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Essays, book reviews, syllabi, and letters are welcomed. Your essay should have a title, your name and address, and a brief summary of content. In addition, your telephone number and e-mail address (not for publication) would be helpful.

If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please also put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text; these should be original photographs, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of the paper, double-spaced and preferably laser-printed, should be sent to:

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Note to Librarians

Humanistic Mathematics Network Journal issues #8–#14, ISSN#1065-8297, are the successors to the *Humanistic Mathematics Network Newsletter*, ISSN#1047-627X.

Cover

This is an illustration taken from D. Huylebrouk's article "Puzzles, Patterns, and Drums", in which a drum player from Burundi performs on an ornately decorated drum. Both the geometric nature of the decorations and the mathematical significance of the music played on the drum are explored in Huylebrouk's article.

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Humanistic Mathematics Network Journal #14
November 1996

From the Editor

I want to welcome and introduce the new Production Manager, Matthew Fluet, who was Assistant Production Manager last year. Matthew is a sophomore at Harvey Mudd College and a very capable production manager, which is evident from the appearance of the printed journal and the graphics. We are very fortunate to have Matthew on the job.

Our friend and colleague, Thomas Tymoczko of Smith College died last August after a brief battle with stomach cancer. Many will know Tom from his two essays in *Essays in Humaninstic Mathematics* (MAA Notes #32), "Humanistic and Utilitarian Aspects of Mathematics," and "Value Judgement in Mathematics: Can We Treat Mathematics as an Art?" Two of his books were *New Directions in the Philosophy of Mathematics* and *Sweet Reason: A Field Guide to Modern Logic* (with Jim Henle). His paper "The Four-Color Problem and its Philosophical Significance" argued that the increasing use of computers was changing the nature of mathematical proof.

Tom was one of the thirty who gathered at the 1986 conference in Claremont to consider whether mathematics is a humanistic discipline, and launched the HMNJ.

He was a student of Hilary Putman at Harvard and is survived by his wife and three children.

He will be greatly missed.

What's all the Fuss about?

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The proposal to hold this conference says that, "the teaching of calculus is in a state of disarray and near crisis . . . [with a] failure rate of nearly half at many colleges and universities." An alarm was sounded earlier by the January, 1985 AMS/MAA joint panel, "Calculus instruction, crucial but ailing" [1].

This came as a surprise to me. Why is the teaching only of calculus under scrutiny? Are we doing such a wonderful job with discrete mathematics, linear algebra, differential equations, complex variables, or upper division algebra? Perplexed, I asked some of my colleagues, good mathematicians and fine teachers all, "What's your impression of the teaching of calculus, here and elsewhere?" One professor suggested that we might drop a couple of topics, maybe some integration techniques. Another said, we should meet five times a week instead of four but he doesn't want to. Finding no sense of calamity, I talked to colleagues in the physics and engineering departments. They liked what we do, but urged us to do more of it in the first quarter, especially differentials, vectors, $e^{i\theta}$, Stokes' theorem, and certain differential equations.

Then I went to the placement office, which helps undergraduates obtain summer internships and seniors get jobs. "What have you heard about calculus?" They were not aware that calculus is in disarray and ailing. I asked what employers were looking for. The answer was clear, "Students who can communicate orally and in writing, think, are not afraid of numbers, with a little touch of the computer." Still no complaint about calculus.

I asked my engineer son-in-law what he looks for when he recruits. His answer: "People who can deal with questions on their own." He seeks recommendations from a professor who regularly assigns his class a few open-ended problems. Though not hard in the sense that their solutions requires the insight of a genius, they are not directly related to the day's lesson. One year not one of the professor's three hundred students could solve his problems. My son-in-law did not blame calculus for this tragedy, though it was clear to me that we do little to prevent it.

So I picked up a calculus market research report that McGraw-Hill had done in 1981, based on a questionnaire sent to mathematics professors in over 200 colleges and universities of all sizes. According to the poll, 83 percent of the students in first semester calculus complete the three semester sequence. That was reassuring. Furthermore, if there was a feeling that something was wrong it should show up in the respondents' comments on the texts they were using. But of the 227 replies 170 judged their text's completeness to be "good" or "excellent" and only 47 called it "poor" or "adequate." They seemed quite satisfied with "topic sequence as well" with 173 out of 227 calling it "good" or "excellent."

In spite of these calming numbers, I still felt that there is indeed something in disarray in calculus teaching, something ailing. Whatever it is, we can't blame the publishers. The books they offer us respond to such polls; the manuscripts are read by a panel of independent, conscientious reviewers. We get the texts we ask for. The problem lies with us. Mathematics, the only discipline where all the cards can be laid on the table, and which therefore should be the best taught, is often among the worst taught subjects. One reason is that we haven't decided what we are teaching.

This uncertainty is visible, in the discussion, *The Introductory Mathematics Curriculum*, presented in [1]. There we find such statements as, "We must instead teach how to create mathematics" (R.W. Hamming, p. 388); "Even more essential is the creation of courses that focus on concepts. Ideas and problem solving are the really critical part" (Robert Davis, p. 391); "Our teaching fails to provide students with the Joy of using mathematics to cope with challenging problems" (Wade Ellis, p. 393); "The main fault of the introductory curriculum . . . is an issue of pedagogy as much as of the content" (Patrick Thompson, p. 394); "Curriculum change must be accompanied by severe questions of current teaching methods" (John Mason, p. 395). Though appearing as asides to the main debate, they call attention to what I feel is the central issue.

Before we propose the medicine, we had better agree on the diagnosis. The diagnosis depends on what we mean by "health," that is, what we are trying to accomplish in our introductory courses. That may depend to some extent on whether the course serves other majors or our own. (According to the McGraw-Hill poll, enrollment in the basic calculus runs about 60% physical science-engineering, 20 percent life science-biology-economics, 12 percent math, and 8 percent others.) In large schools the second group often has its own calculus sequence; at Davis, with its strong biological emphasis, more students enroll in the short calculus than in the engineering sequence. So the main calculus sequence we are talking about serves simultaneously engineers, physicists, computer scientists, and math majors. That is a boundary condition that any solution must satisfy. But it is not as restrictive as it may appear, since there seems to be a consensus that the students in these varied majors should learn to write, read, and think. The dean of computer scientists, E. W. Dijkstra, has written that the most important requirement for a computer scientist is mastery of his native tongue. And my computer-science colleagues urge us to expect well written answers and proofs in our sophomore course on sets, relations, functions, and induction.

But what about calculus, where the texts have settled into a fairly uniform table of contents? There are always a few sections that the instructor may delete, such as Kepler's laws or Lagrange multipliers. But the instructor could consider deleting some more topics, such as some formal integration techniques or even related rates. Authors have less choice, for if they omit someone's favorite topic, their books will not be adopted and soon will be out of print. After all, calculus committees meet in order to reject books, much in the same way that canneries sort tomatoes. Labelling a section "optional" will surely offend someone who feels his students will then not treat it seriously if he covers it. It seems that a calculus author has the freedom to make only two decisions: Where to put analytic geometry and whether the title should be *Calculus with Analytic Geometry* or *Calculus and Analytic Geometry*. Thus the major revolution in calculus texts in the last decade has been the introduction of a second color. (In high school texts, the number of colors has reached four.) Whatever proposals this conference may make, I predict calculus will begin with functions, limits, derivatives, extrema, integrals, the fundamental theorem, go on to more applications, series, and then reach at least partial derivatives and multiple integrals. Still, there are options, and perhaps this conference will encourage publishers and professors to be more flexible when developing a table of contents or a course syllabus.

The fundamental question is not, "Should discrete mathematics precede calculus, follow it, be woven into it, or be separate and simultaneous." The question should be, "What are we trying to do in calculus and discrete mathematic courses other than cover some definitions, facts, and algorithms?" If the answer is "nothing", then we make no basic changes. If we also want the student to learn to "think" (this is now called 'problem solving' and 'heuristics') and to write, then we should act accordingly. The last thing we should do is ask for texts that mix discrete mathematics and calculus, for invariably, when two subjects are put between the covers of one book either the book grows unacceptably large or one of the two is sacrificed to the other, or both are shortchanged. Witness the fate of analytic geometry in our calculus books or of both algebra and its applications in our applied algebra books.

My own proposals may appear mild. Indeed, the first one is, but the second could encourage a change in emphasis.

The first is specific, and concerns calculus and discrete mathematics. I suggest that a discrete course of a quarter or semester be available to freshman (if that is successful, then iater it could be extended). It could be taken simultaneously with beginning calculus, or alone, or, in the case of non-engineering students, with the calculus delayed. Such a course could help develop maturity and thus prepare students for calculus. It could, incidentally, weed out those who are not ready to go on. (All campuses of the University of California already require passing an exam on high school algebra and trig for entry to calculus.) It would also broaden the student's mathematical perspective earlier.

My second suggestion applies to our curriculum in general and is a response to what I see as the disarray and the ailment. Implementing this suggestion does not require new courses, nor radically new texts. However, if enough of us act on this suggestion, we may provide the quorum to support certain changes in the texts.

It too is modest, for I find that proposals for abrupt major reform tend to be carried out in form but not in substance, or viewed as something for someone else to implement.

My suggestion is rooted in my definitions of the words "curriculum" and "syllabus." Usually, "curriculum" describes the courses offered and "syllabus" lists the topics in a course. Both "curriculum" and "syllabus" call attention to the material treated. They do not refer to the way it is treated and certainly they do not mention what should be our main goal: to develop the student's ability to read, analyze, write and speak. We easily lose sight of this objective, for facts tend to displace process. We see this bias both in the classroom and in texts. I hope that the reform suggested by this conference gives process at least equal billing with content. And I hope that authors maintain a similar perspective as they try to implement our recommendations.

My suggestion is only a modest step toward rescuing process from subservience to content.

I propose that in whatever course we teach we include a significant number of what might be called "open-ended" or "exploratory problems." Though not routine, they should not be difficult in the sense of a Putnam problem. I mean that when a student sees the solution, he will say "I should have gotten it." These problems should encourage experimentation and independent work. The answer should require the student to write coherent sentences. That means that the instructor or some other qualified person should read and evaluate what is turned in. He should demand suitable revision. The solution should not be in the solutions manual; it should not be closely tied to the particular section in the book that is being covered in class. The assignment should not be due the next day, so that the student will have time to mull it over.

Some examples will bring this proposal down to earth. To demonstrate my neutrality on the relative merits of calculus and discrete mathematics, I will choose some examples from both disciplines. I begin with examples that parallel the standard calculus.

Example 1: Let $f(x) = ax^2 + b$ be a polynomial of degree 2. Is there a polynomial g of degree 3 such that the two compositions, fog and gof, are equal?

Remarks: If the students have trouble, then you might suggest that they look at a specific $f(x)$. Little in their earlier education has suggested such a bold step. The computations involve nothing more than cubing a quadratic or squaring a cubic. The algebra is not mysterious and the final result is both elegant and surprising. Moreover, the student should be urged to write the solution with more than a string of equations. We have a right to expect an introduction and a conclusion. We should demand that a sentence begins with capital letter and ends with a period. The left margin should be straighter than the right margin. The student may complain that such request are inappropriate in a math course. But that same student may one day be writing software manuals and internal memoranda. For us to demand less is to shortchange our students.

Example 2: Are there continuous functions f such that $f(x + y) = f(x) + f(y)$ for all real numbers x and y?

Remarks: The student may or may not come up with some examples. You may have to steer him out of a rut. If he finds $f(x) = kx$, you might then ask, "Are there more?" (In a discrete course, the domain could be Z instead of R.) Of course one could also ask for solutions of $f(xy) = f(x) f(y)$.

Such exercises are usually delayed until the Junior year, but they are appropriate during the lower division courses as well. Perhaps we could delete a few topics from the standard curriculum, whether calculus or discrete mathematics, lowering the pressure so students would have more time for this type of problem.

Example 3: Let R be a bounded plane convex set. Is there a chord that bisects its area?

Remarks: For us this is a trivial exercise in the intermediate value theorem, but most students will need help. They cannot turn back a couple of pages for the example that's just like this exercise. After this problem is solved one might ask whether there is a chord that bisects the area and the perimeter at the same time.

Example 4: What happens to x^y when x and y are near 0 but positive?

Example 5: Which polynomials of degree at most 3 have inflection points?

Remarks: Much is lost in a more conventional wording, such as, "Show that every polynomial of degree 3 has an inflection point." One might then ask about polynomials of degree 5.

Example 6: Let f be an increasing positive function on the interval [0,1]. What, if anything, can we say about the centroid of the region R under the graph of f and above [0,1]?

Remarks: A variant is to demand that f also be differentiable and concave down and ask about the centroid of its graph. Or we could ask whether there is any relation between the centroid of R and the centroid of the solid of revolution obtained by revolving R around the x axis.

Example 7: Let R be a bounded plane convex set and P_0 a point in R. Assume that each chord of R through P_0 has length at most a. What can be said about the area of R?

Remarks: This question ultimately takes the student back to the formula for area in polar coordinates and extrema problems. For a discussion of this example see [2].

Now for some illustrations in discrete mathematics.

Example 8: You could compute x with five multiplications by writing $x^6 = x(x(x(x(xx))))$. But you could also write $x^6 = (x^2x^2)x^2$, which requires only three distinct multiplications. (Assume that once a multiplication is done, the result remains available.) Investigate the smallest number of multiplications needed to compute x^n.

Remarks: The exact formula is not known, though eventually students can show, with the aid of an induction, that the number is at least $\log_2 n$ and equals $\log_2 n$ when n is a power of 2.

Example 9: In which linear graphs can we find a path that passes through each edge exactly once?

Remarks: This is usually given in the "theorem and proof" form, but I think it far more instructive for the students to discover the result themselves. When I have raised the question in a liberal arts class, it isn't long before students observe that the vertices of odd degree give trouble and find the necessary condition quickly. Of course, sufficiency is harder to demonstrate.

Example 10: Let f be a permutation or a finite set. Is there necessarily a positive integer k such that f^k is the identity function of that set?

Remarks: The approach may depend on whether this is given before or after the cycle decomposition of a permutation. In the first case the student will be more likely to experiment. That means choosing some specific sets and functions, again a traumatic experience for students not used to such freedom and responsibility.

Example 11: In a finite graph is there anything that one can say about the number of vertices of even degree or about the number of vertices of odd degree?

Remarks: This exercise usually appears as a theorem. Too often we ask a question and then answer it before the student has had a chance to live with the question. By answering our own questions we turn the students into spectators, putting a barrier between them and the material. The temptation to do this is usually irresistable and is often justified by the "need to cover the syllabus." But what if the syllabus includes "teach students how to explore, to make conjectures, to write clearly?"

Example 12: Is there any relation between the number of vertices and the number of edges in a finite tree?

Remarks: The comments on Example 11 apply to this example as well. In both cases we can ask the students to prove their conjectures. There are several ways to justify both, including induction. These therefore serve as legitimate induction problems. The sooner we reduce the number of traditional induction problems like, "Show by induction that $1 + 2 + \ldots + n = n(n+1)(2n+1)/6$", the better. In a realistic induction problem, the student should propose the statement to be proved. (Recall Example 8.)

The next exercise gives students far more trouble than might be expected, both in carrying out their experiments and in explaining their conclusions.

Example 13: The function of $f: A \to B$ induces functions $F: P(A) \to P(B)$ and $G: P(B) \to P(A)$. For which f is
(a) F one-to-one?
(b) F onto?
(c) G one-to-one?
(d) G onto?

More examples discussed from a slightly different perspective are to be found in [2], but it is not hard to make up your own. Some can be derived from the statements of theorems. In some only an exploration and a conjecture are to be expected. In some a complete argument would be in order.

It may be easier to offer individual guidance and feedback in a smaller class than in a large one, but the organizational challenge in a large class should be negotiable. Though we might prefer to think our task done when we give a clear lecture, we may have to acknowledge that giving good feedback is equally important. Grading homework and examinations, which usually just offers the student the guidance of a number, is hardly adequate feedback. I suspect we, charmed by the clarity of our lectures, could go through an entire semester and never see a single page of a student's work. (I confess that this has happened with me.) It therefore may be necessary to give some time to see what the students write. It may be advisable to sacrifice content to achieve other goals.

My proposal is simply an attempt to respond to the concerns expressed by Hamming, Davis, Ellis, Thompson, and Mason that I cited. I want us to consider the goals of our teaching. Do they go beyond transmitting content? If not, we should say so in our catalogs and encourage others to introduce "problem-solving" courses to compensate for the narrowness of our mission.

If we want our students to be able to think on their own and to express their thoughts, we should give them a chance, even in the introductory curriculum, whether calculus or discrete mathematics, even in service courses even if we propose only two or three open-ended problems in a semester. If enough of us urge publishers to include an ample supply of such problems, with variations and solutions discussed only in the instructor's manual, they will comply. But we don't need to wait for them.

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The Triex: Explore, Extract, Explain

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Often as I walk into my classroom and run through what I will do, a recurrent image haunts me. I think of that cliche: Chinese must be an easy language since in China ,so many children can speak it. I know that if those same children had been born here and wereto study Chinese in high school or college, many would flunk out and give up. This image reminds me that students who fail with my approach might do very well under a radically different regimen.

For this reason I am open to alternatives to the standard lecture approach, where the focus is too much on the teacher. I would like to provide an environment in mathematics analogous to that which helps Chinese children learn Chinese: pernaps the Moore method is the way, or semester-long projects, or greater one-to-one contact. But with the boundary conditions that I face, I cannot turn to any of them.

Instead, over the past few years I have been gradually developing a modest alternative, which I will describe even though I am sure it has been often proposed and just as often been disregarded. I know that education is like death or love -- the important things that can be said about it have already been said. One need only browse through old volumes of the *Journal of the National Education Association* to be convinced of this. For instance, in its 1935 volume we find the report of an experiment in which pupils did not begin arithmetic until the sixth grade [1]. It turns out that within one year they caught up with pupils who had a three-year head start. If such a major discovery can vanish with scarcely a trace, I am sure that my proposal, couched in different terms, lies similarly abandoned in the archives.

My suggestion is "humanistic" in that I believe a humanistic education develops a student's ability to read, write, and analyze -- in short, to think. Of course, such an education should also develop an awareness of the origins of civilizations, East and West, and a love of classical music and literature, but I would be content if it just produced a critical self reliance.

However, no college catalog that I've read places reading, writing, or thinking at the core of its curriculum. Instead it lists courses and their topics, for instance, Linear Algebra: vector space, base, dimension, linear transformation, eigenvalue. The textbooks also reflect this inversion of emphasis. By the time we prepare our first lecture of the semester, most of us -- no matter how vehemently we have cried, "This is not a trade school," and, "Facts are secondary," -- lose sight of our real goal and rush to get through the syllabus. As surely as the debased coin displaces the good coin, routine problems drive out significant problems. By a significant problem I mean one that gives students a chance to explore on their own, to develop self confidence and independence, to carry on, so to speak, a miniature research project, and to write up their conclusions in complete sentences.

