Experience suggests that achievement among underrepresented minority students in mathematics, science, and engineering disciplines is less associated with pre-college ability than with in-college academic experiences and expectations. The rub is that academic departments must see as part of their work the creation of structures and fostering of attitudes that develop academic talent and promote student involvement.

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Like Poetry, Mathematics is Beautiful

Timidly I ask
each one I meet if they
find mathematics beautiful
or useful, and each one dares to say,
"Useful, of course. I use it every day."
And if I seem to want a proof,
they all go on to tell
that daily they subtract and add
to keep a checkbook; sometimes also
they multiply to find how many squares
they need to tile the kitchen floor.

Mathematics is not only plus
and minus, not just counting one,
two, three. There are rules to bend
defiantly, so parallels
will meet before infinity. Look
at the magic of unending terms
that converge to a finite sum:
start with one-half plus half of one-half
plus half of the last again and again.
Though we go on forever, we never
Pass one. Do you find me difficult? Oh, dear!

Suppose, instead, I ask
if poetry is beautiful
or useful. Will each person say,
"Useful, of course. I use it every day."
And if I seem to want a proof,
will they go on to say that they
use rhymes to call to mind the days
of a month -- like "Thirty hath
September" -- and to remember
how to spell words with "i" and "e."

I have a faint, enduring hope
that someday folks will see
mathematics to be
as lovely
as poetry.

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Toward a Philosophy of Humanistic Mathematics

Bill Rosenthal
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It has been an hour since I finished Sherman Stein's essay "Toward a Definition of Humanistic Mathematics". Over two years since I read in Newsletter #4, "Within the community of mathematicians and mathematics educators who identify with the term, 'humanistic mathematics,' an agreement on its meaning is still under negotiation." Five years since Alvin White did the Martin Luther bit by nailing his analogue of the "96 Theses" to the door of the first Newsletter. Six years since the Claremont conference, which I'd describe as the Fort Sumter of humanistic mathematics—if warlike metaphors didn't compromise the life-affirming spirit of our lifework. Eighteen years since Alvin published "Humanistic Mathematics: An Experiment" in Education. Thousands of years since Plato identified mathematics with *episteme*—with human knowledge itself, thus with the very essence and existence of humanity.

And still we can't agree upon a definition of our reformulation of the so-called Queen of Sciences, this Reformation in the putatively perfect language of nature. Still, those of us to whom the two little words "humanistic mathematics" resonate, invite the scorn of those who declaim our oxymoronity, who excoriate us for oh-so unmathematically developing our discipline before defining our terms.

What is this thing called "humanistic mathematics"? More pertinent, what to do about defining this thing?

Stein says in "Toward a Definition" that after the Claremont conference he "pondered the meaning of 'humanistic mathematics' without arriving at a definition." Instead, he "decided to capture the mood of the meeting." Yes. I applaud Stein"s wise and courageous decision. I think he's on to something. Perhaps the energies of those who struggle to define "humanistic mathematics" are better devoted to pondering the meanings of our embryonic endeavor. Perhaps we can serve ourselves and our students most faithfully by posing as amateur anthropologists who strive to describe the moods, the senses, and the cultures birthed at the confluence of mathematics and humanism. Perhaps the felt need to define "humanistic mathematics" is antithetical to its spirit, which cries for an expansiveness, even and infinitude of meaning rather than the constriction and delimitation of a definition.

I believe that the desire to define humanistic mathematics (no more the self-effacing quotation marks!) arises from the values of a culture rooted in an ideology of *theory*. As conceived and practiced by canonical mathematical culture, theory requires that we unambiguously define any term prior to so much as whispering it in a sentence. The theory-über-alles imperative of mathematical schooling takes this unnatural, denatured conception one misstep further: mathematicians are initiated into a subculture of *theorem-ists* who march in lockstep to the linear beat of Definition,

Perhaps we can serve ourselves and our students most faithfully by posing as amateur anthropologists who strive to describe the moods, the senses, and the cultures birthed at the confluence of mathematics and humanism.

Theorem, Proof. My nightmare: Will a definition of humanistic mathematics be followed by theorems and proofs, technical lemmas and corollaries, generalizations and abstractions? Will there be such 21st-Century bastardized offspring as advanced, applied, universal, and homological humanistic mathematics? How about the the ultimate mathematical-culture accolade: a Bourbaki volume? Is this an absurdist apocalyptic fantasy? Sure. But stranger things have happened to other well-intentioned reformers who too narrowly defined the scope of their movements only to sink in the quicksand of the mainstream.

Pondering the meaning. I offer our community—our gelatinous subculture of humanistic mathematicians and mathematical humanists—the unoriginal idea that to ponder meaning is to practice *philosophy*, not theory. Maxine Greene, the mother of all present-day philosophers of education, draws upon Hannah Arendt to tell us that "to do philosophy means to pose the kinds of questions that empower us 'to think what we are doing.'" By positing that "philosophic thought is that which bears on questions rather than answers," George Steiner recently seconded this emotion. I now bring this notion to you, to us. Inspired by Greene, Arendt, and Steiner, I propose that we engage in the devotion of a philosophy of humanistic mathematics by pondering and questioning its multitudinous meanings and what we are doing with them. May we seek to refine, expand, and characterize rather than to define, constrain, and circumscribe. Let us be mathematical-pedagogical Talmudists who inscribe our shifty and ever-shifting understandings in an infinity of questions and answers and question and....

Let's ponder the meanings of humanistic mathematics by weaving an unbounded quilt of questions. Let's wander about and wallow in its indeterminate, undeterminable senses and our partial results. Let's acknowledge and *celebrate* the reality that humanistic mathematics is to remain an eternal dilemma, not a proven lemma.

I propose that we engage in the devotion of a philosophy of humanistic mathematics by pondering and questioning its multitudinous meanings and what we are doing with them.

Let's do philosophy. And let's be proud of our choice to ponder, question, create, problematize, negotiate, amend, and mediate meanings in lieu of pandering to a positivism that mandates the yoke of a precise, taken-for-granted, non-negotiable definition.

Humanistic mathematics can and should be: a movement devoted to sociocultural synthesis of knowledge; a paradigm founded on the accumulation of differential experience; a world view whose images are formed by inversion of the usual lenses. If we have the courage to resist the invitation to lie in the definitional bed, it can be all these and more. I'm talking indefinite integration, which is a process with no limits. A philosophy of humanistic mathematics—*a humanistic mathematics of philosophers*—will inspire and organize an infinite indefinite integration of humanism, mathematics, and philosophy and now unforeseen, now unforeseeable.

Let the word go forth to all students, teachers, educationalists, administrators, researchers, reformers, liberals, liberators, radicals, reactionaries, humanists, mathematicians, and other life forms. Humanistic mathematicians are philosophers. Humanistic mathematicians make love with wisdom.

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Writing Style—An Editorial Comment

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What is Humanistic Mathematics? This has been the topic of discussion at several gatherings of members of our Humanistic Mathematics Network.

We should, however, set an example and encourage writing designed for clarity and enlightenment rather than for the purpose of impressing people with our erudition.

As those of you who were present know, and the rest of you might well suspect, there is great diversity among our members as to what Humanistic Mathematics is. This diversity of meaning is reflected in the articles of our Newsletter. However, there is one aspect which is common to most, if not all, of our views. That is that mathematics belongs most appropriately in the humanities. In fact, it seems to me that this network started as "Mathematics as a Humanistic Discipline" which I think is more descriptive of what we're about, but that's a topic for another discussion.

The point that I would like to make here is that the philosophy and spirit of the humanities should carry over into our writing. In short, we should write as humans, not as automatons. A recent criticism of a humanistic math article was that it was "informal and conversational". I would take that as a compliment; I think that is how we should be writing. I don't know about you, but I dislike the stilted pseudo-esoteric prose that permeates mathematics writing. I think that is one thing that keeps people from reading about mathematics and drives many capable, intelligent, people away from mathematics.

We should also, as humanists, be tolerant of various writing styles. We should even, I guess, tolerate the writing of the research mathematicians, bless their little hearts, who likely will continue to write in their stilted, stark manner. We should, however, set an example and encourage writing designed for clarity and enlightenment rather than for the purpose of impressing people with our erudition.

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1992 3 10

Prof. Alvin White
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Dear Prof. White:

A good deal has happened since my October letter to you. Series Three of *Philosophia Mathematica* now has the nucleus of an editorial board and I am talking to publishers. The subtitle has evolved to Philosophy of mathematics, its learning, and its application, which I hope will attract many of the HMN folks.

I now have the minutes of the Canadian Society for History and Philosophy of Mathematics meeting last May. HMN members should know about the Society, which costs U.S. $23 a year to join ($14 for students/unemployed/retired), and is thoroughly international. Being a member costs only U.S. $8.50 for subscribers to *Historia Mathematica*, since for members the subscription rate is only U.S. $28. At its annual meeting in Kingston, Ontario, in May 1991, the theme session on women in mathematics was begun by an address entitled 'Women in Mathematics: Historical and Cross-cultural Perspectives' and given by Ann Hibner Koblitz. The scientific programme was completed with a number of contributed papers. I append on the enclosed disk a list in case you would like to print it.

The proceedings of the previous year's meeting in Victoria, B. C., were distributed at the meeting to those attending, others being mailed later. This is the only distribution that they have, enough being printed only for the membership. *Proceedings* of the Kingston meeting will be distributed next May and June. The Society also publishes a *Bulletin* with news like that here.

The business meeting was chaired by the President, Prof. Craig Fraser of the Institute for the History and Philosophy of Science and Technology at the University of Toronto. Among announcements was that of Prof. Robert Thomas of the University of Manitoba, saying that he would be taking over the editorship of *Philosophia mathematica* from its founder Prof. J. Fang of Old Dominion University in Virginia. Another announcement was of the pre-ICME7 meeting of the International Group for the Relations between the History and Pedagogy of Mathematics (HPM). This will be held August 12-14 at Victoria College, University of Toronto. The organizer of the meeting is Florence D. Fasanelli, Chair of SUMMA, Mathematical Association of America, 1529 Eighteenth St. N.W., Washington, D.C. 20036 U.S.A.

The 1992 meeting will be held at Charlottetown, Prince Edward Island, May 28-30. The special theme will be ethnomathematics. The invited speaker will be Prof. Michael P. Closs of the University of Ottawa. Programme Chairman is Professor Jerry Lenz, Department of Mathematics, St. John's University, Collegeville, Minnesota 56321 U.S.A.

Yours truly,
Robert Thomas

Editors Note: Volume one of Series III has already appeared in March 1993. Remittances, subscriptions, and non-editorial correspondence for the Philosophia Mathamatics, Series III should be sent to Journals Division, University of Toronto Press Incorporated, 5201 Dufferin Street, North York, Ontario, Canada, M3H 5T8. Subscription price is $60 for institutionsm $29 annually for individuals. Currency: U.S. outside Canada, Canadian inside Canada (no GST).

Canadian Society for History and Philosophy of Mathematics speaker list

SPECIAL SESSION: WOMEN IN MATHEMATICS/SESSION

Israel Kleiner (York University)
Emmy Noether: Highlights of her Life and Work

Ann Hibner Koblitz, Guest Speaker (Hartwick College)
Women in Mathematics: Historical and Cross-Cultural Perspectives

Sharon Kunoff (Long Island University)
Women in Mathematics: Is History being Rewritten

M. A. Pathan (Aligarh Muslim University)
Lilavati

J. J. Tattersall (Providence College)
Women and Mathematics at Cambridge

REGULAR SESSION

Francine Abeles (Kean College)
A Geometric Approach to Arctangent Relations for Pi

Thomas Archibald (Acadia University)
Potential Theory and the Foundations of Analysis, 1870- 1890

Edward G. Belaga (Université du Québec à Montréal)
On the Enhanced Biblical Value of π

Louis Charbonneau & Jacques Lefebvre (Université du Québec à Montréal)
L'Introduction à l'art analytique (1591) de François Viète:
programme et méthode de l'algèbre nouvelle

Colin R. Fletcher (University College of Wales)
The Fermat-Frenicle-Mersenne Correspondence of 1640

Craig G. Fraser (University of Toronto)
The Technique of Variation-of-Constants in Lagrange's
Theory of Differential Equations

Alejandro R. Garciadiego (UNAM Mexico)
Bertrand Russell's Mathematical Work and His
Personality circa 1901

R. Godard (Royal Military College of Canada)
Condorcet et la mathématique sociale et politique

Hardy Grant (York University)
Leibniz - Beyond the Calculus

Peter L. Griffiths
The Conditions Favouring Mathematical Discoveries up to 1750

Katherine L. Hill (University of Toronto)
Early Set Theory: Dedekind's Influence on Cantor

Alexander Jones (University of Toronto)
Recovering Astronomical Tables for Greek Papyri

Emelie Kenney (Siena College)
"Imaginary Quantities" and Their Role in the Rise of Abstract Algebra in England, 1778-1837

Erwin Kreyszig (Carleton University)
On the Concept of Space in Analysis, Geometry and Physics

Jacques Lefebvre & Louis Charbonneau (Université du Québec à Montréal)
Sur quelques moyens d'accroître la diffusion et le rayonnement social de l'histoire des mathematiques

M. A. Malik (Concordia University)
Mathematization of Motion: Calculus vs. Analysis

P. Rajagopal (York University)
Arithmetic and Algebra: al-Kowarezmi and Brahmagupta

S. Sanatani (Laurentian University)
Mathematics as a Means of Communication

Norbert H. Schlomiuk (Université de Montréal)
An Undergraduate Course in History of Mathematics - Its Short History at l'Université de Montréal

Jonathan P. Seldin (Concordia University)
H. B. Curry, Logic, and Computer Science

Abe Shenitzer (York University)
Survey of the Evolution of Algebra and of the Theory of Algebraic Numbers During the Period of 1800-1870

Sylvia M. Svitak (City University of New York)
The Contributions of the Spearman-Thomson Debates to the Mathematical Theories Underlying Factor Analysis

Siegfried Thomeier (Memorial University of Newfoundland)
Some Mathematical Questions in the Development of Magic Squares and Stifel Squares

Glen R. Van Brummelen (Simon Fraser University)
The Computation of the Chord Table in Ptolemy's *Almagest*

The Role of Faith in Mathematics

Dick Wood
Seattle Pacific University

We typically are so involved with the fruits of mathematics that we neglect its roots! Students are generally aware that science is based upon the "scientific method." But mathematics students have little knowledge of the foundations of mathematics. Is mathematics really as firmly rooted as most people think? Are mathematical results necessarily true? Or, is there an element of faith/belief/trust involved?

We should be honest with our students and discuss metamathematics a bit. They need to be aware that mathematics is rooted in logic and the axiomatic method. Generally, students use a naive (unstated) logic, and this is appropriate. However, they should be aware that there are numerous formal logics and various philosophies as to the meaning of "proof."

We must realize that mathematicians are human—and, hence, fallible.

Suppose we prove that $\sqrt{2}$ is an irrational number, or that the cardinality of [0, 1) is uncountable. These typically are done via proof by contradiction within a mathematical system. How can we be so quick to reject this idea?

We know that a contradiction is a statement of the form "P and not P." Also, we realize that this is a false statement. Further, if one can prove a false statement, then one can prove each statement within the mathematical system; and hence the system is not useful.

Fundamental to the axiomatic method is the requirements that each set of axioms be "consistent" so that no contradiction can be proven within the axiomatic system. But, how can this be that one system is as consistent as another one is. But this still leaves room for doubt. We can try to obtain "absolute consistency" by presenting a model of the axioms within the framework of reality. But this leaves one wondering about the consistency of reality. Further, since the model needs a one-to-one correspondence with elements of the system, can we obtain absolute consistency for infinitely large sets of elements?

For example, the arithmetic of whole numbers is not absolutely consistent. So why do I trust it? Faith is not necessarily blind! Over the centuries we have come to rely on certain axiom systems. Yet we should realize that it really is a matter of convenience/faith. "If we hold to finitary or even classical methods of proof, faith cannot be banished from mathematics: we simply have to believe that PA (Peano Arithmetic) is consistent, since any proof which we could formalize will use methods or principles which are more questionable that those we use in the system itself."[2, p. 214]

Further, we need to point out that the results really do change when various axioms are used. Geometry provides nice examples of this as we move between Euclidean and non-Euclidean geometries.

We must realize that mathematicians are human—and, hence, fallible. You and I are sophisticated enough to have seen errors in the answer book, erroneous statements, and fallacious proofs. But these can really be upsetting to students. Stanislaw Ulam estimates that mathematicians publish 200,000 theorems every year. [3, p. 288] Many of these are later disallowed or thrown into doubt, and most are ignored. The discredited results are typically not the work of incompetent people. Reference [1] indicates a number of prominent mathematicians who have published flawed proofs.

Certainly a good reputation provides increased believability. This is a key reason for a teacher trying to be creditable to the students. We tend to put trust/belief in the results published in a reputable journals and in articles by noted persons. With limited time resources, can we do otherwise?

The computer brings into play another level of faith. We are fully aware of GIGO (Garbage In, Garbage Out). Yet we daily rely on computer results even when we know there are operating system "bugs," possible programming errors, and the problems involved in representing real numbers exactly. We tend to be more skeptical of computer-dependent "proofs" and results. Professors De Millo, Lipton, and Perlis remind us to stay wary in these.[1] Perhaps we each have several degrees of belief.

In conclusion, we cannot avoid the realization that mathematics involves faith. Some see mathematics as the ultimate in logical rigor, and I agree. However, we need to realize all of the basic assumptions, hiding nothing.

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Note: A good source of fallacious mathematics is the "Fallacies, Flaws and Flimflam" section of The College Mathematics Journal published five times a year by the MAA.

Developing a Mathematical Mode of Thinking in an Undergraduate Program

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It is a common notion that arithmetic and computation alone constitute mathematics. Let me quote from a statement that was circulated among the faculty of Elizabeth City State University, Elizabeth City, sometime ago. "At the undergraduate level Mathematics is presented as that discipline that describes the real world by numbers and numerical relationships. A course in Mathematics consists of developing some mathematical skills and some mathematical arguments. Undergraduate courses are more heavily weighted towards the former than the latter." To my knowledge this statement is true of most of our courses. Courses such as Abstract Algebra and Geometry are exceptions. From among the computational skills, some are disappearing before our very eyes because of the introduction of the calculator and computer. Students are trained to think that every characteristic or phenomenon of the real world could be measured and depicted by a number and that they could make their decisions based on that number. In my judgement, this is a mistaken notion about the world and about mathematics. Therefore, I think there should be a change in course content (especially those of service courses) as well as the manner presentation of material in text books written for those courses.

Students are trained to think that every characteristic or phenomenon of the real world could be measured and depicted by a number and that they could make their decisions based on that number.

