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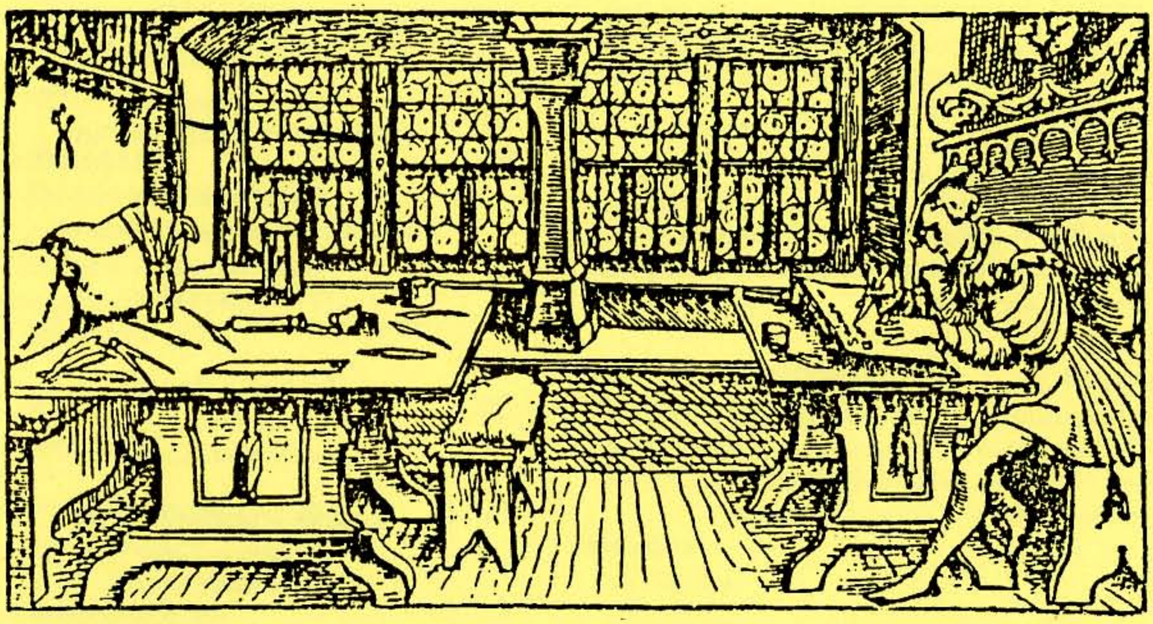

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Cover

This is an illustration taken from Frank J. Swetz's article "The Mathematical Quest for the Perfect Letter", in which a calligrapher from fifteenth century Europe is engaged in harnessing the geometric and mathematical principles necessary to construct a set of "perfect letters".

Publication of the *Humanistic Mathematics Network Journal* is supported by a grant from the

EXXON EDUCATION FOUNDATION.

Humanistic Mathematics Network Journal #13
May 1996

From the Editor

The last issue celebrated the fortieth anniversary of the Exxon Education Foundation by reprinting Allyn Jackson's essay, which surveyed many of the programs that the Foundation supports. These programs have had a great impact on much of mathematical education, for which professors, teachers, and students are grateful.

This issue celebrates the tenth year of the Humanistic Mathematics Network Journal supported by the Exxon Education Foundation. The Network started in March of 1986 with a three day conference at Harvey Mudd College on the topic "Examining Mathematics as a Humanistic Discipline."

Thirty university and college mathematicians and philosophers participated in the discussions. As with many new or rediscovered areas of mathematics, there were ambiguities and unanswered questions. The experience, however, was positive enough to continue the discussions by means of a newsletter (now this journal).

Humanistic dimensions of mathematics discussed included:

- An appreciation for the role of intuition, not only in understanding, but in creating concepts that appear in their final version to be "merely technical."
- An appreciation of the human dimensions that motivate discovery - competition, cooperation, the urge for holistic pictures.
- An understanding of the value judgements implied in the growth of any discipline. Logic alone never completely accounts for *what* is investigated, *how* it is investigated, and *why* it is investigated.
- A recognition of the need for new teaching and learning formats that will help wean our students from a view of knowledge as certain, to-be-received.

Mathematicians and other academics are aware of these ideas at a deep or subconscious level. A few, such as H. Poincare, H. Weyl, and A. N. Whitehead, have articulated the ideas at length. In the tightly constrained situation of formal education the themes and dimensions of humanistic mathematics and other disciplines are sometimes buried or overlooked.

The newsletter that was originally sent to the thirty conferees is now the journal sent to over 1500 readers all over the world — North and South America, Europe, Africa, Australia, the Middle East, China, etc. Authors are as widespread as readers. Academic and public libraries receive and circulate the Journal. While some doubted that humanistic mathematics could be defined, others thought the meaning was obvious.

Although letters to the editor are not new, the journal is starting a new section of comments and letters. Harald Ness, who suggested a Comments section, will edit it. In this issue, two letters comment on an article of the previous issue — each with strong feeling, but on opposite sides.

Your essays, poems, puzzles, polemics, and random thoughts are invited in the spirit of humanistic mathematics.

The Mathematical Quest For the Perfect Letter

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ABSTRACT
In fifteenth century Europe, geometric and mathematical principles were harnessed in the construction and formulation of "perfect letters." These letters, based on the script of imperial Rome, occupied the attention of both mathematicians and artists of the time. This article discusses this search for the perfect letter.

PERSPECTIVE
Among the legendary accomplishments credited to Pythagoras of Samos (ca. 585-ca. 500 B.C.) was an interest in the geometric design and construction of the letters for the Greek alphabet. According to Apollonius of Messene, a second century grammarian and teacher in Alexandria, Pythagoras sought to achieve visual harmony within each letter through a systemic use of angles, line segments and circular arcs [1]. Whether this story is true remains unresolved; however, its existence does testify to a purposeful application of geometric methods for the construction of letters in the ancient world. Careful analysis of civic inscriptions from the Augustan period (27 B.C. - 14 A.D.) of Imperial Rome shows that their aesthetic appeal was based on the use of rather exact geometric designs and fixed proportions [2]. The thickness of the principal strokes of a letter as compared to the letter's height was in the ratio of 1 to 10. Apparently architects of this period who possessed a mathematical training imposed a canon of inscription design on their lapidary workers. This canon was adhered to for several generations. With the subsequent decline of the Roman Empire, the classical forms of the *litterae lapidariae* were temporarily forgotten and replaced by the Gothic styles of the Middle Ages.

With the intellectual and artistic blossoming of southern Europe in the fifteenth century, the prevailing atmosphere which had separated mathematics from aesthetics during the previous millennium began to change. Applications of mathematics to space and form became more widely recognized and appreciated. One episode of this movement was the renewed quest for the "perfect alphabet." This quest involved mathematics, aesthetics, philosophical, and mystical considerations and resulted in a fascinating series of connections between peoples and ideas. As the humanistic spirit grew in Italy, it manifested itself in many forms, not least of which was a renewed reverence for classical studies, artifacts and institutions. Latin, as a written language and conveyor of ideas, achieved a new prominence. Architectural forms and principles of Imperial Rome were resurrected. In particular, the theories of the architect Marcus Vitruvius Pollio (1st century B.C.) once again became popular.

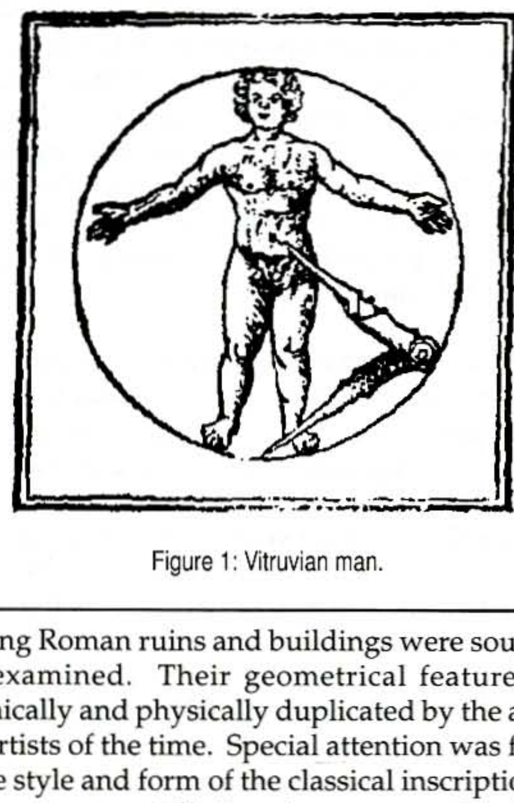

Figure 1: Vitruvian man.

Existing Roman ruins and buildings were sought out and examined. Their geometrical features were graphically and physically duplicated by the artisans and artists of the time. Special attention was focused on the style and form of the classical inscriptions, the *scriptura monumentalis*, found on many such buildings. The resurrected classical Roman alphabet became an object of admiration and speculation and a new genre of theoretical and didactical writing appeared, *trattati delle lettre antiche*, manuals devoted to the graphical design of the Roman alphabet. By 1459, Roman letters began to appear in panel paintings and frescoes. An example was in the work of Andrea Mantegna, who depicted Roman buildings and monuments and emulated Roman letters in his signatures. Ferrarese and Paduan miniaturists also incorporated Roman letter forms in their initials, imparting, so they believed, a classical charm to their products. These early letters were noticeably slender and possessed a dominating verticality. But gradually lettering, particularly that employed in lapidary work and medallion inscription, drew closer to its historical Roman exemplars. This transition was guided by a pronounced scholarly debate as to the form and shape of the "perfect letter", a debate in which the geometry of the ruler and compass and theories of proportion played prominent roles.

THE QUEST

The first noted commentaries on classical Roman lettering can be found in the writings of Veronese calligrapher and antiquarian Felice Feliciano. Feliciano dedicated a collection of such inscriptions copied from existing Roman structures to his friend, the artist Andrea Mantegna, in 1463 noting "I, Felice Feliciano, have revived this in the antique manner after ancient marble tablets such as are to be found in Rome and elsewhere" [3]. As a disciple of the theories of Vitruvius, he used the Roman architect's system of structure and proportion for the design of his ideal alphabet. Feliciano explained:

> According to ancient practice the letter is shaped from the circle and the square, the sum of whose forms rise to 52 [50 units], from which is drawn the perfect number 10. And thus the width [of the stroke] of your letter should be one-tenth of its height. In this way the letter has as much of the circle as the square [4].

Vitruvius, in turn, was influenced by neo-Pythagorean number theory of his time. For the ancient Greeks 10 was considered a very significant number.

It, of course, numbered the digits of the hand, establishing a special numerical relationship to the human body. For the Pythagoreans 10 was the sum of the tetractys, the numbers that represented the four elements i.e. 1 (fire) + 2 (water) + 3 (air) + 4 (earth) = 10, which, in itself, represented the universe. Further, Vitruvius viewed the human form geometrically as symmetrically contained within a configuration of a circle inscribed within a square: *homo ad quadratum et ad circulum*. In this geometrical complex, the human form was then longitudinally divided into 10 units each of which approximated the facial distance from the chin to the top of the forehead. This geometrizing and proportionalizing of the human form captured the imagination of the renaissance artistic community, particularly members such as Lorenzo Ghiberti and Leonardo da Vinci. Da Vinci's famous "Vitruvian Man" drawing depicting an individual, arms and legs extended, reaching for the circumscribing bounds of a confining circle and square, could well serve as a visual metaphor for the spiritual and intellectual climate of the times [5]. The concept of human centeredness was stressed by Vitruvius in his *De architectura* where he noted:

> Then again, in the human body the central point is naturally the navel. For if a man be placed on his back, with his hands and feet extended, and a pair of compasses centered at his navel, the fingers and toes of his two feet will touch the circumference of a circle described therefrom [6].

This physiographic theme was incorporated into the construction of Renaissance alphabets. For example, the writing manual *Luminario* (1526) of Giovam Baptisa Verini, featured its own representation of a Vitruvian man [7] [Fig. 1].

It is a fascinating realization that the humanistic trends in the graphic arts of this period purposefully associated the proportions and symmetries of the human form with the design of letters, the eventual vehicles for the expression of human thought.

Feliciano's suggestions for the design of an alphabet reflected the Pythagorean mystique now resurrected in the early Renaissance that man, music, architecture, and ultimately the world possess beauty based on a harmony of inherent numerical proportionality. This concept of harmony and beauty was now extended to the letters of the alphabet. While earlier European scribes drew their curved lines by visual reckoning, the calligraphers of the Renaissance, seeking higher quality and standardization and spiritual significance for their letters, now employed a ruler and compass constructions and a system of fixed proportions.

Felice Feliciano's designs for the Roman alphabet are contained in a slender codex now found in the Vatican Library (Vat. Lat. 6852) [8]. The collection of twenty-five drawings depicting various letters are rendered in watercolor. Each letter contains shading providing a sense of depth for stonecutters who would consult the model. The instruction provided is purely visual; a calligrapher or craftsman would study the letters together with their indicated constructions and then duplicate the image as required. Explanatory text is also nonexistent. This treatise is believed to have been compiled in about 1460, making it the oldest extant reference on the classical calligraphic reforms of the Renaissance. Feliciano crafted each letter within the confines of a circle and a square; however, time has faded his original construction lines and they have been reproduced in later copies [Fig. 2].

While the Feliciano treatise was influential in establishing new directions for lettering, its format diminished possible impact. The mathematical quest for the perfect alphabet would be taken up and carried forward by others.

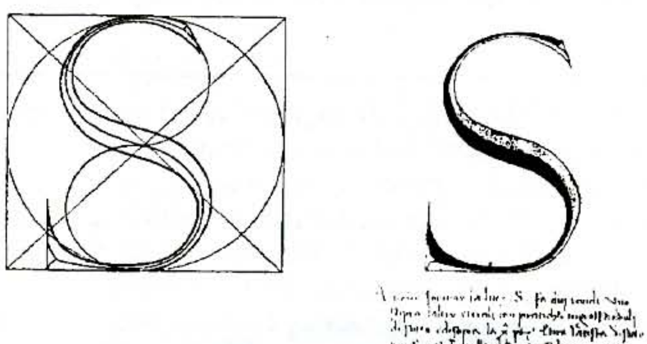

Figure 2: Letters designed by Felice Feliciano.

The first Italian to devise a printed manual on the design of letters was Damiano da Moile of Parma who in 1480 published his *Alphabetum*. This small tome, written for craftsmen, provided specific geometrical instructions for the design of each letter [Fig. 3]. Da Moile also employed the guiding circle and the square constructs, but his letters were not constrained within their bounds. In his designs, geometry provided guidance without exerting a strict control. In the *Alphabetum*, the letter prints were made from wood block impressions. The wood block constructions and carving imposed certain restrictions on letter design features. Da Moile's system of proportions was based on a ratio of 1:12; that is, the thickness of the major stroke of a letter was to be 1/12 its height. Twelve was possibly chosen as a multiple of the perfect number 6.

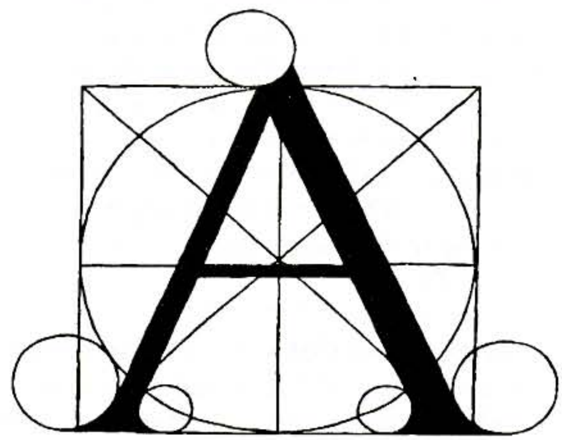

Figure 3: Letter designed by Damiano da Moile.

At the height of the Renaissance in 1509, the last of the early theoretical works on Roman alphabet design appeared. This work was an appendix to a treatise on human proportion contained in the *De Divina Proportione* by Luca Pacioli. Pacioli, a theologian and mathematician, was the geometry teacher of Leonardo da Vinci and a friend and companion to Piero della Francesca and Andrea Mantegna. He was known for his work on geometric perspective and was well respected in the artistic and scientific communities. Pacioli's instructions on lettering were directed at "worthy stone cutters and zealous followers of the craft of sculpture and architecture." All requirements were rationalized in a humanistic light.