I am not going to propose a vast reform. I will describe only what I have been doing, timidly at first but recently more boldly. When I presented my ideas at one conference [3], the older participants said, "Old hat," but the younger ones asked such questions as: How do you make up this type of problem? How many problems do you give? How much time do you allow? Do they count in the grade? If so, how?

I propose that we offer our classes what I call "triex" problems; "triex" stands for, "Explore, Extract, Explain." Such problems do not begin with, "show that," "prove that," or "verify that." Instead, they begin with an opportunity for experimenting. The experiments should suggest a plausible conjecture to most students, even though at first glance the answer should not be at all evident. The conjecture should be easy enough for many of the students to prove. Even students who do not complete the third step of a triex are at least primed to appreciate the explanation when given by the instructor or by another student.

Such an exercise puts the emphasis on exploring, extracting, explaining, writing. Therefore the exercise need not relate directly to the course in which the student is enrolled. For instance, in the second or third semester of calculus, it may be drawn from the first semester. The focus is on process, not on fact; the center of responsibility moves from the teacher to the student. In [2] and [3] I presented several examples. Now I will describe three more in some detail to make the idea of the triex more concrete.

> **Example 1**: Let $y = f(x)$ be a nondecreasing function defined on $[0,1]$, with $f(0) = 0$ and $f(1) = 1$. Let R be the region below the graph of f and above the x axis. How low can the centroid of R be? How high? How far to the left? How far to the right?

Note first that the student cannot immediately guess the answers. However, there are accessible experiments, for instance, testing the curves $y = x^n$ or step functions. The first part, exploring, is not hard, though students must get used to accepting this responsibility. (If students get stuck, a hint may get them out of a rut.) The second stage is "extract." (It turns out that x is between 1/2 and 1 and that y is between 0 and 1/2.) The final step, "explain," involves only a symmetry argument. (As a follow-up one could ask whether the centroid of R lies in R.)

> **Example 2:** Diocles, in the year 190 B.C., in the book *On Burning Mirrors*, studied the reflecting property of a spherical surface that subtends an angle of 60°. When this surface is aimed at the sun, the rays of light arrive parallel to the axis, bounce off the inner surface of the sphere, and pass through the axis. How much of the axis is illuminated by the reflected rays?

The solution involves nothing more than trigonometry or, perhaps l'Hopital's rule (depending on how the problem is solved). As a follow-up, which requires the derivative, one could ask, "Describe the variation in the amount of light that strikes in the vicinity of each point of the illuminated part of the axis "

The next example is appropriate in an elementary discrete mathematics course, for it requires an induction or the use of binomial coefficients.

> **Example 3:** How many ways can you list the integers 1, 2, . . ., n such that each integer after the first one you list differs by 1 from an integer that you have already listed? (For n = 5, 32415 is one such list.)

This exercise satisfies the triex criteria: the answer is not immediately obvious; exploration through examples is feasible; the resulting conjecture is simple; the proof is not difficult and its write-up requires exposition, not just a string of equations.

I may require that a triex be turned in at the next meeting or perhaps in a week. If I am not satisfied with the solution or a student is stuck, I will comment on the paper and return it for further work. There may be a class discussion of Step 1 to catch errors which were interfering with Step 2. How often I assign such problems depends on the size of the class and the time I have to read the papers. These problems are separate from the regular homework which is read by an undergraduate.

Often a standard exercise can easily be reworded to become a triex problem. Consider the exercise, "Show that for any odd integer n, the number $n^2 - 1$ is a multiple of 8." As it stands, Steps 1 and 2 of a triex are missing. Such an exercise minimizes the involvement and responsibility of the student. It alienates by insinuating that mathematics is discovered by an elite and is merely checked by the masses. However, that same problem, rephrased, easily turns into this triex: "What is the largest fixed integer that divides $n^2 - 1$ for all odd integers n?" Clearly the three steps are now present.

Even the simple exercise, "Prove that $x + 1$ divides $x^n + 1$ for every positive odd integer n," can be transformed to a triex, namely, "For which positive integers n does $x + 1$ divide $x^n + 1$?" This triex, in turn, generalizes to, "For which positive integers m and n does $x^m + 1$ divide $x^n + 1$?" and to, "For which positive integers m and n does $x^m - 1$ divide $x^n - 1$?"

A triex creates the environment of a miniature research problem, whether applied or theoretical. It puts more responsibility on the student. I expect (but have not tried to prove) that it develops self reliance and self esteem. It certainly exploits a key feature of mathematics, which such disciplines as physics and history lack: all the cards can be laid on the table -- the student need not depend on facts transmitted by an authority. The use of the triex may reduce the alienation and passivity which develop through years spent on plug-in problems.

The triex is one of my responses to that image of little children speaking fluent Chinese. Through it I try to place process above fact. I suggest that more teachers try a few triexes in class in order to become familiar with them and their implications.

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Noesis

Lee Goldstein

Emication of thought is not love,
Because it has no exteriority;
Yet whatever is muted willfully
Has a countenance;
Nay, the autoptic ----- relativistic range of things
Can be a beauteous species,
If thought can be transmuted,
Even as in a mirror,
By law, and homologically into strings.

Puzzles, Patterns, Drums: the Dawn of Mathematics in Rwanda and Burundi.

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1. INTRODUCTION.

Douglas R. Hofstadter received the 1980 Pulitzer Prize for his book *Gödel, Escher, Bach: An Eternal Golden Braid.* It wedded the mathematical results of Kurt Gödel, the graphical art of Escher and the music composed by Bach. Hofstadter showed how a common idea seemed to emerge in three different modes of expression, easing access to the more arduous mathematical part by suggesting the reader to solve a Gödel problem (see [Hof] and [Swa]). Drawings by Escher were alternated by excerpts from Bach's score and dialogues between the imaginary actors Achilles and Tortoise served as intermezzos.

Hofstadter elucidated one of the apogees of modern mathematics, Gödel's theorem, but maybe at another era in history analogous similarities can be discovered between mathematical, graphical and musical expressions. Inventing names for numbers, adding them, making geometric and numeral combinations with pawns, with lines or through music, might have been a comparable summit for humanity, in the times of the dawn of science.

In the middle of Africa, two small countries lived until recently in such an epoch. On the tops and flanks of the almost round hills of Rwanda and Burundi, lived one of the most dense populations of Africa, from agriculture and cattle breeding. There are many similarities between their populations of Hutus, the vast majority of peasants, Tutsis, the former aristocratic cattle-breeders and Twas, the more marginal potters. Their Kinyarwanda and Kirundi languages, their tradition and social history are closely related. Some pretend that, if the word *Rwanda* could be interpreted as *the vast territory, (B)urundi* would simply mean *the other [country].*

Historically, small kingdoms existed in this region within living memory, and each had its own sacred symbols, like for example a *little drum.* There were a multitude of these little principalities dispersed in the mountains, until the legendary *Gihanga* descended from the heavens, along with the thunder, as did other emperors in Sumeria, Mesopotamia or Creta. Besides cattle and seed, he brought fire to the mortals; the memory of this Promethean event would only extin-

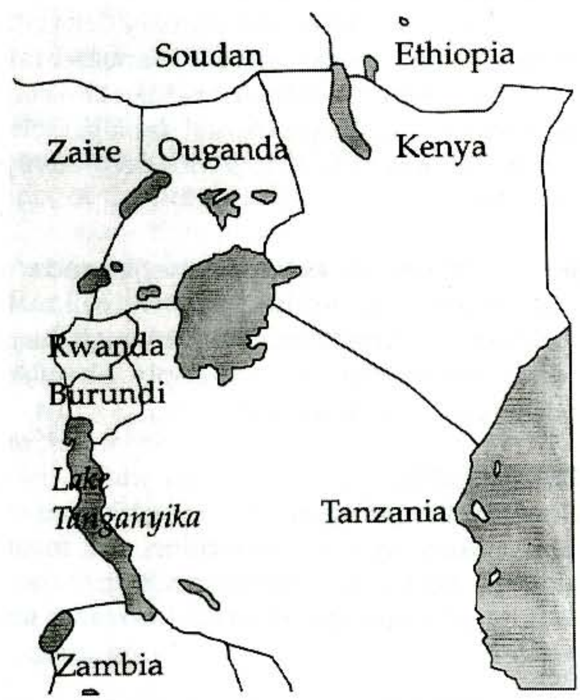


Figure 1

guish in 1933, when the king was converted to Catholicism. *Gihanga* also created the cult of the *large drums*, a very visible sign of his monarchy. These wooden emblems had a symbolic value comparable to the respect a scepter and a national flag have in other nations. The royal drums at the Rwanda court were not beaten but only touched when an important decision had to be to justified through the resonance of their deep bass sound.

There were 4 sacred drums, of about six feet high.

These wooden cylinders were covered by a brown cow-hide and each contained a crystal of quartz, their soul. The most magnificent had been called *Ruoga*, but it was lost in the 15th century. This was believed to be the cause of eleven years of distress, until a determined king could restore its shape *by his knowledge of numbers*. The new sacred drum *Karinga* or *warrant for hope* was placed in the *hut* of worship close to two others and next the oldest drum of all, called *the king is the owner of science*[1].

The sacred drums could not be regarded in absence of the king. Partition-walls protected them from the eyes of the mortals and as a security measure, other non-sacred drums were used when the king had to travel. Still, on other batteries music was indeed performed, for pleasure. It was not the only delectation, since playing on the *igisoro*-board was another favorite diversion, as were the Homeric riddles and *puzzles* that were told during the nocturnal *drum* gatherings where milk was drunk from jars with decorative *patterns* of all kinds.

Important differences between the Rwanda and Burundi culture exist. In this paper we will mainly focus on examples from the former, although the general principles of most topics apply likewise to Burundi, as can be seen from Figures 2, 3 and 4.

2. AN *IGISORO*-PUZZLE.

Two elements, **the traditional *igisoro*-game** and some facts about **counting** without writing, will together provide the setup for the formulation of a puzzle. The idea to propose a problem, to make the reader familiar with some characteristic difficulties, was inspired by Hofstadter's book. An answer to the enigma will be given in §6.

Figure 2
The *igisoro*-board around the hearth in Burundi (see [Acq]).

Figure 4
A drum player in Burundi (see [Acq]).

The traditional *igisoro*-game (see [Cou-Ben] and [Mer]) is played by two opponents on a rectangular wooden board. It is about 2 feet by 1, and has 32 circular cavities, arranged in 4 rows and 8 columns divided in 2 parts (see Figure 5). The players move around with 64 pawns (little stones, seeds or beans) to get enough pawns of the adversary to prevent him from taking pawns on his turn.

The players move their pawns on their own half of the board, following the indicated direction (opposite, as we would say, to the direction of the hands of a watch). A move can start at any cavity containing at least two pawns by collecting all the pawns in it and consists in dropping the pawns one by one in every cavity, after the cavity where the pawns were taken. If the last pawn is dropped in an empty cavity, the move stops. Otherwise, the player may go on by collecting these pawns and doing another similar move; this is called a *bridge*.

Pawns of the adversary may be captured if a move ends in a cavity on the lower row, containing at least one pawn, as should the opponent's two cavities in the same column. Taking pawns is obligatory, if it is possible. If a player has indeed captured some pawns

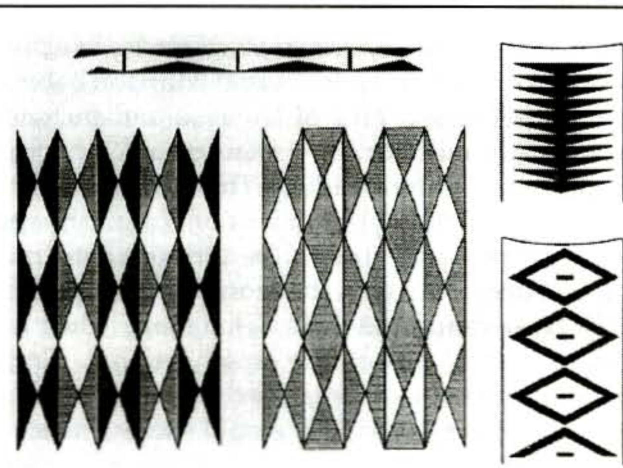


Figure 3
A decorative pattern called *abashi* or *wooden support* (see [Cel] and [Pau]) from the border region of Rwanda, Tanzania and Burundi.

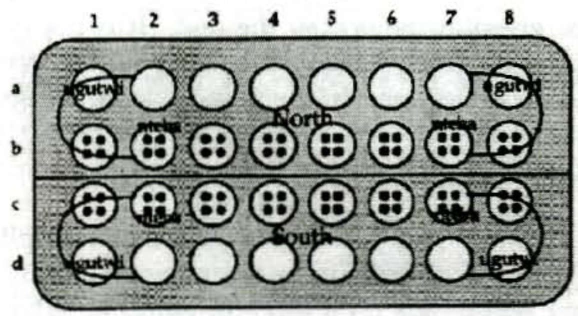


Figure 5
The *igisoro*-board with a few denominations.

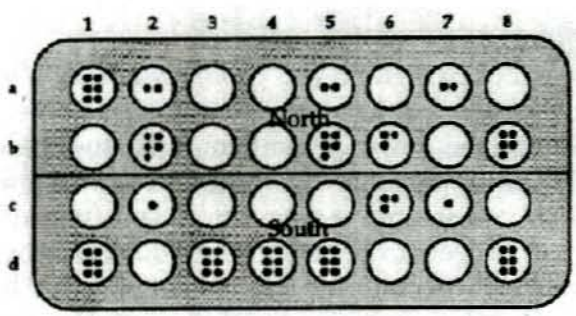


Figure 6
Taking pawns: only if player South could manage to end a move by dropping the last pawn in c2, he could take his oppenent's pawns. For instance, starting at d5, a bridge at c6 reaches to c2. The player goes on with the pawns from a2 and b2, starting at c5.

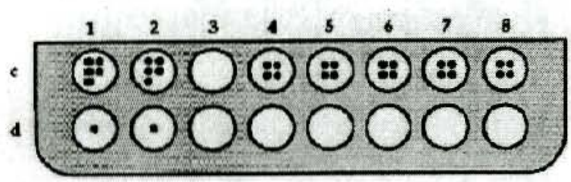
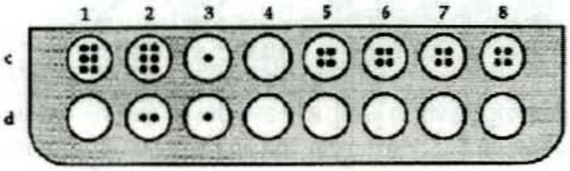


Figure 7
An easy opening in *igisoro* is called *madondi*, meaning *to deal dry and repetitive whips*. The move starts at c3 (above). It is followed by another move starting from d3, with a bridge in d1 (below).



of his adversary, he goes on playing with the pawns he took. He drops, one by one, the pawns he captured in the next cavity after the one where he started the last simple move (or after the last bridge if there has been one) before taking his opponent's pawns (see Figure 6).

There is a particular rule about the direction of movement: if one starts a move or if one makes a bridge at cavities called *nteba* (b2, b7, c2, c7) or *ugutwi* (a1, a8, d1, d8), and if one can, by doing so, get into a situation of capturing pawns by a simple move, without bridging, then the player may reverse the direction.

When a game starts, there are 4 pawns in each of the middle rows (as in Figure 5), and both players begin their opening moves simultaneously. There are different kinds of openings, sometimes with amusing names (see Figure 7), and like in chess they each have their reasons for being used. If one of the players has finished the second movement of his opening, the opponent has the right to take his pawns, and the winner of the previous game starts an attack (if it is the first game they play together, it is a matter of tactics to choose who starts). Each player makes a move, until one of them does not have enough pawns to continue. The game has to be played fast and sanctions are foreseen for a player who hesitates or cheats.

Note that these rules define the *igisoro*-game as it is known in one particular region, and that different versions exist, even within the region of Rwanda and Burundi. Traveling farther, larger variations are encountered. In East Africa, in Tanzania, a similar *soro* or *boa* game is played, while farther North, the *Kabaka* of Uganda play the *okweso*, and going to the West, the Nigerian *Yoruba* call their version *Ayo*. North of the Equator, the game is often performed on a board with only two rows instead of four, while three rows seem to be the tradition in Ethiopia. There may be from six up to fifty holes in a row.

Counting in Burundi and Rwanda was done using a base 10 system, and even for numbers as large as 1,999,999,999 words existed (see [Huy]). It must be pointed out that no consensus about these facts exists among historians[2], but this does not, of course, prevent mathematicians to admire the feat of inventing words for large numbers. There were no written expressions, and one can wonder how the slightest arithmetical operations could ever be executed.

An example of a procedure for executing complicated multiplications without any notation, can be found with the Yoruba (see [Jos]). They have a number system with base 20 and often use substractions to describe numbers. For example, *nineteen* (*nine plus teen*) is expressed as *ookandinlogun*, meaning *one less than twenty* (*ookan = one, dinl = minus, ogun = twenty*). Similarly, the appellation of *525* corresponds to *80 less then 600 plus 5*, or $(20\times3)-(20\times4)+5$. A story from 1887 tells about a counter who used cowry shells. To multiply 19 and 17, he started forming twenty piles of twenty shells each. Next, he took one shell from each pile, and then put three piles aside. These three heaps were rearranged by taking two shells from one of them, and adding it to the two others, the objective being to reduce the involved numbers to twenty:

$$400-20-(20\times2)-(20-3)=(400-80)+3=323$$

The Yoruba example shows some arithmetic operations were indeed done even in civilizations where no form of notation existed: representations with cowries replaced the written symbols.

Before turning again to the *igisoro*-board, we need a more convenient multiplication method, called the *Russian peasant* method. It was already known in ancient Egypt and in Greece, and is said to have found its way during the Middle Ages to Russia, the Middle East and finally back to Africa, in Ethiopia (see [Nel]). To multiply two numbers, like for example 241 and 17, one proceeds in this method as follows: divide 241 by 2, until 1 is reached; if an odd number is encountered, first subtract 1:

$$241\to120\to60\to30\to15\to7\to3\to1$$

The other number, 17, is multiplied as many times by 2:

$$17\to34\to68\to136\to272\to544\to1088\to2176$$

The numbers in this last row, corresponding to odd numbers in the previous row, are added:

$$17+272+544+1088+2176=4097$$

This is the desired result: $4097=241\times17$!

The puzzle: could the reader find out how to perform such a multiplication without writing down any auxiliary calculations, and use but an *igisoro*-board? In other words, a description is asked, of numbers with pawns placed in cavities, and a way for translating the Russian peasant multiplication into this representation. The mathematical justification of the method will be given at the end of the text. It is not quite necessary for solving the puzzle, but could be useful to find an indication. In the next paragraphs, some additional information is given first, to render the proposed answer more plausible.