Mathematics at one level is a mode of viewing the world and a mode of thinking about it. Mathematics has a language of its own for presenting its ideas. But mathematics is not limited to mere abstractions of the real world. We will come to this later on.

"Number" is at the simplest level of abstraction. It is a mode of viewing objects in space. But then we see patterns and colors, hear rhythms and tones, feel heat and cold. Add to this list political, social and economic organizations of societies and all our physical and mental experiences. Capturing these in ideas, expressing them in appropriate languages and creating new ones have resulted in the creation of bodies of knowledge and have deepened our understanding of human existence on this planet Earth. Civilizations have been significantly affected because of such knowledge. Mathematics provides structures or models and theories about those structures. They may be immediately perceived or they may be removed from perception. Therefore it is left to the seeker to find the appropriate structure and apply the theory that mathematics provides to organize what he observes and get additional or perhaps more precise insights that mathematics could provide. The creation of a new structure could also be facilitated by a necessity for developing a model for a particular problem irrespective of the area.

Mathematical thinking is not having an opinion or a notion. It is pursuing an idea with valid arguments. Given a problem, mathematical thinking calls for finding the premises and finding the appropriate reasoning to arrive at conclusions. In certain cases, problem solving calls for discovering new techniques for solution. Therefore seeing, analyzing, abstracting, remembering, reasoning, synthesizing, discovering, representing and questioning, all constitute mathematical activity. Logic is the tool for reasoning. Therefore every mathematical activity rests on having this tool.

Critical thinking is an essential part of mathematical activity. Mathematics has progressed because of challenges in terms of examples and counter examples, in terms of altered hypotheses or an additional hypotheses. What has been established as a piece of knowledge is a spring board for additional knowledge, for imagination, discovery, and creativity.

What is critical thinking in mathematics? It is not doubting the truth of postulates anymore. It is looking at the postulates themselves and the structures they generate. It includes entertaining alternatives to certain hypotheses of methods of solving a problem of arguments used in a solution. It is not questioning arbitrarily but thinking in such a way that it leads to better insights or new insights into the problem or opens a new world of knowledge or better method of solving a problem. Learning to think critically is learning when and how to question. Critical thinking calls for examining underlying principles. It should be a liberating experience for the student. A student should become an active participant instead of being a silent recipient.

Is developing critical thinking ability one of the primary goals of an undergraduate program? The answer is 'yes'. Critical thinking as I have defined is essential for a human being to conduct herself or himself as a free and responsible member of the human society. When this ability develops,

Mathematics has progressed because of challenges in terms of examples and counter examples, in terms of altered hypotheses or an additional hypotheses.

it will pervade all areas in which a person is called upon to make decisions or to be creative. Education at the undergraduate level should not be limiting. By this, I mean, a student after completing an undergraduate program should not be left with no options but to opt for a routine unproductive life.

It is my thinking that mathematics courses, besides familiarizing a student with applications, should open her/his mind to viewing the world intelligently. This calls for the ability to think critically, ability to abstract a given situation and the ability to appreciate and wonder. All of these are mathematical activities and all of these are essential for intelligent and exciting living.

How can this be done?

Here are some suggestions for change.

(1) Text books can be written in a different style. The traditional style is definitions, formulae, examples and problems. Instead, concepts should be presented in an exploratory style so that the students arrive at generalizations or formulae themselves. Examples of solutions to problems should be written in exploratory style so that a student could arrive at a general method of solving a problem to the extent that the problems could be categorized.

(2) Study the content of the undergraduate courses and organize the topics in such a way that the course can be taught in an exploratory way, approaching the concepts in a simple straightforward manner. One of the virtues of mathematics, I think, is simplicity.

(3) Provide an opportunity to the students such as a lab program or individual study program in which the students will be able to cultivate his/her mathematical ability.

Aesthetically appealing results and problems exist—outside the regular syllabus—in Geometry, Number Theory, Graph theory and other ares of mathematics. Lab projects based on these can by written. There are sufficient examples for all levels of mathematical maturity.

<u>What can be done in a lab program?</u>

I. Freshman and Sophomore level students may need initial guidance such as a manual. A manual will consist of

(a) breaking the problem into parts
(b) asking a series of questions that will help the student to see relationships
(c) ask open ended questions that motivate the student to ask his/her own questions to arrive at a solution.

II. A lab manual can also just give a list of problems and topics that can be explored with a bibliography so that more mature students in mathematics can work individually or in groups.

From time to time labs can be turned into seminars in which the students who have arrived at solutions or interesting results can present their findings.

The end result should be that a student appreciates pursuing an idea, sees beauty in the arguments and simplicity of relationships and is struck with wonder at unexpected relationships and results. Mathematical thinking can be better realized with appreciation of its beauty.

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Karl R. Popper: The Open Society and Its Enemies.

Robert H. Ennis: A Concept of Critical Thinking; Harvard Educational Review, Vol 32, Number 1; Winter 1962.

John E. McPeck: Critical Thinking and Education.

Math Nonsence Verse

Helen Lewy
(widow of Hans Lewy)

An astronomer who was named Cecil
Was quite fond of those functions called Bessel
Said his Wifie, "I see
You love them more than me:...
And she boarded a foreign bound vessel!

******

A programmer living in Crisp,
Fell in love with a Will o' the Wisp;
Said his parents (in Cheshire)
"Don't mix business with pleashire"—
But he still did his courting in LISP!

******

Oh, his-tor-y may seem to you
A thing of bygone value;
It's soothing, tho', if you compare
It to an eigen-value.

******

Möbius Strip Labels, Yet?

The tags on scarves, they make me sick;
They always show—they're so conspic:
You know what this is all about?
There's no inside, there's just an out!!

A Reading List for Undergraduate Mathematics Majors A Personal View

Paul Froeschl
University of Minnesota
Minneapolis, MN

This has been a reflective year. Early in September I realized that I was starting my twenty-fifth year of college teaching. Those days and years of teaching were ever present in my mind. One day in class I mentioned a book (I forget which one now) that I thought my students (mathematics majors) should read before graduating. One of them asked for a list of such books—a wonderfully reflective idea!

One list was impossible, but three lists were not. As I thought about the various books I had read and suggested to students over the years I discovered that I could divide them into three, not necessarily mutually exclusive lists. an immediate caveat: I do not claim to have read all possibilities and that these are the winners; there is some bias of ascertainment working here. Moreover, in advance, forgive me if your favorites are not listed. I would love to hear about them.

The three lists are titled: Classics, Bedsides, Larks' Songs.

There are some reasons to the order in each list; I will leave those for you to discover. The lists are annotated here and there.

Classics

1. Alice's Adventures in Wonderland and Through the Looking Glass, Lewis Carroll. How could I not start with these two? Martin Gardner's annotated edition will set these in the various contexts you will need.
2. Flatland, Edwin Abbott. May I suggest the new edition with Thomas Banchoff's introduction. There are other equally interesting modern books on these ideas, but Abbott's is the classic.
3. Elements, Euclid. One does not need to real all the books of the Elements, but do cross the Pons Asinorum until Proposition 29.
4. Science and Hypothesis, Henri Poincare.
5. A Mathematician's Apology, G.H. Hardy.
6. History of Mathematics, Carl Boyer. There are perhaps more thorough histories, but for ease of reference and early accessibility for nascent mathematics majors this history is best.
7. History of Calculus, Carl Boyer. Students, do not read this until after you have finished your calculus sequence.
8. Adventures of a Mathematician, Stanislaus Ulam.
9. The Mathematical Experience and Descartes' Dream, Philip Davis and Reuben Hersh.
10. Mathematics and the Search for Knowledge, Morris Kline.
11. Infinity and the Mind, Morris Kline.
12. Zen and the Art of Motorcycle Maintenance, Robert Persig.

These last six are classics in their own right.

Bedsides

1. A Wrinkle in Time, Madeleine L'Engle. If you have not read this as a child, read it in college.
2. ...And He Built a Crooked House, Robert Heinlein. This is just to get you started with short stories. Others that may interest you are The Purloined Letter by Edgar Allen Poe or A Subway Named Möbius by A.J. Deutsch.

3. The Lives of the Cell and The Medusa and the Snail, Lewis Thomas.

4. The Dragons of Eden, Carl Sagan.

5. Other Worlds, Paul Davis.

6. Grammatical Man, Jeremy Campbell.

These four present views of mathematics and the use of mathematics from outsiders.

7. Breaking the Code, Hugh Whitmore.
A perfect play for one or two nights of reading about Alan Turing. If you want more, read Alan Turing: The Enigma by Andrew Hodges.

8. What is the Name of This Book, Raymond Smullyan.
An enjoyable way to learn about Gödel's Theorem.

9. Metamathematical Themes, Douglas Hofstadter.
Many of the chapters are articles when Hofstadter was serving as the mathematics writer for "Scientific American."

10. Jurassic Park, Michael Crichton.
The hero is a mathematician. The villain is a computer scientist.

Larks' Songs

This list could appear without comments, but I will indulge.

1. The Ascent of Man, Jacob Bronowski.
Although having been around a while, it (the book or films) is not dated. If you can, see the films not the videos. Watching and listening to Bronowski in a darkened room is enthralling. In the same vein is Civilization by Kenneth Clark.

2. The Structure of Scientific Revolutions, Thomas Kuhn.
Kuhn's book and his ideas are so important to science and the philosophy of science that every mathematics major should know them.

3. The Mind Body Problem, Rebecca Goldstein.
To many of you this book may be unknown, but what an eye-opener when you read it.

4. Godel, Escher, Bach, Douglas Hofstadter.
Although some of the information on Artificial Intelligence should be updated, this lark's song is still melodious.

5. The Emperor's New Mind, Roger Penrose.
Penrose has in one book organized disparate areas into a beautiful whole.

So there are the lists. Over a four year period (now maybe five) a mathematics major should be able to complete them with ease. You may suggest different entrants to my lists. I would enjoy hearing from you.

Mathematics and the Arts—A Bibliography

complied by
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January, 1993

Despite its length, this bibliography is just a beginning. **Please, dear reader, send me other items to include,** so that the list will become more comprehensive and more useful. As you scan the present list you will find that some items have been annotated, and others await annotation. I will continue this process and will appreciate whatever assistance others can give to the effort.

INTRODUCTION

For many years, I have made brief mention of visual art, music, and literature in the mathematics courses that I have taught. Students have been surprised and pleased to learn of these links. At length, I decided that I wished to develop a full course on "Mathematics and the Arts." I am grateful to Bloomsburg University for a one-quarter released time to build this list of references on which a course may be based.

This collection begins with a list of items suitable for a general reader, imagined to be a college freshman who knows a bit of calculus and who enjoys mathematics and has interest in examining connections between mathematics and other arts. Selected items from this first general list may serve as texts for the "Mathematics and the Arts" course.

Following the general list, are specialized lists of references for topics listed below; these references will enable students to begin investigations for projects or term papers.

- Aesthetic Standards for Mathematics and Other Arts
- Biographies/Autobiographies of Mathematicians
- Mathematics and Display of Information (including mapmaking)
- Mathematics and Humor
- Mathematics and Literature (fiction and fantasy)
- Mathematics and Music
- Mathematics and Poetry
- Mathematics and the Visual Arts

Themes that recur again and again in these lists are: infinity, non-Euclidean geometry, ratio, repeated pattern, and symmetry.

In developing my list, I owe a debt to three others who have complied previous lists from which I have been able to learn and select. These are:

Hutchinson, Joan P., "Summertime and the living is . . . ," *AWM Newsletter*, Vol 22, No. 4 (July-August 1992), 9-11

Hutchinson focuses on books in which mathematicians (especially women) play a significant role. Her list includes authors Sophia Kovalevsky (*Vera Barantzova*), Virginia Woolf (*Night and Day*), Charles Kingsley (*Hypatia*), Saul Bellow (*The Dean's December*), Rebecca Goldstein (*The Mind-Body Problem*), Marge Piercy (*Small Changes*), Lynne Sharon Schwartz (*Rough Strife*), Scott Turow (*Presumed Innocent*), Charles Baxter (*First Light*), Michael Crichton (*Sphere*), Erik Rosenthal (*The Calculus of Murder* and *The Advanced Calculus of Murder*), Robert A. Heinlein (*The Number of the Beast*).

Koehler, D. O., "Mathematics and Literature," *Mathematics Magazine*, Vol. 55 No. 2 (March 1982) 81-95.

Koehler features works in which mathematical ideas play a significant role in the content. Featured authors include: Jonathan Swift (*A Modest Proposal*), Robert Coates (*The Law*), Thomas Pynchon (*Gravity's Rainbow*), Jorge Luis

Borges (*Death and the Compass*, *The Garden of the Forking 'Paths*, and *The Library of Babel*), Lewis Carroll (*What the Tortoise Said to Achilles*, *Alice in Wonderland* and *Alice Through the Looking Glass*), Douglas Hofstadter (*Godel, Escher, Bach*).

Lew, John S., "Mathematical References in Literature," Humanistic Mathematics Network Journal, vol. 7 (April, 1992), 26-47.

Lew's list contains over three hundred items that he and his father have gathered from their lifetimes of reading; some involve substantive mathematics while many are works in which a mathematical concept receives only a brief mention; he includes novels, plays, poems, science fiction tales, short stories, essays, biographies, and collections.

MATHEMATICS AND THE ARTS—GENERAL REFERENCESE

Blumenthal, Howard J., *The Complete Time Traveler: A Tourist's Guide to the Fourth Dimension*, Berkeley, Ten Speed Press, 1988.

Boles, Martha and Rochelle Newman, "Art, mathematics and nature in the interdisciplinary classroom," *Leonardo*, Vol 21. (1988) No. 2, 182-86.

Bronowski, Jacob, *The Ascent of Man*, Boston, Little, Brown and Company, 1973.

This book was written to complement a BBC film series of the same name; Chapter 5, "The Music of the Spheres," deals with mathematics and harmony and symmetry.

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Buchanan, Scott M., *Poetry and Mathematics*, J. B. Lippincott, 1962.

Crowe, Donald, "Symmetry, Rigid Motions, and Patterns," *The UMAP Journal*, vol. 8, no. 3 (Fall 1987), pp. 207 - 236.

An introduction to "border" and "wallpaper" patterns.

Davis, Philip J. and Reuben Hersh, *The Mathematical Experience*, Boston, Birkhauser, 1981.

Selections of interest in this collection of brief essays include: "The Aesthetic Component" (168-70), "Pattern, Order, and Chaos" (172-9), "Intuition" (391-9), "Four Dimensional Intuition" (400-5), "True Facts about Imaginary Objects" (406-11).

Dienes, Z. P., *Mathematics Through the Senses: Games, Dance, and Art*, New York, Humanities Press, 1973.

Activities for children than enable them to learn mathematics through artistic activities.

Ellis, Havelock, *The Dance of Life*, Riverside Press, 1923.

Reflections on the art of living.

Eves, Howard, *In Mathematical Circles* (1969), *Mathematical Circles Revisited* (1971), *The Other Side of the Equation* (1971), *Mathematical Circles Squared* (1972), *Mathematical Circles Adieu* (1977), *Return to Mathematical Circles* (1988), Prindle, Weber, and Schmidt.

Collections of mathematical stories and anecdotes by an eminent geometer and historian of mathematics.

Fauvel, John and Jeremy Gray, editors, *The History of Mathematics: A Reader*, Macmillan, 1988.

A collection of readings that includes the works of mathematicians, letters, poems, and excerpts from plays and novels; attempts to give an historical outline of mathematical activity from ancient to modern times and to show the role that mathematics has played in culture.

Gardner, Martin, *The Ambidextrous Universe: Mirror Asymmetry and Time-reversed Worlds (Second Edition)*, New York, Charles Scribner's Sons, 1979.

Chapter 1 — "Mirrors"
Chapter 2 — "Lineland and Flatland"
Chapter 3 — "Solidland"
Chapter 5 — "Art, Music, Poetry, and Numbers"
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Gardner, Martin, "Hypercubes," *Mathematical Carnival*, New York, Vintage, 1975.

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Guillen, Michael, *Bridges to Infinity: The Human Side of Mathematics*, Los Angeles, Jeremy P. Tarcher, Inc., 1983.

Hardy, G. H., *A Mathematician's Apology*, Cambridge University Press, 1969.

Some well-written biographical remarks from an eminent mathematician.

Henderson, Linda D., *The Fourth Dimension and Non-Euclidean Geometry in Modern Art*, Princeton University Press, 1983.

Henderson focuses primarily on French art and artists and on Cubist ideas as she explores the effects on art (during 1900 - 1930) of the new, non-Euclidean geometries developed during the nineteenth century.

Hinton, C. Howard, *The Fourth Dimension*. Allen & Unwin, 1904.

In careful detail, Hinton analyzes the changes that take place in perception as one moves from one to two and two to three dimensions, and deduces some properties of perception in a four dimensional space. Using colors to help keep things straight, he devises a model of a four-dimensional solid.

Hinton, Charles H., *Speculations on the Fourth Dimension: Selected Writings of Charles B. Hinton*, with an introduction by R. V. Rucker, Dover, 1980.

Essays by a nineteenth century mathematician who endeavored to teach everyone to *see* four dimensional space and who anticipated some of Einstein's discoveries.

Hofstadter, Douglas R., *Godel, Escher, Bach: An Eternal Golden Braid*, New York, Basic Books, 1979.

Hughes, Robert, *The Shock of the New, Revised Edition*, New York, Alfred A. Knopf, 1991.

This book is a revision of a book originally written to accompany a BBC television series written and narrated by Hughes. Although it provides few references to mathematics or science, this book documents the changes in art (primarily in painting and during the period 1880-1914) that accompanied or followed major changes in scientific thought.

Kapraff, Jay, *Connections: Geometric Bridge Between Art and Science*, New York, McGraw-Hill, 1991.

Introduces the reader to many ways in which geometry underlies the creation of beautiful structures and serves as an intermediary between the harmony of the natural world and the humans who perceive it. The book grew out of a course, developed by the author, that related mathematics and design.

Kasner, Edward, and James R. Newman, *Mathematics and the Imagination*, Redmond, WA, Tempus Books, 1989.

Keyser, Cassius J., *Mathematics as a Culture Clue and Other Essays*, Scripta Mathematica, 1947.

King, Jerry, *The Art of Mathematics*, New York, Plenum, 1992.

What is mathematics? What is beautiful? What is art? Provocative questions and answers are raised in this easy-to-read book by a mathematician who is also a poet.

McLuhan, Marshall and Harley Parker, *Through the Vanishing Point: Space in Poetry and Painting*, New York, Harper & Row, 1968.

A collection of poems, pictures and notes by the authors; sometimes thought-provoking.

Manning, Henry Parker, *Geometry of Four Dimensions*, Dover, 1956. Also *The Fourth Dimension Simply Explained*, Dover, 1960.