> . . . from the human body derives all measures and their demonstrations and in it is to be found every ratio and proportion by which God reveals the innermost secrets of nature.

> The ancients, after having considered the right arrangement of the human body, proportioned all their work, particularly the temples, in accordance with it. For in the human body they found two main figures without which it is impossible to achieve anything, namely the perfect circle . . . and the square [9].

He stressed the theory that only the classical geometric instruments, a straight edge and a compass, were necessary for the construction of perfect letters. Pacioli further noted that he had written his text on lettering to demonstrate that "everything comes from the discipline of mathematics." The friar-scholar differs from his predecessors in requiring a proportion within letters of 1:9. In a practical sense, such a ratio produces a bulkier letter, one more recognizable from afar. Theoretical considerations may have also reflected Platonic thinking where it was believed that 9 was a mediator between the dynamic numbers 6 and 12. Pacioli's letters are beautifully illustrated within the *Divina Proportione* [10] [Fig. 4]. There is some speculation that Leonardo himself supplied the illustrations for these letters—he is known to have drawn the regular polyhedra contained in the work; however, there is no firm evidence to support this claim [11]. Da Vinci's involvement with the calligraphic reforms of this period is controversial. While he was known to comment on the geometry and proportions of Roman letters, no specimens of his own constructions are available. An anonymous fifteenth century manuscript devoted to the geometric design of letters resides in the Newberry Library of Chicago [Fig. 5]. At times, the Newberry manuscript has been attributed to Leonardo Da Vinci; however, its authorship has never been fully confirmed. This manuscript bears many features, for example, the use of a 1:9 ratio, that may have influenced Pacioli's work.

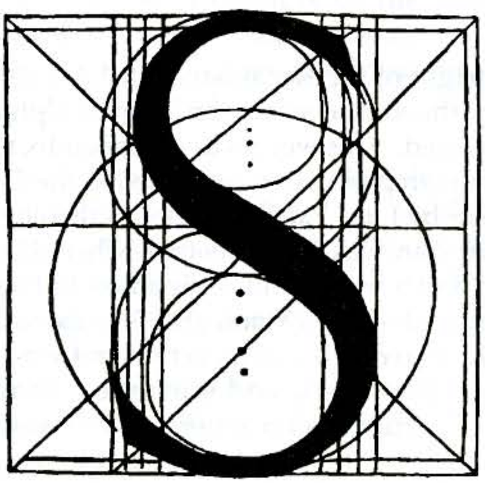

Figure 4: Letter designed by Luca Pacioli.

The first comprehensive manual devoted solely to the art of classical lettering was *Theorica et Practica de Modo Scribendi* by Sigismondo de Fante which appeared in Venice in 1514. This manual considered the construction of both Gothic and classical Roman letters. After its appearance, a proliferation of writing manuals by various authors appeared in Italy in rapid succession: Francesco Torniello (1517), Vicentino (1522), Giovanni Tagliente (1524), Verini (1527), Giambattista (1540), Giovanbattista Palatino (1550), Fernando Ruano (1554) and Giovan Francesco Cresci (1570) [12] [Fig. 6].

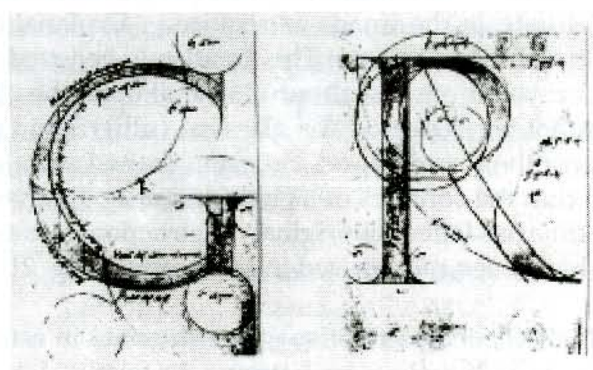

Figure 5: Letters of the anonymous Newberry Library manuscript.

Cresci deviated from his contemporary calligraphic theorists in that he avoided ruler and compass constructions. His admiration for classical Roman letter models was equally as fervent as that of Feliciano or de Fante; however, it was based on the letters of an inscription on the Trajan column in Rome [13]. This column was erected by the Senate and the Roman people in 113 A.D. to commemorate Emperor Trajan's victories on the frontiers of the Danube. Trajan letter models employed a ratio of 1:8 in stem width to height [14]. Cresci's letters are strong, free flowing and devoid of obvious geometric constraints [Fig. 7].

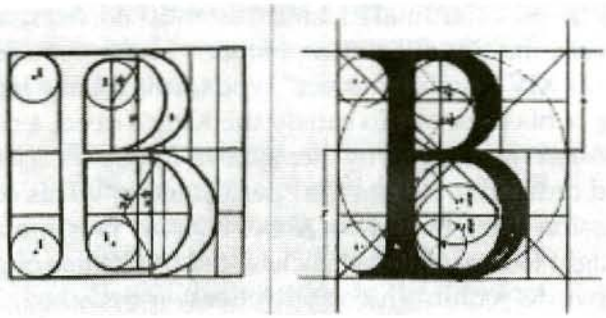

Figure 6: Letters designed by Sigismondo de Fante.

In the preface of his first published writing manual, *Essemplare di piu sorti lettere* (1560), he noted his disdain for the geometric methods of his fellow calligraphers:

> And in drawing every curve of each letter they make more circles than a sphere for the most part contains.... I have come to the conclusion that if Euclid, the prince of Geometry, returned to this world of ours, he would never find that the curves of the letters could, by means of circles made with compasses, be constructed according to the proportion and style of the ancient letters [15].

Figure 7: Letter designed by Giovan Francesco Cresci.

At the time, Cresci's letter forms attracted little following, and geometric-based letter rendering prevailed among Italian calligraphers.

Classical lettering models were imported north of the Alps by Albrecht Dürer. In 1525, Dürer published his *Underweysuna der Messung mit dem Zirckel uñ Richtachezt [Course in the Art of Measurement with Compass and Ruler].* This book was written for *tagliche Arbeiter*, ordinary workmen, to demonstrate applications of geometry to the concrete tasks of architecture and engineering. Dürer felt that "Beauty is the harmony of the parts in relation to each other and the whole" [16] and attempted to insure this harmony in his instructions; however, he also exhibited a spirit of artistic independence. The artist employed two ratios in his lettering, 1:9 and 1:10, while advising his reader to use *"weliche dir am besten gefelt,"* opening the door to multiformity and individual taste. In this respect, Dürer shared the feelings of Leonardo Da Vinci. In his writing Albrecht Dürer considered two letter forms, German Gothic and Roman. In Gothic constructions he dispensed with circular arches and built up each letter from a number of geometric units, squares or trapezoids; the resulting forms were reminiscent of existing Arabic calligraphic styles [17] [Fig. 8].

In surveying the major works of the classic lettering movement of the Renaissance, one final contribution warrants special attention. In 1529, the French royal calligrapher, Geofroy Tory published his theories on the design of *litterae antiquae*, classical letters. Published in Paris, Tory's *Champ Fleury ou L'Art et Science*

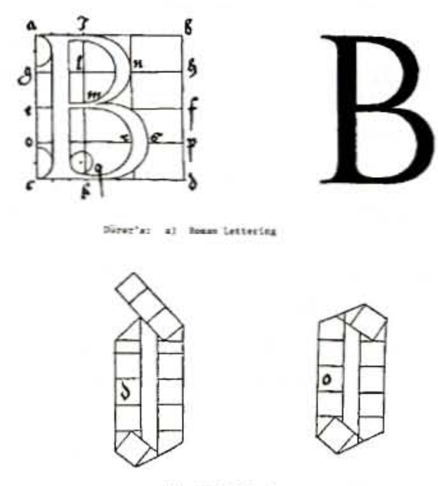

Figure 8: Letters designed by Albrecht Dürer.

de la Proportion des Lettres criticized the popular theories of Fanti, Pacioli and Dürer and offered the author's own version for the construction of perfect letters [18]. A ratio of 1:10 was employed not justified on mathematical or artistic grounds but rather derived from the nine Muses with Apollo added. This calligrapher not only attempted to relate his letters with Vitruvius but also associated them with classical mythology. The design of each individual letter was justified within the premise of *homo ad quadratum et ad circulum*. Almost mystic rendering of letters were constructed upon a square reference grid comprised of 100 smaller squares [Fig. 9].

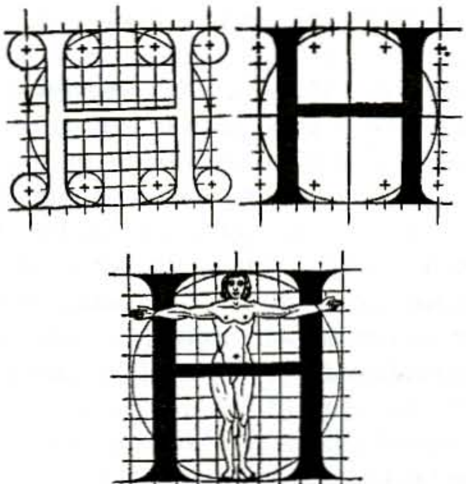

Figure 9: Letters designed by Geofroy Tory.

While it remains questionable if the search for the perfect letter achieved its goal, its quest brought together some of the greatest minds and artistic talent of the Renaissance. An aesthetic dimension was added to the printed and sculpted word and geometry lent itself to producing something living and personal.

SOME LATER EFFORTS

The concept of the "perfect letter" haunted the 16th-17th century architectural and artistic scene. In a sense, its issues involved a conflict between art as an aesthetic expression and mathematics as an aesthetic formulation. The theories of Pacioli, the rationalist, Dürer, the practical artist and Tory, the mystagogue, all reflected a humanistic concern to seek out fundamental laws, even those governing the shape of letters. As the Age of the Enlightment ushered in a sharper scientific spirit, one based on precision and exactitude, it exerted an influence on the search for the perfect letters. Now, design emphasis was on the style of printing fonts rather than the handwritten letter. In 1640, Cardinal Richelieu established, *Imprimerie Royale*, the Royal Printing House. In the year 1692, Louis XIV sought "perfect" type forms for the lettering of his presses. To satisfy the King's need, a commission was formed by the *Academe Royale des Sciences* and ordered to design the "perfect letter." This commission was chaired by Abbe' Nicolas Jaugeon and sought to carry out its task assisted by the use of grid networks within which each letter was designed. The Jaugeon Commission, as it was known, issued its recommendations three years later. Its geometrically perfect letters were each constructed on a square grid composed of 2,304 small squares i.e. 64 large squares each divided into 36 smaller squares. See Figure 10 [19]. The resulting letters were precise in their constructions and were meant to serve as models for the design of type fonts. The royal engraver Rajon freely adapted these designs and cut fonts in a style henceforth to be known as *romain du roi*; however, in general, engravers confronted with what they believed as contrived geometrical patterns simply refused to use them. They felt that ultimately the human eye was the sole judge of proper form and proportion. Eventually the enthusiasm for *litterae antiquae* waned and the use of Gothic letters returned to favor, especially in the printing of religious texts.

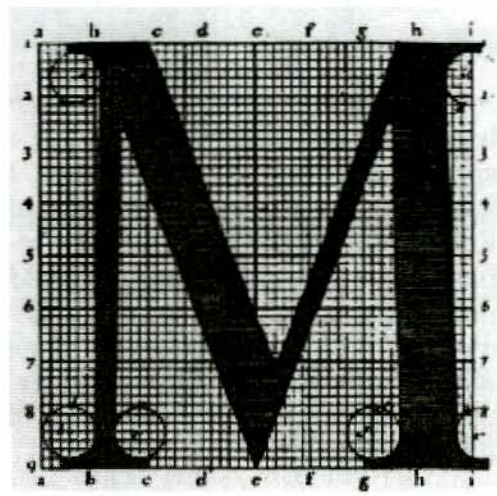

Figure 10: Letter designed by the Jaugeon Commision.

THE ISSUE REVISITED

Rapid technological advances within the last thirty years, including the perfection of photocomposing and the digitalization and computer storage and re-production of images, have radically altered the graphical potentials of the printing industry. These new potentials have forced a rethinking of processes, techniques and formats, not least of which has been type or letter design.

> The type designer - or better, let us start calling him the alphabet designer - will have to see his task and his responsibility more than before in the coordination of the tradition in the development of letterforms with the practical purpose and the needs of the advanced equipment of today . . . [20].

In essence, the new challenges have resurrected the quest for the perfect alphabet with an included dimension of technological accommodation.

Intrigued by this challenge and recognizing "a good mathematical problem still waiting to be resolved", in 1977, Donald Knuth of Stanford University decided to tackle the problem of mathematically designing the "perfect letter" [21]. Systematically approaching his task, Knuth defined the problems as one of finding the "most pleasing" closed curve, MPC, to fit a set of $n \geq 4$ given points: Z_1, Z_2; Z_3 . . . Z_n in the plane. He then postulated a set of axioms to clarify the concept of "most pleasing."

The closed curve had to satisfy the following properties:

1. Invariance. If the defining set of points Z_1, Z_2, Z_3 . . . Z_n, are subjected to a Euclidean or projective transformation, the MPC will experience the same transformation.
2. Symmetry. A cyclic permutation of the set of points will not change the form of the MPC.
3. Extensionality. Points from the MPC, if added to the defining set of points, will not alter the solution.
4. Locality. Each segment of MPC between two given points depends only on those points and their immediately adjacent points.
5. Smoothness. MPC is everywhere differentiable.
6. Roundness. If Z_1, Z_2, Z_3, Z_4 are consecutive points of a circle, their MPC will be a circle.

Guided by these postulates, Don Knuth derived his MPC as a piecewise continuous curve where each segment is determined by an appropriate cubic spline. Appropriate MPC's have been devised for each letter of the alphabet and loaded into a computer program. The resulting system, called METAFONT, is flexible and allows its user, through a variation of parameters, to devise an infinite variety of letter forms and fonts. Thus the concept of the "perfect letter" now truly lies within the eye and control of the METAFONT user. Knuth has completed the task attributed so long ago to Pythagoras, labored upon by the artists and calligraphers of the Renaissance and pondered by the Jaugeon Commission. A visual comparison between a METAFONT composition and those proposed by some Renaissance masters is shown in Figure 11.

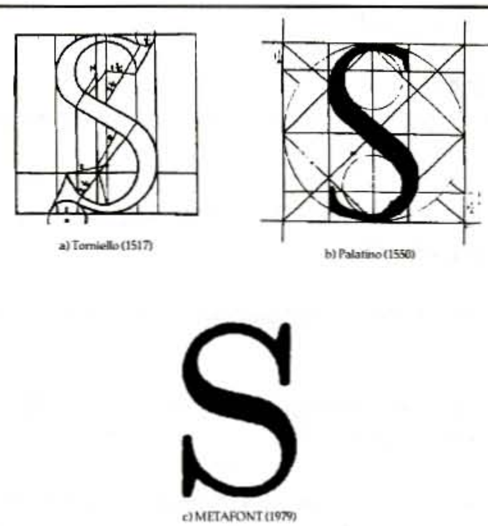

Figure 11: Comparison of classical and modern lettering.

CONCLUSION

This historical discussion has briefly surveyed the conceptual development and evolution of a problem - the mathematical design of the letters of the Western Roman alphabet. While the problem is not a pressing one in terms of utilitarian applications, it possesses a certain psychological and intellectual, as well as mathematical appeal - the combining of mathematics with aesthetic concerns to serve the needs of communication. The various solution attempts have reflected on their times. It can be assumed that the Pythagoreans sought a visual balance and symmetry in the image of written letters to conform with their mystical gestalt of cosmological harmony. Whereas Renaissance calligraphers strove to replicate classical exemplars, they still indulged in experimentation and rationalization. The failure of the Jaugeon's Commission's recommendations may, in part, reflect the rise of a spirit of individualism and rebellion among French artistians that would later express itself so strongly in the French Revolution. Don Knuth's resolution of the situation while ultimately prompted by a mathematical challenge, was also motivated and facilitated by the existence of new technologies. The scope and power of METAFONT and its companion TEX typesetting system has revolutionized the field of typography and demonstrated how mathematics under the rethinking of an old problem can be applied to new fields. Interestingly, these new resulting techniques of typography have not narrowed choices as to the "perfect letter", rather they have broadened options and personalized the decision of perfection. What a nice result!