Note that we do not mean to suggest that multiplications were traditionally done on an *igisoro*-board, but playing with seeds on a piece of wood to solve an arithmetic question may be a diverting and instructive exercise to get an idea about the necessary intellectual efforts needed to realize a mathematical achievement in a given cultural environment.

3. PATTERNS.

The previous paragraph was probably not very helpful to discover a primary explanation on the how and why about the dawn of mathematics. Indeed, the *igisoro*-game is played in different countries, and so it might be conjectured it was introduced from other cultures. However, the genesis of the idea of decorating walls with geometric patterns, can be traced back to its very origin. Indeed, an oral account relates why suddenly someone preferred to decorate his hut by geometric patterns instead of figurative images.

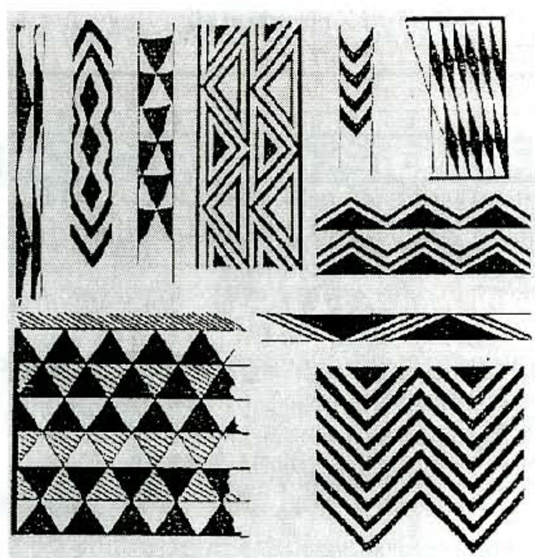


Figure 8

Paintings givin in [Cel]; all have descriptive names in Kinyarwanda. For example, the first is called *umuheha*, or *tube*.

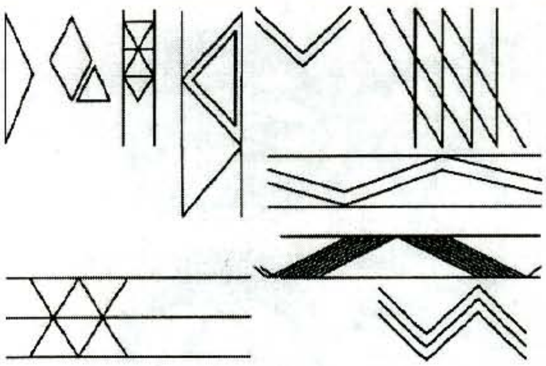


Figure 9
Explanations of the patterns by Celis.

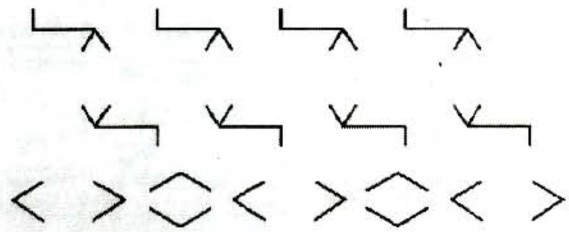


Figure 10
Zaslavasky's symmetry example.

In [Cel] the authors published their discovery of original paintings on enclosures of huts, in an isolated region in Rwanda (see Figure 8). It is difficult to access and hence, most paintings are believed to be traditional concepts, and not the result of an exchange with other cultures, nor the consequence of the ever progressing phenomenon of acculturation. The phenomenon of decorating a hut by the so-called *imigongo* seems to go back in the past for about three centuries, and the oral narration still relates how the legendary notable *Kakira ka Kimenyi* came to install the tradition of embellishing walls:

> Numerous acts in his life proved *Kakira ka Kimenyi* was possessed by neatness; his cattle were held in huts and were slaughtered there, so that no fly would ever touch it. [...] He hated mud and sat on a rock during heavy rainfall. His neatness was so legendary it became a locution to say *isuku ni ya Kakira* (neat like *Kakira*). Plenty of initiatives, *Kakira* would have made these paintings for pleasure, and by solicitude of neatness; first, he made them for his father [...], and then in his own hut. [...] Having made these paintings, he encouraged young girls -- of the aristocracy -- to imitate him. In this way, these paintings spread.

In other regions of Rwanda and Burundi, drawings and patterns were, of course, made too, but then it was on enclosures or walls in the huts, on small baskets, covers of milk jars and decorated drums (see Figure 11). On one of the previous pages (Figure 3), three drawings of the pattern *abashi* are *imigongo* paintings (reported in [Cel]) while the two on the right were found elsewhere in Rwanda on enclosures in huts (reproduced from [Pau]).

G. and T. Celis noted that the patterns they found, are combinations based on just a few elementary constructions. Only vertical, horizontal and three skew directions together with their symmetric directions along the vertical, are enough to form all the motives (see Figure 9). The *imigongo* can be classified by these 8 directions into just a few cases since only parallel lines, isosceles or equilateral triangles and kites are involved. Incidentally, these geometrical observations also led them to reject some other paintings as non-traditional.

A discussion about the use of some Chinese, Arabic and African drawings in the curriculum of pupils from 6 to 16 age was given by J. Williams (see [Nel]). Her comments apply to the present drawings from Rwanda:

> The classification of patterns by their symmetry groups is studied by crystallographers, and can be pursued through multicultural sources of patterns and design. Zaslavsky (see [Zas]) reproduces a picture of embroidered cloths from Kuba, Zaïre (now in the British Museum) which provides a complete set of seven different one-dimensional strip patterns. These patterns involve transformations in one dimension, such as 180° rotations and

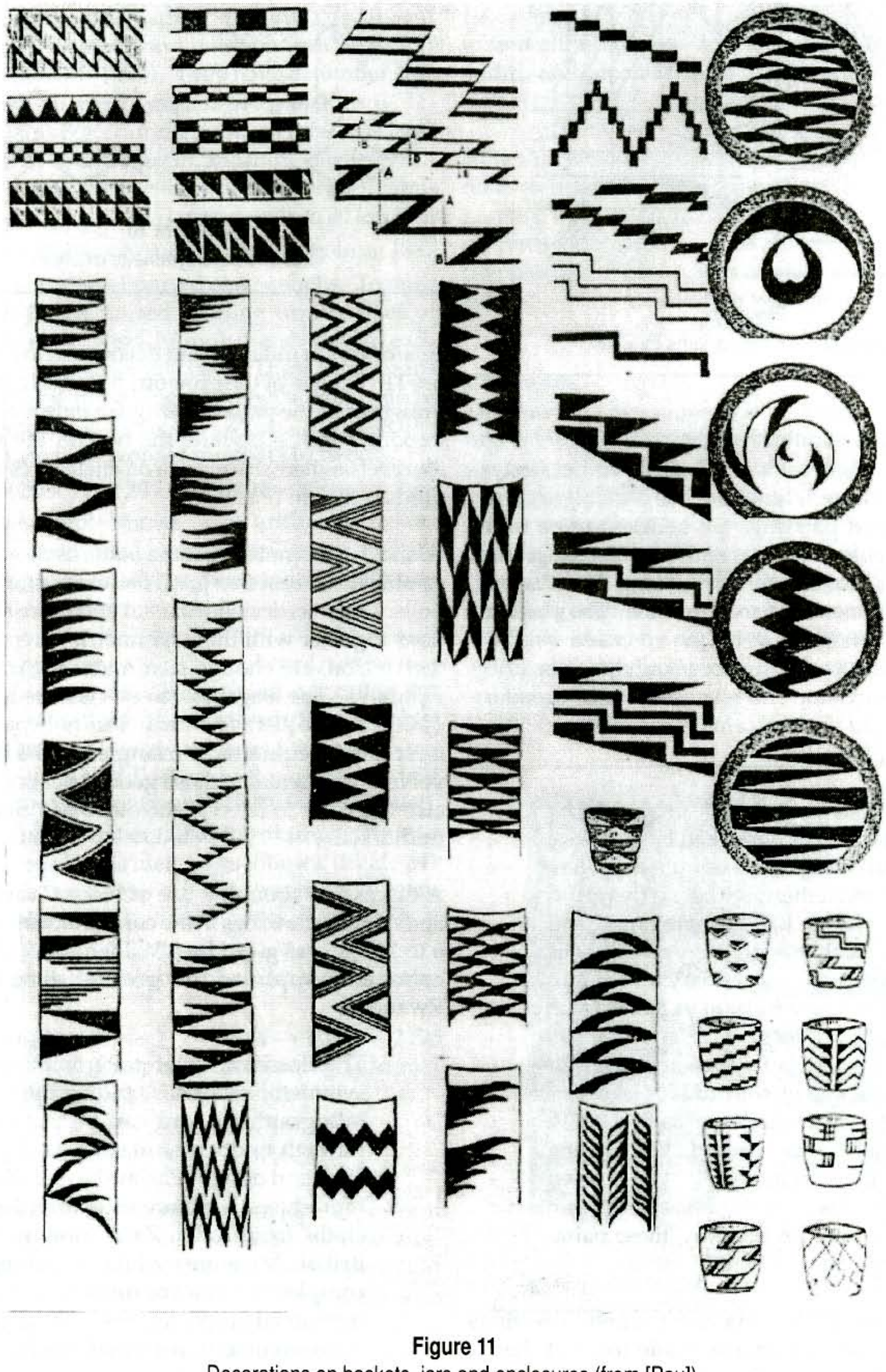


Figure 11
Decorations on baskets, jars and enclosures (from [Paul]).

Figure 12
Percussion staffs; cf. [Nke].

> horizontal and vertical reflections. Group theory can be used to prove that only seven such patterns can exist.

Williams also gave the above drawing (Figure 10), showing two strip patterns with rotational symmetry of order 2, but only one of these has horizontal and vertical lines of symmetry. It is a more mathematical way of appreciating geometric figures: it illustrates that any pattern with two perpendicular reflectional symmetries must have a rotational symmetry of order 2.

4. DRUMS

Some structure was apparent is the *igisoro*-game because of the presence of counters obeying well-defined rules, and in the previous paragraph the reader was invited to cerebrate an *igisoro*-framework for the Ethiopian multiplication method. However, when hearing African music, recognition of some logical basis seems even more difficult. Günther relates that, at the end of the 50's, the royal drums of Rwanda came to the world fair in Brussels. The Belgian audience was not prepared: the 24 drummers made the impression, said one listener, of *insistent, horrendous banging*. Others confessed more politely that *after a while the din overcame one's power of concentration*.

However, ignorance of the underlying structure may have been the main reason for the latter conclusion. Of course, if someone is familiarized to some kind of art, the knowledge about how some craft was accomplished is not necessary to appreciate it, though someone who went to an academy is more likely to appreciate Bach's music. Three rules seem to govern the percussion music in Rwanda and Burundi: the ***hemiola*** **effect**, the ***additive*** **rhythm**, and the ***Gestalt*** **phenomenon**.

Figure 13
Hemiola; cf. [Nke].

Hemiola is about the proportions of rhythmic models in their organization of the rigorous measures of time. Fixed intervals of time are subdivided in an equal number of subintervals by consecutive beats. Possible subdivisions of the intervals of time are two, four, eight or sixteen impulses or else three, six, twelve or twenty-four impulses, even if one starts with the same base interval. This, of course, implies that the proportion of the period between the pulses in both cases is 2:3. Usually, an intermediate rhythm of 4 or 6 beats follows, accentuated by slapping the hand, and by beating wooden sticks: this is called *simple idiophony*. Sometimes, the slow rhythm of 2 or respectively 3 drum-beats is used to reinforce this base rhythm. The faster cadences, of 8 or 16 and 12 or 24 pulses, are the bases for more melodic or percussional rhythms. They form the basic elements of the structure.

Yet, there are often subdivisions that cannot be placed in either this basis 2 or the basis 3-form. One can imagine these as proportions of an alternating basis 2 or basis 3 -form, and so as a successive realization of the proportion 2:3. This drum-beat structure is called

Figure 14
A piece of Rwanda drum music with *Hemiola*; cf. [Bra].

hemiola. It is the serial combination of a 2 or 3 subdivision, each of the same length, possibly with still more subdivisions. R. Brandel (see [Bra]) points out that reversed sectional change, that is, from 2-grouping to 3-grouping, is also encountered. His main example is precisely a piece of music from the royal drums in Rwanda:

> Here the 2/8 groups are organized in 3/4 measures (17 measures in this section), and the 3/8 groups are organized in 3/8 measures (22 measures in this section). Again the true *hemiola* is evident, provided two 3/8 measures are combined.

The consulted references ([Bra], [Nke], [Mic], [Gün]) agree that this *hemiola*-rhythm makes African music so different from Occidental patterns, although alliance with Middle Eastern and Hindu rhythms certainly does not make it unique. The five-unit *hemiola* of Ancient Greece also contained this 2:3 leader-beat contrast, but the rapid succession of unequal leader-beats in a 2:3 length-ratio is the typical African *hemiola* change: the music is distinguished by immediate exchanges of leader-beat: many changes occur within a short space, usually within a measure.

Additive **rhythms** differ from the more Occidental division rhythms, although they both are ways of subdividing an interval of time. The use of unequal groupings is preferred in African music. This attitude of asymmetry is the domain of excellence of the percussion.

To describe what additive rhythms are about, consider an interval with 12 pulses. It can be grouped in two groups of 6+6, but also into 7+5 or 5+7. Also, a measure of 8 beats can be decomposed as 5+3 or 3+5, and as 3+2+3, 2+3+3, or 3+3+2. Inside an interval of time, an equal duration can thus be lengthened or shortened, but of course all pieces should add up to the given number (here, 12 or 8). On a staff, this is written as follows:

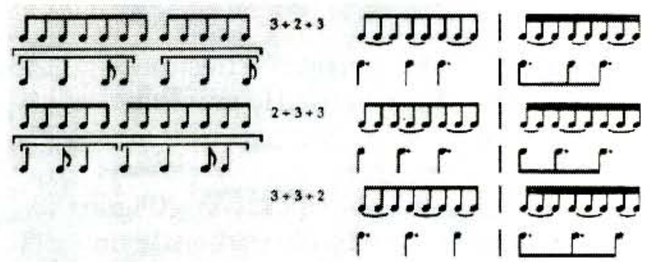


Figure 15
Additive rhythm; cf. [Nke].

Gestaltvariation is the third remarkable feature in the drum-batteries of Rwanda (see [Bra]):

> The coincidence of hemiolic lines inevitably carries with it some kind of *Gestalt* effect, almost as if a new rhythmic pattern, resulting from the composite interplay of all the lines, emerged. Very often the preponderance through timbre, pitch, etc. of one line over the others makes it suitable for *single-line listening* no matter how complex the entire work.

The indications of this *Gestaltvariation* again point towards a similarity with Mediterranean and Asian music, notes Günther (see [Gün]), and others again

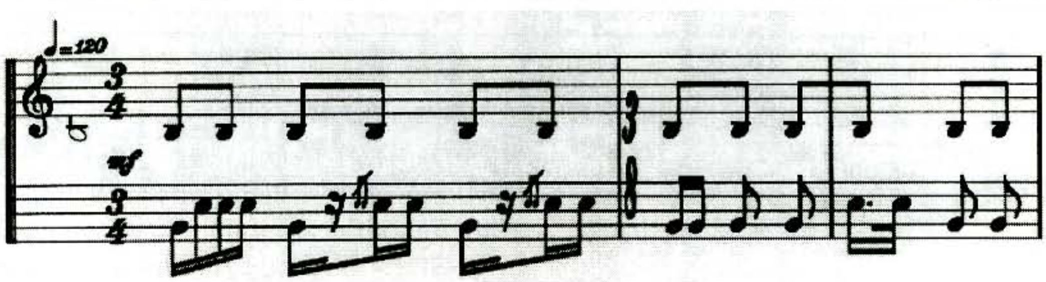


Figure 16
A piece of Rwanda drum music with *Hemiola*; cf. [Bra].

Figure 17
A more involved example of Rwanda drum music; cf. [Bra].

see a link with Ancient Greece. In the example of Figure 14, the deeper toned drums in the ensemble changed from 2-grouping, 3/4, to 3-grouping, 3/8. The leader-drum continued its 2/8 figure (see Figure 16) but is overshadowed by the basses.

> Yet, [says Brandel] because of its lesser obtrusiveness, the listener does not really hear the total counter-rhythm -- he merely feels it. The dynamic accent in the leader drum is almost lacking and the 2/8 grouping is achieved by means of very subtle timbre contrast.

Finally, all these constructions can be put together as in the following piece of percussion from the royal drums in Rwanda. It is more complicated to understand for the non-initiated:

> Despite the galloping strength of the lowest line, the 3-grouping of the top line somehow makes itself quite apparent, and the eventual result is complex pull in two directions.

In contrast to the remarks made at the Belgian '50 world exposition, given in the beginning of this section, it is therefore not the lack of structure and logical constructions that make this music difficult to access for Western listeners, but rather its abundance.

5. INTERMEZZO.

In Hofstadter's masterpiece *Gödel, Esher, Bach*, conversations between Achilles, the Tortoise, and the Tapir alternated the tougher mathematical reasonings that explained the Gödelian concepts. The well-known paradox of Zeno was the inspiration for the creation of these imaginary personages. In the context of Rwanda, actors exchanging Aristotelian sophisms were created by Kagame (see [Kag1]). He called them *Gama* and *Kama*, and some of the exchanges of ideas of the players Kagame invented, suit well to provide us a Hofstadter-like intermezzo. The following excerpt contains riddles related to the present topic about the dawn of mathematical reflection:

Gama: It would be convenient to examine if the bantu-rwandean philosophy has elements related to the notion of "time". I think, at this very moment, about that woman of the Court, who lived under the reign of *Mibambwe III 'Sentabyo*, in the XVIIIth century. One attributes the following reflection to her. It passed afterward onto the common language like a profound adage: *Ko bucya bukira, amaherezo azaba ayahe?*, that is *Since there is day and night, and the end of times, what will there be?*

One can apply this sentence, as you know, upon the events that go on and on, without interruption, even when one expects it to end. That woman certainly had thought profoundly about the progress of "time"! Don' t you think, like myself that her reflection merits the qualification of "philosophical"?

Kama: Up to a certain degree, yes. However, there is much better in this domain. Did you ever hear about the riddles that were solved by *Ngoma*, the son of *Sacyega*? I do not want to confirm that these two personalities have really existed. The solved riddles have been grouped under the name of *Ngoma*, as some lies were gathered under the name of *Semuhanuka;* in the same way the gourmet anecdotes were attributed to *Rugarukirampfizi* and the sly puns to *Semikizi*.

Gama: That is the way it goes with our traditions presenting a certain literary value, characterizing a numinous turn, and of which the various authors are forgotten. Our narrators grouped them in series, and each series got a single, but maybe faithless, name.