First published in 1914, Manning explores the geometry of four-dimensions by taking "natural" extensions of two and three dimensional Euclidean geometry.

Maor, Eli, *To Infinity and Beyond: A Cultural History of the Infinite*, Boston, Birkhauser, 1987.

Geometric maps. tiling the plane, the Mobius strip, mirror reflections, and Maurits C. Escher are among the subjects of this provocative book

May, Kenneth O., "Mathematics and Art," *The Mathematics Teacher*, October, 1967.

Moritz, Robert E., editor, *Memorabilia Mathematica*, New York, Macmillan, 1914.

Morse, Marston, "Mathematics in Our Culture," *The Spirit and the Uses of the Mathematical Sciences*, edited by T. L. Saaty and F. J. Weyl, New York, McGraw-Hill, 1969.

Morse, Marston, "Mathematics and the Arts," *Bulletin of the Atomic Scientists*, vol. 15 (1959), no. 2, Also in *The Yale Review*, June, 1951.

Newman, James R., *Mathematics: A Small Library of the Literature of Mathematics*, presented with Commentaries and Notes by James R. Newman, Redmond, WA, Tempus, 1956, 1988.

Essays of interest found in this collection include: "Topology" by Richard Courant and Herbert Robbins (I, 573-90), "Durer as a Mathematician" by Erwin Panofsky (I, 591-612), "Symmetry" by Hermann Weyl (I, 659-711), "Infinity" by Hans Hahn (III, 1565-83), "Mathematics as an Art" by John William Navin Sullivan (III, 1985-95), "A Mathematician's Apology" by G. H. Hardy (IV, 1999-2014), "Mathematics of Aesthetics" by George David Birkhoff, (IV, 2175-86), "Mathematics of Music" by Sir James Jeans (IV, 2253-88), A selection from "Flatland" by Edwin A. Abbott (IV, 2361-73), "What the Tortoise Said to Achilles and Other Riddles" by Lewis Carroll (IV, 2375-86), "Mathematics for Golfers" and "Common Sense and the Universe" by Stephen Leacock (IV, 2433-48).

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Polya, George, "Some Mathematicians I Have Known," *American Mathematical Monthly*, Vol. 76 (1969), 746-752.

Reid, Constance, *From Zero to Infinity: What Makes Numbers So Interesting*, New York, Crowell, 1961.

Renyi, Alfred, *Dialogues on Mathematics*, San Francisco, Holden-Day, 1967.

Robson, E. and J. Wimp, Editors, *Against Infinity: An Anthology of Contemporary Mathematical Poetry*, Parker Ford, PA. Primary Press, 1979.

These poems use mathematical concepts—zero, infinity, prime, distance, circle, and the like—to create poetic images.

Rucker, Rudy, *The Fourth Dimension: Toward a Geometry of Higher Reality*, Boston, Houghton Mifflin, 1984.

Sections I and II suggests ways to visualize objects in a geometry with a different number of dimensions than three.

Rucker, Rudy, *Infinity and the Mind: The Sciences and Philosophy of the Infinite*, Boston, Birkhauser, 1982.

Chapter 1 gives a short history of the various ways in which the term "infinity" is used.

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Zippin, Leo, *Uses of Infinity*, Washington, DC, Mathematical Association of America, 1962.

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Birkhoff, George David, *Aesthetic Measure*, Cambridge, MA, Harvard University Press, 1933.

The author has developed a systematic means of analysis of aesthetic properties of music, poetry, and visual art using quantitative measures.

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Brooks, Cleanth and Robert Penn Warren, *Understanding Poetry, (Third Edition)*, New York, Holt, Rinehart and Winston, 1960.

Aesthetic standards for poetry that may be compared with aesthetic standards for mathematics.

Bunch, Bryan, *Reality's Mirror: Exploring the Mathematics of Symmetry*, New York, Wiley, 1989.

Explores the importance of mathematical symmetry in modern science, especially physics.

Day-Lewis, Cecil, *The Poet's Task*, Clarendon Press, 1951.

Day-Lewis describes the role and responsibility of poets and poetry; his ideas also apply to mathematicians and mathematics.

Dorwart, Harold L., "Configurations: A Case Study in Mathematical Beauty," *The Mathematical Intelligencer*, Vol. 7, No. 3, 31-48.

Franke, Herbert W., "Mathematics as an artistic-generative principle," *Leonardo*, Vol. 1989 supplement, 25-26.

Frost, Robert, "The Figure a Poem Makes," from *Complete Poems of Robert Frost*, Holt, Rinehart and Winston, 1967.

Gardner, Martin, "A mathematical aesthetic applied to modern minimal art," *Scientific American*, Mathematical Games, November, 1978.

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Hammond, Allen, "Mathematics, Our Invisible Culture," *Mathematics Today*, Lynn Steen, ed., New York, Springer-Verlag, 1978.

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Albers, D. J. and G. L. Alexanderson, *Mathematical People: Profiles and Interviews*, Boston, Birkhauser, 1985.

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Preston, Richard, "The Mountains of Pi, a Profile of David and Gregory Chudnovsky," *The New Yorker*, March 2, 1992, 36-67.

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Ulam, Stanislaw M., *Adventures of a Mathematician*, New York, Scribner, 1976.

Wiener, Norbert, *I Am a Mathematician*, Garden City, NY, Doubleday, 1956.

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Casas, Fernando R., "Polar perspective: a graphical system for creating two-dimensional images representing a world of four dimensions," *Leonardo*, Vol. 17 (1984) No. 3, 188-194.

Dickinson, Carol, "Crossroads of art and science (the art of cartographer Hal Shelton), *Southwest Art*, Vol. 18 (September 1988). p. 44+.

Freudenthal, Hans, *Mathematics Observed*, New York, McGraw-Hill, 1967.

Freudenthal writes about the nature of mathematics and of mathematical thinking; Chapter 1, "Measuring the world," and Chapter 5, "The art of drawing badly," include some of the mathematical ideas used in mapmaking.

Rickey, V. Frederick and Philip M. Tuchinsky, "An Application of Geography to Mathematics: History of the Integral of the Secant," *Mathematics Magazine*, vol. 53, no. 3 (May, 1980), 162-166.

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Tufte, Edward R., *Envisioning Information* (1990) and *The Visual Display of Quantitative Information* (1983), Cheshire, Conn., Graphics Press.

Tufte designs displays of information in ways that help readers to understand and digest it. See also "Up from Flatland" by Phil Patton in *The New York Times Magazine*, January 19, 1992.

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Collections of mathematical stories and anecdotes by an eminent geometer and historian of mathematics.

Fadiman, Clifton, *The Mathematical Magpie*, New York, Simon and Schuster, 1962. Also *Fantasia Mathematica*, 1958.

Magpie is a collection of essays, rhymes, and anecdotes, many of them amusing. *Fantasia* also contains a number of short stories.

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"Accidental You, Fully Covered Me" by Hayes B. Jacobs, 296-297.
"Confessions of the World's Fastest Reader" by Clifford D. Owsley, 251-253.
"The Permanent Traffic Solution" by Harland Manchester, 161-162.
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Peter, Laurence J., *Peter's Quotations: Ideas for Our Time*, New York, William Morrow & Co.,1977.

Quotations about statistics (452-3), logic (307-8), facts (187-8), education (172-7), knowledge (280-2), problems (408-9), science (436-8), and many other topics.

Paulos, John Allen, *Mathematics and Humor*, University of Chicago Press, 1980.

Simpson, Adrian P., "The Infidel Is Innocent," *The Mathematical Intelligencer*, 12 (1990), No. 3, 42-51.

An informal introduction to nonstandard analysis; uses humor.

Mathematics and Literature (fiction and fantasy)

Abbott, Edwin, *Flatland*, New York, Barnes & Noble, 1963.

A fantasy about life in two dimensions that explores the dilemma that human beings have when they try to imagine a number of dimensions other than three.

Asimov, Isaac.

Several science fiction stories by Asimov that include mathematical ideas are found in the collections of Clifton Fadiman and Rudy Rucker.

Borges, Jorge Luis, *The Aleph and Other Stories*, New York, E. P. Dutton, 1970.

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Burger, Dionys, *Sphereland*, New York, Crowell, 1965.

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Carroll, Lewis, *The Annotated Alice: Alice's Adventures in Wonderland and Through the Looking Glass*, illustrated by John Tenniel with Introduction and Notes by Martin Gardner, New York, Brumall House, 1960.

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This book serves as a supplement to the 1960 edition.
For more by Gardner on Carroll as logician and mathematician see also *Scientific American*, March 1960, pp. 172-76.

Carroll, Lewis, A Tangled Tale, New York, Odarki Books (The Third Press), 1885, 1974.

Ten amusing tales, each embodying a mathematical question; written for children.

Doyle, Sir Arthur Conan, *A Study in Scarlet* and *The Final Problem*

Doyle's Sherlock Holmes mysteries contain a little bit of mathematics and many references to logic or "the science of deduction." See particularly *A Study in Scarlet*. In *The Final Problem* one meets Professor James Moriarty, "The Napoleon of Crime," described as an embittered and ruthless mathematical genius.

Fadiman, Clifton, *The Mathematical Magpie*, New York, Simon and Schuster, 1962. Also *Fantasia Mathematica*, 1958.

Magpie is a collection of essays, rhymes, and anecdotes, many of them amusing. *Fantasia* also contains a number of short stories.

Fauvel, John and Jeremy Gray, editors, *The History of Mathematics: A Reader*, Macmillan, 1988.

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On Attracting Mathematics Majors

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Acknowledgement. I would like to mention my indebtedness to all my teachers, throughout my life, and to all my colleagues and students at Potsdam, who showed me that there are a variety of methods of teaching mathematics which work. A special note goes to Clarence Stephens and John Poland who helped me to find the probable factors which are involved in attracting students. I would also like to thank Charles Smith, Benjamin Brewster, Leona Ludwig, Irene Schensted, Dilip Datta, and Vasily Cateforis for reading this manuscript and making many valuable suggestions.

I want to consider what factors are involved in attracting students to mathematics. Several of these factors have become evident to me from observing and participating in the Potsdam College mathematics program, where I have been teaching for the last nine years.

Every teacher first of all is a product of his/her teachers and his/her students. When we are in the classroom sometimes we play the role that our previous teachers played. We usually imitate those we liked the most, but sometimes we imitate the ones that we did not like at all. This is both good and bad. Our students are inevitably exposed to our philosophy of teaching. Our students also influence our teaching. They can encourage or discourage us. As we all know, they can make the classroom a pleasant or miserable experience for us. Hence instead of talking about teaching I would like to talk about the factors that probably make the interplay between student and teacher a pleasant and successful one. One observes that while some students are successful with certain teaching methods they are not as successful with a different teaching method. For example Hector Foisy's method of teaching in Potsdam is as effective as Clarence Stephens' method of teaching. Yet the two of them have different ways of teaching. Each of these methods attracts its own group of students. Therefore it is good that in a given department there are different methods of teaching. Enforcing just one method of teaching in a department is only going to attract certain students, those that respond to that certain method.

Let us first start with the nature and psychology of human beings. Some people seem to be born with a natural ability to survive and be mathematicians, like Ramanujan, Galois, Sophie Germain, etc.... They survive even when academia has not been appealing to them and has not provided a positive environment for them. We must leave this to biologists, to find out why some

If we can inspire our students so that many of them continue taking and pursuing mathematics, then we have created mathematicians.

people are born with this natural ability. Maybe in the future one can walk to a clinic and have his/her brain set in this way. Some others, perhaps not as talented as the first group, have enough energy, motivation, and willpower to go on regardless of the situation and their instructors. Their hard work compensates for their lesser talent. Academia has accommodated them and has been good to them. We will leave this to psychologists to find out why some people become like this. Maybe in the future one can walk into a psychologist's office and get enough energy and willpower and motivation. These two groups need challenge and opportunity.

In contrast to these two groups, I believe there is a third group that will survive if a nurturing, encouraging and understanding environment exists for them. This is the group that depends on us. As instructors, we can attract them to mathematics or turn them away. For people in the first two groups I see, among other things, t an intense love and desire that drives them. We need to create a desire and interest in the third group. To identify some factors that create this interest and desire in students, we will discuss the roles of the following: proper goals, instructor attitude, instructor's maternal/paternal advice and help, department, school, positive publicity, awareness of the student, economic situation, culture, and role models. Some of these factors cannot be influenced by us. We should find out what we can change and control and what we can not change, and must accept. One can see that goals, teacher's attitude toward students, department, positive publicity, role models, awareness of the students and to some extent school can be influenced by us. However, social factors, social values, economic situations and human nature cannot. Now we look at the effect and the role of each of these factors in more detail.

(i) First we would like to clarify the effect of goals:

What do we want to achieve by teaching and what do we want to achieve in particular by teaching mathematics? Is the aim of teaching mathematics to cover certain material and make sure that students know these certain materials, or can there be other goals? My experience at Potsdam indicates there can be other goals. What if we set the following goal of teaching: after taking a course in mathematics the student should wish to go on and choose another course in mathematics or to continue toward graduate school or to continue to be a scholar in mathematics. In other words what if we cultivate the student's interest in mathematics? Let us for a moment, pause and see what we achieve by choosing such a goal. If a teacher goes to the classroom with the goal that he/she is going to cover certain materials and addresses only the brighter students in the class, then my experience says by the end of the course, more than half of the students may drop out and some of the handful that manage to pass the class may not be willing to take another course in mathematics. These students pass the course in spite of the instructor. The probable outcome of this approach is obvious. There will not be many people who want to become mathematics majors. This may be the reason you are reading this article. So, let us look at the other goal: that a majority of the students, taking a course in mathematics become interested in mathematics, choose to take another course in mathematics, and pursue mathematics on their own. Although the instructor has not covered as much material as those with the first goal, the student continues to take courses in mathematics. The student that has been taught one or two chapters fewer in every mathematics course, has survived the courses and become interested in mathematics, may even earn a B.A. in mathematics. Now the student has the mathematical maturity to read and learn quickly the missed materials. Therefore we as teachers have not hurt the student by teaching the student less material but have actually helped the student to become a mathematician. Is not this what we really wanted to do? When we teach a three semester hour course we only have 45 hours for a semester, less than 2 days. The amount of material that we can teach in this short time is very limited regardless of how fast we try to go, especially considering the total amount of mathematics that there is out there. But if we can inspire our students so that many of them continue taking and pursuing mathematics, then we have created mathematicians.

We have students drawn from the best generation that the world has ever had and it is our duty not to mold them but to create an interest inside them, give them confidence and build their brains.

We cannot rely on performance in certain tests and examinations to judge if we have fulfilled such a goal. If a student develops the interest to continue his/her scholarly activity we have achieved our goal. We believe that we have students drawn from the best generation that the world has ever had and it is our duty not to mold them but to create an interest inside them, give them confidence and build their brains. To accomplish these tasks we need to approach them with the right goal and create an atmosphere for them to learn and enjoy mathematics. Testing them is doing nothing more than making them anxious and fearful and nervous and ruining their creativity.

(ii) Now Let us look at the role that instructor attitude plays:

When we as instructors go to the classroom, if our attitude is to find a way that helps our students to learn and enjoy mathematics, to learn how to learn, then the probable outcome would be good. If we go with the intention of creating an atmosphere that is motivating, relaxing and understanding of the student's abilities, then the probable outcome will be to attract students. If we go to the classroom with the intention of creating confidence in them so that they are able to do mathematics and they also can enjoy mathematics, and thereby gain confidence, then the outcome will be great. The attitude that conveys to students that mathematics is only for the elite, that they are unable to be successful since they are making mistakes, the attitude that half of the students should drop out since they can not keep up with the teacher is probably not going to attract students to mathematics. The attitude sometimes seen in our coffee room discussion, making fun of their mistakes, is going to have a probable outcome of scaring students away from mathematics. It is this attitude that engenders comments from the students that mathematics is hard and not for them; it is only for geniuses.

Learning mathematics is like learning to walk or learning to talk. It needs patience, practice, desire, and a relaxing and nurturing environment.

The first day of class can be crucial. If an instructor goes to class and draws the line, puts students down, threatens them, uses grading to create fear in them, makes them worried, gives them the statistics that so many of the students who take this course drop out, tells them that the subject is very difficult, and conveys to them that they are not able, then the outcome is obvious. But suppose from the first class the instructor tells them that he/she is there to help them, to create an interest inside them, to help them to learn how to learn, tells them the grade is not the best thing that they can take out of the class, tells them this class can create a love for learning in them, tells them that to learn mathematics they need to be patient and persistent and have proper scheduling and practicing, tells them that becoming skillful in mathematics is like becoming a wrestler, a runner or a swimmer, tells them that there is also joy in the learning of mathematics. Such an instructor will notice that fear and anxiety have been replaced by a desire for learning. This can be seen in the way that they are sitting (they are all ready to go and learn).

A very bright student in an elementary course drew my attention and I tried to encourage him to take more math courses. The student told me that he knew basic Algebra when he was four years old. But when he got to school, his teacher's attitude turned him off from mathematics. Then he decided not to pursue mathematics any further. The course that he was taking was very easy for him and was part of the requirements; he had to take it. Of course we have had cases that regardless of how much the situation has discouraged students, they have continued and contributed. I do not think I need to mention them here as we all know them. We all know the ones who survived. The ones that have not survived are more hidden, and sometimes it has been because of us. We can change this.

We might be criticized that not all of the students in this system will turn out to be high caliber mathematicians. I would accept this, but I would argue that some of them might go as far as a B.A. in Math, some of them might have another major or another interest, and some of them might go as far as a Ph.D. But a student who is majoring, say, in economics and also has a B.A. in mathematics will definitely be a more effective economist. A representative of a company that was on campus to hire an economics major said that economics majors who are also mathematics majors are very good problem solvers in the sense that in many situations their training in abstract mathematics helps them to look at the problem in a better way. Or somebody who will become an elementary education teacher with a B.A. in mathematics can be a more effective teacher than one with only the minimum requirements. In a traditional approach, these students would not go beyond the courses that were required of them.

(iii) Now let us look at the factor of the Instructor
maternal/paternal advice and wisdom:

The impact that we have on our students is not just in the mathematics that we teach them. A look at classroom surveys and talking to students after classes indicates that we have some other impact on our students. Our words can help them and sometimes can change them, our actions can have an impact on them. Over the years as some of our Mathematics Alumni come back to visit us or write to us, they attribute their success to certain words that a certain professor told them or help that a certain professor provided. Somehow those words or help had an impact on them and created enough fire inside them to help them finish their degree or continue their graduate work or assist them in their professional life.