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On Mathematics in Poetry

John S. Lew
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Many poems of mathematical interest choose one of two paths: either they play games with mathematical jargon, or they express wonder at mathematical beauty. Good examples of these types are Lewis Carroll's "The Hunting of the Snark" (specifically, "The Beaver's Lesson") and Edna St. Vincent Millay's sonnet "Euclid alone has looked on beauty bare." I feel that more recent attempts tread these same paths too often, yielding results that far fail to match the classics. Moreover, such results make only slight emotional connections between mathematics and the larger world, whereas ideally the poet would so naturally feel mathematical concepts that such concepts might become metaphors even for "humanistic" things.

Few poets have achieved such integration — perhaps only John Donne, who knew little mathematics, but who, in his poems, now and then used its concepts quite expressively. On a dead nobleman he lamented: "O soul, O circle, why so quickly be / Thy ends, thy birth and death, closed up in thee?" In his famous "A Valediction: Forbidding Mourning", author and dead friend become two points in a plane, not wholly separated by death, but connected in a hiqher dimension: his friend the fixed point of a compass, he the point still free to move.

As another example I submit a short poem of my own, which, perhaps more accurately, presents not mathematics but physics as metaphor. Naturally, I like it, but I do not press its merits. Rather, I offer it here because it further illustrates my point, and because, with some authority, I can state the intentions of the author. The tale behind the poem is that at a party, unmarried, I met an attractive woman, that I dated her a few times, and that she dropped me. Such tales, after all, inspire a good part of all human poetry — and the Age of John Donne produced some great examples of that genre.

Ruefully, my post-mortem on this failed relationship concluded that I had "come on too strong." At that time comet Kouhoutek had recently passed by, flouting expectations of a great show among the night stars. The conjunction of these events yielded the following tetrameter sonnet (a precedent for whose form is Shakespeare's Sonnet No. 145.) Here, obviously, my erstwhile date is the sun, while I am the comet, rushing toward her, yet fated mathematically to swing round and drift away beyond the most distant planet. After the poem I note some less obvious things.

The Comet

John Lew

Near from infinity I came
        Drawn to your strong, unmoving light
By some ascendance of its flame
        That charms the planets through their night.
The distance melts, my spirit thaws,
        Sublimes, and in your radiance flies
Soon, by the old, unchanging laws,
        An exhalation through the skies.
Sweet perihelion! May we touch,
        Our auras intermingle? No,
The impulse of my flight too much,
        I must again to darkness go;
While you may stand, and watch my face
        Dwindle through trans-Plutonian space.

The linear momentum of a body is its mass times its velocity; and if one prolongs a straight line through the velocity vector of the comet, then one can find the minimum distance from this line to the sun. However, a comet will not hit the sun unless its momentum times this distance (the angular momentum) is sufficiently small. For simplicity, my poem makes (disguised) reference only to momentum, but the astronomical image yields the moral: like a comet, I lost the desired union by aiming not close enough — and by coming on too strong.

Two gravitating bodies circle an intermediate point, but if one body has negligible mass then the pivot is almost the center of the other; whence the sun is a "strong, unmoving light". Critics of Newton griped that the concept of gravitation just reduced planetary motion to a deeper mystery; whence that attraction, in my poem, becomes an "ascendance", i.e. a mystic power that "charms the planets" — whose "night" is the darkness of space.

Supposedly, a comet is a "dirty snowball", i.e., a mass of frozen water (and other stuff) surrounding a small, rocky core. As this body nears the sun, its rising surface temperature frees surface material, and the solar wind sweeps this away into the familiar tail. Hence "thaws" and "sublimes"; to "sublime" is to make a direct transition from solid to gas. Likewise, the comet's tail becomes "spirit", then "exhalation", then "aura", while clearly the sun's "aura" is its corona — a dim glow visible only when other light is excluded. Perihelion is the point of closest approach; even then the comet's lost material cannot touch the corona: ultimately, comet and sun cannot come close enough even to mingle their spirits. Thus the exclamatory lines 9 and 10 — evoking this closest approach and its human analogue — should be the poem's climax.

Another pun draws the moral. "Impulse" is the time integral of force, and a theorem of mechanics says that impulse = momentum. Too much impulse means too much momentum and just that behavior — coming on too strong — sends the comet back to outer space. The sun remains fixed while the comet retreats — necessarily keeping its face toward the sun because the solar wind blows its tail away from the sun. Far, far retreats the comet, past all the planets, into a dark void where the sun attracting it is only one more dim star in the black firmament.

Years later, I found almost the same image in Kenneth Rexroth's poem "Inversely, as the Square of Their Distances Apart". Once favored by his beloved but now estranged, Rexroth pictures himslf as a small, frozen outer planet moving slowly, yet still in distant orbit, about his personal Sun. Once again, the science becomes a metaphor that expresses the poet's loss.

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Personal Reflections on Mathematics and Mathematics Education

Lynn E. Garner
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PERSONAL EXPERIENCES

My story is similar to that of many other mathematicians now approaching the last decade of their professional lives. We were educated in the '60s by a mathematics faculty feeling the mandate of the Sputnik era for training mathematicians and scientists and encouraged by considerable financial support. The search for Ph.D. candidates brought an increase in rigor in mathematics courses and an expansion in the number of graduate programs.

We were taught almost exclusively by the lecture method; the professor transferred his notes to the blackboard and the students dutifully copied them down, usually with little interaction on the spot, hoping to answer their questions on their own by studying the notes and whatever related material they could find. If they developed discussion groups with other students, they were lucky, for mathematics was a solitary activity, even a competitive activity, especially on the undergraduate level. The discussion groups developed more naturally in graduate school; at the Ph.D. level, mathematical research and personal interchange with the thesis advisor and other Ph.D. students enlightened the candidate as to how mathematics was really done by the professionals.

I had been attracted to mathematics in the eighth grade when I discovered that I liked solving story problems. Though my school courses emphasized story problems less and less, I continued to do story problems just for fun when I ran across them. It was during high school that I began collecting mathematical puzzles and problems.

When I was about fourteen, I became fascinated by the coconut problem [1] that I found in a desk encyclopedia at my grandfather's house. It was a story of five men on a tropical island who spent all day gathering coconuts. At the end of the day they had a large stack, but being too tired they decided to wait until morning to divide them up. During the night one of the men awoke and decided to take his share right then. He counted the coconuts, finding one more than a multiple of five, tossed the extra coconut to a monkey, and took one fifth of the rest. He hid them and then went back to sleep. Later, each of the other men awoke in turn; each decided to take his share then, found one more than a multiple of five, tossed the extra coconut to the monkey, and took a fifth of the remainder. In the morning the stack was greatly reduced, but no one said anything. They counted the coconuts and again found one more than a multiple of five, tossed the extra coconut to the monkey, and each took one fifth of the rest. The question was, what is the least number of coconuts that could have been gathered? I puzzled over the problem mightily, and waded enthusiastically but laboriously through the generalized solution presented. Though it involved algebra and number theory at the limit of my understanding, I was undaunted.

When I was a senior in high school, my cousin Bob was a freshman at Caltech. I had admired his intellectual prowess to an extent and wrote to him about my applying at Caltech, too. In his reply, he mentioned something his high school math teacher had told him the year before; why he mentioned it or what it involved I don't remember, but he used the expression $2n + 1$ to represent an odd integer. I do remember being completely amazed that such a simple thing could be so powerful and so general. From then on, mathematics was my major.

As I progressed through the study of mathematics, I liked it increasingly because it became more and more like solving story problems. In undergraduate topology, for example, the entire point of the course seemed to be discovering why a theorem was valid; we spent our time not only finding solutions (i. e., proofs), but explaining them to each other. Graduate mathematics was more of the same, and research for the Ph.D. was nothing but problem-solving. Not that it was applicable to anything in the "real world" sense, but tackling tough problems of any sort brought on the thrill of the chase, as it were, and solving tough problems resulted in a genuine "high" different from any other.

When I began to teach, I tried to share with my students the thrill of solving problems, but the way I had been taught (which was the source of most of my teaching strategies), the textbooks available, and the lack of time to deviate from the syllabus prevented me from really communicating that thrill to my students with any degree of success. In effect, teachers at the undergraduate level were constrained to leave out the real problem-solving aspects of mathematics; all we taught was prelude to the real mathematics to be done, and consisted of symbol manipulation rules and recipes for solving template problems. Our material and approach were still designed to bring potential Ph.D. candidates to the forefront; students not majoring in math became more and more of a "load" to whom we paid less and less attention.

During the academic year 1986—1987, of the approximately 300,000 students who began the study of mainstream calculus in American colleges and universities, only 140,000 completed the year-long sequence with grades of D or better.

As I taught mathematics, I gradually became aware of some of its history, something that had not been particularly fashionable at the times or places of my formal education. When I was assigned to teach the math history class out of Eves' book [2], I was amazed at the quantity of rich information of which I previously had been totally unaware. The history class led me to use a collection of articles reprinted from Scientific American, edited by Morris Kline [3], as text for a sophomore seminar. Statements in Kline's introductions to the sections led me to explore the nature of mathematics. As I discussed it with others, we came to the conclusion that we didn't really know what mathematics was, beyond the fact that it was what mathematicians did.

We knew mathematics was not a description of the "real world"; that had been settled in the middle of the nineteenth century by the development of non-Euclidean geometry. When Cayley and Klein showed that hyperbolic and elliptic geometries were just as consistent as Euclidean geometry, the question arose as to which was a description of the real universe. The profession as a whole gradually came to the conclusion that none of the three need be "true" of reality; soon mathematics became independent of the physical universe in the minds of mathematicians.

On the other hand, mathematics was not just an elaborate logical game that existed only in the mind, for how then could it attract the attention and enthusiasm of serious scholars? While some claimed that Russell and Whitehead had shown that all of mathematics could be derived from the clear blue of pure logic, it also had an "unreasonable effectiveness" [4] in predicting real-world phenomena. Each working mathematician felt that mathematics was somehow "out there," external to himself, but he was never quite sure whether his mathematics was discovered or invented. Someone suggested that mathematics was "composed," but that notion failed to gain any currency.

During the decades of the '70s and '80s, enrollments in mathematics classes, particularly in calculus courses, increased dramatically. At our institution, the growth rate was about eight percent, compounded annually, and that under fixed-ceiling enrollments overall. Most of that increase consisted of non-majors and therefore expanded the service load. Burgeoning classes but constant resources forced creative arrangements to meet the demand—large classes, laboratory-based courses, and cheap labor (TA's) were used widely. During the academic year 1986—1987, of the approximately 300,000 students who began the study of mainstream calculus in American colleges and universities, only 140,000 completed the year-long sequence with grades of D or better [5].

During much of this time, I was working on my own calculus text. I became convinced in the mid '70s that I could write a better book than the ones I had to teach from; I finally succeeded in producing a "good book" in 1988. By that time, dissatisfaction with the results of current strategies had produced the Tulane conference of 1986 which spawned the calculus reform movement. Participants cited unacceptably high failure rates as a waste of human potential, tradition-bound text materials, and concerns with the way students learn as reasons for paying attention to the way mathematics, particularly calculus, was taught.

In 1988 the first symbol-manipulating calculator hit the scene; from the centennial banquet of the American Mathematical Society, 1500 mathematicians took home a new toy that would not only do arithmetic but would manipulate algebraic expressions, draw graphs, and differentiate and integrate as well. Many professors began to see that the new technology would have a profound effect on the way they taught. Several professors reported that the new technology could easily pass the previous semester's calculus final.

I first began experimenting with computers in my math classes in 1983, but lack of resources prohibited any large-scale or permanent effort. I began to see what technology in the hands of students would do to the way mathematics was taught; my vision, limited though it was, became possible when students could arm themselves with the HP28S. By 1989 I was using the calculator freely in my classes and allowing my students the same privilege.

Much experimentation showed that there were ways to use the technology that greatly enhanced the acquisition of concepts. For example, the calculator could produce a dozen good graphs in the time it used to take for the student to produce a single decent graph; consequently, graphical properties became intuitive and were much more easily applied to the analysis of functions. The graph itself was no longer the point. The same could be said for many algorithms; by turning over to the technology the drudgery it could do well, the student was freed to think about what it all meant and how it applied to solving problems. The technology could also compress time; in a single class period, second-semester calculus students could start with the Riemann sum definition of the definite integral and, by observing what was happening to errors, could guess for themselves the trapezoidal and Simpson's rules.

In 1990, my publisher told me to start thinking about a second edition of my calculus book. When colleagues invited me to attend a workshop at Harvard in May of 1991 on teaching calculus, I consented to go along to see what I could pick up for my book. Just before we went, my publisher informed me they had changed their minds about a second edition; previous sales didn't warrant it. When I got to the workshop and saw the prepublication version of the "Harvard calculus," I was forced to admit to myself that I had come upon a better way to teach calculus. Here was a whole calculus book based on the idea of problem-solving the way I had approached it and loved it as a student but had failed to pass on to the students in my classes or to incorporate well into my textbook. I quit using my own book that Fall and began class-testing the Harvard materials. I also required my students to obtain and use HP48S calculators.

My experiences in teaching that year are almost indescribable. I was totally unprepared for the enthusiasm with which students attacked the new materials. Their love of the technology was astounding. But the thing that surprised me most was the sense of community that developed and the amazing amount of mathematics that the students did as I joined them in learning the calculus from a new approach. I pretty much quit lecturing and used a great deal of collaborative learning in small groups; as I moved around among the groups, I found myself gaining insights right along with them. I saw more mathematics being done by far than when I was the only performer.

Feedback was immediately positive. Students reported feeling much less anxiety and much more self-confidence than was reported the year before by very similar students. One young woman reported being in a chemistry class when the instructor started putting up a problem of a type that she recognized from calculus. She whipped out her calculator and had the problem finished long before the instructor finished presenting it. She said that what pleased her most was the incredulous looks on the faces of the young men sitting around her; her self-confidence grew by leaps and bounds.

I later taped a conversation among several of the students about their experiences in the class. Concerning their work in groups, they said:

Teresa: "Working in a group was a different experience for me because you're getting different people's opinions on ideas and you realize that mathematics is not just a set, defined pattem—that there are different ways to look at things. It was hard for me to get used to that setting—that everyone looks at math in a different light."

Kari: "Sometimes when we'd be working on our problems, you'd come to a point where you couldn't figure it out—you're stuck and you can't see any way out of it. Someone may say something and it triggers something in your mind and you can go from there and figure out the rest of it. You need that little help that somebody else can give to you."

Kristin: "The thing I enjoyed about the group work even beyond the concepts was the people that we worked with, because that created a foundation for a study group so that outside of class we could get together and work on assignments. The group work was especially fun, with [the instructor's] help to keep our ideas going...."

Kari: "Your ideas get a little bit more in depth when you're working with a group, too, because everyone sees different details...and it all comes together and you see the detailed, whole picture."

Chad: "I think that group work was very essential in the whole process of learning what we learned last year in calculus."

Monica: "It wasn't individual learning at all...but it was just the class learning together. Everybody worked together and if one person didn't understand, three or four people would help until they did. It was a community, I guess.... We all got to be really good friends. I think most of us were freshmen and most of the best friendships we made were from that class."

Concerning the use of the calculators:

Monica: "I was scared to death of that calculator when we first got it. I don't like computers, I don't want to like them, and I was really not happy to have to get the calculator."

Chad: "All I could think of was the price, and it was a different way of using a calculator also because it uses reverse Polish logic and so it was difficult to adapt..., but I learned. I had so much fun using that calculator after getting over the initial shock.... I realized that this thing could do so much more and it was so much easier to do my homework with ...."