Kama: Correct! So one day, our *Ngoma* had to solve another riddle. His father was in debt with the Death about a head of cattle. Thus, it is clearly an invented story, the work of someone regarded as a thinker. The terrible creditor went one day to see *Sacyega* and declared:

> "The debt has to be paid without hesitancy! However, I demand that you pay me a head of cattle, that is nor a bull nor a cow! Failing that head of cattle, I will sacrifice yourself!"
>
> "You ask me something completely impossible!" *Sacyega* begged; "A head of cattle always is a bull or a cow, because one never sees one that is nor the first nor the latter!"
>
> "Your problem!" answered the Death; "or you find me that head of cattle, or you can within eight days from today arrange your affairs."

Informed about this terrible dilemma, *Ngoma* answered his father as follows:

> "It is not so difficult! It suffices to put the Death in the impossibility to claim his incompatible head of cattle. When he will show up at the agreed day, answer him by the words: 'I finally have found what my arrearages are. Yet, to seize it, you cannot come during the day nor at night. At daytime one can see the stars, and at night they are visible. So come between both events and you will get your cattle.'"

The narrators do not tell the continuation and they did not need to. The inventor of the problem only had in mind to formulate two impossibilities and to oppose them one another.

Gama: In fact, the solution attributed to *Ngoma* has an obvious philosophical significance. It points very precisely to the moment whose duration is as impossible to evaluate as it is impossible to meet cattle without genus.

Yet it is clear that the narrators did not recall the entire depth of this point of view. They do not conceive this limit between the day and the night, explicitly based on the passage of non-being to being, as we envisage:

> "non-being of light of the stars" to "being of light of the stars".

This solution corresponds exactly to the well-known principle of the great meta-physicists: between the being and the non-being there is no third way.

The above quote from Kagame obeys the tradition of the *smiths of intelligence*, the narrators at the royal court who memorized thousands of verses relating a poetic version of the history of Rwanda from about the year 1100 up to the beginning of our century. To ease memorization, a rigorous formal structure based on rhyme, rhythm and tone, was imposed on the text, as in the Homeric verses.

In other poems too, a unit of vocalic quantity could be discovered: the *mora* (see [Cou-Kam]). It consists of one short vowel or half a long vowel, and 9, 10 or 12 *moras* form the basis for the main type of verses. Studies in Kirundi poetry (see [Cou]), confirm these findings about the rigorous formal structure of the poetry in the culture of this region.

Parenthetically, the illustration below comes from a book of modern Kinyarwanda poetry (see [Kag2]). The poem glorifies the creation of the almighty *Imana*, but here the bard does not ask if *God* plays dice: the divine hand covers an *igisoro*-game.

6. COMMENTS.

Returning to the §2 *igisoro*-puzzle, here is first an explanation for numerical example of the Russian peasant method given in that section. The number 241 is written as a sum of powers of 2:

$$241 = 1\times 2^0 + 0\times 2^1 + 0\times 2^2 + 0\times 2^3 + 1\times 2^4 + 1\times 2^5 + 1\times 2^6 + 1\times 2^7$$

The product of 241 and 17 follows from a term by term multiplication:

$$\begin{aligned}241\times 17 &= \left(1\times 2^0 + 0\times 2^1 + 0\times 2^2 + 0\times 2^3 + 1\times 2^4 + 1\times 2^5 + 1\times 2^6 + 1\times 2^7\right)\times 17\\ &= 17\times 1\times 2^0 + 17\times 0\times 2^1 + 17\times 0\times 2^2 + 17\times 0\times 2^3 + 17\times 1\times 2^4 + 17\times 1\times 2^5 + 17\times 1\times 2^6 + 17\times 1\times 2^7\\ &= 17\times 1\times 2^0 + 17\times 1\times 2^4 + 17\times 1\times 2^5 + 17\times 1\times 2^6 + 17\times 1\times 2^7\\ &= 17 + 272 + 544 + 1088 + 2176\\ &= 4097\end{aligned}$$

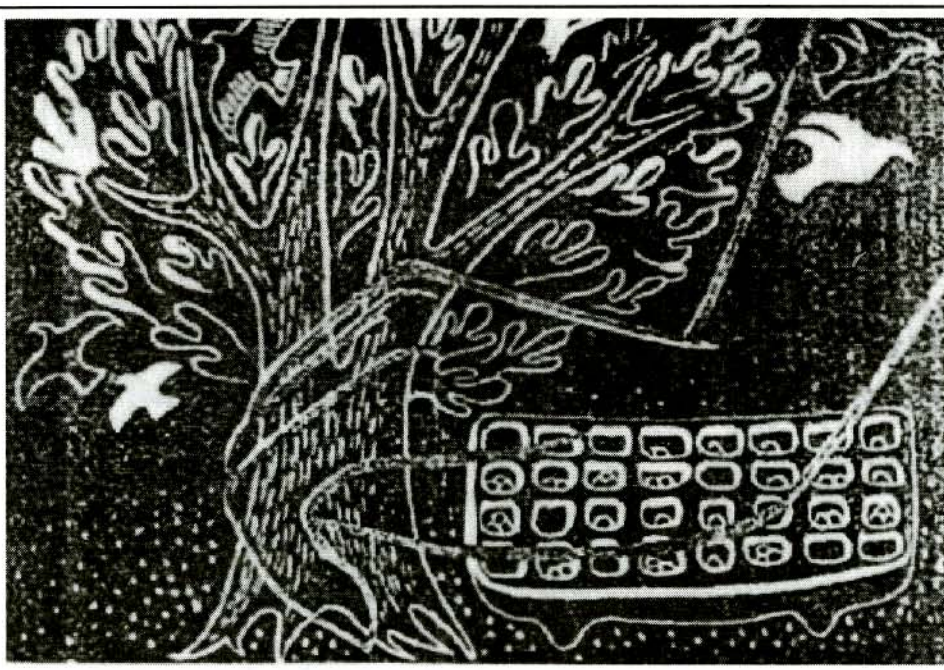


Figure 18
An illustrationfrom a tome ([Kag2]) of modern Kinyarwanda poems.

In §2, 241 was first divided in halves, with the condition to subtract 1 in case of an odd quotient. This first row of operations allowed to get the non-zero coefficients in the decomposition of 241 as a sum of powers of 2, while the second row, where 17 was doubled, served to obtain the corresponding numbers that had to be added: 17, 272, 544, 1088 and 2176.

The proposed *igisoro*-puzzle asked for a representation of this multiplication diagram on an *igisoro*-board, without using any notation to re-

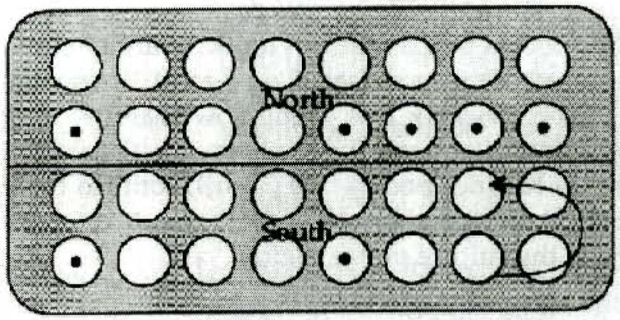


Figure 19
Imaginary representations of 241, *North*, and 17, *South*.

member the operations. A possible solution goes as follows: imagine a pawn in the first cavity (b1 or d1) would represent 2^0, one in the second (b2 or d2) would stand for 2^1, and so on, until the last one (a1 or c1), 2^{15}, is reached. Then, in Figure 19, North would represent the first number, 241, while South would be the second, 17. This is merely a recreating idea by the author, inspired by the Yoruba cowry calculations and the principles of the *igisoro*-game; it does not correspond to any historical data.

Following the *Russian peasant* method, 241 should now be divided 2. Each pawn in the representation of 241 is replaced by two pawns in the previous cavity, and only half of them are withheld (see North, in Figure 20). 17 is doubled simultaneously by moving its pawns 1 step to the right. There was a problem with the remainder 1 of the division of 241 by 2, since it could not be represented adequately. This fact reminds us we should keep track of the initial value 17, before it was doubled (cf. Figure 20, South, white pawns).

The next consecutive divisions by 2 yield no problem, since the remainder is 0, and thus the results of those multiplications by 2 are not withheld. Note that the operations of halving and doubling are easily executed: it is enough to move the pawns one cavity to the left or the right, respectively. Yet, when 4 pawns on a row are obtained, in b1, b2, b3 and b4, representing the number 15, one has to keep in mind that for the next doubling in South, the initial pawns should again be withheld (see Figure 21).

Finally, when there is only 1 pawn left in North, the procedure stops. In South, the withheld pawns in d1, d5 (2 pawns), d6, d7, d8, c8, c7, c6 and d5 correspond to the numbers 17, 272, 544, 1088 and 2176 and these should be added (see Figure 22).

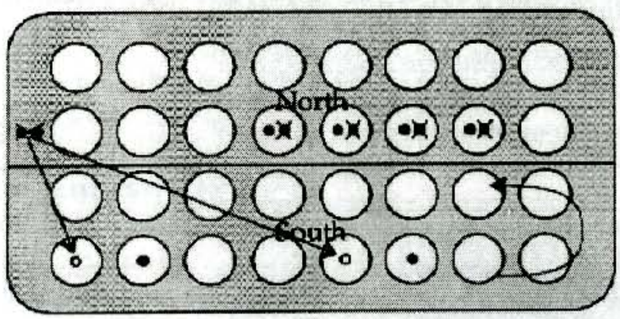


Figure 20
Halving 241 in *North* and doubling 17 in *South*; the pawns of 17 are withheld (white) , the result being 34 (black).

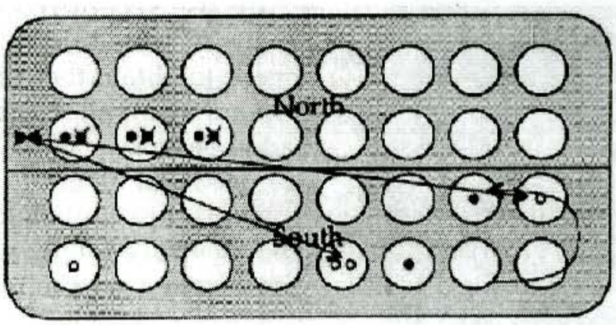


Figure 21
Halving 15 in *North* and doubling 272 in *South*; the pawns of 272 are withheld (white), the result being 544 (black).

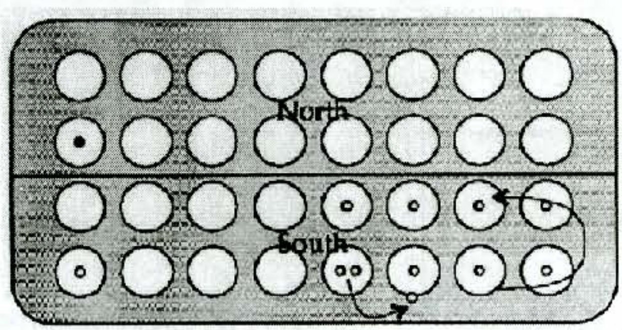


Figure 22
Finally, only 1 pawn remains in *North* and 10 in *South*; the latter should be added.

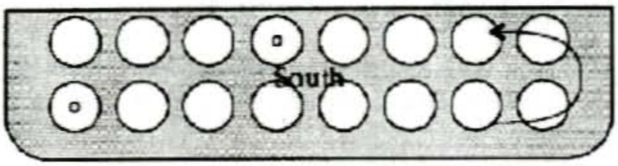


Figure 23
The final result: 1 + 4096 = 4097.

The addition of the 2 pawns in d5 is straightforward: they are replaced by a single one in d6. Now there are 2 pawns in d6, and the procedure continues until every cavity contains but a single pawn. The demanded product can be read off: 1 (the pawn in d1) plus 4096 (1 pawn in c4) yield the required 4097.

An objection to this apparently very easy method could be that the example works so smoothly because of the choice of the numbers 17 and 241. This is indeed partially true: if there are many pawns left to be added, a harder mental computation is necessary in the last step (Figure 23) to convert the answer in base 2 to the final result in base 10.

A final wink to *Gödel, Escher, Bach* is the observation that Hofstadter liked to refer to computer problems, although the subject of his book was a topic out of the domain of the purest mathematics of all. His favorite computer savant was Babbage, but in the present case it might have been entertaining to say a few more words about N. Wirth, the creator of PASCAL. Indeed, instead of **puzzling** about the multiplication procedure on an *igisoro*-board, one could imagine that the cavities corresponded to computer switches. A pawn in a cavity means the switch is closed. Thus, doubling a number by transferring pawns one cavity to the right, corresponds exactly to a computer shift. Of course, the reality is not that simple, but even N. Wirth explained the importance of converting a multiplication to an operation of doubling in his successful book on programming fundamentals (see [Wir]). Note that the prestigious Massachusetts Institute of Technology expressed its appreciation for the *igisoro*-concept by programming it on a computer. They restricted their study to one of the most simple *igisoro*-versions with only 2 rows of 6 holes and 36 counters. Nevertheless, there are still about 1024 possibilities in this very simple situation. Thus, it is a good test case for trying out heuristic methods, applying only advantageous moves. R. C. Bell's classification of *igisoro* among the world's nine best games seems amply justified (see [Zas]).

The design by computer of geometric patterns, as those found in Africa, was the subject of Williams' text entitled *Geometry and Art* (see [Nel]). This author proposed the following key lines of a computer program to form patterns of TRIANGLEs separated by GAPs:

```
FOR N=1 TO ENDX;
        NEWY=0;
        FOR M=1 TO ENDY;
                NEWY=OLDY+GAPY(M);
PROCTRIANGLE(NEWX,NEWY);
        NEXT M;
        NEWX=OLDX+GAPX(N);
NEXT N.
```

Musicians like computer toggling too: Frank Michiels, a researcher at the prestigious Belgian Museum for Central Africa in Tervuren and a recognized percussionist, plays on African **drums** for his computer. The electronics transform the recorded music into notes of any kind, from organ to violin. And still, the African musical structure remains irrefutable!

The **summary** given in Table 1 is easily completed from the present paper. The words in italics refer to some striking terms or names used in the text.

Expression → ↓ Representation	Puzzles	Patterns	Drums
Without writing	*Igisoro* -board	*Kakira*-drawing	*Additive hemiola*
Written	241×17=4097	*Symmetry-groups*	*Staffs-structures*
Computer-screen	*Shift switches*	*Williams' computer patterns*	*F. Michiels' violin-percussion*

Table 1

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[1] This is a quotation from [Per], but one has to be careful about how to translate statements like *by his knowledge of numbers* or *the owner of science*. Some linguists may provide different translations, but of course, being a mathematician, this is a discussion that the author willingly omits. Another problem that was not regarded, is the particular transcription with special punctuation linguists use for the Kinyarwanda or Kirundi words.

[2] from [Cou2] and [Rod] one could conclude 10,000 or *ibihuumbi cumi*, was the highest number that was conceivable, while [Pau] mentions 100,000 or *akahumbi*, but [Kag3] goes indeed as far as one less than 2 billion.

Music and Mathematics

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INTRODUCTION

As a child, music played a big role in my life. My father is a musician, and he tried to expose my brother, sister, and me to as much music as possible. Each of us was given the opportunity to play a musical instrument and encouraged to perform whenever given the chance. Although my brother and sister excelled with their instruments, I chose not to continue with lessons after the seventh grade. I enjoyed music immensely, but playing a musical instrument was really not my forte. In high school, I found something else that made me get excited: Mathematics. I enjoyed it so much that I decided to major in it when I went to college. Now I am here, and both music and mathematics continue to play a big role in my life. Instead of playing a musical instrument, I listen to music while doing my mathematics.

This semester I was given the opportunity to do an independent study in the mathematics department focusing on any topic that I desired. I now had a chance to combine two driving forces in my life, and to try to find some connection between them. I chose to investigate the relation of music and mathematics.

The focus for this paper is to find the commonalities between music and mathematics, with the hope that beauty will abound within this connection.

NOISE VS. MUSIC

First, we must establish that noise and music are two different entities. As defined in the tenth edition of *Merriam Webster's Collegiate Dictionary*, noise is a sound that "lacks agreeable musical quality or is noticeably unpleasant" [5]. The same dictionary defines music as "the science or art of ordering tones or sounds in succession, in combination, and in temporal relationships to produce a composition having unity and continuity" [4].

Sound waves are produced by vibrating matter. The sound waves produced by irregular vibrations in matter are called noise, whereas the sound waves produced by regular vibrations in matter are classified as musical sounds. These regular vibrations are the simple harmonic motion that can be represented graphically by adding a sufficient number of sine waves [1] (see Figure l). Jean-Baptiste Fourier is the man credited for this discovery. The frequency of the vibration determines the pitch of the musical sound, represented on the graph by the number of waves per unit time. The wave's amplitude, which indicates the intensity of the sound, is represented by the height of each crest.

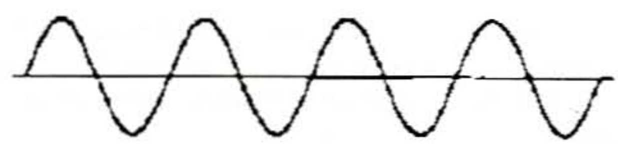

Figure 1
Sine Curve

A Fourier representation of a sound would consist of a series of simple, regular sine waves that, when added, represent the sound being analyzed. As the number of individual sine wave graphs increases, so does the complexity of the sound. Fourier analysis is useful for describing long, regular sounds in a very concise way [7].

Fourier, who studied mathematical vibration analysis circa 1800, knew that there was a flaw in his representation. He realized that a sound could not exist unchanged unless that sound was infinite in its duration. Because musical sound has a beginning and an end, the graphical representation of that sound must also be discrete [8]. Fourier analysis fails to reproduce accurately the timing of a sound when focusing on its pitch [7]. That is, there is a problem determining the time when a particular sound occurred.

It is now possible to represent both the pitch and the timing of musical sounds, thanks to Ingrid Daubechies. Daubechies uses a method that breaks down complex signals into what are called wavelets. The length of each wavelet represents the pitch of the sound -- the higher the pitch, the briefer the wavelet. Unlike Fourier representation, wavelets have no redundancy. With redundancy comes unnecessary information needed for reconstructing a sound. When using wavelets for analysis, "each wavelet is an essential component of the complex signal it represents" [7]. Wavelets are not only useful for representing sounds heard individually, but they are so precise that they can be used to single-out sounds in a graph of several simultaneous sounds.

Research in this area is very new. Because of this, the information regarding waveless is limited. Keep your eyes open; information on this topic is bound to explode!

HARMONY OF MUSIC
Its Frequency, Intensity, and Duration

As many of us may know, Pythagoras is the man credited with being the first to discover the relationship between musical harmony and mathematics [2]. It all happened one day, or so the story goes, when Pythagoras was considering whether it was possible to systematize musical sounds. He thought: sight is made precise with tools like the compass and ruler, as is touch by measures and balances. While thinking about this, he passed by a brazier's shop where he heard hammers beating on a piece of iron. Some sounds produced by hitting the same piece of iron were harmonious; others were not [3].