Let me give you a few examples to clarify what I mean by paternal advice and wisdom. A student who is very frustrated and cannot prove theorems tells the instructor she cannot do mathematics, she is not good enough even to finish the course in mathematics. And consider two instructors with two different approaches. The first one tells the student that it is probably a good idea to drop the course and conveys a message that the instructor and people who do mathematics are indeed very smart and they were just born with this ability. After a few minutes the student leaves the office of the instructor. The student will probably never take any more mathematics courses. Now let us look at some other instructor who is more helpful. He asks the student how long it took the student to learn to speak the way that the student speaks today, how long it took the student to learn to walk. Then the instructor tells the student that learning mathematics is like learning to walk or learning to talk. It needs patience, practice, desire, and a relaxing and nurturing environment. It needs persistence and time. This makes the student feel better and more aware. Now the instructor asks the student whether she is overloaded, whether she has somebody that she can work with, whether the student would like to join a study group. And the outcome: there is a chance that the student will survive the mathematics course and even take more mathematics courses.

Let us look at another case. A student who is taking a mathematics credit on an independent study does not do well in his first presentation. The student complains that he cannot understand the subject. Let us look at two different instructors. One instructor tells the student that if he is not doing well now he will probably get a bad grade or fail and then the instructor does not think about the student anymore. The outcome is probably that the student will fail or drop out and it could be his last mathematics course. Now let us look at another instructor who is more helpful. At first, this instructor may not know how to answer. He starts thinking about the student and starts searching for a way to help him. In the middle of the week the instructor has a brainstorm. He calls the student. The instructor tells the student that he just called to see how he is doing, whether he still feels stuck and gives a few words of fatherly advise. Students respond to this type of specific attention, and this student tells the instructor that he is going to pick up the book as soon as their phone call is over. The instructor realizes that the behavior of the student has changed and soon he is

There are lots of rumors that mathematics is a boring, joyless and useless subject. These can influence our students before they arrive in our courses, and are driving many others away from even attempting college mathematics.

doing much better. Of course different instructors might come up with different methods to approach these problems. But it pays off if we think about our students and try to find some solutions rather than blaming students' low performance on their inability. A mathematician once told me that his teacher, in a geometry course in high school, told him that it is good to take a problem in geometry and think about it an hour or so per day, in a relaxed manner, until the problem is solved. Then take another problem. The instructor told him that this will also help to build his brain. Although this student usually did not take the instructor seriously, he took this advise seriously. He felt it was instrumental in helping him to become a mathematician. We all know of examples of students for whom some teacher made an important difference. We can be effective and make a difference for many of our students, with the right attitude.

(iv) Let us now look at the effect of role model:

We are motivated to do different things by our inner interests or the joy that we get. We can affect our students by our interest in mathematics, provided that we use this interest in a proper way.

I am not saying that mathematics is easy. What I am trying to say is that mathematics is useful, and joyful, and can be learned. Our students need at least to feel the joy of mathematics.

By this I mean using our knowledge not to make them feel inferior and unable, but instead using our knowledge of mathematics to create an interest and desire inside them, convey our pride in mathematics to them, convey our joy of doing mathematics. Imagine what would happen, if our mass media would publicize mathematicians, if they ran programs in which the eminent in the field would convey to the public their joy of doing mathematics and the usefulness of mathematics. This might help to attract more students to mathematics. Many of our mathematics majors do not know of Hilbert, Emmy Noether, Poincare, Sophia Germain, Gödel and so on, nor who the current eminent ones are or even what a Fields Medal is. Many people do not know that doing mathematics can be joyful and rewarding. We can play a good model for our students. Here a true story might help. One of our recent undergraduates said the reason that he decided to go to graduate school and become a mathematician was that he wanted to become like Paul Erdos. The student had read and heard about Erdos, and this story had created a fire inside him. Erdos was his hero.

(v) Effect of awareness by students of opportunities for mathematicians:

Many students are not aware of opportunities for a mathematician to have a job beside teaching. There are also many mathematics majors who are not aware of financial opportunities that are available for them to go to graduate schools. In Potsdam college we announce jobs that our graduates have had and a newsletter about our alumni. Some of us also talk to the students that we see are able to go to graduate school and inform them of the financial opportunity that is there. We find that they are very surprised to hear that they are paid to go to graduate school.

(vi) The effect of positive publicity:

There are lots of rumors that mathematics is a boring, joyless and useless subject. These can influence our students before they arrive in our courses, and are driving many others away from even attempting college mathematics. But we can create a more positive, interesting, useful, and appealing image of mathematics. When we ruin one student who we could have saved, we turn hundreds away by negative publicity. It is rather alarming that when we speak to virtually anybody who is not a mathematician, the person considers mathematics useless, hard and boring, and considers herself/himself unable to do it. It seems that the word "mathematics" creates a kind of fear and anxiety that prevents a student from approaching it with an open mind rather than prejudiced. This prejudiced mind will make it even more difficult for a student to learn mathematics. I can not resist telling this story: once upon a time a teacher told his students that their next test would be difficult and most of them might not be able to do it. Then he gave them an easy test. But he had created an image of a difficult test so that some of the students insisted on giving complicated answers to the simple questions that had nothing to do with the questions. They could not see that it was simple.

I am not saying that mathematics is easy. What I am trying to say is that mathematics is useful, and joyful, and can be learned. Our students need at least to feel the joy of mathematics. They need to be assured that with some effort they can learn to learn independently of whether they get interested in it or not, or whether to devote their life to it or not.

(vii) Effect of culture on education:

Culture creates values and guidelines which can have either positive or negative effects since these values and guidelines create pressure. This pressure sometimes is visible and sometimes invisible; it can be helpful or not helpful. The reason that culture can play this dual role is that it is based on the past rather than the future. I remember one person who told me that she wanted to become a mathematician and liked mathematics very much, but her family wanted her to become a chemical engineer because they thought chemical engineers had higher status. She became a chemical engineer just to keep her family happy. Another student reported his parents told him that either he should go to the best mathematics department or he should not go to graduate school at all. He went to the best department but could not survive. Then he quit graduate work in mathematics altogether, although he might have survived if he had gone to a less competitive, smaller school.

There is pressure on students from family and from society. The weaker the students the more vulnerable they will be to this pressure. A female student of ours said that when she was a student in high school her teacher constantly made negative remarks to the effect that female students do not succeed in mathematics. Fortunately this student did get her B.A. in mathematics; she was among our better students. I am sure that this high school teacher discouraged many other female students from following their interest in mathematics. We have had female students that wanted to go to graduate school to continue mathematics, but were being discouraged by friends and family: They were told they should get married or find a job rather than go to graduate school. With some effort from some of their instructors, some decided to go to graduate school but some did not. One of our students said she did not want to go to graduate school because she was under pressure: going to graduate school meant she could not have a family and children. But after her instructor told her that there is a life after graduate school and she could have a family and children after she got her Ph.D., she became convinced to go to graduate school. A guideline of the past was misdirecting her. Or consider the simpler case of a student who did not want to study that much because she did not want to miss having fun. However, she was told first that studying can be fun and second after school there is also life and she can have fun then, too. This was a revelation to her. After this conversation she was a more serious student. Pressure or culture of the dorm was not letting her see other values. Yet another example: A group of friends were not doing well, because they were trying to have fun together primarily by getting together to drink. One of their instructors suggested instead they have fun by studying together, or they could get together and discuss an agenda together and have fun this way. These words were a revelation to them. Some of them became better students afterward. What I want to convey is that we have to counterbalance harmful social customs.

(viii) Now let us look at the role that the department can play:

The effect of departmental policies is not something that can be overlooked. The department needs to create a proper environment in which the faculty can develop both in their scholarly activity as well as in their teaching methods. The department needs to provide help, encouragement and orientation for faculty members in need. In a department in which faculty energy is consumed by politics and bureaucracy, and faculty is stressed by guidelines then true scholars may be looked down on, and faculty will be less able to help students. In a department where the faculty is overloaded and not respected, where faculty is expected to do the unexpected, one cannot hope that faculty will function properly. One uncomfortable faculty member can make hundreds of students

In a department where the faculty is overloaded and not respected, where faculty is expected to do the unexpected, one cannot hope that faculty will function properly.

uncomfortable. The thing that drew my attention to Potsdam College Mathematics Department, when I was there the first year, was to hear the chairman, Clarence Stephens, constantly saying that "we need to cultivate a culture of using our brains to help students by providing a proper environment for faculty". Charles Smith, a senior faculty member who kindly took charge of orienting me assured me that the Department was very considerate of the faculty and an example of this helpfulness was that those few of the tenured faculty that did not have a Ph.D. in mathematics were encouraged and given proper help and the environment to work toward their Ph.D. This made me feel at home right away. What we want to convey here is that a proper and humane management of the department is very necessary and should not be overlooked. Of course, a department in a school whose administration is not understanding is bound to be less effective. The administration needs to be in communication with the department. The administration needs to be helpful and in turn, faculty need to be ready to orient and guide the administration when necessary.

(ix) The effect of economic factors:

The following true story might clarify the effect of the economy on mathematics. One day, one of our better mathematics students, who had applied to graduate school and also had an offer for a good paying job, went to the office of one of his instructors. He wanted to get some advice about whether he should go to graduate school or join the work force. The instructor tried to convince him to go to graduate school. At one point he asked the professor about the beginning salary of an assistant professor. It was less than what the student was being offered. The professor could see that as soon as he found out about the salary of an assistant professor the student had no interest in going to graduate school any more. This might be one reason, among other reasons, that we do not have too many mathematics Ph.Ds. What do you think would happen if the beginning salary of an instructor in mathematics was $200,000 and he/she had a personal secretary and a fancy office?

I would like to conclude with the statement that there is no known perfect system or method of teaching, managing, advising, and caring for students. Every method regardless of how humane could alienate some students. Therefore, different methods, in a given department, will accommodate a wider spectrum of students. In addition, students need to be informed that the teacher is not the only one responsible for their education. They need to take responsibility too in order to make the event of student-teacher interaction a successful one. By this I mean that they need to be treated like mature adults and given respect. A classroom is not just a teacher who is standing there, anxious to teach and to finish certain materials. It is a student-teacher event. Therefore the teacher should avoid conveying to the students that she/he is the one that feels the entire responsibility for the students to learn and the students need not feel any responsibility. I have noticed that when a teacher mentions to the students in class that it is the students who are the ones that are paying money and the teacher is the one that is paid, students become more aware and act more responsibly. The first year that I was in Potsdam College, I visited some of Clarence Stephens' classes. I noticed that among the many good things that he was doing, one was that he was allowing his students to feel that they were responsible for their own learning.

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Editors Note: For historic and pedagogic interest, two essays about the Potsdam Program that appeared in HMNN#2(March 1988) and HMNN#3 (December 1988) are reprinted following this essay.

The Basis for the Success of the Potsdam Program

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Based on a visit to Potsdam College, 13-15 April 87
(reprinted from HMN Newsletter #2, March 1988)

Much of what distinguishes the program at Potsdam is not what the department "does" so much as the way it "thinks"—it is a matter of attitude. I believe that these points are the fundamentals of the Potsdam model:

1. The department members adopt the view that—contrary to the prevailing belief in this country, but consonant with that in most other industrialized countries—success in mathematics is due much more to hard work than to innate talent. Many can achieve success in mathematics by persevering—it is not limited to an elite class of geniuses. Faculty must personally accept this view, as well as press it on students in both formal and informal ways, e.g., advising, pamphlets, bulletin boards.

2. The faculty must also be willing to "suspend disbelief" with respect to students whose past records have been undistinguished. There have been many success stories by those whose early records were abysmal, starting with Albert Einstein. Who is to say which of those students who have not yet shown promise are incapable of blossoming later? Don't wait for the "good" students. Again, advising, pamphlets, bulletin boards, can press this point of view on the students. Case histories can be assembled to prove this point—preferably alumni of the institution, but others would do also.

3. The secret of getting students to succeed is to keep up morale. Therefore students must constantly be given things they can do. They should be challenged, but each challenge should be at the appropriate level. Once the student's confidence is shot, he's lost to the discipline. The teacher who presides over failure excuses himself by saying the students "didn't work hard enough." But they didn't because he didn't inspire them to. Avoid all temptation to "inspire" by threats, abuse, competition, impossible problems, guilt trips, invidious comparisons, anything negative—it won't work.

   Thus there should be nothing called "remediation" and no placement tests. No one is ever "ready." Let everyone feel the pride of trying a high-prestige subject. Throw then in and let them learn to swim. Believe in them, and they will probably do it. They can be given simple problems at first, so they will succeed and gain confidence, and they can be led on to greater and greater levels of achievement with problems of constantly increasing difficulty.

   Every success should be recognized. Every formal and informal method should be employed to see that achievements are publicized and publicly recognized and appreciated.

4. Abandon the traditional lecture format of teaching. It rarely works. In our own experiences as students, we didn't learn from listening. We learn by explaining, or otherwise getting actively involved. Students should be learning in the classroom, which means not just listening passively. They should be solving problems then and there. Helping each other—good for both helper and helpee. There should be formal ways for students to help each other—such as a Math Lab.

   Everybody knows the professor can do the proof. No one benefits from him rehearsing it, no one needs to see him do it to believe he can. Students benefit from discovering it themselves and explaining it to others. An instructor must learn to "bite his tongue." The "lecturer" never penetrates the student's mind, never shares his confusion. And the student is quickly left in the dust.

Some teachers say "I taught them—but they didn't learn it." Imagine a car salesman telling his boss— "I sold it to them—but they didn't but it."

Some teachers expect the students to learn how they teach. The teacher should teach the way the students learn. (This isn't always the same. It may never be the same. Good teachers are above all flexible.)

5. The important thing in the math curriculum is not racing through a long syllabus that students are largely not going to absorb anyway and leaving them panting and breathless and overwhelmed and discouraged after the final exam; but learning enough of the subject and learning it well enough to understand the point of it, the philosophy, the general strategy, the essential idea. Emphasis should be on maturity and technique, not merely content. Students will enjoy math when they can say "I understand." They can read and learn on their own after that. They will become life-long learners. The mental skills they learn will transfer to any subject they want to learn. They can even become teachers of others. At Potsdam the math major curriculum is an eight-semester mega-course in independent learning, in thinking, in conceptualizing, in intellectualizing. In later life, this sort of skill will be much more valuable than specific knowledge of specific mathematics. If and when the time comes that they need to know some particular mathematics, they will have the capacity to learn it on their own if they have been properly trained to it. The old proverb "Catch fish for a man and he will eat today; teach him to fish and he will eat all his life" has some application here

There can be honors sections for large enrollment courses to challenge those who can learn faster or have stronger backgrounds. But they should be allowed to take standard tests so they are not penalized for trying the honors level.

For courses which are prerequisites to later courses, a certain minimal syllabus should be established and agreed on; but the emphasis should be on minimal. Some flexibility and good will is necessary between instructors.

6. All courses should be oriented toward pure mathematics, and the joy of doing it and understanding it. There is plenty of chance elsewhere and elsewhen to apply what one learns if and when it is necessary—other courses, or later work experience. This even applies to students other than math majors. There should be no "service courses." A good course for a math major is a good course for anybody else, and vice versa.

7. Most important of all, an atmosphere must be engendered of total support for the student. The function of the educator is to serve the student, to meet him wherever he is and help him grow, help him achieve his goals, help him prepare to flourish in later life however he defines this. The educator must be deeply committed to this task and must constantly convey to the student his direct personal concern for the student's welfare. There must be a loving, supportive, almost familial atmosphere in the department, a sense of community, of mutual support, everyone helping everyone else, everyone proud of everyone else's achievement. What benefits one benefits all; what one achieves is an achievement for all. There is no place for competitiveness, except to the extent that every faculty member and student in the institution are on the same team. (This is very much the way an Eskimo village operates to succeed against the elements.)

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See "A Modern Fairy Tale?" Amer. Math. Monthly Vol. 94 (1987) pp 291-295.

A Humanistic Academic Environment for Learning Undergraduate Mathematics

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Teachers of undergraduate mathematics work under conflicting professional responsibilities. In strong Ph.D. granting mathematics departments undergraduate enrollment in mathematics forms the major support for graduate students as well as the regular mathematics faculty. At these universities much undergraduate mathematics is taught by graduate assistants who have their primary obligation to their graduate studies and research. Most regular mathematics faculty at these universities have no interest in teaching undergraduate mathematics. (It is rare for undergraduates to understand or to participate in the research of their mathematics teachers.) If regular mathematics faculty teach undergraduate mathematics, the lecture method is used most often with very large classes so that the lecturers have almost no knowledge of the hopes, anxiety, or growth in mathematical maturity of their students. Mathematics faculty at these universities expect a large number of their best graduate students to be foreign students and very few will be selected from the undergraduates they teach at their own university. Research grants and fellowships are sought in order to relieve a faculty member from teaching, and in particular, undergraduate teaching.

In group M and B Departments (Departments granting a Master's or Bachelor's degree as the the highest degree) the teaching loads, support for graduate students and research imply that the primary responsibility for the mathematics faculty is to teach undergraduate mathematics. However, research publications compose an important part of the qualifications for tenure and promotion, although the academic climate for high quality research is not very favorable at these colleges and universities

One estimate is that more than 50% of the current enrollment in mathematics courses at public four-year colleges is for remedial course and pre-calculus courses. Also on average fewer than 10% of undergraduate credits in the mathematical sciences are in post-calculus level courses. Often students are assigned to remedial and pre-calculus courses by placement examinations which may not correlate very well with the preparation and ability of students to learn college mathematics, but may correlate with the academic environment for learning undergraduate mathematics at that particular college or university. In general, students do not enjoy studying these courses and teachers do not enjoy teaching them. Frequently, part-time teachers are employed to teach these courses. Perhaps the failure rate in regular calculus courses is an indication of an unfavorable academic environment for learning undergraduate mathematics, as well as the lack of effectiveness in teaching remedial and pre-calculus courses in such an environment.

In large group M and B Departments with few mathematics majors many regular faculty members may have an opportunity very rarely to teach a post-calculus course for mathematics majors. Some faculty members believe that only those students who have the ability to continue their studies in mathematics to the graduate level and to become research mathematicians should be encouraged to major in mathematics. Since research in mathematics is very competitive, mathematics majors should be limited to an elite class of geniuses, and a particular college or university may have very few or no members of this class. Thus many mathematics faculty members are encouraged to process students through undergraduate mathematics courses for supply departments and/or general college mathematics course requirements, and this processing helps to establish an unfavorable academic environment for learning undergraduate mathematics.

At two-year technical colleges and community colleges most of current enrollment in mathematics course is for remedial courses and pre-calculus courses. Very limited or no opportunity is provided for many faculty members at these colleges to teach across mathematics curriculum and rarely is there provided a humanistic academic environment for learning undergraduate mathematics.