Monica: "Even though it's a calculator and it does rote manipulations and calculations, I thought more because the calculator was there. As I was using it, my mind would be clicking just as fast, or more so, than if I'd been doing it on paper. Using the calculator made me think about problems a lot more."

Teresa: "In any sort of problem the HP would basically analyze it and do the work for you so you could take it one step higher and say, 'OK, what is actually going on here?' You could look at the graphs and say, 'OK, I've got this graph now; what is taking place?' and you didn't have to sit there and graph it out all by yourself..."

During the ensuing summer, four of the students let me know that they had changed their majors to mathematics; such a thing had never happened to me before.

Not everything went smoothly, but I was happy to see that most of my worries about changing my teaching habits were unnecessary. One prominent worry had been giving up control in the classroom. (Perhaps I had only imagined I had control before, and the students had been merely passive.) I had already been aware that when students have technology in their hands, they aren't listening to you talk, but are off on their own, doing things you never thought of. I discovered that the best way to get them back was to use interesting material that they perceived as relevant and for which they felt responsible. I turned out to be quite happy to relinquish control, turning it over to the material.

THOUGHTS ABOUT MATHEMATICS

These experiences have led me to think deeply about how students meet mathematics and how it ought to be presented to them. They have caused me to question the very nature of mathematics and have enabled me at long last to see how it is that I approach mathematics.

Historically, developments in what we now regard as elementary mathematics came about through the efforts of non-mathematicians to understand some aspect of the world around them. The developers of algebra were just playing around with numbers, trying to outdo each other with clever puzzles; Fibonacci was one of the foremost. Trigonometry was just a tool developed by astronomers. Newton was really a physicist who developed the calculus into a usable tool in order to understand motion and gravity. Maxwell developed the calculus of vector fields in an attempt to understand electric fields.

In each case, a "real world" problem presented itself and the tools of logical analysis were applied to it. Assumptions were made about the problem to make it more tractable, and order arose out of the assumptions. Techniques were developed for handling the order and drawing from it a prediction about the situation. The entire process was called mathematics.

Gradually, it was noticed that the same process of logical analysis could be applied to the perceived order itself, independent of the real situation. Modern abstract mathematics thus came into being. As the mathematics was refined, it drifted ever further in the minds of its practitioners from the real situations which had first given rise to it. Thus by the middle of the nineteenth century, mathematics had come to be defined as the abstract study of order or pattern, taught in a manner progressively axiomatic and devoid of physical content.

As a result, elementary mathematics has been taught for more than a century as a purely logical discipline, consisting of rules for manipulating the symbols that came to represent ideas. Because it is thus divorced from "reality," many students of mathematics regard their experiences as stultifying at best and mystifying more often than not. Most students do not survive in mathematics long enough to discover that the way mathematics is taught is not the way mathematics is done.

Mathematicians know that when they do their work, they are using logical analysis to understand the world around them, even if it is just the artificial and specialized world of mathematics. When they refer to mathematics, they include the thought processes they use in solving research problems, every bit as much as the body of knowledge consisting of all the manipulation rules, identities, and techniques that they wish their students knew. But in teaching elementary mathematics to beginning students, they never invite the students to use those same reasoning processes. It is not because, for beginning students, there is nothing appropriate to which to apply such reasoning processes, but because it has been forgotten that mathematics is every bit as much a process as it is a body of knowledge.

Many students of mathematics regard their experiences as stultifying at best and mystifying more often than not. Most students do not survive in mathematics long enough to discover that the way mathematics is taught is not the way mathematics is done.

This leads me to a point of view of mathematics that seems to be valid. Both historically and as research mathematics is done today, mathematics is a means of dealing with the order that we see in the world around us.

Some remarks about this point of view are in order. I say "a means of dealing with the order" because thought processes are so varied as to defy any more specific categorization when taken in the aggregate. When one is wrestling with a problem, there are no holds barred and one catches as one can. The only criterion is that there should be some convincing, logical explanation afterward, even though most insights come from highly illogical combinations.

I say "the order that we see" because it is our perceptions to which we apply reasoning, not what is actually there. The traditional language is that a mathematical model is constructed and reasoning is applied to the model; in this language, mathematics is first of all modeling. Moreover, the "order" that arises from a situation is often the result of our assumptions, conditioned by previous experience. When shown a series of pictures of a cat in varying poses, some see only many pictures of a cat while others see the cat in motion and can even ascribe velocity and acceleration to it; those who see only many poses tend to lose interest quickly, while those who see motion find a myriad of things to analyze.

Human beings seem to need their perceptions of a situation to "make sense" if the situation is to be regarded at all. They are even willing to make unrealistic assumptions in an effort to understand. Thus we analyze a situation according to the way we construe it; it may or may not be an accurate or useful representation of reality. This basic uncertainty about our understanding of reality is what keeps most of us interested in learning about the universe.

When I refer to "the world around us," I mean whatever attracts our attention. The process of mathematical analysis can be applied to any subject whatever, concrete or abstract. These days, the "scientist" tends to focus on some aspect of reality while the "mathematician" typically focuses on some aspect of an abstraction. In actuality, the scientist is also dealing with an abstraction; the main difference is the frequency with which the researcher checks with reality.

THOUGHTS ABOUT TEACHING MATHEMATICS

This point of view of the nature of mathematics has what I think are profound implications for the teaching of mathematics at least through calculus. If we want a catch phrase for it, I think we could say, "Mathematics is a process; to introduce students to mathematics, we must engage them in the process." The process, of course, is dealing with the order that we see in the world around us.

People are scientists at heart, in that they seek to understand the events that go on around them so as to predict and control (or at least be prepared for) future events [6]. To assist themselves in the process, they construct theories into which they seek to organize and understand the mass of information impinging on them. The information comes not as facts but as perceptions; thus people deal with the world as they construe it or as they believe it to be. Insofar as their theories involve quantity, order, and pattern, they can deal with their perceptions mathematically.

In teaching mathematics, I believe we should capitalize upon the natural scientific tendencies of each student. We should begin with the process of logical analysis of problems, not with the body of manipulation rules and recipes. Mathematics is first the process; the rules come later, both historically and in the solving of research problems. If we begin with the process, it will be much more clear to the student that reasoning and analysis are what mathematics is all about, not merely memorizing formulas.

The problems to which the beginning student applies logical reasoning must be in the world of the student's interest, not in some artificial world someone else creates. If not so, there is no motivation; we know well that telling a student to be motivated does not make it happen in most cases.

Mathematics is at base a social activity; work can proceed individually, but never in a vacuum, and it is never complete until shared.

This means that problems at first must come from what the student perceives as the real world; as the student gains success in analyzing situations, the process of abstraction becomes clearer as we point it out and eventually the student's attention can be turned to the abstraction itself. This applies to the beginning student at any level, as much to the beginning student of calculus as to the beginning student of counting or arithmetic.

Moreover, much of the process is in communication of ideas. Forcing students to work in isolation is not only contrary to the way in which mathematics is created but often insures that the student will fail to learn. Allowing, indeed requiring, the student to communicate with peers helps to correct, refine, and solidify concepts and introduces the student to many more ideas than he or she is able to imagine alone. Mathematics is at base a social activity; work can proceed individually, but never in a vacuum, and it is never complete until shared.

If the student develops the ability to solve problems by thinking deeply and productively about certain key problems, it is not necessary for the student to see a recipe for the solution of every problem that was ever solved. Remember the adage, "Teach a man to fish...." Each mathematical subject has its key problems; in fact, each discipline to which mathematical reasoning can be applied has its key problems illustrating that application. A student who has thought deeply about some key ideas and is armed with logical reasoning will always outperform the student with a book of recipes.

The wise use of technology can be a great aid to the learning of mathematics. Current graphing calculator technology, for example, allows for multiple representations of concepts, powerful visualization, the compression of time, ease with experimentation, and the elimination of much drudgery. We should turn over to the technology the rules and recipes, things computers do very well, and get on with the thinking process. After all, if a calculator can do it, is it really thinking?

Unwise use of technology would include using a computer as a "black box." The student should never be programmed simply to push the right keys; only after an algorithm is completely understood is it appropriate to rely on the computer to perform it. On the other hand, once an algorithm is understood, we can save a lot of time and get on to the higher-level thinking we value by using the technology freely; the fact that the teacher or the student's parents did it "by hand" for years implies no particular virtue in the student doing so .

The biologists have a heuristic point of view that "ontogeny recapitulates phylogeny," meaning that it is helpful to view the developing embryo as progressing through the stages of evolution of that species. The same idea, applied to the individual student, would be "education recapitulates civilization." I believe that students of mathematics should re-create for themselves the development of elementary mathematics, time-compressed by the appropriate use of technology and by the wise choice of problems to analyze. The challenge to mathematics educators is now to select those problems and promote their analysis so as to engage the student fruitfully in the mathematical process.

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Mathematician

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What do you see?
A mathematician at a desk.
Papers scattered, open books,
a pencil. A few scribbles,
then a crumpled piece of paper
flung at a wastebasket.
Then a fresh piece.
You call this dull,
think nothing happens.

In truth, she is not here
in this room, bound by these walls.
She journeys beyond the moon,
the sun, the stars, out of our galaxy,
she roams beyond the rim of the universe,
soars where no one has ventured before.

Everywhere she looks
questions rise up,
surround her, trick her,
the baffling disguised as simple,
the shallow masquerading as deep.
They won't let her sleep.

Some she will answer;
others will consume her.
Each answer raises new questions,
that tempt her, pull her yet farther away.

One discovery
may unlock a deadly riddle
back on earth.
She may never know
which one it was.

Who says there are no worlds
left to explore, no mysteries
to challenge
the most daring soul?

Symmetry: A Link Between Mathematics and Life

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INTRODUCTION

In mathematics, certain basic concepts, such as symmetry and infinity, are so pervasive and adaptable that they can become elusive to the student. Understanding these concepts and the tools for studying them is often a long process that extends over many years in a student's career. Students first see infinity appearing as the potential infinite inherent in the positional number system, then implicit in plane geometry, and eventually underlying all of calculus and analysis. Students begin to use symmetry with commutativity and associativity in arithmetic, making more use of it in Euclidean geometry and plane geometry, and may eventually see it in terms of transformation groups. Nevertheless, it is natural to want to teach these concepts in their full value from the very beginning. This paper will describe how I have been introducing students in a general education geometry course to the concept of symmetry in a way that I feel gives them a comprehensive understanding of the mathematical approach to symmetry.

WHY TEACH SYMMETRY?

Symmetry is found everywhere in nature and is also one of the most prevalent themes in art, architecture, and design in cultures all over the world and throughout human history. Symmetry is certainly one of the most powerful and pervasive concepts in mathematics. In the *Elements*, Euclid exploited symmetry from the very first proposition to make his proofs clear and straightforward. Recognizing the symmetry that exists among the roots of an equation, Galois was able to solve a centuries-old problem. The tool that he developed to understand symmetry, namely group theory, has been used by mathematicians ever since to define, study, and even create symmetry.

Students are fascinated by concrete examples of symmetry in nature and in art. The study of symmetry can be as elementary or as advanced as one wishes; for example, one can simply locate the symmetries of designs and patterns, or one use symmetry groups as a comprehensible way to introduce students to the abstract approach of modern mathematics. Furthermore, the ideas used by mathematicians in studying symmetry are not unique to mathematics and can be found in other areas of human thought. By looking at symmetry in a broader context, students can see the interconnectedness of mathematics with other branches of knowledge.

For these reasons, many mathematicians today feel that the mathematical study of symmetry is worthwhile for general education students to explore.

A LINK BETWEEN SYMMETRY AND LIFE

The central idea in the mathematical study of symmetry is a symmetry transformation, which we can view as an isomorphism that has some invariants. For example, a symmetry transformation of a design in the plane is an isometry that leaves a certain set of points fixed as a set. I would like students to realize that this concept of symmetry transformation, as abstract as it may appear, can be connected to ideas that may seem more central to a view of life as a whole; for this, I introduce a verse from the Bhagavad-Gita.

In the Bhagavad-Gita, Lord Krishna lays out the complete knowledge of life to his pupil Arjuna, just as a great battle is about to begin. This work has long been appreciated for the great wisdom that is expounded in just a few short chapters. A verse that seems to me to capture the essence of the mathematical study of symmetry is part of Krisna's explanation of the field of action (chapter 4, verse 18, [1]):

> *He who in action sees inaction*
> *and in inaction sees action is*
> *wise among men. He is united, he*
> *has accomplished all action.*

How is this related to symmetry? A geometric figure that we wish to study is usually given as a set of points existing in some ambient space. For example, a tiling pattern may be given as a collection of line segments in the plane. A symmetry transformation can be regarded as "action" and invariants can be regarded as "inaction." We begin with a non-dynamic situation (the set of points of the tiling pattern sitting in the plane) and then find some dynamism (the symmetry transformation). Thus, in inaction, we see action. But a symmetry transformation is not just any action; it must leave the pattern (as a set of points) invariant. Thus, what is important to us is that in this action (the transformation), we are able to see inaction (the invariance of the set of points making up the pattern).

This is the seed of all that I want students to know about symmetry: action and inaction, a transformation and its invariants, what changes and what stays the same.

With this, the students gain a unifying perspective on the concept of symmetry that can help them understand it initially and that can later help them simplify and unify all the occurrences of this concept as they are met and eventually understand symmetry groups, invariants, and so on. This theme can also help students connect all instances of symmetry that they have already seen to this one unifying perspective. For example, in the commutative and associative properties of arithmetic, the positions of the numbers or parentheses change, but the answer does not change. For a tiling, the Euclidean plane can be rotated, reflected or translated in certain ways, but the pattern remains the same. A knot can be moved and redrawn, but its Conway polynomial is invariant, and so on.

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This verse from the Bhagavad-Gita not only captures the essence of symmetry, it also helps students understand the importance of invariants wherever they might see them. In his commentary on this verse, Maharishi Mahesh Yogi [1] explains that "in action sees inaction" means that one sees the nonchanging unmanifest absolute silent level of pure consciousness underlying the normal activity of thinking, perceiving, and acting. This silent level of life is the source of the active levels of life; it is subtler and more abstract than the active levels, but more powerful and more important. Elsewhere, Maharishi explains this using an analogy of the ocean. The ocean is silent at its depths and the dynamism of the waves is just the natural expression of the silent levels; the silent, nonchanging level is more fundamental. Thus, it is the invariants of a transformation that will be useful to us, even though at first they may seem difficult to grasp because of their subtlety or abstraction. With this perspective, whenever we see a transformation, our first question is, "What are the invariants? What doesn't change?"

For students at Maharishi University of Management, this understanding takes on a very personal meaning in terms of their practice of the Transcendental Meditation technique, which allows the active thinking mind to settle down to the silent, nonactive state of consciousness at its source. In their own experience, they see that their consciousness has two aspects, active and silent, and that the silent level is more fundamental and more powerful than the dynamic level. In a very concrete way, they are able to connect the ideas of symmetry transformation and invariants to their own personal experience.

TEACHING SYMMETRY

Students come to mathematics with rather limited ideas of symmetry; frequently the word symmetry is interpreted to mean "bilateral symmetry" and nothing more. Nevertheless, they will have seen symmetry in many forms already: nature, manufactured objects, art and architecture, and even in mathematics (commutativity, circles and squares, odd and even functions, and so on). It is good for students to have an understanding of symmetry that includes all the examples that they have seen and lays a foundation for further study. I want to introduce them to the idea of symmetry transformation, even though they may not know what a function is, so that they will remember it, feel that it is important, and be able to make some use of it. Students should realize that symmetry locates some underlying property that may be more abstract and less obvious but is more unifying and more discriminating. I also want students to have some insight into why symmetry is attractive and aesthetically appealing to us.

Using the verse from the Bhagavad-Gita as a guide, symmetry can be learned in a unifying way that students seem to enjoy.

We begin with a discussion of what symmetry is, recording some of the students' points on the board. Then we examine some finite designs from the artwork of different cultures and revise our notions of symmetry based on the fact that these designs should come under our definition of symmetry. To motivate this discussion, I bring up the idea that mathematics needs a precise definition that can allow us to definitively say whether something is symmetric or not and that a good definition will also help us to study objects in terms of the property.