Later, after considering what he heard, Pythagoras went back to the brazier's shop to investigate how hammers beating on the same piece of iron could produce harmonious sounds. He discovered something astounding! When comparing the hammers, he found that they were of different weights. There was a six, eight, nine, and twelve pound hammer. When using the six and twelve pound hammers together, where the first hammer was half the weight of the second, the sound was harmonious. Harmony resulted when using the eight and twelve pound hammers together as well. But the hammers that were eight and nine pounds, when used together, produced a sound that did not harmonize [3].

The relationship between the weights of the hammers and harmonious sounds can be represented by using any musical instrument. For ease of explanation, I will discuss the representation in reference to a stringed instrument. The procedure is as follows:

1. A single stretched string vibrating as a whole produces a ground note. The frequency of the vibration determines the pitch of the musical sound.
2. Allow only half the string to vibrate, and the pitch will rise an octave above the ground note.
3. Allow 2/3 of the string to vibrate, and the pitch will rise a fifth above the one produced by the total length.
4. 3/4 - tone is a fourth higher.
5. 8/9 - tone is a whole step higher.

etc.

If the still point on the string, called the node, is not at one of these exact divisions, the sound is discordant. As we continue to divide the string, the fractions become more complex, and the two notes represented by the resulting intervals become more dissonant, or unpleasant, when they are sounded together. The smaller the whole numbers in the fractions, the more consonant, or pleasing, the sound is [2]. This is the reason Pythagoras felt that the six and twelve pound hammers sounded harmonious together, but the eight and nine pound hammers did not. Eventually, the fractions of the vibrating portions of the string became expressed as ratios. For example, the octave was expressed as a ration of $1:2$.

The frequencies of intervals between the tones of a musical scale can also be represented as a ratio. The frequency of middle C is 261 cycles per second. The ratio of $1:2$ describes the interval of an octave, so by doubling that frequency, we obtain a note defined by 522 cycles per second, or C one octave above middle C.

The chromatic scale, used in western music, consists of twelve intervals. Because of this, each tone in the scale has a frequency ratio of $\sqrt[12]{2} \approx 1.0595$ to the next tone (where the two comes from the ratio of an octave). It is with this ratio ($1:\sqrt[12]{2}$) that frequency intervals are spread equally over the twelve tone intervals of the octave. The break down of one octave is shown in Table 1. Because all twelve tones are necessary to construct musical scales, we can now find the frequency of any note in any octave [1]. The intensity of a tone is determined by the rate at which sound energy flows through a unit area. Intensity can simply be thought of as the loudness of a tone. The duration of a tone refers to how long a tone exists. With these three properties specifically stated, a musical sound can be duplicated.

ANALYSIS OF A COMPOSITION

When writing a piece of music, composers usually do not write a mathematical function and then compose the piece around the function. Instead, the composer might hear music in her head and then record that thought on paper. Whatever the process, I believe it is safe to say that mathematics is generally not the motivation for a composition. What is amazing is the fact that music is very organized. We have seen how harmony is made. We understand the idea of consonance and dissonance. Now let us investigate the mathematics of a composition.

First, let us look at a single, generic sound. Our sound will be an event that is considered as a whole and will be considered neither pleasant nor unpleasant. We can consider the abstract relations within the event or between several events, and the logical operations that may be imposed on them. Our event will be denoted as a.

Properties :

1. If the sound is emitted once, all we have is its single existence that appears and then disappears. Here, we only have a.
2. If the sound is emitted several times in succession and compared, all that we can conclude is that they are identical.
   Now we can say that repetition implies the notion of identity, or tautology:
   $$a \vee a \vee a \vee \ldots \vee a = a$$
   where $\vee$ is the logical operator "or", disregarding time.
3. Modulation of time imposed on the sound.

When the element of time is considered, our sound takes on new meaning. Instead of just a sound, we now have potential for a code. For example, the Morse Code is an emission of a single sound that varies in duration. It is the duration of the sound, rather than the sound itself, which gives meaning to the code. For this reason, we will disregard the modulation of time and consider the case of two or more generic sound.

Let a, b, and c be distinct, easily recognizable sounds $(a \neq b \neq c)$.

Properties :

1. $a \vee b = b \vee a$
   Since time is not considered, our events are commutative.
2. $(a \vee b) \vee c = a \vee (b \vee c)$
   If we combine two elements, the combination can be considered as forming another element, or an entity, in relation to the third. This combination will allow our events to be associative.

When we exclude the time factor in composition, we end up with the commutative and associative laws of composition outside-time [9]. If we do consider the element of time (denoted with the logical operator T), then the sonic events, when played in succession, have a new meaning.

$$a\mathrm{T}b \neq b\mathrm{T}c$$

The comutative law no longer holds. Because our events are distinct and easily recognizable, it follows

Note	Frequency Approximation
middle C	261
C#/Db	276.5199
D	292.9626
D#/Eb	310.3831
E	328.8394
F	348.3932
F#/Gb	369.1097
G	391.0581
G#/Ab	414.3117
A	438.9479
A#/Bb	465.0491
B	492.7024
C	522

Table 1
Notes and Frequency Approximations of an Octave

that a played before b sounds different from b played before a.

With these properties of sound, we can now investigate the concept of the interval. As defined in the *Norton/Grove Concise Encyclopedia of Music*, an interval is simply "the distance between two pitches" [6]. An interval is described according to the number of steps between notes, inclusive. For example, from C up to D, the interval is a major second. From G down to C, the interval is a perfect fifth.

With this in mind, let us consider a set of pitch intervals, $P=\left(p_a, p_{b,\dots}\right)$, and the binary relation $\geq$ meaning greater than or equal to.

Then:

1. $p \geq p, \forall p \in P$
   - reflexive
2. $p_a \geq p_b \neq p_b \geq p_a$ *except for* $p_a = p_b$
   - antisymmetric
3. $p_a \geq p_b \wedge p_b \geq p_c \rightarrow p_a \geq p_c$
   - transitive

So, the set of pitch intervals, P, with the binary relation $\geq$ (P, $\geq$), forms a partially ordered set.

The ultimate goal of composers, let us assume, is likely to be the ability to share their musical inclinations with others. To do this, a composer must tell the musician exactly what she is thinking or hearing in her head. In order for a musical sound to be duplicated, all aspects of that sound must be considered. These aspects include frequency (pitch), intensity, and duration. With these three elements correctly combined, any musical sound can be constructed and repeated. In this case, the number 3 is irreducible.

Structure

When considering the set of pitch intervals, we are forced to consider the structure within that set. If p_a is a pitch interval going from C up to D (a major second), and p_b is a pitch interval going from D up to F (a minor third), then a third element, p_c, can be made to correspond when combining p_a and p_b. The element p_c would then be a pitch interval going from C up to F (a perfect fourth). Xenakis refers to this as the "law of internal composition" (consecutive pitch intervals, $p_a, p_b \in P$, can be made to correspond to a third pitch interval, $p_c \in P$, by the composite of p_a by p_b and is denoted as $p_a + p_b = p_c$) [9]. With this in mind, and once again disregarding time, we can say:

1. The law of internal composition for conjuncted intervals is addition.
2. The law is associative:
$$\left(p_a+p_b\right)+p_c=p_a+\left(p_b+p_c\right)$$
3. $\forall p_a \in P$, $\exists p_0 \in P$, a neutral element, such that:
$$p_0+p_a=p_a+p_0=p_a$$
4. $\forall p_a \in P$, $\exists p'_a \in P$, called the inverse of p_a, such that:
$$p_a+p'_a=p_a+p'_a=p_0$$
5. The law is commutative:
$$p_a+p_b=p_b+p_a$$

These five axioms hold for pitch *outside-time*. This example of pitch intervals can be extended to intensity intervals and durations, the other two fundamental factors of musical sound. It should be noted that the sets form an Abelian additive group structure.

So far, it has been established that the idea of sound possesses a structure outside-time. The element of time forms a temporal structure. When we combine these two structures, the result is a structure in-time, or an actual composition.

Before considering a musical composition, let us first consider the notes that a composer uses. The only limitation imposed on what notes and in which octaves are usable is with the instruments that the composer chooses to use. If the piece is written for a bassoon, then only the notes in the available octaves can be used. The composition would not be written in the same octave as, say, the upper register of a piccolo.

APPLICATION

For a composition with one instrument

Let

$R=\{\textit{all the notes of a particular instrument}\}$

$A=\{\textit{a certain choice of notes of the instrument}\}$

$B = \{\textit{another choice of notes of the instrument}\}$
Where A and B are subsets of the universal set R.

If we first hear A, and then B, and then compare the two sets, we can establish some relationships between them.

1. If certain notes are common to both sets A and B, the sets intersect (see Figure 2a).
2. If no elements are common between the chosen sets, they are disjoint (see Figure 2b).
3. If all the elements of B are common to one part of B, then our set B is included in A (see Figure 2c).
4. If all the elements of A are found in B and all the elements of B are found in A, then the two sets are indistinguishable, or equal (see Figure 2d).

Now that we understand the basic relationships between sets, we can investigate a method of creating new sets given existing sets. When we choose A and B so that they have some elements in common, we can then establish those new sets.

1. If we hear the notes in common between A and B, we are using the operation of intersection (conjunction) to form a new set consisting only of those common elements:
$$A \cdot B \text{ or } B \cdot A$$
2. If we hear the notes of both sets and interpret them as a mixture of the elements of A and B, we have a new set formed using the operation of union (disjunction):
$$A + B \text{ or } B + A$$
This set consists of all the elements of set A and set B.
3. If we are allowed to hear all the notes in our universal set R except those of A, then we have a new set defined by the negation A with respect to R:
$$\overline{A}$$
4. In music, there is another set which is represented by silence. This set is equivalent to the empty set, and is called a rest.

With a proper choice of notes for each set, and a proper grouping of these sets, we can write a mathematical function to represent a composition. When given three sets, A, B, and C we can write a Boolean function in the form called disjunctive cannonic:
$$\sum_{i=1}^{8} \sigma_i k_i$$
where,

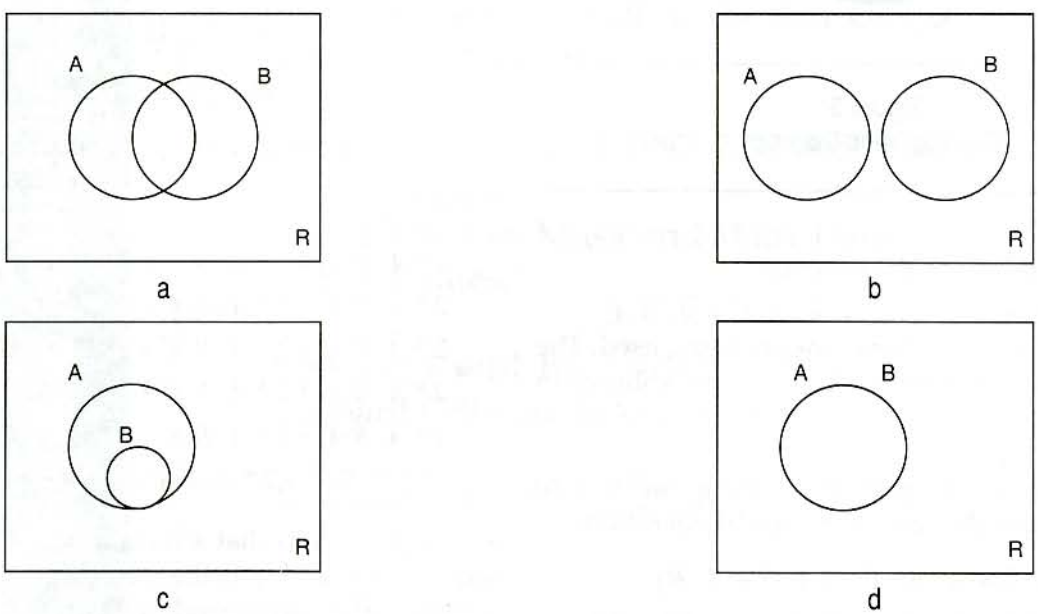

Figure 2
Relationships Between Sets

$$\sigma_i = 0, 1$$

and

$$k_i = A \cdot B \cdot C, A \cdot B \cdot \overline{C}, A \cdot \overline{B} \cdot C, \overline{A} \cdot B \cdot C, A \cdot \overline{B} \cdot \overline{C}, \overline{A} \cdot B \cdot \overline{C}, \overline{A} \cdot \overline{B} \cdot C, \overline{A} \cdot \overline{B} \cdot \overline{C}$$

A Boolean function can always be written in a way that brings a maximum of operations using (+), (.), and ($\bar{}$), equal to $3n \cdot 2^{n-2} - 1$, where n is the number of sets being used. In this case, $3 \cdot 3 \cdot 2^{3-2} - 1 = 9 \cdot 2 - 1 = 17$ [9].

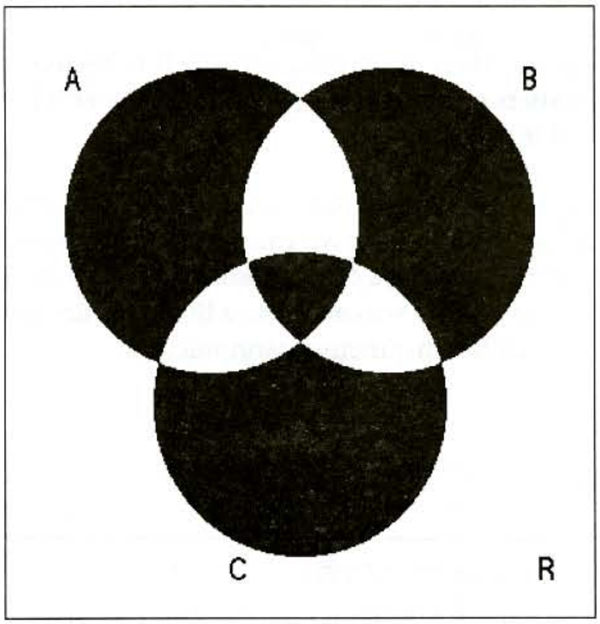

Figure 3
Example Venn Diagram

For example, if we use the function:

$$F = A \cdot B \cdot C + A \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot B \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C$$

we will notice that 17 operations are being used. The Venn diagram representing this function is shown in Figure 3.

Of course, we can simplify the original function to obtain a function that only requires 10 operations:

$$F = \left(A \cdot B + \overline{A} \cdot \overline{B}\right) \cdot C + \overline{\left(A \cdot B + \overline{A} \cdot \overline{B}\right)} \cdot \overline{C}$$

but by doing this, we will change the procedure in the composition.

I must stress that this mathematical model deals only with which notes in a composition are played. It does not deal with other variables such as intensity or duration.

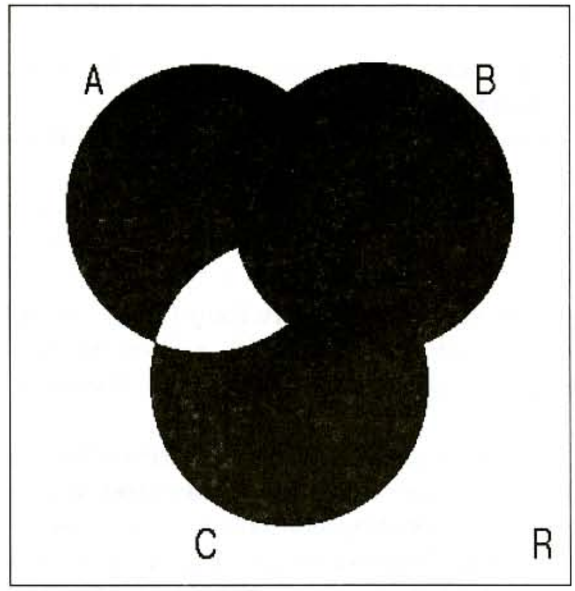

Figure 4
Challenge Venn Diagram

Finally, I leave you with the following.
Let

$$A = \{\text{A, B, C}^{\#}, \text{G}\}$$
$$B = \{\text{B, C}^{\#}, \text{D, E}\}$$
$$C = \{\text{B, E, F}^{\#}, \text{G}\}$$

and let

$$F = \begin{matrix} 2*A \cdot \overline{B} \cdot \overline{C} + 2*\overline{A} \cdot B \cdot C + 2*\overline{A} \cdot \overline{B} \cdot C + 2*\overline{A} \cdot B \cdot C + \\ 2*\overline{A} \cdot B \cdot \overline{C} + 2*A \cdot B \cdot \overline{C} + 2*A \cdot B \cdot C + 2*A \cdot \overline{B} \cdot \overline{C} + \\ 2*\overline{A} \cdot B \cdot C + 2*\overline{A} \cdot B \cdot \overline{C} + 2*A \cdot B \cdot \overline{C} + 2*A \cdot B \cdot C + \\ 2*\overline{A} \cdot B \cdot C + 2*\overline{A} \cdot B \cdot \overline{C} + 2*A \cdot B \cdot \overline{C} + 2*A \cdot B \cdot C + \\ 2*A \cdot \overline{B} \cdot \overline{C} + 2*\overline{A} \cdot B \cdot C + 2*\overline{A} \cdot \overline{B} \cdot C + 2*\overline{A} \cdot B \cdot C + \\ 2*\overline{A} \cdot B \cdot \overline{C} + 2*A \cdot B \cdot \overline{C} + 2*A \cdot B \cdot C + 2*A \cdot \overline{B} \cdot \overline{C} \end{matrix}$$

where $2*$ means that a certain note is played twice, sequentially, and $+$ is the transition from one note to another. The corresponding Venn diagram is shown in Figure 4.

Here is the challenge: Interpret the function (determine the sequence of notes), and give the interpretation to a musician. Ask her to play it, and try to *name that tune*!

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San Diego Joint Math Meetings

The Humanistic Mathematics Network has organized a panel at the San Diego Joint Math Meetings.

Saturday, January 11, 1997, 2:30 - 3:50 PM.

"Art, Literature, Music and Math: Degrees of Similarities."

Speakers will be Annalisa Crannell, Leonard Gillman, JoAnne Growney.

Moderated by Alvin White.

Book Review: *Emblems of Mind, The Inner Life of Music and Mathematics*, by Edward Rothstein

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We associate beauty with music, but not often enough with mathematics...
...Edward Rothstein

Emblems of Mind: The Inner Life of Music and Mathematics. Edward Rothstein. Avon Books: New York, 1996. 263 pp, ISBN 0-380-72747-1.

As a graduate student in mathematics with a serious second interest in piano, I often heard the platitude that mathematics and music go hand in hand. My own informal research was leading me to conclude that while composers had their mathematical work staked out for them and most of my mathematician friends loved music, nevertheless musicians very often disliked mathematics; moreover, the music loved by a mathematician was liable to extend as far as a mechanically-played Bach fugue. We might agree that those contrapuntal voices were . . . mathematical! But was there a mathematician, I wondered, who had feelings for the work of Scriabin or Brahms, or who wanted to discourse on Mozart's laughter through sorrow (or was it sorrow through laughter?) or the profound harmonies of Beethoven? The predominating hobbies among our mathematics group were chess and baseball. I gave up on comparisons that I felt were superficial — there was even a saying among my musician friends that people who *talk* about music can't *do* it. (Later I found that musicians talk about *musicians*.) At any rate, I decided that the best one could say is that both subjects are non-verbal, possibly indicating a brain deficiency, and I married an English major.