Some talented and dedicated teachers at each type of school described above are able to obtain some good results in teaching students undergraduate mathematics. However, the writer contends that most students study undergraduate mathematics in academic environment which are dehumanizing for both students and teachers. Too many future elementary and secondary school teachers study mathematics in such environments. It is important for mathematicians and mathematics educators to discuss such things as the content of the calculus course and how calculus is taught, concrete vs. abstract in mathematics education, as well as the role of problem-solving. The writer challenges chairs of Mathematics Departments and other responsible academic administrative officers to provide a humanistic academic environment for learning undergraduate mathematics. If this can be done, the lay public may view mathematics more favorably, and give mathematics education and research in mathematics more support.

During the past 20 years, the Mathematics faculty at SUNY Potsdam has made a determined effort to establish a humanistic academic environment for learning undergraduate mathematics, and some unfavorable national trends in mathematics education have been reversed. Our average number of bachelor's in mathematics during the last three years is 193 and the average percent is 24% while the national average is 1%. The gender imbalance in mathematics seen nationally is not a factor at Potsdam College. A little more than 54% of our college graduates during the last 18 years have been women and a little more than 55% of our Bachelor's in mathematics have been women during the same period. At SUNY Potsdam the completion of mathematics as a major is gender independent.

Most students enroll in mathematics courses on a voluntary basis and not as a requirement for a major or minor in some other subject. Our college has no mathematics requirement as a condition for graduation. For example, one year with a freshman class of less than 1000 students, more than 600 students enrolled in beginning calculus. No more than 100 of these students came from the supply departments. The issue of teaching algorithms vs teaching thinking or concrete vs. abstract in mathematics eduction is not a problem at Potsdam college. Students consider the study of mathematics as an important part of a liberal arts education and not necessarily as a way of making a living using mathematics primarily. For example, some bachelor's in mathematics in the class of 1987 completed a second major in the following subjects: Anthropology, Biology, Chemistry, Computer and Informational Sciences, Education, Economics, English, French, Geology, History, Political Science, Psychology and Physics. Also, they completed minors in the following subjects: American Politics, Business Economics, Business of Music, Directing, and Health Science.

The number of bachelor's in mathematics who entered Potsdam College with high school averages of 90 and above increased more than 9 times during the past 18 years. In our graduating class or 1987 more than 40% of the students who graduated summa cum laude or magna cum laude where bachelor's in mathematics. More than 50% of our undergraduate credits in mathematics are in post-calculus courses, while the national average is less than 10% We have good cooperation from supply departments, our bachelor's in mathematics choose many different career options, and they are repeatedly hired by the same company or government agency as industrial mathematicians. They make many professional choices for graduate study.

A brief of the steps we took to establish our academic environment is given below.

Entering freshmen students with high school averages in mathematics of 90 and above, quantitative SAT scores of 550 and above, good general high school averages and aptitude test scores are invited to elect our honors calculus course during the fall semester of their freshman year without regard to their intended major in college. Each student is sent a personal letter of invitation. We make clear to students invited that they will not be penalized by the grades they will receive as a result of electing the honors calculus course rather than the regular calculus course. We explain that all students enrolled in calculus will be given the same final examination. Also the teachers who teach an honors calculus section teach a regular calculus section and are well able to compare the achievement of students in different calculus sections.

From an entering freshman class of 1,000 students or less, we usually invite about 130 to 150 students. A little more than one-half of the students invited elect the honors calculus section. In the fall of each year, we usually offer two sections of honors calculus with an enrollment of 35 to 45 students. Some of the students invited who do not elect the honors calculus course do elect the regular calculus course. Many of our best mathematics majors complete our honors calculus course, although our mathematics faculty never discussed or made any particular effort to increase the number of mathematics majors.

We established in 1970 a BA/MA program in which able students can complete both the bachelor's and master's degrees in four years without attending summer school. Some of our most able students with advanced placement credit in calculus complete our courses in Linear Algebra I and Set Theory and Logic during the first semester of their freshman year. Students who do well in theses courses are invited to apply for admission to our BA/MA program. We believe that a long period of preparation is not necessary in order to be successful in advanced courses in mathematics.

We do not give placement examinations in order to assign students to mathematics courses.

The Honors Calculus courses and our BA/MA Program give us an opportunity to recognize early our entering students, as well as their high school teachers, for their excellent achievement as high school students. We use our love and respect for the students we invite to lead them to an enjoyment of the study of mathematics, to understand the meaning of a mathematical proof and respect for a mathematical proof, to learn how to learn mathematics, to develop the ability to read a mathematics textbook for pure enjoyment, and to study independently. These students serve as role models to help us provide an intellectual climate where the mathematical potential of all students who elect to take mathematics courses in the Department can be identified and nurtured. Some of these students tutor in our mathematics lab and provide leadership in our large chapter of Pi Mu Epsilon. An opportunity to teach students in our BA/MA Program aids faculty members to teach mathematics courses close to their research interest.

Faculty members choose the teaching methods which have been most effective in maximizing the development of the mathematical potential of their students. While some teachers use the lecture method as their primary teaching method, others use many different methods of teaching which include active learning method. Each teacher has an opportunity to teach across the mathematics curriculum. Every effort is made to provide the most favorable working conditions possible for faculty members so that they can maximize their working conditions possible for faculty members so that they can maximize their teaching effectiveness and professional growth.

The writer helped to develop a similar humanistic academic environment for learning undergraduate mathematics during the 15 years he served as chair of the Mathematics Department at Morgan State University, Baltimore, Maryland with similar success in the achievement of students in mathematics. Therefore, the writer conjectures that similar environments can be established in many colleges and universities. If such environments are established, perhaps over time, most of the lay public will no longer regard mathematics as its most hated and feared subject.

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Liberal Education in the Liberal Sense

Speech for the Brown University Graduate Commencement Ceremonies
May 25, 1992
Annalisa Crannell
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This past spring, the Brown University Graduate Commencement featured two student speakers, of which I was one. My speech was inspired by the contrast between my own devotion to intellectual pursuits in every field (but especially in mathematics and poetry) and the very different, disparaging views of academia which I've encountered in my students and colleages alike.

The premise of my speech was that professional and personal reasons abound for pursuing knowledge in multiple disciplines. Being a graduate student with little exposure to the existing literature, when I first set about writing this speech I believed that I was a modern pioneer. I quickly realized that this was not at all the case. Within the past decade treatises have been written on this subject, conferences have been held, books have been published. The ten minute time limit on my speech, despite my concentrated efforts at paring, was exceeded.

I was struck by the response that I received—I seem to have struck a chord that was eager to resonate. People approached me with further examples, personal anecdotes, and practical and philosophical arguments supporting my views. I do not take this as a sign that the majority is "on my side" or even that times are changing; but I am indeed encouraged by the strength and support that is available for those of us who dare to be diverse.

Members of the Board, Distinguished Deans, Honored Guests, but especially to all those who are graduating today—congradulations! Today, after these ceremonies are over we will spill out into Thayer Street like so many marbles spilled out of a bucket, and we will roll off in our own directions to take our places in the world. What I would like to talk about—briefly—is a matter that touches our lives as academics and scholars, and that is the question of how others perceive our work. By "others" I don't mean just colleagues or administrators, I mean the public, our students, our friends and family. And by that token, I'll also discuss the way in which we view fields other than our own.

When mathematicians tell people what we do, more often than not we hear, "Oh, I never could do math!" While it's true that the nation is not as mathematically literate as we'd like it to be, what is especially disturbing is the implicit pride in their ignorance. Would these same people say, "Oh, I never could learn to read!"? And yet this theme of noble disdain is heard across the disciplines. "I never could memorize dates, balance my checkbook, learn to spell, read poems that don't rhyme".

Yet another disturbing aspect of these replies is their tendency to reject the deepest and most beautiful aspects of a field for a superficial stereotype. I wish I could say that with advanced education, the prejudice against delving beneath the surface of a subject not one's own disappears, but

There is a perception that human beings are more wholesome when ignorant and more ingenuous when they shed their excess intellectual baggage.

I'm afraid I can't. The Council for the International Exchange of Scholars reported last year that many American scholars see foreign travel as irrelevant to their fields—and *therefore* they don't attempt it[1]. As another example, Harold Howe, former Vice President of the Ford Foundation, recently lamented that in "a country with the greatest and most pervasive commitment to education in all its forms of any other in the world," the subject of pedagogy is still disparaged[2]. "Pedagogy" is not only the method but also the art of teaching.

Why do these attitudes persist? In some fields, such as education, it is a matter of snobbery. In other, especially the sciences, the answer often contains some aspect of fear. Consider for example Shelia Tobias' hit *Overcoming Math Anxiety*[3]. Newly emerging fields are considered frivolous or overly political (Women's studies and Semiotics are good examples of this). Pecuniary decisions play an important tole: what does one do with a degree in archeology? But most frequently, there is a perception that human beings are more wholesome when ignorant and more ingenuous when they shed their excess intellectual baggage:

"I took four years of French in High-school and now I don't remember a word." If somebody said that to you, would you be more likely to say, "I'm sorry to hear it," or "Yeah, me neither"?

That these attitudes persist even among well-educated persons within liberal arts institutions is often laid at the door of efficiency. Today's graduate institutions, the argument goes,

By the time we reach graduate school, the impetus to expand our intellectual horizons has been replaced with demands that we focus our knowledge.

are becoming more-and-more institutes for advanced research and less-and-less centers dedicated to broad-based knowledge. A scholar of early European literature who wants to learn more about issues of race and ethnicity, or a physicist who wishes to devote part of her summer to improving her teaching, soon learns that these 'extra-curricular' activities hamper research and thereby the time-to-dissertation or the chances of getting tenure. In terms of reward systems, it becomes apparent to graduate students and struggling young that the university hires molecular biologists and classicists rather than scholars in the broad sense. The argument concludes: We have ceased to be a nation of universities, and instead have become a nation of multiversities. Because so much of this argument focuses on the question of whether these phenomena are new or whether they are deeply entrenched in tradition, I'd like to give a highly-expurgated) version of the history of graduate education.

At the end of the eleventh century, itinerant teachers began settling near monastic schools in Bologna, Salerno, Paris, Oxford, and Cambridge. The first universities were not intentionally designed to be centers of knowledge; they were actually formed as guilds or trade unions which was associated with the nearby school (*Universitas Magistrorum et Scholarium*). Paris in 1200, was the first University to receive a charter, which meant that they were permitted to offer degrees which conferred upon the recipient the right to teach anywhere in the kingdom. As an aside, the first student to flunk did so out of Paris in 1426. It may not surprise you that he sued the University. (His suit was unsuccessful)[4].

Graduate education in the United States is incredibly new. Prior to the 1870's, colleges were often directed by the clergy, who were more concerned with orthodoxy and decorum than with learning. The first president of the University of Chicago, Dr. Harper, was an avowed critic of the American educational system, saying that it had "actually destroyed the intellectual growth of thousands of strong and able men."[5] And, of course, that wasn't even half the problem!

Although there were places that offered limited graduate education (Brown had graduate students as early as 1859), there was no place which had a program devoted entirely or in the main to graduate education and research. As late as 1874, the President of Harvard University declared it would be impossible to "deliberately undertake" such a program[7]. But two years later in 1876, John Hopkins University opened the first American graduate school and likewise a new era in American education. When Brown University boasts of a graduate school that is over a hundred years old, it places itself among the innovators of that era.

It becomes apparent that graduate education was formed in the United States not so much out of lofty ideals as out of a more fundamental pragmatism. Graduate schools were designed to meet the needs of a society that was ready to blossom academically, but which was notoriously ill served by the current state of affairs.

One of the unusual aspects of American education today is its emphasis on the liberal arts, modelled after the German principles of *Lehrfreiheit* and *Lernfreiheit*. A college student in the U.S> is not required to declare a major until almost half-way through college—the sophomore year. Compare this with systems in other countries where students begin specializing in high-school if not earlier. An undergraduate education in the U.S. emphasizes and at some schools mandates curricular diversity. And yet, in 1978 Joseph Duffey, then Chair of the National endowment for the Humanities, estimated that seventy percent of our undergraduates are pre-law or pre-med, and that only one in sixteen major in the humanities[8].

By the time we reach graduate school, the impetus to expand our intellectual horizons has been replaced with demands that we focus our knowledge. The rewards for specialization at the expense of extra-departmental scholarship are tangible: passing prelims, promotion, tenure. And when you see the light at the end of the tunnel, it's hard to change course. It has been said that the definition of originality in graduate students is "the capability to present their professors' ideas back to them in a way that they'd never thought of before."

What view does this give us as we leave? I'm afraid I have painted a much drearier picture than I wanted to. The rewards for academic diversity may be less forthcoming and less tangible, but they are not less. Reaching beyond the academic borders of one's discipline results in "bridges built, inertia combatted, old icons broken"[9]. What is often lacking in external impetus is made up for in buckets by our individual aspirations, which are in turn made more feasible by the rhetoric, if not the finances devoted to scholarship and liberal education in their most liberal sense. In the most ideal and idealized worlds, we truly become the "teachers of the love of wisdom", doctors of philosophy. I would like to close with a quotation by Kenneth Boulding, former president of both the American Association for the Advancement of Science and the American Economic Association:

> It may well be that the only answer to this problem [of generalization versus specialization] is redundancy, inefficiency, extravagance, and waste. [But] One could indeed argue that the main reason for getting rich, that is to say, for economic development, is to permit the human race to indulge in these last four delights.[10]

Thank you very much.

FOOTNOTES

1. C. D. Goodwin and M. Nacht, ("Missing the Boat: the failure to internationalize American higher education", Cambridge University Press, 1991)

2. Harold Howe II, *Foreword*, (American Graduate Schools of Education, 1982), p. *vi*

3. Sheila Tobias, ("Overcoming Math Anxiety", Houghton Mifflin: Boston, 1980)

4. W. H. Rouse Ball, ("A Short Account of the History of Mathematics", Dover Publications, 1960), pp. 139-143.

5. W. Carson Ryan, ("Studies in Early Graduate Education", The Carnegie Foundation for the Advancement of Teaching, 1939), pp. 3-14.

6. ibid.

7. ibid.

8. Joseph Duffey, *Graduate Education: A Case for the Public Interest*, ("The Philosophy and Future of Graduate Education", The University of Michigan Press, 1978), pp. 1-6.

9. C. D. Goodwin and M. Nacht, *op. cit.*, p. 49.

10. Kenneth E. Boulding, *Graduate Education as Ritual and Substance*, ("The Philosophy and Future of Graduate Education", The University of Michigan Press, 1978), pp. 143-155.

The Popular Image of Mathematics

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The popular image of mathematics is that it is difficult, cold, abstract, ultra-rational, important and largely masculine. Many persons operating at high levels of competency in numeracy, graphicacy, computeracy in their professional life still say 'I'm no good at mathematics, I could never do it.' They perceive mathematics to be alien to themselves and their professional concerns.

For many people the image of mathematics is associated with anxiety and failure. When Brigid Sewell asked adults on the street if they would answer some mathematics questions, 50% fled. (She was gathering data on adult numeracy for the Cockcroft Committee of Inquiry.) Extreme mathephobics are undoubtedly a small minority in Western societies, and may not be significant in other countries, but their existence, and that of the popular image of mathematics raises a number of important questions. How widespread is the popular image described above? Does it correctly describe mathematics? What causes it? Can any change in educational practices alleviate it? Is there any hidden agenda behind the popular image?

For many people the image of mathematics is associated with anxiety and failure.

It could well be that the popular image of mathematics is the single most important issue of concern for the Philosophy of Mathematics Education network—important in terms of social significance—for mathematics serves as a 'critical filter' (to use Lucy Sell's term) controlling access to many areas of advanced study and better-paid and more fulfilling professional occupations. If its image is an unnecessary obstacle which blocks popular access to mathematics, then it is a great social evil. Of course, changing the image alone may do little to solve the problem. That is the politicians's and advertiser's view. It may be that the nature of the populace's encounters with mathematics also needs to be changed, to be humanized.

These insights are increasingly widespread. Alvin White has founded the Humanistic Mathematics Network and has been actively promoting mathematics as a humanistic discipline through the network's conferences and newsletter. ICMI sponsored a conference on the popularization of mathematics in 1989 in Leeds, England. An outcome was the volume The Popularization of Mathematics edited by A.G. Howson and J.-P. Kahane in the ICMI Study Series, Cambridge University Press, 1990. This book offers valuable insights about the problems of mathematics described above, and a range of possible measures to address them. The upcoming conference ICME-7 has a Working Group 21 on the Public Image of Mathematics and Mathematicians, with Thomas J. Cooney as chief organizer.

Given this attention, what can PoME uniquely contribute to the understanding and solution (or rather initial steps towards the solution) of this problem? It could be argued that even if PoME cannot offer something unique, the problem is of such importance that all efforts directed at it are valuable. However PoME does have something unique to contribute. On the one hand, the image of mathematics, the nature of mathematics, conceptions of mathematics all find their most systematic treatment in the philosophy of mathematics. On the other, their promulgation, dissemination and re-creation is largely affected through education. Hence the study of the intersection and interaction of these two fields, which is the concern of the Philosophy of Mathematics Education, has a central role to play.

To return to the questions listed above: How widespread is the image of mathematics as difficult, cold, abstract, ultra-rational, important and largely masculine? To answer it, first the distinction must be drawn between mathematics as a discipline (what professional mathematicians understand as mathematics) and school mathematics. As it happens, both of these can share the popular image described above, at least to outsiders. Such an image is associated with negative attitudes to mathematics.

However, research on children's attitudes toward mathematics in the past two decades shows fairly widespread liking of the school subject, certainly in the years of elementary schooling. In the later years of schooling attitudes become more neutral, although extreme negative attitudes are relatively rare. Presumable this downturn in attitudes is due to such things as adolescence, peer-attitudes, the impact of competitive examinations, not to mention the image of mathematics conveyed in (and out) of school. According to this image, school and the discipline of mathematics are all of a piece, beginning in school, and then rising like a ladder to dizzy heights of abstraction. In contrast, numeracy, contextual mathematics, even ethnomathematics are perceived to be quite distance from 'academic mathematics', presumably because of the differences in context and surrounding practices.

If its image is an unnecessary obstacle which blocks popular access to mathematics, then it is a great social evil.

Does the popular image indicated above correctly describe mathematics? The answer to this is both Yes and No. First the Yes part. The experience many learners have, and certainly in the West virtually all citizens go through the educational process for many years, confirms this image. Teachers and others and the experience of learning itself all confirm this view.