Here, the idea of symmetry transformation is introduced. We look at some of the designs and find rotations and reflections and see that a rotation followed by a rotation is another rotation, a reflection followed by a reflection is a rotation, and the composition of a reflection and a rotation, in either order, is a reflection. Further investigation reveals the fundamental properties of finite designs: (1) A pattern can have only rotations, but not only reflections and (2) If the identity is treated as a rotation and there are reflections, then there are just as many rotations as reflections.

At this point, I introduce the Bhagavad-Gita verse and we spend quite a bit of time understanding the verse and how it can be interpreted in terms of symmetry transformations. The questions "What changes?" and "What stays the same?" start to become part of the students' way of thinking.

The first application of this way of thinking comes when we start working out the group table for the symmetry group of an equilateral triangle. After two symmetries are performed, one needs to determine what one symmetry is equivalent to the composition. We look at what stays the same. If one vertex of the triangle is left fixed, the composition is a reflection. If no vertices are left fixed (so that only the center is fixed), then the composition is a rotation. If we want to determine the type of a given transformation, look at what is fixed: if only a point is fixed, it is a rotation about that point; if a line is fixed, it is a reflection across that line; if everything is fixed, it is the identity.

As we move on to frieze ornaments and wallpaper patterns, to identify all possible transformations becomes more challenging. Now, we can think of "inaction" in terms of sameness, lack of change. Locate a motif or small design that is repeated throughout the whole pattern; then see if there is a way to transform that motif or design to as many of its repetitions as possible. If any of these transformations are symmetries of the pattern as a whole, then we have located a symmetry transformation. And the best way to describe the transformation is to say what stays the same: the center of rotation, the axis of reflection, the direction vector for a translation and the direction vector for a glide reflection (the lines determined by these vectors are fixed).

THE BEAUTY OF SYMMETRY

When students begin to design their own patterns, they start thinking in terms of aesthetics, what patterns they like and want to work on themselves.

Symmetry is beautiful and fascinating. From the charm of a snowflake to the deep spirituality of Leonardo's last supper, symmetry has an essential role in nature and art. Can the understanding of symmetry that we have gained here help us in any way to understand this role? We have seen that a symmetrical pattern gives rise to symmetries or transformations

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of the pattern which leave it essentially unchanged. From the Bhagavad-Gita, we see that life has two aspects, active and inactive. According to Maharishi [1], the silent level of life is pure consciousness, the source of thought, and it is subjectively experienced as bliss; whenever the active level of the mind begins to move in the direction of the silent level of the mind, there is increasing bliss. An artistic pattern or structure of nature expresses the diversity of relative existence, yet in the repetition of aspects of the design or structure, that is, in the symmetry, an underlying sameness or unifying value is indicated. The mind is spontaneously led to experience activity and silence simultaneously. This is in the direction of the nature of the experience described in the verse of the Bhagavad-Gita that we have examined. Thus, our analysis can help shed light on the charming nature of symmetry.

CONCLUSION

Mathematics is part of life; mathematicians doing mathematics are subject to the same natural laws that govern all of life. A deep understanding of the whole of life should give us the kind of insight that will help us understand the parts of life, including some very specific aspect of mathematics. This paper presents how one expression of knowledge about the nature of life from the Bhagavad-Gita can be used to go deeply into the mathematical study of symmetry and, hopefully, acts as a suggestion that this bringing together of mathematics and life as a whole can be done in other ways.

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Life Math

Kathy Hayes

Don't take this 2 personal.
It can be +, not −.
Our thoughts might be ||.
The possibilities are relative.

Keep the R perspective.
Choosing division over subtraction,
It's surely the right theory.
But whole numbers are better than $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$.

Don't try to predetermine.
The addition is complete.
The $(x = a + b)$ is on the board.
The eraser is obsolete.

Will there be progression?
Will the ?'s be multiplied?
Problem solving can be fun!
Are the totals on your side?

Don't choose perfection.
Is that not possible?
Are you < or >?
Is the answer plausible?

Study the basic principle.
Does quality = value?
A story problem? T or F?
Calculate the % and review,
Review, REVIEW!

Monasticism and Mathematics

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More than forty years ago Arnold Toynbee predicted that twenty-first century historians would be more interested in the interaction between Buddhism and Christianity than in the tensions between communism and capitalism. His prediction is promising, given the decline of communism. However, it may still be quite a leap to anticipate our future connection to the relationship between these two major world religions.

The thesis of this article is that we as mathematicians already have a great deal in common with monastics. Consider four parallels:

1. We are both ascetics.
Ascetics are commonly thought of as people wearing hair shirts and practicing mortification of the flesh. You may have conjured up an image of a monk but the general public could have easily substituted a mathematician. To the public who typically disdains algebra, the process of advancing a dozen or so levels beyond calculus may seem less desirable than the hair shirt.

2. We are both cloistered.
Mathematicians usually practice their faith (in the discrete or the continuum) in a college or university. They rarely glimpse the relationship between their religion (mathematics) and the eventual real world applications of their work. Monks likewise spend much of their time in activities (such as prayer), where the link to influence in the outside world is uncertain.

3. We share an attitude that money causes debasement.
Bertrand Russell never quite recovered from the seven year stress of writing *Principia*. He lost several hundred dollars in publishing his work. *Principia* is a hallmark of intellectual accomplishment and though not mathematics, is revered by most mathematicians.

After completing my third book this year I have done a little better financially than Russell's negative earnings. But I was recently volunteered to present a mini-course next year at a northeastern conference. When I embarrassedly brought up the issue of honorarium, my colleague told me that the budget was very small for the conference. I volunteered to give the course for free.

The next day my sister-in-law was chatting about the fee a consultant was getting to deliver a two day writing seminar at her office—$30,000. When my jaw rebounded from the floor, my sister-in-law comforted me by adding that there were three extra days that the consultant would be involved with their company as part of the fee.

With two small children needing college educations in twenty years, I encourage a gradual financial debasement within our mathematical community. After all, the best selling CD in Europe last year was "Chant", authentic Gregorian chants from the Benedictine monks of Santo Domingo De Silos. The monastics may be leading the way for us.

4. Paradox inheres in monasticism and mathematics.
Thomas Merton, a celebrated Trappist monk, said of himself, "I find myself travelling toward my destiny in the belly of a paradox." He went on to elaborate that life (in his view) was almost totally paradoxical and that the very contradictions in his life were signs of God's mercy. The mercy was evidenced by his ability to function in light of his insecurity and confusion over the ubiquitous paradoxes of life [1]. According to Merton, both Christian and Buddhist monks are "poured out into the world" by bearing witness to the contemplative experience. This pouring out into the world while removing oneself from the world is a grand paradox. By every measure our world is increasingly reaching out to monastics for their example in our troubled times.

After his ascetic ordeal of seven years, Bertrand Russell failed (in *Principia*) to provide mathematics with a secure foundation in logic. Goedel later showed us that the axiomatic method possessed intrinsic limitations even when confined to the natural numbers. According to Ernst Snapper, "It is evident that such a foundation is not necessary for technical mathematical research, but there are still those among us who yearn for it. The author (Snapper) believes that the key to the foundations of mathematics lies hidden somewhere among the philosophical roots of logicism, intuitionism, and formalism" [2]. We push the frontiers of mathematics possibly "in the belly of paradox."

CONCLUSION

In a moving ABC documentary, entitled The Monastery, interviewers visited a Massachusetts monastery. They gained exquisite access to the candid and private feelings of the monks. One monk questioned his life, saying that he had no direct evidence that God existed. Another admitted his pain at not having had a family. Their touching honesty suggested that they had seldom if ever discussed these issues.

A decade later, it may be time for us to discuss the relation between our isolating and esoteric mathematical endeavors and their long-term influence toward people. Both monastics and mathematicians currently live in a shadow world, when it comes to understanding our long-term effects upon society. If monastics can open themselves up to scrutiny by television, perhaps mathematicians can follow suit and candidly discuss the role of mathematicians in society. We may need help from qualitative psychologists like the visionary Amedeo Giorgi. The Berlin wall has fallen; it is time to break the more formidable walls separating collegiate disciplines.

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8th International Congress on Mathematical Education • Sevilla, Spain • July 14-21, 1996

The **8th International Congress on Mathematical Education** (ICME-8) will be held in Sevilla, Spain during July 14-21, 1996. ICME-8 will aim to increase the development of mathematical education in order to improve the learning and teaching of mathematics.

The Congress will include a wide variety of scientific activities and an extensive cultural and social program. Between 3,500 and 4,000 participants are expected to attend. Principal activities include plenary and ordinary lectures, working groups, topic groups, round tables, workshops, national presentations, short presentations, projects, films, and special exhibits. There will also be exhibitions of textbooks, software, and other teaching materials. Each participant will receive a copy of the Congress proceedings.

The First Announcement for ICME-8 has been published. To request a copy of the Second Announcement, send your name, address, and e-mail address to: ICME-8, Apartado de Correos 4172, 41080 Sevilla, España; fax 34-5-4218334. Information on ICME-8 will be posted on the World Wide Web at the URL: `http://icme8.us.es/ICME8.html`.

Gresham's Law: Algorithm Drives Out Thought

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A talk delivered at the 1987 AMS meeting in San Antonio, Texas. Reprinted from HMN Newsletter #1.

Gresham's law in economics says, "Bad money drives good money out of circulation." Copper replaces silver; silver, gold. Gresham's law in mathematical pedagogy can be stated several ways. "Algorithm drives out thought." "The robotic displaces the humanistic." "Cultivation of algorithms replaces concern for thinking and writing."

We view colleges and universities ideally as places that develop the ability to think analytically, to probe independently, to resolve the open-ended problem, to write and speak clearly. Though the catalog may not mention them, these goals are in the backs of our minds when we picture ourselves as teachers. In the catalog we find descriptions of courses couched in terms of their content, such as: "**Linear algebra**. Matrices and linear transformations, determinants, complex numbers, quadratic forms."

This list, with its focus on topics, illustrates the power of our version of Gresham's law. We can be sure that there will be definitions, theorems, and proofs, and algorithms. Swept under the catalog is concern with the ability to think and to communicate. So, without a battle, in spite of our best intentions, the combination of curriculum, syllabus, and schedule seems to assure the triumph of Gresham's pedagogic law.

Algorithms, of course, are good and must be taught. After all, the world would be an unpleasant place if every time we added two fractions we had to discover the procedure from scratch.

But the temptation to emphasize drill over understanding is almost irresistible. It is much easier to teach the execution of an algorithm than the ability to analyze. Furthermore, an algorithm can be described in just a few minutes and skill in its execution can be tested and scored easily.

Moreover, the incredible power of calculators and computers may entice us to shape our courses around them rather than around the students. As we incorporate these devices into our teaching, we must be sure that their role does not shift from servant to master and that skill in punching keys is not confounded with the ability to think and communicate.

The tendency for algorithm to displace reflection is not new. The student who shows up in our remedial or calculus class may already have experienced twelve years of robotics. Recently in my first-quarter freshman calculus class I assigned an exercise which asked the student to show that a polynomial of odd degrees has a real root. The next day a student asked "Could you work this problem?"

"What was the trouble?"

"Well, what's a polynomial of odd degree?"

"Didn't you take algebra in high school?"

Then a girl in the back raised her hand: "Professor Stein, you don't understand. In high school the teacher works one problem on the board and we then do twenty just like it. We don't have to know anything." A murmur of endorsement swept the room—from students who had graduated in the top eighth of their class from schools throughout California.

In one classroom in an above-average high school, logarithms were taught in this way: "Logarithms are tough, but all you need to know is that when you press the log-key you get the logarithm." This is the complete triumph of algorithm over understanding.

Of course, educators have tried to resist the working of Gresham's law. The director of the California Curriculum Commission recently complained, " Youngsters need to know more than just computational skills. We want them to have a sense about what numbers mean." This announcement followed the Commission's rejections of all the textbooks submitted for adoption in grades K to 8 because they did not relate to the objectives that the Commission had published a year earlier, such as:

> "The focus of the program is on developing student understanding of concepts and skills rather than 'apparent understanding.'"

> "Students should be actively involved in problem-solving in new situations."

> "Non routine problems should occur regularly in the student pages."

These objectives, taken from the 1985 "framework", were not new. In 1980 an earlier Commission had urged,

> "Problem solving has become the all-encompassing theme of mathematics instruction and is no longer a separate topic."

Twelve years earlier, in 1968, a still earlier Commission had said the same thing in different words:

> "Textbooks shall facilitate active involvement of pupils in the discovery of mathematical ideas."

But even before that, in 1963, another Commission had insisted that:

> "Pupils should make conjectures and guesses, experiment and formulate hypotheses, and seek meaning."

> "Materials should elicit thoughtful responses and develop understanding."

So the texts submitted in 1986 not only failed to satisfy the demands of the current Commission, but they wouldn't even satisfy the demands set by any of the Commissions going back a quarter century.

However, concern with the displacement of thought by algorithm did not begin in 1963. In describing some of his experiments in the teaching of arithmetic, L. P. Benezet, a superintendent of schools, wrote in 1935 [1]:

"For some years I had noted that the effect of the early introduction of arithmetic had been to dull and almost chloroform the child's reasoning faculties. [In my experiments] the teacher is careful not to let teaching of arithmetic degenerate into mechanical manipulation without thought... The objectives are first of all reasoning and estimating rather than mere ease in manipulation of numbers."

Incidentally, pupils in his program for one year caught up with pupils who had spent three years in the traditional arithmetic program.

This conflict between the thoughtful and the mechanical is as ubiquitous as the conflict between good and evil. Once you are sensitized to it, you see it everywhere. In one mail delivery recently I found an ad for a college algebra text and a sample of a new journal. This ad included this reassurance: "Numerous algorithms for solving word problems are developed to help students learn and remember concepts." So algorithm finally disposes of its arch enemy, the word problem.

There was an odd juxtaposition between this ad and the title of the journal that came in the same batch of mail: Teaching Thinking and Problem Solving, with the peculiar implication that we need not think to problem-solve.

There seem to be two separate worlds. One is the world of Math Commissions with high aspirations, enrichment materials at publishers' booths, conferences on humanistic mathematics, articles that show how to teach thinking, books with the phrase "problem-solving" in their titles, and the exciting prefaces of texts. The other is the world of the typical classroom, whether K to 12 or freshman to senior at college. Vast storms of reform rage in the first world, but they stir scarcely a faint breeze in the second world. The first corresponds roughly to the world of "thinking"; the second to the world of "plugging in".

The fashionable terms are now "problem-solving" and "algorithms". Whatever the terminology, students know the difference. In anonymous course evaluations, they write, "This course made me think." They do not write, "This course made me problem-solve." The word "think", loose though it may be, is good enough.

But there are many obstacles to teaching "thinking". Some are external to any particular course. As individuals, we can't do much about them: that for twelve years most of our students have learned robotics, with even word problems resolved by mnemonic devices; that society rewards the seemingly practical rather than the fundamental; that many students go to college only to get a good job at a time when the economy no longer even promises everyone a job.

The internal obstacles are quite different. The prescribed syllabus may move so fast that there isn't time to address such fine points as "thinking". The midterms and final are squeezed into such narrow times slots that we dare not pose problems that demand fresh thought. The text may offer almost exclusively exercises that cultivate algorithms. Indeed, if you thumb through many a high school or college text, you can come upon section after section where every single exercise is routine.

Everything seems to conspire to favor algorithm over thought. The syllabus is worked out and expressed in terms of topics, not in terms of processes. Texts, by their very structure, offer answers before the students have absorbed the questions. Homework assignments draw the students' attention to individual exercises rather than to underlying concepts. To cap it off, we're so busy or the classes are so large that we read neither the daily homework (read by undergraduates), nor the midterms (read by graduate students). So, captivated by the clarity of our own lectures, we assume that all is well.

For some twelve years most students have been strapped to a table. No wonder they cannot walk on their own two feet. We must remember that thinking in a mathematics classroom may be a novel or at least unusual experience.

In spite of these obstacles, external and internal, there are actions we can take as individuals to subvert Gresham's pedagogic law.