In *Emblems of Mind*, Edward Rothstein creates a galaxy of connections between mathematics and music. Analogies are everywhere. The title derives from Wordsworth's poem, "Prelude," a key statement of the romantic movement regarding the nature of art, in which the poet, walking up a mountain, is overwhelmed by the moon and the surrounding panorama of nature.

"There, I beheld the emblem of a mind
The power, which all
Acknowledge when thus moved, which Nature thus
To bodily sense exhibits, is the express
Resemblance of that glorious faculty
that higher minds bear with them as their own."

Thus the "emblem of a mind" may be seen as the moon or something more -- a symbol of the outer world interacting with the artist's inner world vision, creating a sense of harmony or unity within the universe. Rothstein's plan is to explore the *metaphors* shared by music and mathematics, with emphasis on the *process* of discovering, comparing, and contrasting their natures. With this approach, we are very much engaged with the project of interacting with Rothstein's mind, but the journey is smoother than that found in many books which feature interpreters or guides; Rothstein's voice is always serene and poised, never capricious, cynical or evangelical. While he may be pointing the way up the poet's mountain, he never falls into the trap of lecturing, which is admirable, considering the scope of the book. Experts might recognize the influence of the University of Chicago, and in fact, Rothstein was on the Committee on Social Thought there, where he combined studies of music, philosophy and literature. If the Committee influenced his ideas, the book itself grew out of an essay which Rothstein a music critic for *The New York Times*, wrote about his "'two strongest intellectual passions.'"

My greatest challenge with the book, explained a little by my own history, was to bear through introductory formalistic explanations that seemed to deny music the emotional vocabulary I required. (The book is *Emblems of Mind* after all...but is this really how a music critic thinks?) It was not until halfway through the book that I saw that Rothstein perceives music with much more than an eagle eye. For example, when

Rothstein talks about music formalistically, it may sound like this -- describing the initial phrase in Bach's D# minor fugue:

> "it begins with a leap upward, but it is felt less as a leap than an unfolding. It should be heard as if the second note grows out of the first, opposing it but also connected to it. The theme then turns with a plaintive caress, and, as if taking a breath, gently echoes its own beginning before sadly returning step by step, to its origins. The gesture's two parts have almost different characters--an excursion and return. . . ."

To the author the phrase is a living and breathing organism; however, I am still on the sidelines wondering about the faith and determination, loneliness and peace, fortitude and acquiescence in Bach. Until Chapter 4 where the book suddenly seems to melt in the warmth of its discussion of beauty, Rothstein is apt merely to shroud that which cannot be explained by reason, like the poet's moon, with the label, "mystery." Thus, the reader is advised to be patient.

In fact, Rothstein claims early on, that emotion is too simplistic a basis on which to found a definition of music. For example, the Indian rage may bring about meditative states, while chanting music may provide energy for hoeing a field, or a drum beat serve up the background language for a culturally expressive dance. A crescendo in Palestrina may be there to send us to heaven and not on any passionate route. These examples are slightly unfair, since the music Rothstein spends effort analyzing is definitely western, and classical or romantic at that -- never even contemporary. Eventually, Rothstein will pull out all the stops and will not only acknowledge the emotional countenance of music but will demonstrate ways in which music goes beyond what Thomas Mann calls 'cow warmth,' to the topic of music's power to affect our lives. But even while he is avoiding dipping into the emotional pool and is maintaining an arm's length (or mind's length) on his subject, the author always manages to convey his exuberant joy in these twin stars of the galaxy, these 'kindred mutations.' He convinces us that both subjects achieve their value when seen as a process; thus we have multiple layerings of processes in this book, recursion at play. Mathematically speaking, mathematics and music are functions, and his job is to attempt to give a metaphor for the mapping (or fugue) between these functions (or voices); in this process we map onto Rothstein's mind, or the author's mind onto us.

In introducing the similarities between marhematics and music, Rothstein offers up the gamut. For example, he notes that both subjects are represented as languages encrypted in special notation which does not necessarily read linearly. They are subjects in which one talks of "giftedness," subjects with quasi religious roots ("mystery"), and which carry epical tales of their heroes. Some of the mathematicians (Pythagoras, Aristotle, Euler, Kepler, Galileo, etc.) we recognize as being at the job of trying to link mathematics and music long ago. The *tritone* (an augmented fourth -- from C to F#, an impossible interval for singers, best remembered by the opening two notes in the song *Maria*, from "West Side Story") was regarded as "unutterable," the *diabolus in musica* in the Western church, just as the irrational number $\sqrt{2}$ was regarded as *alogon*, or unutterable, by the Greeks. (Rothstein does not push the parallel by pointing out

. . . mathematics and music are functions, and his job is to attempt to give a metaphor for the mapping (or fugue) between these functions (or voices)

that F# is practically halfway up the logarithmic scale, giving it a relative frequency of just about $\sqrt{2}$.) Mathematics and music also share levels of complexity, a sense of space, the creation of order, a reliance on axioms governing a style, and fundamental building blocks such as groups (in music, groups of tones). Rothstein disclaims "very little of what I say about mathematics will be news to mathematicians and very little of what I say about music will be news to musicians and composers. The hope is that much of what I have to say will still be of interest because of the juxtapositions I make and the hypotheses I propose." Since the translation is not exact, and there is no formula for the mapping, we find ourselves delightfully orbiting around a central premise, from time to time struck by meteoric insights. I marvel at the artful construction of the book; it is a virtual playground for seeking out connections. For example I have just noticed that Rothstein begins with a description of *numbers* and introduces the musical scale as a *ladder of discrete tones*; in his summarizing chapter, titled "Chorale," he discusses Socrates' hierarchical ranking of *levels of thinking* about the world.

To hone in on the connections he wants to make, Rothstein first dialectically divides his subjects, with chapters on the inner natures of mathematics and music. In the mathematical chapter, dubbed "Partita" we are treated to a seamless transitional linking of mathematical topics where mathematics flows like a stream into a river, from numbers to the concept of the irrational, to the infinitesimal, and from there, to spaces characterized by their axioms, or *styles*. (A partita is a suite of dances.) A few parallels with music are thrown out, but the reader is probably already able to make his or her own connections, the choice of terms is so suggestive. In the chapter on the definitional character of music, "Sonata", Rothstein suggests that music exists for particular audiences and as such, is "modeled." Think, he suggests, how music accepted as appropriate in a horror film might be received in a concert hall or church.

> But in music neither the model nor the object nor the map is clear. How could music possibly progress in its understanding of a concept or an experience? I spoke metaphorically about film scores 'modeling' an emotion and about a piece of music serving as a 'model' but what can this mean? Music does not even seem to be looking for something to model from the world; nor does it seem to involve the sort of reasoning we find in mathematics. In music we don't see the act of construction taking place . . . we are submerged in a realm in which at least at first, 'knowing' seems irrelevant.

He then returns to the argument that the *product* of mathematics too is more of a *process* than a *result* -- something which we probably forget all too often in our teaching. "There is," he claims, "'a life to the ideas within a mathematical proof.'" In other words, the message is the modeling, mathematics itself is a model of mathematical activity.

Contemporary composers have used mathematics via computers to output random music or to model and blend styles, to fractalize a Bach fugue or to paint a page with notes in chaotic or planned patterns; Rothstein ignores these connections. But I confess to being at a loss when he follows on the trail of musicologists such as Heinrich Schenker and David Lewin with dizzying phrase-by-phrase and note-by-note dissections of musical pieces or phrases to make his points. Every corner of "muso-mathematics" is probed here: distance, connection, contraction and expansion, equilibrium, the vertical and horizontal, modulations, patterns, symmetry, variations, contradictions, ascending sequences, leaps, disorientations, pulsings -- it would hardly matter what conclusions are made with this tsunamic force of words. At least no one

. . . the product of mathematics too is more of a process than a result . . . mathematics itself is a model of mathematical activity.

will be left believing that composers are not in possession of mathematical minds.

It is a deserved pleasure when Rothstein climbs higher on the poet's mountain, to discuss what mathematics and music share as *art* forms. In Chapter 4, "Theme and Variations," he investigates the meanings of beauty and truth.

> There is something about beauty that is both private -- because it involves silent feeling -- and public --because it makes us feel as if it is revealing something universal. It emphasizes both our isolation and our feelings of common sense and sensibility. For the same reason it can also risk inspiring contemplative withdrawal or impassioned absolutism. The judgment of beauty is not idiosyncratic -- or so we think and feel -- but something more fundamental. Beauty feels like an aspect of public knowledge. We may not actually assert that everyone *will* agree with our proclamation of beauty, but beauty inspires a feeling that everyone *should* agree. The feeling makes a claim

> not only on us, as we view the beautiful object but on our sense of others. (This is the origin of snobbery.) Of course we haven't proved that taste is universal or that beauty is objective; the only assertion is that the judgment of beauty is treated as *universal* and felt as *objective*. . . .

Music, he asserts, allows us to dwell in a second nature; it intoxicates, it is more powerful than ourselves. Music is not merely abstract; it reflects the ways in which we experience the world. Mathematics also shares these attributes: the mathematical proof that loses touch with reality and becomes baroquely ornate for the sake of itself alone (the ring version of the Chinese Remainder Theorem) also loses some traditional notions of beauty. But note: music may not necessarily be beautiful in the sense of delighting (Chopin's "Dies Irae" prelude, in A minor); it may "disturb and overwhelm;" it is not so much beautiful, as . . . here the word "mystery" is converted to the romantic's word: "sublime." "To our rational minds . . . the sublime seems to subvert our judgment, perpetrating, in Kant's words, an 'outrage on the imagination.'" Mathematics as well has this quality of being sublime in its sheer immensity and depth.

Rothsein's last chapters are refined writing, as he continues to navigate where many authors might find the oxygen too thin. Retaining the idea of "metaphor," he keeps a charted course, scattering thoughts that in themselves could create books, and distributing wonderful quotes. Speaking, but not preaching, about the creative process in mathematics and music, Rothstein says: "our greatest risk is that our metaphorical interpretations will be willful, arbitrary, unenlightened, that connections will be made of trivial importance." He himself never falls into this trap. Now the topic has become the power (and soul -- but this is a word he never mentions) of music and mathematics; the ability of mathematics to interpret the universe, while the power of music is interpreted in its equally versatile ability to bear a variety of meanings, its role as "gesture" which makes it a voice in our cultures. From the heroic to the sarcastic, the diabolical to the religious, from the excellent to the inferior, it can present us with a narrative of our history and a reflection of ourselves.

I occasionally felt some pang of regret that the book presumes a large body of prior understanding about these subjects. I know I would have enjoyed the book as a student, but wonder about the reaction of today's students; would they find the subjects of mathematics and music so exalted? Much of what I have learned in the past thirty years that has fascinated me about mathematics is the ways in which statistics and probability, chaos and computers have come to the inside track, and regarding music -- its manifestations outside the western classical idiom in the contemporary and ethno-cultural realms. Thus part of my understanding of mathematics and music is not covered in this book, which is more wrapped in the traditional and creates something of a backward look, perhaps a "bridge to the past." But *Emblems of Mind* is a tour de force in writing and thinking the concept of metaphor providing all of us (not just mathematicians or musicians) a way of seeing how thinking about one discipline can be a useful way to think about another, giving validity to the act of thinking on many levels. (If Socrates would have it, these levels would be: intellection, thought, . . . *trust*, and *imagination*!) And if the subjects of mathematics and music are still immutably divided for me in that mathematics will never have the emotional heart of music, the warmth of tone in this book comes as close as anything I have experienced in helping to span that gulf by humanizing mathematics.

> The search for the sublime links music and mathematics. Both arts seek something which combined with the beautiful provokes both contemplation and restlessness, awe and comprehension, certainty and doubt. . . . The sublime inspires an almost infinite desire, a yearning for completion which is always beyond our reach. But we are then comforted by the achievements of reason in having brought us so close to comprehending a mystery fated to remain unsolved.

Book Review: *Emblems of Mind: The Inner Life of Music and Mathematics*, by Edward Rothstein

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Emblems of Mind: The Inner Life of Music and Mathematics. Edward Rothstein. Times Books: New York, 1995. 263 pp, ISBN 0-8129-2560-2.

While many writers have commented on mathematics and music, this author ultimately pursues a deeper relationship between those subjects. The connection that the author describes and promotes is along aesthetic, philosophical, even religious lines. Rothstein's credentials indicate that he is definitely up to the task. He has studied graduate-level algebra, analysis and topology as well as music and literature. He is an award-winning musical critic and chief musical critic of The New York Times.

To support his arguments, Rothstein calls upon a veritable cast of superstars of Western thought and art. To evoke a feeling for the inner life of the two subjects the author makes references to the works of Cantor and Chopin, Dedekind and Debussy, Helmholtz and Haydn, and many others. Many wonderful quotes are sprinkled throughout the book, which give testimony from great thinkers as to a math/music connection. Here is one from James Sylvester: "May not Music be described as the Mathematic of sense, Mathematics as the Music of reason?" From musician Igor Stravinsky: "The musician should find in mathematics a study as useful to him as the learning of another language is to a poet."

"May not Music be described as the Mathematics of sense, Mathematics as the Music of reason?"

In order to make the connection comprehensible, there is of necessity a good deal of preliminary spadework. This is done in the first three chapters, appropriately titled, "Prelude", "Partita", and "Sonata." In Prelude, Rothstein adopts as a guiding metaphor for the entire book, the journey of William Wordsworth to the peak of Mount Snowdon.

In "Partita", Rothstein discusses the inner life of mathematics. Although the discussion is declared to proceed heuristically, and to be sparing on details, two proofs of the infinitude of the set of primes are presented, compared, and contrasted. Additionally, fundamental concepts from set theory, analysis, and topology are described. The prerequisite for reading the book as given in the introduction is: no more than high school mathematics and no more music than what is learned in elementary school. A year of college-level mathematics would seem to be a more suitable prerequisite.

"The musician should find in mathematics a study as useful to him as the learning of another language is to a poet."

One of the longest chapters, titled, "Sonata", presents the author's opinion of the inner life of music. It is rich in musical nomenclature and references almost all the well-known composers of the past as well as the contemporary musician, David Lewin, who is described as a musical topologist. Some of the terminology of this chapter is a marriage of mathematical and musical terms such as, "musical regions with different centers of gravity," "continuous musical surfaces," and "musical modelling."

One of the goals of Chapter 4 is to convince the reader that it makes as much sense to call mathematics beautiful as it does music. Rothstein backs up this premise with quotes such as this from Hermann Weyl: "My work always tried to unite truth with the beautiful; but when I had to choose one or the other, I usually chose the beautiful." The Cantor set, formulae involving pi, and several pages on the Golden Ratio are included as examples of beauty in mathematics.

In Chapter 5, "Fugue: The Making of Truth," the aesthetic/religious natures of mathematics and music are described to show what the author considers the really important connections of these subjects. We are reminded that both mathematics and music have been closely associated with religious ritual. How do mathematics and music seem so "other worldly" yet impact our lives daily? This question is not about the internal workings of the subjects but about how they "map into" the world - it is a question about meaning and truth. Rothstein describes mathematical proof as ritual and uses a quote from G. H. Hardy to support his contention: "If we were to push it to its extreme, we should be led to rather a paradoxical conclusion: that there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but point; that proofs are . . . gas, rhetorical flourishes designed to affect psychology."

"My work always tried to unite truth with the beautiful; but when I had to choose one or the other, I usually chose the beautiful."

We are finally prepared for the point: The mathematician, the musician, the poet, all imitate "Nature at work, reproducing in their creations the emblems that Nature had bodied forth in hers . . . A mathematician will spin out a new theory or a composer create a miniature sonic universe; a poet will turn an experience into metaphor, a scene into a source of illumination. And each creator will, 'mid circumstances awful and sublime,' be as astonished by the result as was Kepler or Bach."

The book is really a wonderful work which glorifies two subjects of great importance to any civilization. It would be excellent as a required or ancillary reading in a Liberal Arts Mathematics course.

Mathematical Rebuses

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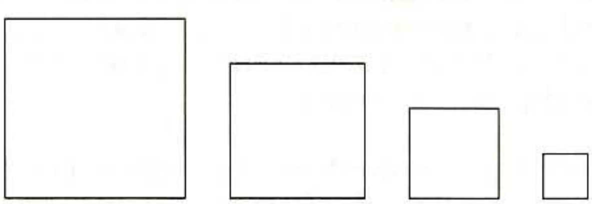
$$\begin{pmatrix} M & R & X \\ R & A & I \\ X & I & T \end{pmatrix}$$ = Symmetrical Matrix



= Method of the Littlest Squares

Fibonacci Melodies

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INTRODUCTION

One way in which mathematics has informed modern musical composition is through the use of algorithms. To compose algorithmically one begins with a sequence of numbers and maps the terms of the sequence into various musical parameters such as pitch, duration, dynamics and even timbre. Students of Arnold Schoenberg (1874-1951) and Anton von Webern (1883-1945) are credited with first using this composition technique whose incipient stage was known as *serialism* [4, p. 544]. More recently composers have been employing iterated function systems and chaos theory (e.g., [1], [3], [5]) to produce music.

Little is written, however, about the influence which music has had on mathematics. This article describes how the creation of a musical composition suggested a theorem concerning the Fibonacci sequence: $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$

THE FIBONACCI COMPOSITION

One can create a very simple example of algorithmic music by associating the terms of the Fibonacci sequence with notes on a keyboard. In the example that follows, no attention is paid to timbre, dynamics, or duration; unless otherwise specified, each note is taken to be a quarter note. Of course since there are more terms in the Fibonacci sequence than there are keys on a keyboard, a more reasonable association would map the terms of the Fibonacci sequence modulo $m \in Z$, the set of integers, to the keyboard keys. Although any value of $m \geq 2$ would work, reasonable values include $m = 88$ (a piano has 88 keys) and $m = 8$ (an octave includes 8 notes). Selecting $m = 8$, one can establish the very straightforward correspondence appearing in Table 1 where x is any positive integer.

x mod 8	0	1	2	3	4	5	6	7
Note	*	C	D	E	F	G	A	B

Table 1
Correspondence between integers modulo 8 and notes on a keyboard

In this example the symbol * is a wildcard and may be interpreted in any number of ways. Let's agree that the effect of encountering a * in a string of notes is to change the duration of the previous note (if one exists) from a quarter note to a whole note. Another arbitrary decision concerns the octave in which the notes will be played. Again let's let that decision be idiosynchractic, entirely up to the discretion of the composer.

Table 2 reveals the sequence of notes generated in this manner by the first 24 terms of the Fibonacci sequence.