Secondly the No answer. The image of mathematics is not as described in many enlightened schools and colleges, and certainly does not have to be that way. This pertains largely to school and college mathematics. What about the discipline of mathematics itself? This is where the philosophy of mathematics enters directly into the picture. Philip Kitcher has described a 'maverick' tradition in the philosophy of mathematics which emphasizes the practice and human side of mathematics. This has been termed variously Quasi-empiricist, Fallibilist, and Modernist thought in education, philosophy and the social sciences. Such mathematicians and philosophers as Lakatos, Putnam, Hersh, Davis, Tymoczko, Kitcher have been at the forefront of these developments. The maverick tradition is represented by Tymoczko's anthology <u>New Directions in the Philosophy of Mathematics</u> (Birkhauser 1986) and more recently has found expression in <u>Philosophica</u> volumes 42 (1988) & 43 (1989) edited by Jean Paul Van Bendigem (to be expanded and reprinted in book form in Sal Restivo's series <u>Science Technology and Society</u>, SUNY Press).

The point of this story is that this maverick tradition rejects the image of mathematics described above as unnecessary, mistaken and downright false. To use Reuben Hersh's image: mathematics has a front and a back. In the front, the public are served perfect mathematical dishes, like in a classy restaurant. In the back, the mathematicians cook up new knowledge amid mess, chaos and all the inescapable associated human striving, successes, failures (and displays of ill temper!).

Can any change in educational or other practices alter the popular image of mathematics? Presumable change is always possible, or else we would all give up! The first step must be to raise consciousness about the future of mathematics, and about the fact that there are alternative and competing conceptions of it. Promulgating such views within educational circles and beyond in society at large are vital. But the final question must be asked. Is there any hidden agenda behind the popular image of mathematics?

If there are, then strong resistance to change can be expected. The status quo always has its own momentum, and is difficult to change. But there is a more radical view that the kind of popular image of mathematics described here serves conservative interests in the mathematics community and in society in general. For if mathematics is viewed as difficult, cold, abstract, ultra-rational, important and largely masculine, then it offers access most easily to those who feel a sense of ownership of mathematics, of the associated values of western culture and of the educational system in general. These will tend to be males, to be middle class, and to be white. Thus the argument runs that the popular image of mathematics described above sustains the privileges of the groups mentioned by favouring their entry, or rather by holding back their complement sets, into higher education and professional occupations, especially where the sciences and technology are involved.

This argument is quite radical, and may involve assumptions unpalatable to some. It may not be accepted that the popular image of mathematics has a hidden agenda or serves particular interests. Even so, it should be conceded that the type of popular image of mathematics described obstructs the full participation of all sectors of the population in higher education and professional occupations involving mathematics, especially in science and technology. It may also prevent citizens in modern society from developing critical numeracy and the mathematical confidence needed to understand the social uses of mathematics and to question statistics, whatever their source. Thus even from a traditional liberal perspective it can be argued that the common popular image of mathematics impedes both industrial and technological development and the full expression of democracy in a mathematically empowered citizenry.

Poems by Lee Goldstein

Pythagoreanism

Virtual reckoning:
The accounts of the world
Require an inconsistency
To be countermanded
By the presentiment
Of numbers
And by inchoatively
Reforming the tempest
Into files of noological charts.

- 1987

Impedimenta Mathematica

When I am in the differential abandonment
In which I could grasp the probationary thing or facient;
As the thing is less its manner,
Then it may be seized in order
To satisfy the necessary condition;
But, ah, there remains yet that nagging prototaxic bent
To recollect one of English's innominate,
To boot, the set of all nonce elements;
Alas, it mimics itself typically in paradox

- 1989

Mathematics

Naming is a porism
Or the corollary of incidence to the demonstration of the form;
Naming is the producing of that which is, anon, disposed
Or that that communicatively is at the very proven discovery aborn;
And of the mathesis,
For which these denominative typifications of the idea do casually change,
There exists, projectively, the unnamed, particularly, the innominate
manifoldness, or for the nonce, the innominable prospective . . . ,
To which is the mathematical "leap of faith",
Or, ergo, that that symbolically bounds the genetic, nominal world,
insulate, forlorn.

- 1990

Momentaneousness

'Adverb' nounizes;
Mathematics is as the adverb
To the noun per the consciousness of the nounizing,
For the abstract quality
Would be forgotten,
Where it not for the mathematical mnemonic
Of the adverbial side of the instant,
For mathematics is the realism
Of the adverb.

- 1991

Proto-tonality

Manufactory mathematics:
The hand which would write-
Of the abstract structure of things
May miss the measure of its actualizing,
For mathematics is like a drive-
Of the locus of the sense of which words or symbols have already named
things, i.e.,
A 'logaesthesia'- that prior to the annominating,
(Which may happen to the variable a rehearsal), is the fit mathematical
languaging-
Of an esse prior to calling, or through the manuscript, or per a tone of the
symbols.

- 1992

∃ (There Exists)

Mathematics is concerned with the programs
Of sundrily sonant objects,
And it has fascination ever
Because of its general resemblantness,
Where the programs
Are not affinely dictionarial,
And where such invention is manifest
Through the modular de-individuation,
Miscellaneously, of phenomenal appearance.

- 1992

Psychology

Mathematics begins with the after-mathematics,
Where the latter is prior to the former,
And one may intuit the backwards going under a forward language
That should substitutively be in order.
Indeed, the very expresses of the impassioned representations
In the evolution of a scientific mentation
Are 'over' the after-mathematics as a "mirror" to the fore-mathematics,
Which is the doer,
In 'back' of its inscription as a science.

- 1993

On Sense And Reference

Mathematics disagrees with nominant consciousness,
Where the latter might vie to make mathematics incognizant;
Mindedly is there a consciousness possible in the innominology?
It is the prototaxic of unnamed objects,
Of which the mathematics might arise of,
That mathematical consciousness is as feasible
As the sense of the objects of speculation, not having been named, admit a clear approach,
But where, instead, or in the competing, the nominant consciousness is the more often spoken:
In mathematics, innominology precedes nominance, when namedness precedes the innominate.

nominance: the sense in which consciousness relates to things named
innominology: the sense in which consciousness or unconsciousness relates to things not named.

- 1993

A Multicultural Matrix for Mathematics Education

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A major goal in mathematics education is to make learning more connected and meaningful for students. Supporting this goal is an emphasis on the integration of applications into curricula. These emphases provide many benefits. By connecting school learning with the world in which students live, we have the opportunity to pique student interest and motivate learning. We also provide the chance to deepen student understanding of the world around them. They may see that real learning is connected to many disciplines and also to many strands in mathematics.

One significant area of applications is the use of mathematics by the many peoples and cultures of the world both currently and in the past. Around the world, cultural groups, both native and immigrant, are concerned that youngsters recognize and value their cultural heritage. Often mathematics is embedded in cultural phenomena without members of the culture even aware of it. (Ascher, 1991). As families around the world become more mobile, it becomes more important for students to recognize the contributions to mathematics within their own heritage as well as that of others. Furthermore, there is a growing hope worldwide that through global understanding we may build peace. Thus, youngsters need to appreciate the diversity and insights of the many cultures that have and continue to contribute to human history. Indeed, for future progress it is vital for today's youth to see themselves connected to a continuum of human endeavor. Others throughout time have counted, measured, represented, located, reasoned, predicted, explored and used their minds to improve their existence ... so must we today. It is from these views that this project stems.

The matrix. To assist mathematics curriculum planners and teachers around the world in developing multicultural learning materials, this project offers a matrix to stir creativity. The two dimensions of the matrix are *cultural features* and *mathematics topics*. The entries in the cultural dimension have been adapted from various analyses of cultural attributes. The six categories are language, history and geography, economics and politics (including resources, technology, transportation, communication, and government), social features (including customs, beliefs, family, food, education, health, welfare), aesthetics (including art, music, drama, dance, pottery, textiles, architecture) and recreation (including sports, games, and entertainment). The entries in the mathematics dimension have been adapted from the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics (1989). The eight categories are communication; reasoning; number and numeration; measurement; patterns, functions,

As families around the world become more mobile, it becomes more important for students to recognize the contributions to mathematics within their own heritage as well as that of others.

and algebra; geometry; statistics and probability; and discrete mathematics. Problem solving permeates the entire matrix.

Following this essay are two versions of entries in the matrix. The first matrix identifies generic suggestions, ideas relating the mathematics topic to the cultural feature without specifying a culture. Indeed, several suggestions are intended to be cross-cultural. The second matrix offers specific suggestions, many of which are already in print, but not in a form for school children.

Methodology. The two dimensions of the matrix very naturally suggest two approaches to filling the cells: examining mathematical strands for cultural applications and examining cultural phenomena for mathematical aspects. Both of these strategies were used. Other curriculum developers should consider these two strategies as ways to conceive even more possibilities. Broadening the list of categories on either dimension may also increase the inspiration of more ideas.

Limitations. There are many cautions in presenting a matrix such as this. First, the matrix has a limited scope: curriculum, not instruction or evaluation. Its major intent is to make suggestions for teaching materials in the form of topics, questions, activities, and avenues for student research. It does not address classroom climate, teaching strategies, or the structure of school programs. These factors, too, are culture-bound. Research suggests that altering these factors can make more mathematics more accessible to more students. A second limitation is the small number of categories on each dimension. Some reduction was needed to make the task manageable. This, however, reduces the possible number of ideas generated. Third, the categories themselves reflect the cultural perspective of the author. As Marcia Ascher points out (1991, p.3) "how people categorize things is one of the major differences between one culture and another."

Finally, by its nature, a two-dimensional matrix highlights relationships between pairs of factors, but overlooks other connections. This limitation has many ramifications. First, the categories are not entirely disjoint. For this reason, many items in the matrix could have been categorized in other locations. Next, by being two- and not three-dimensional, the connections made are limited to pairs and not triples or n-tuples. Also, the cellular structure tends to particularize ideas, rather then create networks of connected ideas. This masks important connections within both mathematics and cultures. Another approach might have been to begin with a cultural context and draw out as much of the possible mathematics from it as one could. This would produce a more thematic learning environment which has many educational benefits. This last limitation reveals the western influence of the matrix structure itself: it separates rather than integrates.

In its favor, the matrix is a starting point, from which more interrelationships may be generated. The intent of the matrix is to seed creativity, not restrict it. It is a beginning, a point of departure. It succeeds to the extent that it inspires. Its value will be measured by the extent to which it assists teachers and curriuclum planners in identifying, creating, and integrating multi-cultural experiences for youngsters to make mathematics learning more global, humane, and meaningful.

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A Multicultural Matrix for Mathematics Education Generic Ideas

	Language	History & Geography	Economics & Politics	Social Features	Aesthetics	Recreation
Commun-ication			Coding or graphing economic information, transportation schedules, communication systems.		Coding systems used for dance, music, architecture, crafts, etc.	Coding systems used for games.
Reasoning	Proofs without words to share among students with different languages. Logic puzzles re distribution of languages in a multi-lingual classroom.	Proofs of famous theorems from different cultures, e.g. Chinese proofs of Pythagorean Theorem.	Reasoning within a country's legal system.	Implications applied to costumes and traditions.	Analysis of designs.	Strategies for winning games.
Number & Numeration	Numeration systems. Calculation devices. Gestures for numbers, measures, estimates.		Strengths or limitations on economic system caused by the numeration system used (e.g. limitations of Roman numerals).	Using proportions to resize recipes. Beliefs about numbers (values, superstitions).	Making scale drawings of artifacts.	Counting or scoring systems used in games, e.g. scoring in tennis, bowling. Magic squares in different cultures.
Measure-ment	Units and tools for measurement. Translating a table of measures by using the numbers as clues, e.g., 24 __ = 1 __.	Historical time lines. Historical development of measurement within a culture. Estimation of areas of lakes, etc. from maps.	Systems of measures. Unification of countries via creation of standard measures, e.g. Napoleon in France and Qin in China.	Calendars. Concepts of time.	Sizes of tools & materials needed for crafts.	Sizes of sports fields, game equipment.
Patterns, Algebra, & Functions	Interpretation of graphs in other languages where data and photos provide clues for context.	Use of coordinates for maps, locations. Climatic conditions as a function of latitude.	Categorization of economic phenomena as linear, exponential, polynomial. Simulation models for economic phenomena.	Groups to model family relation structures.	Transformations (reflection, scale change, rotation) applied to native designs.	Models for international sports records.
Geometry	Translation from another language of a glossary of geometric terms by use of diagrams. Interpretion of a geometry problem and solution presented in another language.	Comparisons of different systems of navigation using local or global parameters.	Geometric properties of trademarks (Metz, 1981).	Beliefs about shape, space. Home designs. Symmetry as a value.	Architectural design features, e.g., arches. Design of musical instruments. Shapes used in jewelry, pottery, etc.	Design of game materials and sports equipment
Statistics & Probability	Distribution of languages over the globe or by continent.	Analysis of data on land use, demographics, etc.	Analysis of data on resources, technology, commerce, production. Expected values for economic phenomena.	Analysis of data on health, education, religious preferences.	Differentiation of cultures based on analysis of design features e.g. frequency or absence of certain frieze patterns.	Probabilities related to games and lotteries. Design of fair games.
Discrete Math'cs	Analysis of grammars.		Network analyses of political structures, transportation and communication networks. Markov chains for analysis of economic phenomena.	Network analysis of use of space e.g. in homes and public buildings. Sociograms.	Network analysis of weaving designs.	Networks and recursion in recreational drawings and design of algorithms to reproduce them.

A Multicultural Matrix for Mathematics Education Specific Ideas

	Language	History & Geography	Economics & Politics	Social Features	Aesthetics	Recreation
Communication			How do some nations code transportation schedules more effectively than others? (Tufte, 1990)		How can we encode dance movement? (Tufte, 1990) How do musical time signatures encode numerical information?	How are chess moves coded for a computer? What other games can be similarly coded?
Reasoning		Use a time line to identify intersections of important periods and disjoint periods whose intersection might have had important implications.		Reconstruct sacred symbols, e.g. yin-yang and 7-12 New Jerusalem design (Kappraff, 1991).		Analyze the *mu torere* game of Maori of New Zealand (Ascher, 1991).
Number & Numeration	How do different communities in Africa use gestures to communicate numbers? (Zaslavsky, 1973) How do Hindus compare strenghs to people, animals and gods to build powers of ten? (Schultz, 1982)	How did the system of Roman numerals hamper development of Roman civilization?	How did Incan quipus encode economic information?	How are numeration systems related to beliefs in a culture? E.g. the Babylonians used 20 and 60 as bases. Why?	Make scale drawings of symbols, icons, artifacts (Krause, 1983, p. 42). Indian chain making builds links in 3s, then 7s, then 15s. Why these numbers?	
Measurement		Qin in China (221 B.C.) and Napoleon in France (early 19th century) created standardized measures to assist unification. Why? Today Europeans are planning for a single monetary system. How will this work? What are its advantages?		The Mayans had two calendars, one of 260 days, and one of 365 days. How long a cycle did they need before they both began together? How did different cultures solve the problem of leap years?		Design a scale model of one of the Olympic stadia or athletic fields encompassing multiple sports.
Patterns, Algebra, & Functions		Find growth rates of different nations at different periods of time. Project world population. How have sizes of major cities changed over time? How is mean temperature for one season a function of altitude?		Show how group theory explains family relationships of Warlpiri of Northern Australia (Ascher, 1991).	Categorize Incan strip patterns into 7 transformation categories (Ascher, 1991) and 2D patterns into 17 categories (Crowe, 1987). Find symmetries in folk arts (Krause, 1987; Bradley, 1992; Zaslavsky, 1979, 1990). Model music with trig'c functions and log'c properties (Maor, 1979). Analyze musical transformations (Schultz, 1982).	How many steps are needed to solve the Tower of Hanoi most efficiently? Create models and predict winning times for Olympic events for the next games.
Geometry		What geometric properties are embedded in the wonders of the ancient world? modern wonders?		Compare visions of the line and the circle from western and Native American perspectives (Ascher, 1991); Why do some cultures build round houses? (Zaslavsky, 1989) How do Japanese design homes with tatami mats in 2:1 ratios? (Boles & Newman, 1987) Why does Taj Mahal have two mosques? (Schultz, 1982)	How do you put a hemispherical dome on a square building as in the Aya Sofia in Turkey? (Blackwell, 1984) How are different shaped arches built? (Heafford, 1959, pp. 128-129)	What patterns of dimples are used on golf balls? What properties must these patterns have? What tesselation does a soccer ball have?
Statistics & Probability	What nations are most multilingual? What continents? A cryptogram might be French or English. Use frequency distributions of letters in each language to decode the message.	Create and compare population pyramids for different nations or continents.		How are religions distributed over the globe? What continents or countries have the most diversity? the most homegeneity?	Make inferences about Bach and the acceptance of his music from a graph (Tufte, 1990) on the writing, publication, and debuts of his works.	Determine expected values for outcomes in Iroquoian dish game (Ascher, 1991) or Hopi game (Krause, 1973). Find correlations for home altitude of athletes and Olympic sports records.
Discrete Math'cs	Translate the labels, given these set relationships: duos: {2, 6, 28, 104 {10, 40, 50, 70 pentos: {5, 35, 75, 95	How do matrices and distance measures help classify lost Mayan groups and the Anasazi of Chaco Canyon? (Meiring, 1992; Crowe , 1987).	Analyze transportation networks (Zaslavsky, 1981).		Analyze, using recursion, the pattern of the roof tiles in the Sydney (Australia) Opera House.	Design algorithms to reproduce sand tracings of Malekula, Bushoong or Tshokwe (Ascher, 1991). How many dominoes must there be? Why? Suppose dominoes were numbered to 12, how many pieces would there be? Write computer code for scorekeeping in bowling, tennis, etc.

Are There Revolutions in Mathematics

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Thomas Kuhn's Theory of the Structure of Scientific Revolutions, first published in 1962, heralded both a renaissance and a shift in the philosophy of science. The main tendency had been towards Logical Positivism and its successor Logical Empiricism, with an emphasis on the logical structure of scientific theories, shown in the work of Carnap, Frank, Hempel, Nagel and others. This was revitalized with the English publication of Popper's Logic of Scientific Discovery in 1959. It was not until after the impact of Kuhn that the philosophy of science became thoroughly cognizant of developments in the history of science (although there were precursors, such as Hanson). Kuhn offered a powerful new synthesis of pre-existing elements (some perhaps unknown to him) such as Wittgenstein's notion of a 'paradigm' and Bachelard's concept of 'epistemological rupture' in the history of ideas. He constructed profound theory in the philosophy of science, the influence of which, controversy notwithstanding, had reverberated through many other fields of enquiry since.

According to Kuhn, science does not grow by a simple accumulation of knowledge. Instead, it alternates between periods of 'normal' and 'revolutionary' science in its development. During a period of 'normal' science, new knowledge is accumulated by accretion, as a dominant theory and paradigm of inquiry are followed and used as a model. Anomalies and contradictions in the dominant paradigm lead to a period of revolution in which competing camps of scientists promote alternative theories (including the falsified old theory). A new theory comes to be accepted and gradually becomes the new paradigm of explanation and enquiry. In the shift to the new theory many of the concepts involved change meaning (e.g. mass and length in the transition from Newtonian Mechanics to Relativity Theory). Kuhn's controversial claim is that the old and new theories are 'incommensurable', and that their supporters may not be able to understand each other.