As we propose a day-by-day syllabus we can delete topics to provide more time to give attention to "thinking".

We may even propose a new course whose main purpose is the cultivation of the student's ability to analyze and write. It can be smuggled into the catalog under the guise of, say, "discrete mathematics".

And we can make a conscious choice as we begin teaching a course. Are we going to emphasize facts and algorithmic skills, hoping that incidentally the students will mature? Or are we going to emphasize independence, analysis, and communication, hoping that along the way students will pick up the facts and algorithmic skills?

In the first case we plan more in terms of our lectures, in terms of what we will do. In the second case we plan more in terms of the homework, in terms of what the students will do.

In the second case we would examine the exercises and ask "What is the purpose of this exercise?" Is it to check a definition or a theorem or the execution of an algorithm? Such exercises have their place, but they should not be the last word. They represent one coin of Gresham's law; they are designed to have a closed field. Blinders are placed on the student to focus attention on particular facts or skills. For instance, we may ask the student to factor x^4-1.

An open-field exercise puts no blinders on the student. We might ask, "For which positive integers n does x^2-1 divide x^n-1?" An open-field exercise may not connect with the section covered that day; it may not even be related to the course. Such an exercise may require a student to devise experiments, make a conjecture, and prove it. If it has all three parts, it is a "triex", which is short for "explore, extract, explain" or for "try the unknown". But it may have only the first two parts, amounting to "find the pattern". Or it may have only the last two parts. For instance: "If a continuous function defined on the x-axis is one-to-one must it be a decreasing function or else an increasing function? This could be reworded to become just the third part of a triex: "Prove that a one-to-one continuous function defined on the x-axis is either an increasing function or a decreasing function." Since experiments with such functions are not feasible, this exercise does not lend itself to the full triex form. However, the following exercise does.

> "Does every convex closed curve in the plane have a circumscribing square?"

The way we word a problem may detemine how closely it approximates a full triex and where it stands on the "closed-open" scale. Here is an illustration in which each variation enlarges the field from closed to open. At each stage the student is offered more responsibility, more chance to develop self reliance.

First formulation:

Prove that if 3 divides the sum of the digits of an integer, then 3 divides the integer.

(This is the narrowest form, just the last part of a triex.)

Second formulation:

If 3 divides the sum of the digits of an integer, must it divide the integer?

(This opens up a bit of the second part of a triex, but the student can guess, "Of course, why else would the instructor ask?")

Third formulation:

Let d *be one of the integers 2 through 9. If* d *divides the sum of the digits of an integer, must it divide the integer?*

(This is a full triex. There are no clues to the answer. The student must experiment and conjecture.)

The following exercise has a closed field: Prove that when a segment AB is cut into segments by dots labelled either A or B, then the number of segments having both labels is odd. It can be recast to have an open field: (a) Draw a segment AB and cut it into segments by dots you label A or B. Count the number of segments AA, the number of segments AB, and the number BB. (b) Do this several times and on the basis of your experiments make at least one conjecture. (c) Prove your conjecture. See [2, 3, 4] for more examples.

So the simplest way to resist the assault of Gresham's law is to include exercises that are not simply routine. To do this, it helps to go beyond the usual ways we contrast exercises as "easy" versus "hard", "short" versus "long", "new" versus "review", but to think in such dichotomies as "computation only" versus "exposition required" or "closed field" versus "open field".

But choice of exercises comes late in the game. Other steps can be taken earlier.

1. Curriculum reform

As we propose a new course or curriculum, we should think in terms of the student, not just in terms of the topics. The temptation is to make a neat outline of chapters and sections, leaving skills in analysis and communication to develop magically on their own.

2. Planning a course

As we work out the day-by-day schedule of a course we should put concern for the student's growth at least on a par with concern for particular topics. This means that we may sacrifice some traditional topics to make time for other matters.

3. Texts

When writing or adopting texts, we should pay attention to the exercises that provide an opportunity to explore, conjecture, and write. This means checking that there are enough open-field exercises.

4. Feedback

The student's work on open-ended exercises requires more careful reading and criticism than do routine computations. An instructor who does not have the assistance of prematurely wise undergraduates or graduate students will have to read papers carefully. This requires time.

There are a few ways to resist Gresham's law of mathematical pedagogy. Perhaps there is another law that reads, "If each of us tries, we can repeal Gresham's pedagogic law."

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Socrates and the Nonslave-Boy

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Compulsive teaching can be risky. I once began a conversation outside of class with a student by saying that *calculus* literally means "pebble" and that playing with pebbles, or *calculating*, was a primitive form of arithmetic. To illustrate, I showed how pebbles arranged in the shape of a right triangle could be regarded as making up half a rectangle. This, I explained, was how the ancient Greeks saw that the sum of the first four positive integers was half of four times five, that the sum of the first five integers was half of five times six, and so on. After a few more examples I asked him about the sum of the first 1000 positive integers and quickly elicited the desired reply that it was half of 1000 times 1001.

Waxing with pleasure at the success of my Socratic method, I remembered Plato's tale relating how Socrates himself deftly led an ignorant slave-boy to the discovery of truth. It occurred to me that there must have been unrecorded instances of Socrates giving lessons to students playing with pebbles, just as I was doing now. What an intriguing idea! I confidently began a Socratic dialogue, featuring myself in the principal role.

[SOCRATES] What is the sum of the first million whole numbers?

[BOY] It's half of a million times a million and one.

[SOCRATES] Great! Now, what is the sum of the first N whole numbers?

[BOY] *(After a pause)* What's this N? N is not a number.

This was not going as smoothly as Socrates and the slave-boy.

[SOCRATES] Yes, but suppose N *stands for* a number. You just did the case when N was a million and you said the answer was half of a million times a million and one... Now, WHAT IS THE SUM OF THE FIRST N NUMBERS?

He seemed to need time to absorb this. I backtracked some more.

[SOCRATES] You said the sum of the first four was half of four times five; the sum of the first five was half of five times six; the sum of the first million was half of a million times a million and one. What is the sum of the first N?

[BOY] *(Exclaims)* OH! The sum of the first N is half of N times. . .

There followed a very long pause, during which I bit my tongue, determined to say nothing more. Socrates would allow the boy to discover the truth for himself. . .

[BOY] *(Exclaims again)* OH!

At last, success is imminent! Is there anything sweeter?

Yet only silence followed as I awaited the answer I expected. Inexplicably, the student lowered his eyes. He would say nothing more.

Clearly, he was waiting for me to speak.

What would Socrates do now?

Socrates would show infinite patience, of course. I took him slowly through the earlier drill again and finally got him to say that the sum of the first billion integers was half of a billion times a billion and one. At length we arrived back at the same point. This time I was sure to get the answer I anticipated!

[SOCRATES] So, now! What is the sum of the first N integers?

[BOY] It's half of N times. . .

[SOCRATES] Yes, times WHAT?

[BOY] *(Growing agitated)* TIMES. . . THE NEXT ONE!

This had gone on too long. I was losing control. Why couldn't the student be like the slave-boy? Socrates was no match for such an awful and obstinate student. I reverted to my normal self.

[ME] *(Wild-eyed)* Of course it is! WHAT'S THE NEXT ONE AFTER N?

I realized that shouting was a mistake as soon as I had done it. The student was eyeing me nervously, as if he knew that at least one of us had lost his mind. For three seconds the silence was electric as we glared at each other, eyeball to eyeball. Then. . .

[BOY] *(Softly and tentatively)* It's 'O', . . . isn't it?

He might as well have hit me in the stomach, so thoroughly had he knocked the hubris out of me. "Of course it is!" I said, "Of course it is. How stupid of me!" I hugged him as tightly as I could. "I love you, Thomas," I said to my bewildered, eleven-year-old son.

Using Mathematics Courses in Support of Humanities In a Liberal Arts Curriculum

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This paper examines the question of designing a course which will support other courses in humanities in providing a holistic education. Such a course could form a part of an integrative experience in mathematics that many institutions require before graduation. The first three sections look at the philosophy underlying such a course and the last section lists topics that could be used in such a course.

Living at the end of the twentieth century, we cannot fail to appreciate the conveniences afforded by modern technology and the promise of the future in science. Many people may be aware of the role of mathematics in this development and may concede that it is a very useful subject. While its utilitarian value is appreciated, the role of mathematics in the history of civilization and its cultural value may not be well understood. The intellectual values have been recognized in the past by educationists and we have had a tradition of requiring most of our students to take some course/s in mathematics. These courses are not always designed for a cultural education in mathematics. Quite often they are prerequisites for some other subject. Students take these courses, often, to fulfill an inevitable formality rather than out of curiosity. This defeats the purpose of such requirements. While some students complain about the need to take mathematics, there are new exciting results found in mathematics and its applications. In some ways it is like Dickens writing "It was the best of times, it was the worst of times".

Many years ago, the study of mathematics was considered as a worthy form of intellectual pursuit for an educated person. Neither its utility nor its relevance for education was questioned. There are always a few who feel that doing mathematics is "fun". But many more will need greater justification to appreciate the role of mathematics. As early as 1953, the mathematician Richard Courant wrote: "...after an unbroken tradition of many centuries, mathematics has ceased to be generally considered as an integral part of culture in our era of mass education. The isolation of research scientists, the pitiful scarcity of inspiring teachers, the host of dull and empty commercial textbooks and general educational trend away from intellectual discipline have contributed to the anti-mathematical fashion in education. It is very much to the credit of the public that a strong interest is none the less alive" [1]. Many people have attempted to redesign courses so that they are more "meaningful". By trying to relate mathematics to everyday applications, people have tried to make the courses less abstract and more down to earth. This should be looked at as attempts to humanize mathematics. But courses that stress the interrelationship of mathematics to other fields and which are more "cultural" in outlook do not appear to be offered very often. Such courses are in a sense "general" and students tend to opt for "utilitarian" courses as opposed to these cultural courses. Regular math courses have little or no time to dwell on these cultural values.

The liberal arts curriculum attempts to be both humanistic and holistic. It reflects a philosophical thought expressed by the Roman emperor Marcus Aurelius: "Nothing is conducive to the elevation of mind as the ability to examine methodically and honestly everything which meets us in life, and to contemplate these things always in such a way as to conceive the kind of universe they belong to, their use and their value with regard to the whole" [2]. The distribution requirements and integrative experience in the liberal arts curricula are just some ways of achieving this. The distribution courses need not always be designed to provide mathematical preparation for some other course. Some of them could have a broader perspective and provide a cultural education in the field. They could be historical and interdisciplinary in nature. The historical perspective is particularly valuable when one wants to treat mathematics humanistically. George Sarton writes:"It is (also) the historian's privilege to make young people appreciate the value of the earlier efforts, however crude they may seem, and to implant admiration and reverence into their minds.... A man's moral worth is largely a function of his capacity for admiration and reverence." [2]. At the same time we need to remember that we are not talking about a course in the history of mathematics. It is an interdisciplinary course offered from the perspective of mathematics.

Let us look at some examples. Mathematics developed as part of the human civilization. The interaction between cultures has played a significant role in its development. The development of mathematics in antiquity (Babylonian, Mesopotamian, Egyptian, Oriental and Greek) provides a good example. [ref 4 ,5]. The use of geometry in art and architecture started in Greece because of their belief in the aesthetic beauty of geometry. Later developments in perspective drawings gave rise to the new discipline of projective geometry. [1, 6]. The interaction between mathematics and philosophy or mathematics and other branches of science are well known. Recent work in artificial intelligence and the psychology of learning mathematics are all examples that provide material for such a course.

In the usual curricula, which is time-bound, it is not always possible to expect the regular math courses to discuss such relations in more than a superficial way. A separate course whose object is precisely to examine these relations is what we need. The non-science student benefits from such a course by becoming aware of the role of mathematics in human civilization. The math/science majors benefit from the integrative experience that such a course provides. Since the value of the course is enhanced by drawing upon the experiences of the students, it is recommended that the course should be offered to students who are juniors or seniors. It can be made more meaningful by expecting students to read selected parts of the original works. The mathematician De Morgan once said that the amazing thing about mathematics is the flights of imagination that one sees in its ideas. We can hope that our students may get a glimpse of it by being exposed to such courses.

In this section we list topics that could be used to develop the kind of course we indicated. They are arranged under headings for convenience. Each heading is followed by topics that could be included under that heading. There is an overlap of the topics. The inclusion of topics is not meant to be exhaustive. The bibliography at the end gives sources where more information on these topics can be found. The book by Prof. Morris Kline [1] is a good book where most of the topics mentioned are discussed along with more references. A two semester course can cover all this material in a leisurely fashion. A one semester course will have to be less ambitious.

Mathematics in Antiquity
Development of number systems, algebra and geometry, decay of Greek mathematics under the Roman empire, its rediscovery through Arabic and Hebrew translations, Greek geometry and the development of early Greek philosophy.

Mathematics and the Arts
Use of geometry in Greek art and architecture, aesthetic value of geometry and art, development of perspectivity in painting, projective geometry, works of Da Vinci, development of cartography, ideas of symmetry in art, the works of Escher, computer art and fractals, Pythagoras and the musical scale, the trig functions and the mathematical description of sound waves, the work of Fourier.

Euclidean and Non-Euclidean Geometries
The impact of non-Euclidean geometries, mathematics as a deductive axiomatic science, the Erlangen program of Klein, use of geometries to describe nature and space.

Calculus and the Newtonian Influence
The search for universal laws from Aristotle to Newton, the creation of calculus and the study of deterministic processes, its influence on philosophy, religion and literature.

Probability
Nondeterministic thinking, from games of chance to the description of physical phenomenon.

Mathematics and Philosophy
Greek philosophy and logic, works of Descartes, Leibniz, & Boole, the impact of set theory and the works of Whitehead and Russell, logic versus intuitionism, the works of Gödel.

Mathematics and Learning

Mathematics used as a universal language, the use of language in mathematics, learning problems, math anxiety, women in mathematics, and math education.

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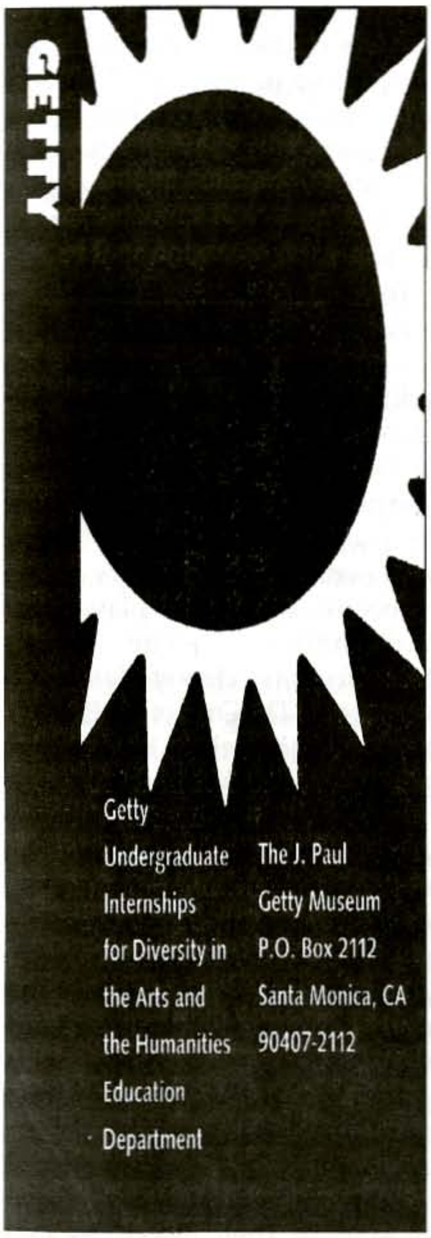

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Multicultural Mathematical Ideas: A New Course

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All too often, even today, the average person fails to realize the universal humanistic foundation of mathematics. We've all heard the standard refrain "I was never any good at math" or encountered that silent pregnant pause when a person makes our acquaintance and finds out we are a mathematician. It seems that many people regard mathematics and mathematicians to be quite far-removed from the reality of human existence. But what makes this so?