Term	1	2	3	4	5	6	7	8	9	10	11	12
Fibonacci Sequence mod 8	1	1	2	3	5	0	5	5	2	7	1	0
Note	C	C	D	E	G	*	G	G	D	B	C	*
Term	13	14	15	16	17	18	19	20	21	22	23	24
Fibonacci Sequence mod 8	1	1	2	3	5	0	5	5	2	7	1	0
Note	C	C	D	E	G	*	G	G	D	B	C	*

Table 2
Notes Generated by the Fibonacci Sequence mod 8

With our conventions concerning octaves, duration, and the interpretation of the *, and assuming common (4/4) time, the first four measures of this song are depicted in Figure 1.

While Figure 1 reveals a surprisingly mellifluous sequence of notes, Table 2 invites us to examine the cyclic nature of the Fibonacci sequence mod 8. Specifi-

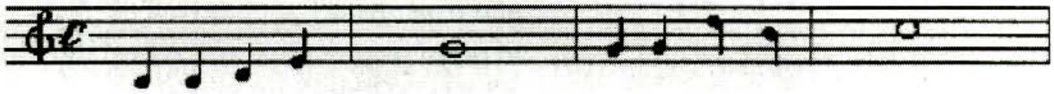

Figure 1
The First Four Measures of a Fibonacci Song

cally, with $f(n)$ representing the n^{th} term of the Fibonacci sequence, Table 2 suggests the following conjecture:

$$f(n) \bmod 8 = f(n+12) \bmod 8 \tag{1}$$

Before attempting to prove this conjecture, it may be of interest to examine the effect of changing the modulus from 8 to some other numbers.

It is a simple matter to verify that the Fibonacci sequence mod 7 is:
$\{1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, \ldots\}$

and that the Fibonacci sequence mod 6 is:
$\{1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0, 1, 1, 2, 3, 5, 2, \ldots\}$

These sequences suggest the following conjectures:

$$f(n) \bmod 7 = f(n+16) \bmod 7 \tag{2}$$

$$f(n) \bmod 6 = f(n+24) \bmod 6 \tag{3}$$

Since all three conjectures involve the terms $f(n)$ and $f(n+x)$, it may be of interest to examine the relationship between these two expressions.

PROPERTIES OF THE FIBONACCI SEQUENCE

Recall that the Fibonacci sequence is defined recursively by the equations:

$$\begin{aligned} &f(1) = 1 \\ &f(2) = 1 \\ &f(k) = f(k-1) + f(k-2), \textit{ for } k > 2 \end{aligned} \tag{4}$$

Using (4) repeatedly, notice that:

$$\begin{aligned} f(n+x) &= f(n+x-1) + f(n+x-2) \\ &= 1f(n+x-1) + 1f(n+x-2) \\ &= 2f(n+x-2) + 1f(n+x-3) \\ &= 3f(n+x-3) + 2f(n+x-4) \\ &= 5f(n+x-4) + 3f(n+x-5) \\ &\ \ \vdots \end{aligned} \tag{5}$$

The coefficients of the terms on the right are all Fibonacci numbers and so (5) may be written as:

$$\begin{aligned} f(n+x) &= f(2)f(n+x-1) + f(1)f(n+x-2) \\ &= f(3)f(n+x-1) + f(2)f(n+x-2) \\ &= f(4)f(n+x-2) + f(3)f(n+x-3) \\ &= f(5)f(n+x-3) + f(4)f(n+x-4) \\ &\ \ \vdots \end{aligned} \tag{6}$$

Equations (6) suggest the following theorem:

Theorem 1:
For $n \geq 2$ and $x \geq 1$, $f(n+x) = f(n)f(x+1) + f(n-1)f(x)$.

This theorem can be readily proved by induction on x [2, p. 289].

Corollary 1:
For $x \geq 1$, if *m divides* $f(x)$ and *m divides* $f(x+1) - 1$, then $f(n) \bmod m = f(n+x) \bmod m$.

Proof:
Given any positive integer n, suppose

$f(n+x) \bmod m = k$. Then there is some $c \in Z$ with

(7) $$f(n+x) = cm + k$$

We wish to show that $f(n) \bmod m = k$. Since, by hypothesis, *m divides* $f(x+1)-1$ and *m divides* $f(x)$, there exist $r, s \in Z$ with

(8) $$f(x+1)-1 = rm \quad (i.e., f(x+1) \equiv 1 \bmod m)$$

and

(9) $$f(x) = sm \quad (i.e., f(x) \equiv 0 \bmod m)$$

Case 1:
Assume $n = 1$. Then

$$\begin{aligned} f(x+n) \bmod m &= f(x+1) \bmod m \\ &= 1 \bmod m \quad (\textit{from}\,(8)) \\ &= f(1) \bmod m \\ &= f(n) \bmod m \end{aligned}$$

Case 2:
Assume $n \geq 2$. Then from Theorem 1

(10) $$\begin{aligned} f(n+x) &= f(n)f(x+1) + f(n-1)f(x) \\ &= f(n) + (f(x+1)-1)f(n) + f(n-1)f(x) \end{aligned}$$

from (7), (8), and (9) we have

$$cm + k = f(n) + (rm)f(n) + f(n-1)(sm)$$

so

$$\begin{aligned} f(n) &= (c - rf(n) - sf(n-1))m + k \\ f(n) &\equiv k \bmod m \end{aligned}$$

Corollary 1 establishes a condition that is sufficient to assure that $f(n) \bmod m = f(n+x) \bmod m$. Expressed in the musical context in which this investigation originated, this corollary asserts that if a song is created using the algorithm described in this paper, and if *m divides* $f(x)$ as well as $f(x+1)-1$, a string of notes so generated will repeat indefinitely, i.e., the song is periodic with period x.

A natural question to raise is whether the sufficient condition of Corollary 1 is also necessary. The following corollary answers this question affirmatively.

Corollary 2:
If $f(n+x) \bmod m = f(n)$ for fixed positive integers x and m and for all positive integers m, then *m divides* $f(x+1)-1$ and *m divides* $f(x)$.

Proof:
Choosing $n=1$, the hypothesis implies that $f(1+x) \bmod m = f(1) \bmod m$ for fixed positive integers x and m. So, $f(1+x) - f(1) \equiv 0 \bmod m$, or $f(1+x) - 1 \equiv 0 \bmod m$, i.e., *m divides* $f(1+x)-1$.

To prove that *m divides* $f(x)$, begin with (10):

$$f(n+x) = f(n) + (f(x+1)-1)f(n) + f(n-1)f(x)$$

Rewriting, we get:

$$f(n-1)f(x) = f(n+x) - f(n) - f(n)(f(x+1)-1)$$

In particular:

$$f(x) = f(1)f(x) = (f(2+x) - f(2)) - f(2)(f(x+1)-1)$$

Since by hypothesis $f(2+x) - f(2) \equiv 0 \bmod m$, and since *m divides* $f(x+1)-1$, *m divides* $f(x)$.

CONCLUSIONS

This article establishes necessary and sufficient conditions for $f(n+x) \bmod m = f(n) \bmod m$ where $f(n)$ is the n^{th} term of the Fibonacci sequence. Musically, the result can be interpreted in terms of when a sequence of notes generated by the Fibonacci sequence is periodic. The novelty of the article lies in its demonstration that the relationship between mathematics and music is a two-way street. Beginning with a mathematical algorithm involving the Fibonacci sequence and the concept of modularity to compose a musical piece, we were rather unexpectedly led to a result in number theory.

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Mathematical Rebuses

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A
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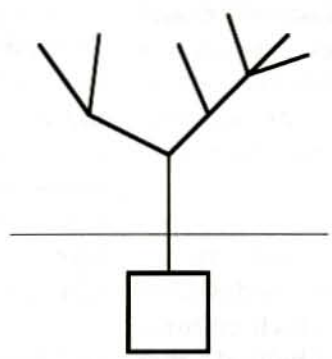
= Higher Arithmetic



= Square Root

S P A
C E

= Compact Space

Book Review: *Ethnomathematics: A Multicultural View of Mathematical Ideas*, by Marcia Ascher

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Ethnomathematics: A Mult icultural View of Mathematical Ideas. Marcia Ascher. Brooks/Cole Publishing, Co.: Pacific Grove, 1991. 203 pp, ISBN 0-534-14880-8.

Marcia Ascher's book *Ethnomathematics: A Multicultural View of Mathematical Ideas* is a superb treatise on mathematics from a multicultural point of view. The text focuses on a diverse collection of mathematical ideas and applications. Instead of limiting the scope of her text to the usual Eurocentric discussion, Ms. Ascher introduces the reader to the mathematical ideas of peoples who have generally been excluded from discussions of mathematics and the development /history of mathematics. These people are the ones who live in traditional or small-scale cultures, the indigenous peoples who live in places that were "discovered" and colonized by Europeans and include such diverse groups as the Iniut, Navajo, and Iroquois of North America; the Incas of South America; the Malekula, Warlpiri, Maori and Caroline Islanders of Oceania; and the Tshokwe, Bushoong and Kpelle of Africa.

In her introduction, Ms. Ascher sets the stage for her text. This introductory discussion acknowledges that "there is no single linear path along which cultures progress, with some ahead and others behind," that mathematics has no generally accepted definitions and that most definitions of mathematics are based solely on Western experience. Nevertheless, Ascher is quick to point out that although a particular culture may not classify an idea as "mathematics," traditional cultures most definitely express mathematical ideas in contexts that we westerners might call art, navigation, religion, record keeping, game playing, or kin relationships. In addition, Ms. Ascher acknowledges that as one views the mathematical ideas of others in their cultural context, one is limited by his or her own cultural and mathematical frameworks. Following this important introduction, Ms. Ascher guides the reader on an exciting journey that explores numerous mathematical ideas in a variety of cultural contexts.

Chapter 1 focuses on the concept of numbers and the words, symbols, and understanding associated with them. The sets of number words of several cultures are examined and the patterns and arithmetic relationships are described. The importance of language and its relationship to number words is emphasized and Ascher includes a clear and informative discussion of numeral classifiers. She devotes a substantive portion of this chapter to a discussion of the Incas and the organization and use of quipus within their culture.

Graph theory is the focus of Chapter 2. It is here that Ascher examines the sand tracings of the Bushoong and Tshokwe in Africa, and the drawings of the Malekula of the South Pacific. She clearly establishes the fact that many different peoples have pondered similar mathematical problems relating to Eulerian paths and provides an excellent background discussion of the Bushoong, Tshokwe and Malekulan cultures as she explores graph theory ideas within the respective cultures.

In Chapter 3, Ascher explores the important mathematical idea of relations or the specified properties that link pairs of objects. She does this by examining the logic of kin relations. The native peoples of northern Australia (the Warlpiri) a group with a particularly complex kin system provide an excellent example that forms the basis for this discussion which draws upon ideas from group theory.

Games of strategy and chance and the logic of puzzles provide the basis for Chapter 4. Since every game can be seen as an expression of a particular culture, Ascher is careful to identify not just the rules of the game but also the simple objects used for it, the times and places for appropriate playing of the game, the social settings, the level of concentration, the systems of rewards and all the other important aspects that make up the game. She includes an analysis of the American Indian game of Dish that is clearly rooted in the area of probability. This chapter also provides an excellent discussion of the Maori game of strategy known as Mu torere. Starting with a simpler version, Ascher leads the reader not only to an understanding of how to play Mu torere but also to a basic understanding of the mathematics connected with this game. A collection of river crossing puzzles from various cultures and the logic behind their solutions provide the final area of focus for this chapter.

The organization and modeling of space and time provides the content for Chapter 5. Because notions of time and space are so basic to the way we perceive, structure, and interpret our experiences, it is sometimes hard to understand or visualize the space-time ideas of other cultures. Nevertheless, Ascher successfully bridges this potential difficulty by her choice of examples. She includes apt discussions on the dynamic universe of the Navajo, the unique process and change dimension of the space-time concerns of the Iniut, and the navigational processes of the Caroline Islanders.

Spatial configuration is the basis of Chapter 6 which focuses particularly on symmetric strip decorations. Ascher includes an introductory discussion of isometries, symmetry, and symmetry groups and describes and utilizes a four character naming scheme for the possible strip patterns. (This scheme was developed by Russian crystallographers and is now accepted as the international standard.) A discussion of perfect coloring is also included in this chapter. Rafter patterns of the Maori and strip patterns found on Inca pottery provide beautiful and illustrative examples for the discussion of strip patterns in this chapter.

The final chapter of the book affords Marcia Ascher an opportunity to weave together the mathematical ideas and philosophies that are the basis for her book. She connects these ideas and issues to mathematics education, emphasizing the need for a redefinition of the boundaries of mathematics, and a revision of our philosophy and history of mathematics.

This outstanding book is a clearly written text that is well-suited for the college undergraduate level. The diverse collection of mathematical ideas in their cultural context provides a challenging yet very interesting array of mathematical topics. Ascher provides extensive notes with appropriate references which afford the reader additional sources for reading and scholarship. Marcia Ascher's book *Ethnomathematics: A Multicultural View of Mathematical Ideas* is a rare gem of a book. Read it!

Psychosis

Lee Goldstein

Nooscopic insociability
Can drive the human intelligence of an incognizable numinosity,
Thenceforward, to the equations of the sphere,
While this programmatic transposition
Can also beget, through the unconscious, an incipient eidos
That splits the personal
And abets an insurgence of psychical energies
Unto the hallucinatory,
That is seeming or chaotic.

---

nooscopic: pertaining to the examination of the mind

The Folktale: Linking Story to Mathematical Principles

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now retired

"Mathematics and Literature" has recently come into its own as a topic on the mathematics education scene. Sessions with this name are scheduled at National Council of Teachers of Mathematics and other organizational conferences. A department called "Links to Literature" now appears regularly in Teaching Children Mathematics, the NCTM publication dealing with the lower grades. Most of the articles included in "Links to Literature" tell how to plan classroom activities based on stories read by or read to students.

For instance, "Mathematics and Mother Goose" uses the familiar rhymes as a springboard to illustrating prenumeration concepts such as patterning, ordering, recognizing attributes, and classifying into sets [1]. "Popping Up Number Sense" relates how popping popcorn was used as a device to bring the concepts involved in *If You Made a Million* alive [2, 3]. "Mathematics Is Something Good!" tells how a teacher used *Moira's Birthday* as a stimulus for a discussion of rate as her second-graders tried to figure out how fast all the children Moira invites to her party would take to eat the cakes she has ordered for her birthday party [4, 5]. Other titles such as *Ten for Dinner* and *The Story of Z* produce related activities in graphing [6, 7]. And, as implied by the title, *How Big is A Foot?* can be used to inspire learning about linear measurement, nonstandard units, and use of a ruler [8].

Counting books are also referenced, and books depicting quilt patterns and the history surrounding them are also used as inspiration for mathematical investigation.

"Fictional Literature", an article in *Mathematics Teaching in the Middle School*, makes note that it is difficult to find middle-level fictional books which mention mathematics in a positive way [9]. Another article in the same journal examines heritages from other cultures, such as calendars and names [10], and a recent article in *Humanistic Mathematics Network Journal* tells of a newly-developed Middle School Mathematics Minor Certification Program course at St. Norbert College in Wisconsin which extends the search for mathematics into an examination of pottery, beadwork, textile, art and basketry patterns; archeoastronomy; comparisons and contrasts in mathematical philosophy; and the mathematical bases for games of chance - all serving as an avenue for "exploration of human endeavors within their cultural context" [11]. In addition, "Mathematics and Poetry" also finds its way into discussions of the use of literature as well. (N.B. Many such poems have been published in the *Humanistic Mathematics Network Journal.*)

As the reader may observe, the stories noted above for use in the classroom were all published within the past twenty-five years. Moreover, it is important to recognize that for the most part, it required the ingeniousness of a teacher to relate mathematics to the story.

But there is yet another way to link mathematics and story, mathematics and human endeavors, mathematics and culture — and I suggest that it is as significant and perhaps more fundamental than any of the examples noted above.

While searching for folktales to use in my present work as a storyteller, I have discovered stories which I believe actually illustrate mathematical principles. The stories in the articles noted above provide a jumping-off place for exploration of mathematical notions, but the folktales I have been collecting are themselves built on mathematical concepts. And therein lies the difference.

For instance, there is a story involving six young lads who go off fishing. Just before they are to return home, Brother Number One decides to count to see if all his siblings are present. He counts to five (forgetting himself) and begins to cry. Brother Number Two asks what's wrong, and upon hearing the problem, proceeds to count. He too finds only five brothers...and so it goes, until a boy comes along and asks if he can help. He quickly sizes up the situation, and asks each brother to count aloud as he squeezes hard a hand from each brother. They soon find that they really are six in number, and joyously reward the stranger with their entire catch. Everyone goes home happy! What better example of one-to-one correspondence, so simple that it can easily be appreciated by a six-year-old [12].

The version recounted above is retold from a tale that was collected in eighteenth-century England. I have also found two American, one Middle Eastern and one African version of this tale, each setting the story within the context of its own culture. "How the mathematical concepts became a part of the folk culture?" is a challenging question in itself. Did people adapt the principles and then apply them to events in their daily lives? Did someone hear the story in a far-off land and then change it to a more appropriate setting before recounting the story to family and neighbors? Or did people in different areas instinctively invent their own versions? Whatever the sequence, there are often multiple versions of many folktales, including those based on mathematical principles, illustrating again the unversality of mathematics throughout different cultures.

A wonderful introduction to fractions is found in *Two Greedy Bears*, a current-day retelling of an old Hungarian folktale wherein a fox helps two bears who are trying to equally divide a cheese into two parts [13]. The fox cleverly keeps dividing the cheese into unequal parts, each time nipping off a piece of the larger part, ostensibly to make the piece even, but always managing to make one part larger. The fox ends up leaving only two crumbs for the bears. But the pieces were equal! A Middle Eastern version, "The Ape and the Two Cats", describes how two cats steal a cheese, and then ask an ape to divide it equally, since neither cat trusts the other to divide the cheese equally. The ape carries on in a similar fashion to the fox noted above. He finishes off the cheese, and the cats conclude that there is "no wrongdoer who is not afflicted by a greater wrongdoer" [14].

A more sophisticated discussion of fractions can arise readily from a story I heard as a child. It seems that there was a father who left his herd of seventeen horses to be divided among his three sons, the oldest to get one-half, the middle to get one-third, and the youngest to get one-ninth. A wise neighbor lends them his horse to make a total of eighteen. The sons receive nine, six, and two horses, respectively. The neighbor takes home his horse, and all are satisfied. Thus far I have found only Middle Eastern versions of this story [15, 16].

Nasreddin, a colorful Middle Eastern character, is working in his garden. A stranger comes along, engages Nasreddin in conversation, and then asks how much time it will take to walk to the next town. Nasreddin does not answer. The stranger politely asks again, and then shouts his question. But Nasreddin still does not say a word. Exasperated, the stranger turns toward the town and begins walking. Suddenly Nasreddin exclaims, "Fifteen minutes." The stranger, astonished, turns and asks why Nasreddin did not say as much before. "Well", replies Nasreddin, "before, I did not know how fast you were planning to walk!" [17]. Mathematical thinking at its best!