The Kuhn-Popper debate in the philosophy of science hinged on the issue of rational versus irrational criticism of scientific theories. Popper's position is prescriptive, and he posits falsification as a rational criterion for the rejection of a scientific theory. Kuhn, on the other hand, proposes a more descriptive philosophy of science, which while treating the growth of objective knowledge acknowledges that rational features are neither necessary nor sufficient to account for theory acceptance or rejection.

Although it is beside the point, there is a fascinating analogy between Kuhn's theory of normal and revolutionary development, and Piaget's theory of assimilation and accommodation

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in cognitive growth, respectively. This lends some support to the thesis that individual conceptual developments mirrors that of humankind as a whole (the Phylogenetic Law). It represents the application of the evolutionary maxim 'ontogenesis recapitulates phylogenesis' to the intellectual plane. This is a strongly heuristic analogy which provides a rationale for the use of history in the teaching of mathematics and science (although ultimately the analogy breaks down).

The claim is made in this and earlier issues of the newsletter that the philosophy of mathematics is currently undergoing a Kuhnian revolution, with the rationalist Euclidean paradigm of mathematics as an absolute, incorrigible and logically and hierarchically organized body of knowledge increasingly under question. A number of mathematicians, philosophers and educators are taking mathematical practice and history as central to any account of mathematics, in place of the traditional narrow focus of the philosophy of mathematics on the foundations of pure mathematical knowledge and the existence of mathematical objects. This new 'maverick' tradition, as Kitcher terms it, regards mathematics as quasi-empirical and fallible, a view which is supported by an examination of the history of mathematics.

A key question concerns the applicability of Kuhn's theory of scientific revolutions to mathematics. Is this theory applicable to mathematics? Does mathematics have revolutions? H. B. Griffiths (1987:71) questions the applicability of the notion of revolutions to mathematics, and argues 'it is doubtful whether Kuhn''s notion of a paradigm applies to mathematics in the same way that it does to other sciences'. Griffiths makes this point in the context of an extended review of a book on mathematics education. He argues that incompatible theories and indeed paradigms can coexist in mathematics, unlike in science, where all the theories purport to describe the same underlying objective reality. This is a point well made. Any over-facile parallel with Kuhn must fall foul on this issue. He argues that the 'overthrow' of the paradigm of Euclidean geometry by that of non-Euclidean geometry does not force mathematicians to reject it, as physics reject Newtonian theory in favour of Relativity theory.

Incompatible theories and indeed paradigms can coexist in mathematics, unlike in science, where all the theories purport to describe the same underlying objective reality.

On this basis, it must be accepted that not all major changes or developments of new theories in mathematics deserve the epithet of 'revolutionary'. Nevertheless, I still want to argue that some radical changes or global restructuring of the background epistemological and scientific context of mathematics can be described as Kuhn-type revolutions. Such changes result in a profound re-orientation of mathematics, which can lead to as much 'incommensurability' as is found in science.

Some possible candidates for mathematical revolutions are the following. First of all, infinitesimal based proofs in analysis were universally accepted, despite Berkeley's (1734) pungent criticism, until they were banished by new standards of mathematical rigour in analytic proofs introduced by Cauchy, Weierstrass and Heine in the nineteenth century. This change reflects a shift in the nature and standards of proof from those based on geometric intuition, to those of arithmetical argument (Boyer, 1968). Another chapter in this story is the re-introduction of infinitesimal based arguments in the proofs of non-standard analysis (Robinson, 1966). This reflects a further change in the nature and standards of proof accepted in analysis, from those based on arithmetic to those of axiomatic first-order logic (Lakatos, 1978; Robinson, 1967). Many other such examples can be sighted. These include in the late nineteenth century, the shift of geometric demonstrations from those relying on spatial intuition to a reliance on an axiomatic logical basis (Hilbert, 1899; Richards, 1989); the move to an axiomatic basis in arithmetic proofs (Peano, 1889); an the axiomatic rigorization of deductive logic itself (Frege, 1879).

To dwell a little longer on an example, a further example of a 'revolution in mathematics' is the shift of standards of proof in algebra in the nineteenth century. These changed dramatically from intuitive generalizations of arithmetic to a deductive axiomatic basis (Richards, 1987). The conceptual difficulties in making this transition should not be underestimated. The rigid attachment to the field-structure of number, crystallized in such laws as Peacock's 'principle of the permanence of equivalent forms' constituted what Bachelard terms an 'epistemological obstacle' to reconceptualizing the nature and epistemological basis of algebra. It took the mathematician Hamilton over ten years to overcome this obstacle in inventing his non-commutative ring of Quaternions. In doing so, he enabled a reconceptualization which heralded a revolution in the nature of algebra and the basis of proof in the subject.

This and the above examples illustrate a global restructuring of a branch of mathematics that might in my view legitimately be termed a 'revolution in mathematics'. What they illustrate is not the replacement of one mathematical theory by another. Instead they record a revolutionary shift in the background scientific and epistemological context, its constituent proof criteria and paradigms, and the associated meta-mathematical views. Changes in the background context can involve a changed pool of problems, concepts, methods, informal theories, the language and symbolism of mathematics, proof criteria and paradigms. It will also include a shift in the meta-mathematical views accepted by the mathematical community, including accepted standards for proof and definition, views of which types of inquiry are valuable, and views concerning the scope and structure of mathematics. Such changes can result in a profound re-orientation of mathematics.

The outcome of this radical restructuring is a new or revised scientific and epistemological context for mathematics.

The outcome of this radical restructuring is a new or revised scientific and epistemological context for mathematics. In particular, it represents a global restructuring of the epistemology underlying mathematics, and the way truth, proof and meaning are conceptualized by the mathematical community. In the examples cited, not only did the standards of proof change. In addition the criteria for evaluating mathematical theories changed, for these themselves are largely based on the proof and definition standards employed in the formulations of the the theory. Such shifts do seem to correspond well to Kuhn's notion of scientific revolution, and would not appear to admit multiplicity as in the case with mathematical theories. In other words, like scientific theories, multiple epistemological frameworks cannot consistently coexist in mathematics, justifying the extension of Kuhn's theory to mathematics.

A number of other authors have also suggested that there are revolutions in mathematics, including Kitcher (1984), Gillies (forthcoming), and McCleary (1989). Overall, whilst agreeing with Griffiths that Kuhn's Theory of Scientific Revolutions cannot be directly applied to mathematics, my claim is that a transformation of it directed at the underlying epistemological contest, instead of just at mathematical theories, does offer a valuable insight to the history and philosophy of mathematics.

A final aside is that the above argument offers grounds for a criticism of Lakatos (1976). Lakatos' Logic of Mathematical Discovery only treats mathematical innovations at the micro-level, and does not accommodate macro-level changes such as the mathematical revolutions described above. Elsewhere, in recognition of this deficiency, I propose a Generalized Logic of Mathematical Discovery, see Social Constructivism as a Philosophy of Mathematics, forthcoming, SUNY Press.)

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On Learning in the Mathematical Sciences: Statistics 200 as a Paradigm of Everything Wrong in Mathematics Education

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One of the hazards in mathematics teaching is that it is possible to communicate without teaching. The litmus test of learning should be whether the student knows anything six months after the course is over. Too often, there is no intellectual evidence that the student ever took the course; and too often it is the fault of the course. This hazard can exist in any mathematical course at any level (and with *good* students) but a superb example is the elementary course in statistics, which I will call *Statistics 200*. The distilled thesis of this essay is that many courses try to teach too much material too fast. To illustrate this point, it is useful to dissect a particular course and that course will be Statistics 200. Note that this essay is intended partly as a sequel to, but is independent of [1] where I tried to address the problem of communication in the classroom. Note, that I agree with most statisticians that statistics, unlike probability, is outside of mathematics proper; nonetheless it is a mathematical discipline and statistical courses suffer most of the problems of mathematics courses.

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Virtually every university and college in the United States has a sophomore introductory course in statistics. This course is usually required of business majors and social science majors and is frequently taken by *hard* science majors who may later take courses in mathematical statistics. A legitimate question is whether any learning at all ever takes place in the course. *Does as many as one student in fifty leave the course with any reward commensurate to the effort expended?*

It can be argued that the above question is inherently unfair. How often is the efficacy of any specific course tested post-course? GRE's and SAT's and that ilk of tests target broad areas. But individual courses are not subjected to this sort of analysis. Nor are instructor evaluations given post-course. It would be interesting to acquire evaluations of the teacher and the course (say) six months after a course; time can dramatically change the focus of perceptions. No more will be said here about teachers; our concern is with curriculum design, and in that area we do test the student's post-course knowledge of the material whenever we give a course that has the other as a prerequisite. Note, that teachers nearly always feel that their students have inadequate knowledge of the prerequisites!

The Statistics 200 course is used here as a paradigm of what goes wrong in mathematics teaching. I am not interested in the problems of teaching that course per se, but since I am using it, I will have something to say about teaching statistics and probability.

Statistics 200

The primary problem with teaching Statistics 200 is that the course has a huge mass of ideas that students with little background are expected to master. They are expected to learn the rudiments of probability theory. Then they are expected to learn data collection and data analysis through hypothesis testing, linear regression, and analysis of variance. *Many* teachers expect more: for example, nonparametric tests and Bayesian methods. The only prerequisite for the course is having passed algebra. As a rule none of the students will have any prior knowledge of probability or statistics. Furthermore, those students who are taking the course as a requirement for a degree in psychology or business, usually have not had algebra for a while, and are weak in that area too. My experience at several institutions and the experience of others I have talked to, is that even the engineering and science majors retain little of the course, despite the fact many of them do not have a difficult time with it.

The Problem of Probability

Usually Statistics 200 starts with two or three weeks devoted to probability. The idea is that the student needs probability to understand the concepts of statistics. Let us look at a few of the laws of probability that we try to impart in this short time:

The law of addition: $P(A \text{ or } B) = P(A + B) = P(A) + P(B) - P(A \text{ and } B)$

The definition of conditional probability:

$$P(A \text{ given } B) = P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

The law of multiplication:

$$P(A \text{ and } B) = P(AB) = P(B)\cdot P(A|B) = P(A)\cdot P(B|A)$$

The definition of independence: Events A and B are independent if any of the following *equivalent* conditions are true

$$P(A) = P(A\backslash B)$$
$$P(B) = P(B|A)$$
$$P(AB) = P(A)\cdot P(B)$$

Bayes' rule: Given that the events B_i are exhaustive and exclusive

$$P(B_j\backslash A) = \frac{P(A|B_j)\cdot P(B_j)}{\sum_{i=1}^{n} P(A\backslash B_i)\cdot P(B_i)}$$

Notice that I have not said anything about sample spaces and events. I always introduce these things informally by writing out the full 36-point sample space of the outcome of throwing a pair of ordinary die. I then use events within this sample space to illustrate all the rules above except Bayes' rule (which I usually do not cover). I generally define the following events:

S_i is the event that the sum of the dice is i, i = 2,3,4,...,12.

R_i is the event that the red die is i, i = 1,2,3,4,5,6.

These two sets of events are sufficient for illustrating all the rules above.

Many teachers give so much attention to the addition rule for disjoint events that the students do not realize it is a special case of the general rule above, and they do not realize that the above rule is *always* true. At this stage, most students struggle

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to differentiate between *or's* and *and's* and have a major difficulty distinguishing between *A and B* and *A given B*. The simplest type of environment to give problems on this is the probability matrix such as:

	C	D	E
A	2	2	1
B	2	3	0

In this setting each event is either a row or a column. The numerical entries are precisely the probabilities of joint events. To solve for $P(A\backslash D)$ we can use the formula directly to get:

$P(A\backslash D) = \frac{P(AD)}{P(D)} = \frac{.2}{.5} = .4$. Or we can legitimately resort to arm waving: the event A makes up .2 out of the event D which has a probability of .5 and that gives A a probability of .2 out of .5, or just .4. Not only do probability matrices give the simplest sort of problems, but they tend to demonstrate the difficulty the students have with the concepts. No matter how simple such concepts seem to the teacher, they are difficult to digest for the students.

In the probability matrix above, the only events that are independent are A and C, and B and C. I stress independence is a matter of *information:* two events are independent if the occurrence of one event does not effect the probability of the other event, that is it yields no information about the other. This in fact is a literal restatement of parts two and three of the definition of independence. I find that I have to stress repeatedly that the three statements are equivalent and I have to stress precisely what that means. Events A and B above are dependent precisely because they are disjoint, if one occurs, then the other can not occur. We can verify this by recourse to the definition. Yet, no matter how many times one repeats this type of example, many students will cling to the idea of disjoint events being independent events. To them, *disjoint* seems to be *independent*. An example that I have found to get a lot of student response is by using the experiment above of throwing a pair of fair dice. The example is that the events S7 and R4 are independent but the event S6 and R4 are dependent. I have found this example to generate a lot of interest and questions.

A further example that I like can be found in Hamming [2 p. 23]. The problem of the gold coins is as follows: There are three drawers each containing two coins. One contains two gold coins; one contains two silver coins; and one contains a gold and a silver coin. Having picked a

There are so many ideas inherent in the above material that by covering it all in a short time, the student can wind up absorbing none of it.

drawer and a coin at random (with equal probability) and found the coin to be gold, what is the probability that the other coin in that drawer is gold? What is wonderful about this problem is that it is counterintuitive and can be solved by a simple application of the definition of condition probability. Let G_i and S_i ($i = 1,2$) be the events that the first or second coin is gold or silver. Our question is to find P(G2|G1). By definition:

$$P(G_2 \backslash G_1) = \frac{P(G_1 G_2)}{P(G_1)} = \frac{\frac{1}{3}}{\frac{1}{2}}$$

Given the difficulty of basic concepts in probability, I find it hard to understand why a beginning class (at the sophomore level) would be given Bayes' rule. It is irrelevant to basic statistics and it is notationally and conceptually more difficult than the concepts discussed so far. I have in the past taught decision theory to students that did not have a background in probability and were not particularly mathematical. In that case the difficulty is the same, but Bayes' rule is necessary. Rather than formally stating Bayes' rule, I have taught them to solve Bayesian problems using probability trees. I will not go into this further since it is not relevant to my main thrust. Bayesian problems can be rendered so simple through probability trees that virtually any student can master the technique.

Discrete probability frequently comes down to a matter of problems in counting. Needless to say, few of the students will have any background here. One topic that we might find useful here is the binomial coefficient. It lends itself to many counting problems such as probabilities in poker. Since it is also required for the binomial distribution, we might as well introduce it during discrete probability and just before the binomial distribution. Normally we define the binomial coefficient *n choose m* as $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

This definition is easy to motivate by beginning with the number of permutations of m out of n items. It needs to be stressed that the above definition is not a good computational definition. We normally calculate binomial coefficients by the usual device of judicious cancelling. This is in fact roughly the way we would do the computation by computer. To compute n choose m, we replace m by n - m if n - m is smaller than m. We then use: $\binom{n}{0} = 1$; *else* $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$ We are now prepared to go into the binomial and related distributions.

The Central Limit Theorem

The Central Limit Theorem (CLT) is not only one of the most elegant and surprising results in probability theory, but it is the foundation of much of statistics. As a result, many textbooks and teachers give CLT extensive attention. Furthermore there are physical as well as computer aids to demonstrate CLT. I myself used to use spreadsheets. I would have a spreadsheet column containing 200 entries each of which would be a sum of uniform (pseudo) random numbers. I would use the spreadsheet's built in frequency tool and graphing capabilities to graph the frequency distribution of the 200 sums. This would be approximately normally distributed.

It is not surprising that many instructors will devote an entire lecture to CLT. It is a subject many of us love, and any good instructor is likely to emphasize its beauty and importance. Here we touch upon the core problem with Stat 200, and the theme of this essay. A month after the course is over, how many of the students can state or paraphrase CLT? How many even remember it, if reminded of its content? If you say 10% or more, you are either a truly great teacher or you have great students.

Further Comments on Probability

If I have spent too much time on probability, consider that in Statistics 200 we are expected to give all of this information in three weeks or less (and there are topics I left out). There are so many ideas inherent in the above material that by covering it all in a short time, the student can wind up absorbing none of it. For example most people take a while to learn to calculate probabilities with binomial coefficients. Understanding the theory is one thing but there is an art to calculating discrete probabilities. In the recent novel *House of Cards* by Conall Ryan [3] (who is himself a computer scientist) the main character calculates that there are 41 hands that will beat four Kings in five-card poker. We can count these ourselves: there are forty straight-flushes and one hand of four aces. However, there are in actuality forty-eight hands of four aces, reflecting the forty-eight possibilities for the fifth card. Later he calculates that there are forty-three hands that will beat four jacks: the same mistake. In fact this is still not quite correct since we have not considered how the cards showing effect the counting.

The Vos Savant Affair

Discrete probability is somewhat less theoretical than continuous probability but can be quite a bit trickier. Any person who puts toc much faith in their probabilistic intuition is probably a fool. There is an abundance of counter-intuitive probability problems that stump nearly everyone the first time they see them. Such a problem is the Vos Savant teaser. Marilyn Vos Savant (who is reportedly listed in the book of Guiness records as having the world's highest I.Q.) has a column that appears in Parade magazine that comes with many Sunday papers. In one issue [4] a problem occurred that achieved a great deal of controversy and led to subsequent columns and then attention in academic journals [5]. The problem is, like all good counter intuitive teasers, quite simple. A game show consists of choosing which of three curtains hides a great prize as opposed to the other two curtains which will conceal turkey prizes. The contestant having chosen one curtain, the game show host reveals one of the other two curtains to conceal nothing and then offers the contestant the chance to choose the remaining curtain. The question is, should the contestant switch? Vos Savant gave the correct answer, which is yes. Typical reasoning says there is no advantage in switching, that there are two remaining curtains and the chances are fifty-fifty. Vos Savant's reasoning is that the initial probabilities were one-third to two-thirds and that nothing has changed. The host used his knowledge to pick a curtain that would be empty and the probabilities are

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unchanged, that is, the originally chosen curtain has probability of one-third to contain the great prize, and the remaining curtain has probability of two-thirds. The problem also lends itself to simple simulation which can even be carried out as a mental experiment. (This is exactly how Hamming solves his problem I quoted above.)

After publishing the problem and solution, Vos Savant received a quantity of condemning letters from scientists many of whom felt it necessary to reveal that they had Ph.D.'s (and therefore couldn't be wrong) and that she was apparently stupid to make such an idiotic mistake.

I find this whole episode to be disturbing. The Vos Savant problem is not a particularly counter-intuitive problem and it has one element that should given any problem solver pause; the game show host possesses and apparently uses information, specifically he knows which curtain has the prize and which is empty.