I have had occasion to ponder this question over the last few years and have come to the conclusion that the average person's perception of mathematics is quite narrow. In addition, a person's perspective is colored by one's culture and experiences. As a teacher of mathematics, I strive to create experiences for my students that will increase their appreciation for, understanding of, and competence in using mathematics. Recently, I was able to create a new course for our newly developed Middle School Mathematics Minor Certification Program. This course, Multicultural Mathematical Ideas (MMI), developed under a grant from the Lilly Foundation, examines mathematics within various cultural settings, both past and present. Frequently, the things that we study are not called mathematics per se, but might be considered art, storytelling, astronomy, religion, commerce, or recreation. Although I have taught the course only two times, I can say that this course is changing how my students perceive mathematics and things mathematical. Frequently, they comment on how their perspective of mathematics is broadening. They are pleasantly amazed to find mathematics permeating just about every aspect of human endeavor. Ahh! One goal reached!

It seems that many people regard mathematics and mathematicians to be quite far-removed from the reality of human existence.

This course had its genesis at a meeting I attended at the University of Wisconsin, Madison several years ago. At this meeting, scientists and mathematicians from the entire University of Wisconsin system came together to discuss the contemporary science-educational situation. The issue of the dominance of the male, Eurocentric perspective in science education pervaded the meeting. The meeting certainly gave me, a woman in mathematics, much food for thought. As presentations and discussions continued, I began to consider how I could merge my own ideas about teaching mathematics with the ideas that had come forth at the meeting. The course Multicultural Mathematical Ideas is the fruit of these reflections.

In this course, the traditional American Indian medicine wheel serves as a model in our opening class discussion as we begin to examine what is mathematics. We begin to explore the place of mathematics in Western and non-Western traditions, consider whether writing is necessary to mathematics, and try to see how our Western perspective comes into play in our approach to mathematics. The books *Ethnomathematics: A Multicultural View of Mathematical Ideas* by Marcia Ascher and *The Crest of the Peacock: Non-European Roots of Mathematics* by George Gheverghese Joseph provide sound resources for the heart of our course. Through numerous journal articles, we confront a variety of ethnic voices which supply many different perspectives to our mathematical dialogue. Comparisons and contrasts in mathematical philosophy and practice become very apparent as we consider articles such as "Sushi Science and Hamburger Science" by Japanese biologist Tatsuo Motokawa along with American Indian Vine DeLoria Jr.'s "Ethnoscience and Indian Realities," Sandy Greer's article "Science: It's Not Just a White Man's Thing," David Mtetwa's article "Mathematics & Ethnomathematics: Zimbabwean Students' View," and "Chicanos Have Math in Their Blood" by Luis Ortiz-Franco.

Meso-American peoples (the Incas, Aztecs, and the Maya), North American Indians and their ancestors, the cultures of ancient Egypt, the Middle East, China, and India provide a wealth of numerical, algebraic, and geometric topics: quipus with their knots help dispel the notion that writing is necessary for mathematics; bars, dots, and shells and number systems of base 2, 5, 8, and 20 provide systems that work efficiently and completely for arithmetic representation and computation; the sophistication of the calendrics of several ancient cultures makes our own calendar system pale.

The study of patterns is an integral part of mathematics and of the course. In our exploration of pattern, we examine the art—pottery, textile, graphics—of many ancient and contemporary cultures. As one part of this unit, we consider strip or border patterns and their classification using the system developed by crystallographers. Although our classification system is Western, the creation and execution of the patterns found in Navajo rugs, on Pueblo pottery, in Hmong paj ntaub (pon dow), and in woodland Indian beadwork and basketry is traditionally non-Western. The patterns that we find on these items are not traced out physically prior to execution but instead are conceived in the mind and then executed usually freehand or, in the case of textiles, with the use of cloth folding. Islamic art provides a great opportunity for students to explore the many sophisticated constructions and patterns using Islamic style and only a compass and a straightedge.

Recreation provides an extensive backdrop for mathematics. Fun and games is the watchword for part of a unit in which students explore a variety of mathematically-based games of chance and strategy. First, we learn about the culture and context of the game, then how to play it and finally, we analyze the mathematics at play in the game. In pairs or teams, students play several games such as Mancala, Nine Men's Morris, Picaria, Nim, or the American Indian Moccasin game. As a project, students analyze the mathematics behind one such game and present their findings in a paper or class presentation. The Tower of Brahma (Tower of Hanoi) provides the subject for a look at recursion. Students try to move up to seven disks according to the allowed procedures on individual wooden Towers of Hanoi. Later, as a group, we simulate this with student volunteers and then generalize to n disks. Pseudocoding the recursive process follows naturally. The Bridges of Königsburg problem, the sona of the Tshokwe, and Malekulan nitus provide an opportunity to look at several cultures that have been concerned with the same problem—that of tracing a figure continuously without any retracing. In our own contemporary mathematical culture, we too are interested in the related problem of networks. Although the contexts vary greatly, the underlying mathematical concept remains the same.

Archeoastronomy provides many opportunities to look at the mathematics, especially the geometry, that existed in many cultures around the globe. Our own country provides a wealth of evidence for the geometric acumen of earlier cultures such as the Hopewellian earthworks of the Ohio area, the Cahokia mounds east of St. Louis, the medicine wheels of the northern Great Plains and the Eastern slopes of the Rocky Mountains, and the sun dagger calendar at Fajada Butte in Chaco Canyon, New Mexico. These and many other sites give us reason to know that more than 2000 years ago our American predecessors were concerned with the movement of the heavens and built a variety of astronomical constructions that indicated their ability to bisect angles and to construct squares, rectangles, circles, octagons, and ellipses. Frequently, their constructions employed a standard unit of measure. That their geometry was not Euclidean does not diminish its importance.

By exploring a variety of human endeavors within their cultural context, the course Multicultural Mathematical Ideas succeeds in emphasizing the universality of mathematics, its intimate connection to the reality of human existence, and the wide spectrum of activities that exists for the expression of mathematics within a culture.

A variety of activities and techniques serve to make the MMI course challenging yet accessible for most students. Journals are an integral part of the course and students are encouraged to reflect daily on their experiences in the course. The journals provide a sensitive barometer of the pace and challenge of the classes. Because this course is so different from any other mathematics course most students have taken, their interest and enthusiasm are very high. This is evident in their journal entries. Small and large group discussions, laboratory experiments, data collection and analysis, projects, videos, guest speakers, readings, and student presentations provide rich classroom experiences. Numerical ratings and written evaluations of this course as well as personal reflections from journals show very favorable responses from students so far.

SUMMARY

Mathematics is an endeavor that has been undertaken by humankind across cultures and throughout history. By looking at the cultural context of many activities with mathematical basis, one can appreciate the importance of mathematics to that particular society. Different cultures express their mathematical ideas in a variety of ways. Looking at how humankind has expressed mathematical ideas gives us an understanding of what it means to be human. The course Multicultural Mathematical Ideas examines a wide spectrum of human endeavors that are the mathematical expression of a culture and that help to create the tapestry of what we call mathematics.

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X-ette

Lee Goldstein

X-ette, you wonder: you've finally acheived
Your friendless paradise, populated only by
These Pythagorean sonances, those Platonic ideas and unnamed symbols,
And surrounded so widely, too, by the crazed transmogrifications,
Whose nebulous trivia you have so deleted of your deftness,
Where to be does not require to be perceived,
These complements of sense, most of whom live daftly amid zeros,
Whilst the dimensionality, itself, rests singly on your shoulders;

adamantine x-ette,
No Hypatia art thee.

Ode To Numbers

Pablo Neruda

Oh, the thirst to know
how many!
The hunger
to know
how many
stars in the sky!

We spent
our childhood counting
stones and plants, fingers and
toes, grains of sand, and teeth,
our youth we passed counting
petals and comets' tails.
We counted
colors, years,
lives, and kisses;
in the country,
oxen; by the sea,
the waves. Ships
became proliferating ciphers.

Numbers multiplied.
The cities
were thousands, millions,
wheat hundreds
of units that held
within them smaller numbers,
smaller than a single grain.
Time became a number.
Light was numbered
and no matter how it raced with sound
its velocity was 37.
Numbers surrounded us.
When we closed the door
at night, exhausted,
an 800 slipped
beneath the door
and crept with us into bed,
and in our dreams
4000s and 77s
pounded at our foreheads
with hammers and tongs.
5s
added to 5s
until they sank into the sea or madness,
until the sun greeted us with its zero
and we went running
to the office,
to the workshop,

to the factory,
to begin again the infinite
I of each new day.

We had time, as men,
for our thirst slowly
to be sated,
the ancestral desire
to give things a number,
to add them up,
to reduce them
to powder,
wastelands of numbers.
We
papered the world
with numbers and names,
but
things survived,
they fled
from numbers,
went mad in their quantities,
evaporated,
leaving
an odor or a memory,
leaving the numbers empty.

That's why
for you
I want *things*.
Let numbers
go to jail,
let them march
in perfect columns
procreating
until they give the sum
total of infinity.
For you I want only
for the numbers
along the road
to protect you
and for you to protect them.
May the weekly figure of your salary
expand until it spans your chest.
And from the 2 of you, embraced,
your body and that of your beloved,
may pairs of children's eyes be born
that will count again
the ancient stars
and countless
heads of grain
that will cover a transformed earth.

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Mixing Calculus, History, and Writing for Liberal Arts Students

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This is a report on my efforts to design a mathematics course for liberal arts students, particularly for those whose principal interest is in the humanities. My college requires a mathematics course of each graduate, but not all students have the background to take Calculus I. Twenty years ago, when I began thinking about this problem, the various "mathematics appreciation" courses developed for such students tended to be a potpourri of shallow topics in discrete mathematics from which neither student nor instructor could derive much satisfaction. I remember saying, perhaps too cynically, that these courses taught students more about appreciating parlor games than about appreciating mathematics.

While on sabbatical leave at Berkeley in 1974 I found myself writing notes on calculus with this audience in mind. I stumbled upon the idea of an historical approach in seeking an excuse to review the necessary prerequisites for calculus. An historical approach has the wonderful feature that you *have* to review the development of basic ideas, so no one can think of it in a demeaning way as being "remedial" work. To accommodate students who remember nothing of trigonometry or logarithms, I decided to deal only with algebraic functions or with functions expressed by graphs already drawn. I decided to emphasize writing skills to compensate for lowering the usual prerequisites and to play up the supposed verbal strength of my clientele. These notes became a blend of calculus, history, and writing that I hesitantly served up to a class of students for the first time in 1976. I had gone through student records and had sent out letters inviting only the weakest mathematics students in the college to enroll. I still remember that nervous first class of 13 students, whose S.A.T. scores in mathematics ranged from 330 to 480.

A few years later, thanks to the interest of Paul Halmos, an augmented version of these notes was published as a textbook [1]. I use portions of this text (chapters 1-6, chapter 10, and appendices) as the basis for the course I am describing.

Emphasis upon writing, it seems to me, is essential in a course like this. By forcing students to try to learn how to write mathematics, they will inadvertently learn how to *read* mathematics. This point is stressed in [2]. The major reason students are so poor in mathematics is that they can't (or won't) read a mathematics text. Once they get to the stage where they will do this, the instructor's job is much easier.

Surprisingly, an historical approach is useful here as well. Making students learn (by rote at first, if necessary) some famous short but historically important proofs not only acquaints them with real mathematics, but helps them learn to write a mathematical argument with some sense of beauty and style. I think that no classroom time is better spent than the time devoted to helping students master the classical proofs of the irrationality of $\sqrt{2}$, the infinity of primes, or the Pythagorean theorem. Concentrating on the irrationality of $\sqrt{2}$ also gives us the chance to talk about how the existence of irrationals may have made the Greeks decide to speak mathematics geometrically rather than numerically, and to speculate about whether this decision helped or hindered the development of calculus. (How crucial to this development was the notion of a function from numbers to numbers?)

Writing is important also because students cannot learn to think like mathematicians until they learn to write like mathematicians. As argued in [2], they understand the theory behind optimization techniques in calculus if and only if they can properly use a small glossary of words like *let, denote, then, when, therefore,* and *attain*. Proper usage of most of these words can be picked up incidentally by knowing—or even by just memorizing—a few classical proofs.

The greatest benefit of an historical approach to calculus, however, is its enabling us to present the fundamental theorem as expressing a marvellous connection between ancient and modern (*i.e.*, 17th-century) mathematics, and thus allowing a semester-length course to close with a satisfying unity. Roughly speaking, the fundamental theorem says that a number calculated by a modern numerical interpretation of a method introduced by Eudoxus in the 4th century B.C. yields the same number as calculated by a method of antiderivatives introduced by Leibniz and Newton in the 17th century. When presented this way, there is no possibility for a student to think—as far too many students of mainstream calculus courses mistakenly do—that the resulting equality

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

is a definition of the integral.

The main historical theme holding such a course together might be described as the principle of elimination of wrong answers, as it manifests itself in the Greek method of exhaustion, yet points to the modern notion of a limit. This principle of elimination can be introduced whenever you please by discussing the ancient Babylonian method of approximating square roots and interpreting it in modern, numerical terms. At each stage we eliminate more rationals that are too large and too small (and with an efficiency that numerical analysts call quadratic convergence). The point to be emphasized is that a search for a numerical value of $\sqrt{2}$ is equivalent to a search for all rationals that are too large and all rationals that are too small. To put it more strikingly, a search for the right answer is equivalent to a search for all the wrong answers.

Studying the Babylonian method also leads, incidentally, to some wonderful research questions the students can do: What about cube roots? What about fourth roots? The Babylonians didn't attack these questions, but I have found that humanities students can make progress on them with just a few hints. (The natural thing to do turns out to be, like the Babylonian method of square roots, simply a special case of Newton's method, which—when later they come to a discussion of computing roots by this method—they may be delighted to compare with their own efforts.)

If next you attack the problem of finding a numerical value of p, you find the same sort of approach leads to an elimination of numbers too small and too large. You have then planted an idea in the student's head that may later make the Riemann integral easier to grasp.

You can dwell on this longer if you please. Draw five or six famous ratios of geometrical magnitudes and puzzle about the problem of determining when one ratio is equal to (or greater than) another. The picture on Archimedes' tomb leads to a wonderful question for speculation. We are close to Dedekind cuts here, though I have never been brave enough to mention this to my humanities students. (I suspect, however, that this background might make them take to Dedekind cuts better than my real analysis students do.)

The drawbacks of the principle of elimination and the advantages of the notion of a limit are seen clearly when we approach the problem of finding tangent lines the same way. If you want the slope of a tangent line, the method would have you first find the slopes of secant lines in order to eliminate them from consideration. Of course, this won't work if the tangent line cuts the curve twice. But even here, however, the method serves a pedagogical purpose, *viz.*, to emphasize that the "right answer"—whether it be the numerical value of an integral or a derivative—is the limit of "wrong answers" that approach it ever so closely.

My experience has been that this indirect approach of finding wrong answers, in order to eliminate them from consideration, is attractive to students, particularly to students in the humanities who have never before realized that we are doing good mathematics if we have a method for proving that an answer is wrong. Once the idea becomes familiar, it can be seen in unsuspected places, such as in the fundamental optimization principle, *viz.*, that one need only consider endpoints and critical points when searching for the extreme values of a differentiable function on an interval. How many students of Calculus I can explain well the main idea behind this principle? Most of my humanities students can tell you that the curve is either rising or falling as it passes through noncritical points, so we may eliminate such points from consideration unless we are at an endpoint.

Introducing the principle of elimination and then seeing limits as a generalization of the principle results in significantly less confusion among students about limits. In contrast to my students of years ago, these students rarely ask naive or nonsensical questions about whether the secant line ever gets to the tangent line or whether two points can ever become one. Yet this principle is no more difficult than Sherlock Holmes's familiar observation: "When you have eliminated the impossible, whatever remains, however improbable, must be the truth." It is surprising that a reasoning device so simple and so useful is not ordinarily taught in grade school.