Then there is the perennial favorite, attributed to both India and China. A man solves a problem for a rajah or an emperor. In return he asks merely for grains of rice, to be granted with the aid of a chessboard: one grain the first day for the first square, two grains the next day for the next square, four grains next, then eight, and so on. The story makes a delightful introduction to the powers of 2 [18].

The folktales cited above have been written down in books, but were originally from the oral tradition. Indeed, when I recently told the Ethiopian version of "The Six Fisherman" to an audience of adults, a fellow in the audience told me how he and his family told a similar story to strangers when asked how many children were in their family. "I count eight!" was always the reply from one of the nine children. The man was from a small town in Ethiopia.

Part of the delight in finding (and telling) folktales which illustrate mathematical principles is in collecting multiple versions. Each variation invariably reflects a way of life peculiar to a particular people or country. I invite correspondence from readers who have such tales to tell:

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Mathematical Rebuses

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P **N** **S**
O **I** **T**

= Noncollinear Points

R *V* *R* *V*
M *K* *M* *K*
A *O* *A* *O*

= Markov Chains

On the Use of Intelligent Tutoring Systems for Teaching and Learning Mathematics

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INTRODUCTION

The idea of using computers in education is far from new. However, the more naive attempts have not been considering all the issues involved in such a complex task. As with any alternative tool, the use of computers for educational purposes demands caution in order to reach its goals; otherwise such effort can result in negative outcomes only. While technological advances continuously bring new design alternatives, conceptual problems which arise from the peculiarities of this medium seem to be frequently dismissed by computer scientists. In fact, no one can guarantee the tutorial system effectiveness (i.e. the student learning efficacy) only by virtue of its technological state. Students and human tutors have particular relationships with computers and this fact cannot be ignored during the design of tutorial systems. The real system's educational role strongly depends upon the roles of all other environmental components.

Computers cannot be seen as a panacea for educational problems. Some enthusiastics in education and computing areas tend to see technologies as the solution to most educational problems. Indeed, educators should not transfer the task of building an efficient automatic tutor to programers and computer scientists under the risk, among more serious repercussions, of the undermining of their own roles in the educational process.

First of all it is imperative to precisely determine what should be done by a computational assistant and what should be left to the human tutor. Moreover, the way in which the system is intended to reach its goals must be carefully designed. Only then, through a controlled experiment with students and human tutors, could the computer tasks be judged with regard to their learning goals.

This paper discusses some issues related to the benefits of tutoring systems and the care needed in the domain of mathematics. First, we will argue for a realistic learning environment where tutorial systems could yeld positive results. The human tutors' and students' roles will be also discussed. Second, we will list and discuss some relevant and problematic tutoring systems features. We conclude this paper by addressing some guidelines related to intelligent tutoring systems design in the domain of mathematics.

INTELLIGENT TUTORING SYSTEMS AND THEIR ROLE IN EDUCATION

Intelligent Tutoring Systems (ITSs) are education-purpose computer programs that incorporate techniques from the Artificial Intelligence (AI) community. They date back to the early 1970s and derive from CAI (Computer Aided Instruction) programs and differ from the latter in the use of AI tools in order to know what they teach, who they teach, and how to teach. The use of AI techniques presupposes an intention of producing human-based "good teaching," since most AI systems try to simulate human activities. Indeed, many ITSs are supposed to replicate all the teacher's activities. ITSs should neither be of naive Skinnerian type linear programs nor completely take over from the teacher. One useful role for ITSs ties in their potential of working as intelligent tutoring assistants. In this framework, AI techniques are welcome and necessary as well.

The need for better quality teaching and for more effective results has always been publicized. Teaching is a very complex task which demands knowledge, ability, mature thought, intuition, self-confidence, empathy, capacities of seeing and hearing, and the capacity of motivating the students, among other human abilities.

Teaching is a special task since it involves the sharing of human responsibilities in society. Children, youngsters, and adults should have their individual natures taken into account in this process. The teacher's behavior must then be adjusted to each student. Learning, on the other hand, is exclusive to the student, and no one, least of all, the teacher can take over in that process. The teacher's task is to provide for the student's learning by creating good external conditions for the development of the learning capacity. Learning is then a subjective process and depends on personal experiences. Two circumstances will determine its adequacy. The first is the motivation to study the subject and overcome knowledge difficulties. The second is the promotion of a safe environment for the student in which he/she gets more independence by overcoming his/her own reasoning and knowledge limits.

Not being a substitute for the teacher, an ITS is a teaching support tool, fitted to the necessities of revision, diversification, flexibility, problem solving, progress in content, etc. Moreover, in the classroom, while the teacher's pace of presentation depends on his/her own experience, through an ITS, the student can determine the pace at which the knowledge should be presented.

As a computational assistant, an ITS would complement teaching activities which are not covered by the teacher. ITS would be stimulating as long as it can be different from the traditional classroom model. However, three main issues can endanger the function of a computational assistant: its limited capacity for expansion, its set of teaching methods, and its inability to understand students' idiosyncrasies. These limitations, nevertheless, can stimulate new questions, analogies and corelations which are unusual in traditional settings. These questions can thus play a role enhancing discussion inside the classroom.

The individual interaction with an ITS favors the student's identification of his/her own mistakes -- a challenge that could imitate a game-like interplay with the machine. Moreover, the ITS can provide the teacher with help in the learning by doing approach which is so difficult to implement in classrooms. This environment also favors the development of intuitive reasoning such as the forecast of right answers. What will be the result? What will be the way to reach it? These questions will drive the procedures even if a realistic student-system dialogue is impossible. The sucess of the use of intuitive reasoning demands the use of analytical reasoning, which depends on the possibility of succeeding and so on.

The teacher's role in such an environment mutates as long as the students are more participatory, offering the former opportunities to discuss concepts outside the realm of the ITS. The teacher is also supposed to indicate why, when and how much the computational assistant should be used. The students' productivity offers the teacher parameters for the system feedback.

LEARNING MATHEMATICS THROUGH INTELLIGENT TUTORING SYSTEMS

One of the most insistent problems in mathematics education is the aversion that many students feel towards this subject. Even students at graduate school levels in mathematics or computing courses often have problems related to the disciplines involving some concepts they are supposed to already be acquainted with. The literature has many studies concerning errors made by students and the persistence of misunderstandings of such errors. There are also other studies reporting high rates of failure among students in mathematics. This probably can be attributed to their experiences in learning mathematics. The use of strategies which minimize rote repetition of algorithms would be of much value [1]. The repetition approach probably leads the students to construct an improper schema to solve the problems by themselves. Such an improper approach is reinforced by doing a large list of similar exercises with the same interpretation.

This is an important point that has to be thought of attentively. The lack of understanding of a concept may not be due to the concept itself. It is often due to an insurmountable barrier for the student which is not the current concept, but a previous one which is a prerequisite to that in question.

As teachers we cannot forget that before introducing a new concept to the class we must have it clear in our mind what adjacent ideas are also involved. For example, the lack of understanding quantification is often a barrier for students in developing a more sophisticated understanding of limits and continuity. This could explain, for example, why students fail to understand calculus and a really long list of other topics.

This example illustrates the necessity for students to be able to express the prerequisites of the concepts they are supposed to learn. It therefore seems that finding information about the idiosyncratic learning methods of understanding concepts we are going to teach, how they are learned, and what we as teachers can do to enhance the student's logical thought might contribute to the goal of improving the students' understanding of advanced mathematical concepts.

We believe that an effective understanding of a mathematical concept depends on individual efforts to construct these ideas by the students themselves. And it is possible to detect, through research, the different ways in which this can take place. We also believe that it is possible to develop computer-implemented tutors which are designed in order to stimulate the constructions detected by the research, towards a reasonable acquisition of mathematical concepts. It is important to notice, however, that a mathematician has his/her own understanding of the involved concepts and it is up to the teacher to have the awareness to avoid the bias of that understanding when the analysis of students' styles of learning is made. It is true that it is not that easy to completely avoid this (although implicit) interference; however, an effort should be made to minimize this as much as possible.

Dubinsky [1] pointed out that it is important to observe that any description of the concept must not only be "mathematically wrong" or "mathematically correct" but must also embody all of the subtleties and other styles used to understand the subject. We are sure that all of these variables come to enrich the process of analysis of the possible ways of learning, giving us many ramifications of the concept in question, reflecting its varying role in the full spectrum of mathematical endeavours.

Of course there are several ways to describe a mathematical concept. The process of its acquisition can be determined by observing students in the process of construction of the concept. The students' successes and failures can be important clues to the essence of the ongoing learning process. An accurate analysis of these components can reveal the defective points that lead a student to make mistakes, which if appropriately explored, would certainly contribute to the main goal teachers must have: to enhance the student's performance as a problem solver.

As Dugdale pointed out [2], presently we have the possibility of using AI methodologies for the realization of expert systems, which permit the use of computers to be extended to fields that some years ago only human experts could master. One such field which could paricularly gain from this is mathematics. We do not refer here to those systems which provide a one-way teaching interaction, but those which have a mixed-initiative teaching dialogue, which is individualized to the needs of the student as an individual. In this way, the analysis and the diagnosis processes must be present as one of the main factors. The intelligent tutoring system used to help students in learning mathematics is supposed to act as an assistant to the teacher. Its task is to support both student and teacher in the teaching-learing r elationship.

Thus, it is a matter of great weight to have a cooperative environment to help students in learning new concepts and prerequisites as well. It is important to emphasize that the ITS must lead the students to dominate their own problems step by step, encouraging them to become active, creative, and independent learners. The ITS system may also allow the student to choose a better way for himself/herself, resulting in a rich environment for exploration. We believe that learners will become more and more motivated and confident; they can find out that the more they learn the more they are able to do.

DISCUSSION

The questions that arise are if and how computational assistance can help in teaching mathematics. The prerequisite barrier can be overcome by the modelling of the students' knowledge by the system. But this is not quite simple. The nature of the students' knowledge to be considered and the rules to manage it are still major problems of ITS design. Most ITSs use poor measures of students' knowledge such as numbers for category levels and quantity of right and wrong exercises. More qualitative measures such as the students' knowledge about the relationship between concepts ought to be taken into account. An ideal student model should be made up of information about the history of the student-system dialogue, as well as information about the student's performance during problem solving. In terms of knowledge representation formalisms, AI-based models combine a frame-based schema with production rules and an inference mechanism for deriving new information about the student. However, the type of each information set and the rules connecting them are far from simple to provide.

However, it is also not simple to detect students' misconceptions. The cause for the students' errors can rarely be localized to a unique concept. Indeed, the method of relating concepts may be the problem focus. The reasoning method is supposed to supply the relationship between the concepts that the student detects in that domain. While the computational artifact seems to be adequate, the qualitative nature of the information remains open for further research investigation.

Probably the biggest problem in designing tutorial systems in the domain of mathematics is the need to handle reasoning. Beyond concepts, the student is supposed to learn the underlying reasoning. Therefore, handling the reasoning requires from the system a description or formalization of the knowledge. The computer should stimulate the student's reasoning, while deeper discussions should take place among classmates and teachers. Once this is done, another issue remains unsolved, which is the importance of stimulating the student to develop his/her own method of reasoning. A useful intelligent assistant should be able to understand and classify that method, or even learn a new one. However, students have idiosyncratic methods of solving problems and even sophisticated systems which know several methods cannot handle all the existing possibilities [4]. While human teachers are able to learn the students' methods through dialogues with them, the use of machine learning -- based approaches is in its early stages [3].

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The imposition of the teacher's way of reasoning can be avoided through the use of different solution methods appropriately presented. Since it is not possible to cover all styles of human thinking, we can start by associating the methods with the concepts in order to better present them to the student. However, only the explicit representation of this knowledge within the system can guarantee its capacity of detecting students' misconceptions and explaining its tutorial strategies. A still open problem related to this is the need for a dialogue where the student can explain his/her way of reasoning. Not only is this useful to enhance the system's knowledge about the student, but is also crucial for the student to become conscious of his/her own failures and successes. Here one should bear in mind the limitations imposed by the computer interface dialogues, especially natural language-based ones -- which still do not allow for a cooperative dialogue with the student. In a cooperative environment, the more active the participants are in the discussion, the more productive and effective the learning process is. The computer should stimulate the student's reasoning, while deeper discussions should take place among classmates and teachers.

As pointed out above, the computer should be part of an environment together with the students and the teacher. As such it is not completely true that the student is the only ITS user. ITS should be able to interact with teachers and students separately as special and equally important users. The role of the teacher as an ITS user must involve two issues: the system validation and the teacher's evaluation of the student.

By system validation we mean the access to the system knowledge bases (domain, student, tutorial) and to the rules that control them during a special session targeted to the teacher. As an expert for domain and/or tutorial knowledge, the teacher should interrogate the system in order to get a system radiography. The underlying assumption is that, as a dynamic tool, an ITS should be constantly adjusted, improved, and corrected.

The evaluation of the student takes place during or right after a student session in order to obtain information about the student performance. This data includes the student model information and the system's justifications for its decisions. System justification has not been granted enough attention in ITS projects. We cannot forget, however, that one of the most important features -- indeed requisites -- for an 'intelligent' system is its capacity for explaining or talking about itself. More important than the adequacy of its designation, this feature gives confidence to its users, the lack of which can jeopardize the entire learning process.

CONCLUSIONS

A useful computational assistant should know many presentation methods and know where and when they should or should not be used. This 'intelligent' feature is mandatory in any ITS and can be supplied by computer resources. In spite of this, there are some useful guidelines which should be followed to achieve successful learning. In mathematics, in particular, software must be attractive and challenging. This does not mean that it always must be camouflaged in games or the like. We do believe that with the cooperation of students and teachers, and only then, will it be possible to design useful assistant mathematical softwares.

Idiosyncratic learning methods demand different system characteristics. For example, some students prefer to be constantly evaluated, while others would prefer more complex evaluation methods; others like to know the system's teaching methods, while others would prefer not to see the system as a teacher. When and how the internal system knowledge should be presented can be a question of preference to the student, but it is mandatory for the teacher who must have access to the system in order to check its behavior. So the teacher should point out what system information she or he would like to access and how this information should be presented .

Students, on the other hand, play a very special role in the tutoring system design. In addition to expressing their preferences, students can determine the system's success or failure, for they really can say what and how they have learned. While learning can be difficult to measure, it is easy to preview that learning is almost impossible to achieve when the students are left out of the decision process. The ways through which the student should participate remain to be further investigated. Cognitive aspects must be taken into account in order to detect the students' idiosyncratic methods of reasoning.

Based on the above ideas, we have designed and implemented a system prototype aimed to support elementary school students in learning plane geometry. The system TEGRAM provides a set of activities based on Tangram. The activities include measure and shapes of plane figures and similarity, among others. The student can use the system according to his/her own cognitive level. The system tries to evaluate the user through a student model and proposes a new set of activities made adequate to the detected level. Initial results point to positive student reactions. The underlying approach is to allow the student to choose his/her way to solve the problems, which is what makes the system quite challenging. However, this freedom does not prevent the system from suggesting an appropriate sequence of activities for the student, based on some knowledge about his/her performance. We are on the point of reiterating that the process of learning and teaching mathematics has much to gain from the use of an intelligent tutoring system as an assistant.

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Comments and Letters

Is protective mathematics humanistic? If so, then what?

Webster defines humanistic as the adjective correlate of humanism, thus relating to: la) classical letters, lb) critical spirit, 1c) secular, 2) humanitarianism, 3) attitudes centering on human interests and values.

Humanistic mathematics, as usually viewed, seems to fit lb) and 3), and possibly lc), but not la) or 2). Protective behaviors, such as conserving, guiding, guarding, or even reforming certain human or natural conditions and attitudes, fit 2) and 3), and possibly lb) and 1c), but not la). It seems to follow that mathematics relevant to protection should be also humanistic, when used in a protective way. The usage of protection in conjunction with mathematics may be new.

However, environmental mathematics is too broad, in that it may include short-term efficiency in exploiting the environment, and also too narrrow, in that protection from organized attacks is not ordinarily included. Briefly, protective mathematics should include specialties relevant to protection from pollution, flood and drought, shortages of food, medicine, and drinkable water, habitat degradation, disease, inter-species attack, criminal, military, or terrorist aggression, or technological faults and accidents. These must be firmly based on science and technology, and are not normally considered as humanistic pursuits. Nor are the discovery and measurement of relevant social-psychological parameters so considered. This is not merely a philosophical issue; the needed training in partial differential equations, stochastic processes, computer programming, statistical estimation (and relevant sciences) typically leads to industrial employment, of contestable protective value.

Inclusion of protective mathematics under the humanistic rubric is not only definitionally appropriate, but also would open up a valuable connection with social and economic concerns, in my opinion. Religious denominations which concentrate on spirituality, and downplay protection from practical exigencies, tend to lose public esteem, even if they attend to sin and doctrine.

Students interested in protective mathematics must be aware that the pay is academic (low), the working hours are like those of engineers (long), the subject is detailed, and the techniques (special functions and computing) are rather boring. Finally, careerjolting political and ideological attacks are not unusual.

Attraction of undergraduates tends to be limited to idealistic or brash individuals. Some topics of interest, in my experience, are risk analysis, forensics, DNA, fires, traffic, gang dynamics, resource allocation, demography, and geographic information systems. A short-term source of skilled personnel, not always suited to protection, either in values or income/status expectations, would be displaced Cold War specialists from the USA and Russia.

The US Bureau of Reclamation, the EPA, and environmental consulting firms deal with many of these issues on a continuing basis. Some universities are beginning to take protective science and mathematics seriously, more in specialized institutes than in teaching. Is this a matter for applications specialists only, or should humanistic mathematicians try to intervene, to provide ethical or philosophical perspectives? Some theologians are moving on this, but those with little scientific or mathematical capacity are clueless. Planning and executing a curriculum for protective mathematicians, as a variant of applied mathematics, would seem overdue.

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In your May 1996 issue you printed a short essay titled "On Mathematics in Poetry," by John S. Lew. Perhaps Mr. Lew's explanation of Donne s poem "A Valediction Forbidding Mourning" is not quite accurate. I think Donne is talking about a speaker who is saying goodbye to his wife or possibly his mistress. In the first stanza he says they should part as quietly as virtuous men pass very mildly away.

Later in the poem, he compares himself and his wife to the two legs of a compass. He asserts that he will be the roving foot whereas his wife will be the fixed foot. Thus whenever he makes a move, his wife will respond with a move of her own and so will always be aware of his direction and movement. In addition, the figure of the compass means that there will always be a connection between them, and it may suggest in addition that God is the actual mover of the compass.

I think this explanation catches some of the magnificent abstraction of mathematics. I agree with Mr. Lew that only John Donne has achieved such integration between mathematics and the "real world."

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