Another disturbing point is the number of *authorities* who were arrogant and the number who even failed to consider the problem worth thought. Whether you agree with Vos Savant or not, the problem always deserved thought. Her solution can still be quibbled with, for example see [5] or [6]. I myself am disturbed by the apparent manner that so-called experts present themselves to others. I think they reflect badly on the entire mathematical community. They hold up their Ph.D.'s as talismans: *I have great magic and the mathematical spirit speaks through me and if you disagree with me you are damned.*

Probability is clearly not an easy subject to assimilate for anyone. Yet in Statistics 200 we expect students who have little training and who often are not mathematically inclined to absorb the great body of the subject in three weeks.

Statistics

So far we have looked at the probability that usually comprises the first two to four weeks of Stat 200. Again, I am not saying that it *should* comprise the beginning of the course. I think maybe we should consider an elementary course in probability (and probabilistic reasoning) as a prerequisite for the elementary course in statistics. But for the time being most of us are constrained by our current curriculum to provide both topics in one course.

Our first problem is that the students generally have no idea what statistics is. They are taking the course because it is required. Oddly enough we tend to consider a statistics course successful if and only if the students become somewhat adept at applying statistical formulae. However, this ability does not imply any real appreciation of statistic's role in society and in science. Furthermore such an appreciation can be given without learning any formulas. A book that does this well is *Statistics: A guide to the Unknown* [7]. I believe that this book, or some equivalent book, should be required in every introductory course on statistics.

Now, I give you the prime exhibit of this essay, its raison d'etre. We are going to motivate the formula for a confidence interval. To make things easier we will assume a sample from a normal population with known standard deviation; hence we make no appeal to the Central Limit Theorem nor do we have to use the **t** distribution. The population consists of independently distributed observations from a normal population with known mean and unknown standard deviation s. Our sample is randomly drawn and of size **n**. We use the fact that the sample means (for samples of size n) are also normally distributed with the same mean and with standard deviation $\frac{s}{\sqrt{n}}$. Using the fact that a normal distribution contains 95% of its population within 1.96 standard deviations of the mean we have:

$$P\left(m-1.96\frac{s}{\sqrt{n}} < \bar{x} < m+1.96\frac{s}{\sqrt{n}}\right) = .95$$

. Subtracting m from all three terms inside the parentheses we get:

$$P\left(-1.96\frac{s}{\sqrt{n}} < \bar{x}-m < +1.96\frac{s}{\sqrt{n}}\right) = .95$$

.

Now by subtracting $\bar{x}$ from each term, multiplying by minus one and reversing the order, we get:

$$P\left(\bar{x}-1.96\frac{s}{\sqrt{n}} < m < \bar{x}+1.96\frac{s}{\sqrt{n}} = .95 .\right)$$

The last equation is of course the confidence interval itself. This is frequently the first confidence interval in Statistics 200 because it is the simplest. Putting it slightly vulgar, it is lousy. Notice that we have not used any of that preceding mass of probability that makes up the first part of Statistics 200. We have used the fact the a sum of i.i.d. normal variables is normal. We have used the fact that the variance of a sum of independent random variables is the sum of the variances. We have used that the sample means of a normal distribution are normally distributed with

By trying to teach too much, we frequently end up teaching too little.

the same mean as the underlying population and with the same variance divided by the sample size. We may view the last fact as equivalent to the prior, but to the student, it is a big step. The concept of a random variable that is itself the sum of random variables is abstract to the student. The concept of a distribution of sample means is even more abstract to the student. (Note that students will use random variables that are sums of other variables with ease; this does not mean that they can explicitly handle the concept.) We have also used the concept of the standard normal distribution and we have used the normal table. Lastly we have used equalities of the form: $P(x<c) = P(ax + d < ac + d), a<0$. This is a revelation to the student. I myself have tended to use the grisly rationale that goes more or less as follows: *Let us suppose that the mean height in the NBA is 6'10". Then if we pick a player, X at random, P(X>6'10") = .5. Now suppose that we amputate twelve inches of leg from each player. Isn't it clear that P(X-12">6'10"-12") = P(X-12">5'10") = .5?*

However we motivate and teach the confidence interval above, it entails a lot of new ideas for the student and uses little of the previously introduced ideas. Normally, this is where we lose any remaining students. The student proceeds through the rest of Statistics 200 trying to understand how to use the formulas and to survive the course. Again the typical course will include both ANOVA and linear regression. There can be no attempt by the student to understand any underlying theory or why the topics are important. The reason is simple: Statistics 200 goes too fast and has too much content. The result is that six weeks after the course is over, the vast majority of the students remember little if anything of the course content.

The Source of the Difficulty

There are many reasons for a course like statistics 200 which is required of so many students but teaches very little to most of them. However, I want to concentrate on one problem. All of us that teach in the mathematical sciences tend to forget how many concepts are actually involved in a course. With time, the substance of any course that we master seems to diminish into a nice tidy package. It seems reasonable that we can teach the material, to capable students, in very short time, and when the students start to choke on the material it is easy to conclude that the difficulty is that the students are not capable.

The Statistics Specific Solutions

Since I have used Statistics 200 as my prime exhibit of a course that includes too much and teaches too little, I will begin by discussing solutions to that specific case. An obvious solution is to split the material into two courses: Probability 200 and Statistics 200. On the other hand, what if we tried teaching the basic non-mathematical statistics course without much probability? Could we do it properly? I say yes. I think a text that would serve for such a course and which shows how to do it is *Statistics for Research* by Dowdy and Wearden [8]. The focus of a statistics without probability would be data collection, data analysis, and experimental design. However, often when we have statistics 200 as a prerequisite, it is for the probability. For example, courses in operations research generally have Statistics 200 as a prerequisite but for its probability content not its statistics content--and teachers generally find that their students have to be retaught the probability from the beginning. We could instead consider teaching probability without statistics. This would give more time for probability concepts to sink in and would provide an excellent foundation for statistics. For some reason there are very few sophomore level courses in probability.

Another solution to the statistics problem is to teach a data analysis course based upon the new computer intensive methods. These methods are conceptually much easier than the established techniques that we generally teach. They enable the student to learn techniques and not lose sight of the underlying problem. In fact, Sir Ronald Fisher may have been thinking along these lines (see the second chapter of Box, Hunter, and Hunter [9] which is, I am sure, derived from Fisher). Techniques such as ANOVA were developed partly because computer intensive methods were not feasible. Teaching an elementary course based on these techniques is the subject of [10] and [11]. A good readable introduction to these methods is given by Noreen [12].

A Last Irrelevant Remark on Statistics Education

I can't help but add one comment on statistics education. It is generally the case that when we learn statistics at a deeper level than Statistics 200 that we study *mathematical statistics*. Here our methods are calculus based and we spend much of our time doing and studying analysis. It is quite common for courses at the first year graduate level to be measure theoretic. What is remarkable to all this is that when it comes to doing statistics and doing statistical design, most of that is irrelevant. Furthermore, it is not necessary even for an appreciation of theory and fundamentals. The just mentioned book by Box, Hunter, and Hunter [9] is superb in substance and theory and does not rely on calculus. The book by Snedecor and Cochran [13] is a magnificent volume of statistical methods and makes no use of calculus. It must be admitted that both books do require substantial mathematical maturity.

The General Solution

I have stated the general problem throughout this essay: *By trying to teach too much, we frequently end up teaching too little*. The solution is obvious: sometimes at least, *Less is More*. We can obviously teach too little, and we can obviously fail to challenge the students, but if we go beyond their capacity to keep up, their tendency is to stop learning at all, and concentrate on just passing the tests and surviving the course. What little they do learn goes into short term memory and is lost. How do we decide how much material is appropriate? I know no easy solution to that problem. However, there are far too many teachers in the mathematical sciences who do not recognize the problem.

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in fin ity

How can any thing
Made up of no thing
Amount to some thing?

(A whole is the limit of
the sum of its parts)

And infinite sum of the little bits
each bit a little bit less
until it is less that the least little bit
you can imagine
and yet still less
but never zero
never nothing
always something

imagime something
smaller than the smallest
you can tell me how small
I'll tell you how close
and I'll get even closer
and then get even
closer
How close til I'm actually there?

Do we jump
or glide
from here to there
form now to then?

Is time a sequence of moments?
A stream a collection of drops?
What is it waves
in the ocean?
in the boundless sea
of infinity
where things happen
that don't

April, 1992
Bonnie Shulman
Univesity of Colorado

A Fairy Tale:
Being
A Pseudo-History of Mathematics With Special Attention Given to The Evolution of the Number System

which humanistic professors of mathematics tell their students under the illusion that this will turn them into humanists

Peter Flusser
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In the beginning the Mother of Mathematicians (MOM) created sets. But sets were without form and void, and darkness lay upon the face of the Universe of Discourse. And MOM said: "Let there be numbers" and there were natural numbers. For MOM created the natural numbers: all else is the work of MAN. But the natural numbers were without order and harmony. So MOM created the music of the spheres, but nobody could hear it. And MOM saw that all was good. And that was the evening and the morning of the new age of mathematics.

And Peano opened his mouth and said: "Let the Induction Axiom be postulated and let it be a BURDEN AND IMPEDIMENT TO LEARNING FOR MATHEMATICS STUDENTS who shall neither recall nor understand it."

And this was the first BURDEN AND IMPEDIMENT. And MOM saw that it was good for the induction axiom caused much weeping and gnashing of teeth among mathematics students.

And Peano defined addition and multiplication inductively on the natural numbers, and the natural numbers could always be added and multiplied, but they could only be subtracted and divided some of the time, but not all of the time. So MOM invented equivalence classes of pairs of integers. And she called them the set of rational numbers, but they were really fractions. That was the second BURDEN AND IMPEDIMENT for nobody likes fractions.

And MOM saw that the set of rationals was good, for it was the smallest ordered field. But it was big enough for all practical purposes. And MOM opened her mouth and said to all peoples: "All numbers ye shall add and subtract and multiply, and by all numbers ye shall divide but by zero ye shall not divide. And this commandment I give unto you, and it shall be a commandment unto you and unto your posterity, even unto the last generation. And it shall be called the Eleventh Commandment: 'Thou shalt not divide by zero!' For whosoever shall divide by zero shall eat of the Fruit of the Tree of Infinity and of the Worm of Indeterminacy that liveth in the Fruit of the Tree of Infinity. And of that Worm and of that Fruit ye shall not eat lest errors and inconsistencies invade your work and ye shall be scorned and derided by your colleagues and successors forever." And that was the third BURDEN AND IMPEDIMENT, and MOM saw that it was good.

And it came to pass in those days that all ancient mathematicians lived contentedly somewhere in Greece, unless they lived in Egypt or Babylon. And they added and subtracted and multiplied and divided, but by zero they did not divide because the Greeks did not have zero. But the Babylonians did perhaps have zero, but they did not divide by zero because the Babylonians did not divide: they multiplied by reciprocals. and this was indeed a sign and a mark of distinction between Babylonians and Amoebas, for whereas Amoebas multiply by dividing, Babylonians divided by multiplying. Now the Babylonians might have invented zero, or they might have obtained it from the Indians or the Chinese, for there lived ancient mathematicians in ancient India and in ancient China; but in modern Western books ancient Chinese and ancient Indian mathematicians do not live; except that the ancient Chinese Remainder Theorem, and the ancient Indian mathematicians are credited with inventing the modern Arabic numerals.

At this time there lived in Crotona, on the toe of the boot of Italy, a set of Greek mathematicians known as Pythagoreans, who made Music and Mathematics as MOM commanded. The Pythagoreans saw music in mathematics and mathematics in music. For when the strings of their lyres were as the ratios of small numbers the sounds produced were becoming and harmonious, but if the rations were not so, then the sounds came forth that were harsh and grating. When the Pythagoreans saw these things there saw that numbers were very good and they opened their mouths and said: "All is number!"

And it came to pass in those days that MOM said unto Pythagoras, the Lord of the Pythagoreans: "Come up to me unto the mount and be there: and I will give thee a tablet of clay, which the Babylonians have written; that thou mayest teach it." And Pythagoras rose up and girded his loins and went up into the mount and MOM delivered the tablet unto him. And lo! there were inscribed upon the tablet fifteen sets of Pythagorean triples. And the name of the tablet was Plimpton 322. And when Pythagoras saw the tablet he rejoiced exceedingly with a great joy, and opening his mouth he said: "This shall be known as the Theorem of Pythagoras, for verily, verily I say to you: Unless you know that $a^2+b^2=c^2$ you cannot enter Plato's academy." And this was not a BURDEN AND IMPEDIMENT, and so MOM's anger waxed exceedingly hot.

And so it came to pass that there lived in Metapontum a Pythagorean Blabbermouth, and he came to Pythagoras and said: "Master, thy theorem implies that the square root of two is not a number. Neither is it a ratio of numbers. It is a surd." And Pythagoras' anger waxed hot and he cast Plimpton 322 out of his hands and he brake it upon the ground. For this was indeed the fourth BURDEN AND IMPEDIMENT. And then Pythagoras opened up his mouth and said unto the Blabbermouth: "Verily thou hast said it. But proclaim not this message to the people lest thou leadeth them into confusion, for they think that a surd is absurd."

But MOM darkened the Blabbermouth's mind, and he kept opening his blabber mouth and blabbered to one and all. And thus he confused the people. So MOM made a big fish, and the big fish came to the Blabbermouth and opened its mouth and said "Gulp!" and swallowed the Blabbermouth. And MOM was well pleased.

And then there came Theaetetus and Democritus and they fixed up the number system and called it **R** because it was real even though Plato and all Platonist mathematicians, in other words, almost all mathematicians, that is all mathematicians except for a set of measure zero, said that **R** is a mental concept and hence ideal and not real. And Euclid wrote it all down in Book X of his "Elements", and Book X was the fifth BURDEN AND IMPEDIMENT for nobody could read it. And MOM saw that this was good.

But the Romans did not like BURDEN AND IMPEDIMENTS. So they conquered the Greeks and there fell upon the earth a thousand years of darkness when there was no light to do mathematics by. The old mathematics was forgotten, and Euclid I.5 was called the PONS ASINORUM, the bridge of asses, because you could not cross that bridge and understand Euclid I.5 while sitting on your donkey.

But the Arabs had light in those days, for they invented Algebra and brought Arabic numerals to the Darkened West. And there came Omar Khayyam, and he opened up his mouth and said:

"A book of verses underneath the sky,
A loaf of bread, a jug of wine, and I
Can solve the cubic, if I try!"

And Omar, the poet and algebraist, solved the cubic geometrically.

But Cardano, the sooth-sayer, Tartaglia, the stutterer, and other Italian geometers solved the cubic algebraically. For after thousand years of darkness there came the light of the Renaissance and all men could see what the Ancients had done. And the light brought Fermat and he discovered the Last Theorem. And when MOM saw that Fermat may have proved the Last Theorem she opened her mouth and said: "Truly if Fermat proveth the Last Theorem then woe be unto me and unto all mathematicians for there shall be naught left for us to do and we shall suffer death from boredom."

So MOM sent an angel to shrink the margins of Diophantus' book so that Fermat could not write the proof there. And she sent guardian angels with flaming swords to guard the proof from all generations even unto this day so that no one might rediscover it. And Fermat write in the margin of Diophantus' book: "Behold! I have discovered something the ancients did not know: the margins of Diophantus' book are too small to write in."

And then came Descartes, and he stayed in bed even in the light of the Renaissance, for Descartes, the soldier of fortune, was weak and infirm. And Descartes opened his mouth and said: "Cogito ergo sum!" And then he got out of bed to teach Queen Christina Analytic Geometry and caught cold and died. And then Newton stood on the shoulders of giants and played with pebbles on the shore of the unexplored ocean and discovered the Calculus, and then Leibniz invented the Calculus and Newton and Leibniz fought over who discovered this, and who invented that, and who copied from whom, and what it all means, and Newton lost the argument and therefore British mathematicians did not know how to use Leibniz's notation. But Euler did know how to use that notation, and he discovered lots of mathematics and he made lots of mistakes and so Cauchy, Bolzano and Weierstrass opened their mouths and said: "Mathematics is like a house without a foundation built upon sand, against which series and infinitesimals beat vehemently, and immediately it falls and the ruins of that house are great. So let us dig deep and lay a foundation for mathematics on the rock of the real numbers: then series and infinitesimals cannot shake it."

And there came a Wise but Unknown Mathematician and he opened his mouth and said: "The real numbers are infinite decimals." And that made sense, and it was not a BURDEN AND IMPEDIMENT and students did not weep nor did they gnash their teeth. And MOM saw that this was not good. So MOM made Cantor and he opened his mouth and said: "Real numbers are equivalence classes of Cauchy sequences." And MOM also made Dedekind who, opened his mouth, said: "The real numbers are Dedekind Cuts." And there came Shakespeare and he said unto Dedekind: "That's the most unkindest cut of all." For this was the sixth BURDEN AND IMPEDIMENT. And MOM saw that this was good, for now mathematics students came unto MOM and said unto her: "What is a real number, really?"

And MOM said: "The real numbers are a complete ordered field, and all complete ordered fields are order-isomorphic so that there is only one complete ordered field; Cantor and Dedekind and the Wise Mathematician notwithstanding. And that is good." For this was the seventh BURDEN AND IMPEDIMENT. And it should have been the last BURDEN AND IMPEDIMENT, for on the seventh BURDEN AND IMPEDIMENT, MOM wanted to rest.

Yea verily, this should be the end of the story except for Gauss, who discovered the complex numbers, which are ordered pairs of real numbers and which Gauss discovered before the real numbers were invented. Only Wessel and Argand invented the complex numbers before Gauss discovered them, which is strange because usually Gauss discovered things before other people invented them. But then Gauss discovered non-Euclidean geometry before Bolyai and Lobachevski invented it, and that was important but also sad, because before the discovery of non-Euclidean Geometry mathematicians had been SEEKERS OF TRUTH, but after this invention they became hewers of wood and DRAWERS OF NECESSARY CONCLUSIONS FROM ARBITRARY ASSUMPTIONS and therefore FORMAL MANIPULATORS OF MEANINGLESS SYMBOLS.

And then came Gödel, and he opened his mouth and said: "You cannot proved that the axioms of a complete ordered field are complete or consistent or categorical. So the foundation of mathematics is indeed built upon sand." And this was a BURDEN AND IMPEDIMENT FOR ALL MATHEMATICIANS. And when Hilbert heard of this BURDEN AND IMPEDIMENT he tore his clothes, covered his head with ashes, wrapped himself in sack-cloth and retired from mathematics. And this was not good. So MOM transformed all mathematicians into humanists who can overcome most BURDEN AND IMPEDIMENTS. And this was very good, and so everyone lived happily ever after.

In Memoriam:
Morris Kline
1908-1992

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