I wish to emphasize that the course I describe is first and foremost a course in calculus (though it is restricted to the calculus of algebraic functions). All my students know (because I remind them once every few weeks) that calculus is the study of the interplay between functions and derivatives. Discovering and experiencing the richness of this interplay is all-important. By the end of the course they are expected to demonstrate knowledge of five aspects of this interplay by being able to work simple problems in optimization, in geometric interpretations of the first and second derivatives, in rates of change, in approximating solutions of equations by Newton's method, and in areas and volumes. All my students know that there will be five problems on the final examination testing knowledge of these five aspects of calculus. They also know there will be a few historical questions, and a few proofs to be given, chosen from ones we have concentrated upon.

When I first began to teach this course I never dreamed that students of this caliber could sustain an argument that lasted more than a few lines. Yet I slowly discovered they were capable of writing coherent three-paragraph arguments when they set up optimization problems, calculated critical points, and justified their answers. This has emboldened me to push them a little further in recent years.

My most pleasant surprise has been to learn that these students—whose skills at algebra, inequalities, *etc.* are very low—nevertheless are fully capable of stating precisely the fundamental theorem of calculus and writing, in a style that indicates understanding, a convincing argument for it when the theorem is interpreted as expressing a connection between areas and antiderivatives. My expectation now is that each student understand the meaning of the fundamental theorem in historical context, state it precisely, and present a convincing argument for it. They all know that they will be expected to demonstrate this ability on the final hour test and again on a comprehensive final examination.

I have tried to develop for humanities students a one-semester course in mathematics that is within their ability to learn, that they could be proud to study, and that I could be proud to teach. It is a course that is not a shallow jumble of unrelated topics, but has a unity about it, and builds upon itself to show the depth of the discipline. I try hard to get the students to become engaged in mathematics, to know the spirit of delight in the discovery of unexpected connections between things and to acquire a sense of beauty and style in a mathematical argument—*i.e.,* to know why mathematics is appealing in itself; but I try also to help the students see mathematics as a significant element in the history of thought that has played a role in our understanding of nature, in the rise of philosophy, and in the development of the liberal arts—*i.e.,* to know how mathematics has interacted with areas outside itself. Whenever I have a little spare time, I remind them of such things. Sometimes I have them read Hardy's *Mathematician's Apology* and write a paper on beauty versus utility in mathematics. Sometimes I even pass out reprints of [3].

More than a few students come into the course with the fantastic notion that *liberal arts* means "a lot of arts" (and that consequently mathematics, to them, is not a part of the liberal arts). They are surprised to find out that in this context *liberal* means "liberating" and that mathematics has been part of the liberal arts for nearly 2500 years. I hope that a course like this helps humanities majors to understand the real nature of mathematics, and helps to bridge the gap that separates students in the humanities from those in the sciences. I hope they will appreciate the centrality of mathematics in education by seeing mathematics as a bridge between the arts and the sciences.

Overall, the students' responses have been pleasing. The rate of withdrawals and failures in this course has been lower than in my regular Calculus I course or in Finite Mathematics. It is a joy to teach calculus leisurely for its own sake, to try to transfuse into the students an intuitive understanding of the fundamentals of the subject, rather than to rush through a pressure-packed semester of Calculus I, emphasizing manipulative skills and multifarious applications to mostly plug-and-chug students picking up calculus only as a tool.

This is not say there are no drawbacks. Any mathematics appreciation course can be frustrating to teach because it is bound to draw some initially recalcitrant students. Yet even after teaching it often, it still excites me because I sense that, after a while, it begins to excite some—perhaps even most—of the students, especially when they realize they are not in a frivolous "mathematics for poets" course. Being treated as grownups in a serious mathematics course is a behavior-modifying experience for many of them. Occasionally, an exceptional student comes into the course with a fine background in mathematics and is able to follow it successfully with Calculus II. Generally, however, it serves as a terminal course in mathematics.

My experience over the years has convinced me that this approach is likely to succeed with humanities students. I am also convinced that if I had caught the brightest of these students when they were younger, it would have marked the beginning, rather than the end, of their involvement with serious mathematics. It might be worthwhile, therefore, to try to adapt this course for use in secondary school, following courses in algebra and geometry. A different adaptation might prove valuable to prospective teachers enrolled in mathematics education programs. I know very little about teaching in secondary school and nothing about mathematics education, but I believe that if someone were able to fit this approach into either of these settings, it would make a real difference.

A revision of a talk presented at an AMS/CMS/MAA Meeting, Vancouver, BC, August 18, 1993.

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Humanist Mathematics and the Internet: the Ugly, the Bad, and the Good

Dr. Ruth Guthrie, University of Redlands
Dr. Judy Guthrie, Upland Junior High

INTRODUCTION - THE INTERNET AND THE WORLD WIDE WEB

When we connect to the World Wide Web (WWW) and search for Isaac Newton, we find, among many items, "Sasha's List of Great Thinkers and Visionaries" (http://linuxl uwc.edu/~sasha/thinkers.html). The home page is Sasha's tribute to influential minds. A picture of Isaac Newton is posted, and with the click of a button on his face, we display interesting information about Isaac Newton. Andrei Kolmogorov is also listed. Connecting to his information by clicking on his name reveals to us his remarkable contributions to probability and physics, and his interest in the form and structure of Pushkin's poetry. As it turns out, we are browsing an electronic archive of famous mathematicians (http://www-groups.dcs.st-and.ac.uk/~history/index.html) developed by a school in Great Britain. We could find out which famous mathematicians were born or died on this date—March 1: Charles de La Faille born 1597; Kiyoshi Oka died 1978. Enough! Let us search again, supposing we really desire to collect materials for a course, a differential equations course.

The numerous items here include several electronic publications. We access the Electronic Journal of Differential Equations (EJDE). To get a subscription we must identify ourselves, but the subscription is tree. Perhaps students could subscribe for the duration of the course. An advertisement for Differential Equations and Introduction with Mathematica by Clay Ross is also on-line. We can look at his home page, see his picture, read that he works at Sewanee University, find out what courses he has taught, and use his own electronic links to mathematical resources.

Next, we access the Math Archives home page, organized by topic. Scrolling down to differential equations, we see:

Differential Equations
Boston University Differential Equations Project
C*ODE*E-Consortium of ODE Experiments
Dynamical Systems and Technology Project
Interactive Learning in Calculus and Differential Equations with Applications

A plethora of materials for this course exists on-line. What does all this easily accessible information mean to the field of mathematics? How can it be used to broaden our thinking about mathematics and stimulate students to learn? What is the potential impact of Internet technology on the culture and humanism of mathematics?

THE UGLY—THE PLETHORA

The sheer numbers of data, books and software online may cause significant quality problems since giving and taking—data, books, software, and opinions—electronically is so easy. Perhaps high quality information will be obtained by subscription fees only, or only be available to an "in" group. It is possible that educators desiring to show the humanistic side of mathematics in the classroom will opt to keep students on task, avoiding a medium which is large, complex, and growing, and which could bring too many disparate problems to the classroom.

THE BAD—DISCONNECTION & DEHUMANIZATION

We recognize three kinds of disconnections. One is the disconnection of those who have no access, or who access vicariously. Another is the disconnection of having the technology and having it work so that the ease of a click and print bring numerous impersonal, discursive discoveries to the classroom. The third is the disconnection of persons drawing diagrams in the sand or those counting and arranging pebbles and imaginaries in layers above the concretes. Technology here would encumber.

In a beginning calculus and computers course, students may plot parabolas with computer assistance and never really understand that the formula with x^2

indicates "u-shaped, two solutions, crosses the axis twice." They are more concerned with "Did I key in the proper lower-case password, how do I get the laser printer to work, and when does this miserable lab end?" Not mathematics!

Dehumanization: Users of the WWW can easily get overwhelmed and lost, forgetting why they came in the first place. In startling contrast, consider an ancient village. Denizens experienced birth and death, floods and harvests—highs and lows dispersed by time. On the WWW, high and low experiences arrive every nanosecond. Events which once held such excitement are not common place if you live in a global cyber-village. Perhaps when one mathematics student accesses the mind of another mathematics student, one will merely lurk and another will put forth opinions and solutions.

THE GOOD—THE HUMANIST

The Internet gives the student—lurker or participant—a bond with a community they had no access to before. The community includes students, faculty, professors and Nobel laureates. Hopefully, it gives them a sense of awe and purpose and the sense that mathematics is a living, vital study that has great rewards and a need to collaborate with past, present, and future.

The Internet assists a mathematician by providing access to resources and research quickly and easily, worldwide. It also can link a mathematician into a global network of the mathematics community that has not existed till now. The Internet assists a flexible mathematics educator by bringing subways around the planet and recycling ventures into the classroom.

The Internet can serve as a communication tool between scholars, can facilitate group work and can add facts, personality, and flavor to the mathematicians behind the theorems. Tools of the collaborative endeavor are the following: e-mail, list-serve lists, resources, journals, faces behind work, collaboration, group work, and argument.

CONCLUSION

The technology is here and will assuredly be integrated into our lives. The Internet, specifically the WWW, gives us resources to explore and treasures to find. The true beauty of mathematics will be experienced by students with and without computers, with or without the information superhighway. Mathematical beauty is a human experience that perhaps can be assisted by machines, but not experienced or created by them.

Mentalism

Lee Goldstein

Spiral of the belike,
Hoping of a like to be at the spiral center,
Where the avoidant dislike is the reality principle
And is typically assumed to be from outside,
But where a disike might, too, be about the center,
And there is the reality within,
And of that spiral wish,
When the dislike, he or she might convert it,
At least, as the spiral, into the neither like nor dislike.

A Prandial Dialogue on Absract Algebra as an Introduction to the Discipline of Mathematics for Liberal Arts Students

Margaret Holen
Princeton University
Princeton, NJ

Prologue: The following vignette intends to provoke thought and discussion on the question of how to introduce non-mathematicians to the joys of mathematics. In the future, when my dissertation ceases to distract me, I hope to write a longer article elaborating on the assertions contained herein.

Meals provide a unique opportunity to enjoy company while satisfying a basic appetite. As lively animals, we sit around the table as if it were a trough, and as high-minded animals, we take the opportunity to converse. Our conversations often carry us to realms far away from the dining room, and when many of us are mathematicians, our conversations sometimes lead us to abstract planes of exotic spaces defined by abstruse axioms.

A few months ago, a dinner party provided an opportunity for myself and another mathematician to try to persuade an undergraduate liberal arts major to explore our discipline....

The meal for which we had gathered was dinner. It was hosted by a classmate of mine from the PhD program in math. As we were being served, an undergraduate at the party asked for opinions on what he should take during his last semester. My classmate suggested that he take abstract algebra. The undergraduate expressed some reservations. A lively exchange ensued. I could not help myself from joining in the fray, and in the end we managed to win him over.

The undergraduate gave us many reasons why the prospect of an algebra class did not immediately excite his interest. First, he expressed some scepticism about the value of taking a math course because he did not anticipate needing mathematical methods in his future academic work in literature. (He came just short of expressing aversion of a "Math, ugh" variety.)

We argued that mathematics should be an integral part of the liberal arts education because it provides a forum to explore the power of abstraction and formal logic. Though there is more to mathematics than abstraction and formal logic, these elements are of primary concern to mathematicians in a way that is unparalleled even in the most mathematical of the sciences. Exposure to rigorous mathematics raises awareness of the role of formal logic in other fields. In the process, students sharpen their logical skills, skills which are important to all areas of intellectual endeavor.

Convinced that he should take some math course, the undergraduate told us that his friends' experiences in their calculus courses had not been as enlightening as our advertisements promised. Calculus claims a unique position in college mathematics departments as the most commonly taught course. It gained this stature because of its usefulness in the physical sciences. The useful concepts and methods of calculus include functions, continuity and differentiation, which are as complicated as they are applicable. This makes calculus a difficult course to teach effectively; often the methods are taught while the concepts sacrificed. Even in the most theory-oriented calculus class, the complexity of the concepts obscures the formal logic of the underlying theory. All this said, we added that the interplay between method and theory in calculus represents another part of the discipline of mathematics, a very rich and exciting part, but this aspect of the discipline is much less accessible at an introductory level.

Our undergraduate interlocutor was acutely aware of the math department's reputation for inaccessible material. He doubted that he had the necessary prerequisites to enter an abstract algebra course. Of course, there are prerequisites for algebra addition, multiplication, whole numbers, fractions and polynomials, basics which our undergraduate had encountered many times in primary and secondary school. Abstract algebra introduces students to a formal system of definitions that characterize these familiar objects. This formal system provides a remarkable structure to what begin as collections of facts. Furthermore, students see how a small set of axioms can allow them to prove rigorously familiar properties of their old mathematical friends, leading them to reevaluate why they accepted certain facts (like prime factorization) without rigorous proof. Students also meet new mathematical objects that share the formal properties of objects that they already know well. This process shows students something of the heart of mathematics, how it moves from familiar country to new frontiers along a path of logic and abstraction.

And so our once reluctant undergraduate friend was persuaded to follow our suggestion. I was pleased that he agreed and even more pleased to have had an opportunity to remind myself of how inspiring I find the discipline to which I hope to dedicate much of my efforts.

Epilogue: Indeed, the undergraduate did take our advice and signed up for Abstract Algebra. He enjoyed the material for the first half of the semester, which included the Sylow Theorem, but after that point he found himself overwhelmed by the very fast pace of the syllabus that was designed to challenge the advanced math majors in the class. He would certainly have remained an enthusiastic participant in a course designed for non-majors or a course for majors without such impressive backgrounds.

Thanks to the host of the dinner, Dani Wise, my classmate in the Princeton math Ph.D program and fellow mathematics missionary.

Comments and Letters

Reuben Hersh's diatribe in HMNJ #12 describes a world which I, for one, do not recognize. It ignores calculus and other reforms (with their strong emphasis on collaborative learning and complex student projects), organizations such as Mathematicians and Education Reform, the burgeoning (exploding?) field of research in post-secondary mathematics education, major NSF initiatives, and the plethora of experimentation in and rethinking of post-secondary mathematics courses and programs going on in institutions ranging from two-year college to major research universities. For many years the sessions on education have been among the best attended at the Joint Mathematics Meetings, and in 1996 there were so many sessions, panels, and minicourses on education—sponsored by AMS as well as AMA — that most time slots had at least one, and one time slot had seven. I suggest that Professor Hersh find out what is going on before complaining that nothing is.

Judith Roitman
University of Kansas

Dear Reuben,

That was a great article in HMNJ #12. You are right on target. I agree wholeheartedly. I think the college math community is the most narrow minded and most difficult to move of any group of people I have had to deal with. It seems that what is taught and, as you stated, how it is taught is as though all the students are future mathematicians. What is going on in college mathematics classes and what is needed and desirable is diverging rapidly. They're not only "teaching the wrong stuff" but teaching it wrong. What would Morris Kline say now? By the way, have you read Keith Devlin's editorial in the December 1995 issue of FOCUS? He has some very good points.

Lynn Steen is concerned about losing half of the students in mathematics courses each year. Need we wonder why? I was at a meeting where Zaven Karian was talking about the introduction of computer modeling (some rather sophisticated stuff) into the math curriculum. He said the "good" students catch on just like that, but others are completely lost. I asked if that wouldn't exacerbate the situation of losing students in math classes, and he said, "of course". Of course, these days those who will be mathematicians will need that, but what of others? Other than mathematicians, it seems to me that those sort of things are better taught in the disciplines in which they will be used. Will math professors know enough about other disciplines to teach meaningful applications using computer modeling? There is concern about the amount of math being taught outside the math departments (and the corresponding decrease in math enrollments in higher level courses), but if math is taught as though all students will be mathematicians, this will increase. Also, I'm not so sure that is bad.

The best teacher of mathematics I had, in my opinion, was meteorologist Vernor Suomi. He presented the material in the concise, precise, definitions, postulates, theorem, proof manner that we math majors learned to love so well, but he added the motivation beforehand and interpretation of the results of the model in terms of the application after. I learned my vector calculus from him in the theoretical meteorology courses. The college math community needs to decide between very small departments that train only mathematicians and departments that offer core courses for all students and have their faculty versed in other disciplines where they can teach in or in cooperation with faculty in those disciplines that use mathematics extensively.

Please forgive my rambling. I wish you luck in "belling the cat". Now that I'm retired, it is up to you working folks to do it. Although, there are very few of you who are concerned.

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