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Humanistic Mathematics Network Newsletter #4

December 1989

He took the golden compasses, prepared
In God's eternal store, to circumscribe
The Universe and all created things.

One foot he centered, and the other turned
Round through the vast profundity obscure,
And said, "Thus far extend, thus far thy bounds;
This be thy just circumference, O World"

COVER The Ancient of Days Sriking the First Circle of the Earth
by William Blake

VERSE From Milton's *Paradise Lost,* VII. 225-31.

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EDITORIAL

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The improved appearance of this newsletter is a result of the efforts of Susie W. Hakansson, Director of Mathematics Programs at the Center for Academic Interinstitutional Programs (CAIP), Graduate School of Education, UCLA, and her staff, who typed, designed and prepared the pages through desk-top publishing technology. The address labels were also produced by CAIP.

Mutual support and collaboration are part of the *raison d'etre* of the Humanistic Mathematics Network. The Network also allows us to share ideas about teaching, learning and the nature of mathematics. Last year the Network collaborated in the workshop on Non Routine Problems in Routine Courses led by Sherman Stein at California State University Northridge arranged by Elena A. Marchisotto of CSUN. This year Prof. Marchisotto has arranged four workshops at CSUN, co sponsored by the Network with major funding from an Academic Program Improvement grant. The four workshops will be: Problem Posing led by Stephen Brown, Calculus led by Robert Davis, Instilling Confidence and a Sense of Achievement in students led by Clarence Stephens and New Directions in Teaching Statistics.

Exxon Education Foundation has awarded the Network funds to acquire desk-top publishing hardware and software, to support two conferences, and to sustain the Network and Newsletter for the year.

The Newsletter needs associate editors to solicit and referee essays, report on local and regional events and edit sections such as book reviews, etc. The Network needs regional or state representatives to coordinate conferences or other activities and to be part of a steering committee. I hope that you will volunteer and/ or nominate others.

With a minimum of publicity, after three years, the Network has grown from thirty to over five hundred. The following letter is typical of many we have received.

"I recently read the Humanistic Mathematics Newsletter Number 1 and my enthusiasm for the philosophy of mathematics as a humanistic discipline has prompted me to write to you . . . . It is difficult for me to express in a formal letter how delighted I am to find a group of people who have the same perspective on teaching that I do. The isolation to which you refer is very real. I look forward to meeting you and others in the group so that I can express my enthusiasm directly and share ideas. Definitely add my name to your mailing list. Please send me reprints of the previous newsletters and the dates for upcoming meetings. I would like to present a paper at one of your meetings."

As one of the Year of National Dialogue special events, the Network is sponsoring a panel discussion on Humanistic Mathematics January 17, 9:30 a.m. at the Louisville Math meetings. The panelists are Dr. Lynne Cheney, Head of the National Endowment for the Humanities who will speak on "The Role of Mathematics in the Liberal Arts." Other panelists are Philip Davis of Brown University, Ubiratan D'Ambrosio of Univ. Estadual de Campinas, Brazil, and Alvin White, Moderator.

From Newsletter #1

HARVEY MUDD COLLEGE
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DEPARTMENT OF MATHEMATICS

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August 3, 1987

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by The Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings".

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other, and how they might better come to understand mathematics as a meaningful rather than an arbitrary discipline were among the idea of the first theme.

The second theme was focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, relate discovery to verification, mathematics to science, truth to utility, and in general, to relate mathematics to the culture in which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical".

b) An appreciation for the human dimensions that motivate discovery – competition, cooperation, the urge for holistic pictures.

c) An understanding of the value judgements implied in the growth of any discipline. Logic alone never completely accounts for *what* is investigated, *how* it is investigated, and *why* it is investigated.

d) There is a need for new teaching, learning formats that will help wean our students from a view of knowledge as certain, to-be-received.

e) The opportunity for students to think like a mathematician, including a chance to work on tasks of low definition, to generate new problems and to participate in controversy over mathematical issues.

f) Opportunities for faculty to do research on issues relating to teaching, and to be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures, . . . , the network might formally support writing, team-teaching, exchanges, conferences,. . . .

Please send references, essays, half-baked ideas, proposals, suggestions, and whatever you think appropriate for this quarterly newsletter. Also send names of colleagues who should be added to the mailing list. All mail should be addressed to

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This issue contains some papers and excerpts of papers that were presented at the conferences.

Hassler Whitney 1907-1989
Some recollections, 1979-1989

by Anneli Lax

Some time, in 1979, I talked to Gail Hirsch about my work with NYU undergraduates, and my curiosity about their mathematics schooling before college. Gail asked me if I knew that Hassler Whitney, in recent years, had been doing important work in mathematics education with young children and their teachers; she had met him and seen him work at Bank Street College for Teachers, was impressed with his ideas, and suggested that I contact him.

I knew, of course, of Hassler Whitney, one of the creators of modern topology and also well known as mountaineer and Alpinist, member of the Princeton Institute for Advanced Studies; but neither Peter nor I knew that he had been working in mathematics education, mainly with elementary school children and their teachers.

Hassler's second career occupied him during the last two decades of his life. Many of his mathematical, first-career colleagues and admirers wondered why such an incisive, original researcher would abandon his seminal work to tackle the complex intractable problems of education. And many of his second-career collaborators wondered why he gave up his prestigious academic position for activities that seemed lacking in scientific stimulation and challenge, full of potentially frustrating bureaucratic and political impediments to meaningful changes. The mathematical community bemoaned the "loss" of its stellar members, while the educational community feared that the rough realities would dampen—perhaps burn out—the new spirit who, some thought, had descended from the ivory tower to join their ranks. Members of each of these communities wanted to know what had led Hassler Whitney to change careers, what accounted for such a transformation or conversion in Hassler. These questions are still being asked, even after his death.

Since I did not know Hassler before or during his career change, I shall leave descriptions of that period to those who knew him at the time; his mathematical biographers, who will report on his scientific contributions, and his educational biographers who witnessed his second career from its beginnings, while I note here the answers he himself gave when people asked him about his transition within my ear shot. These answers, together with pedagogical views and their mathematical illustrations—as well as his personal actions and reactions I am about to recall—may well throw some light on how his two careers can be perceived: not antithetical and conflicting, but the second as being a quite natural outgrowth of the first, resolving rather than creating conflict in Hassler, the whole person.

To the question "Why did you leave Harvard?", he replied that he had some trouble thinking of good dissertation problems for his Ph.D. students. To the question "What made you become interested in education, especially working with elementary school children and their teachers?" I heard him give several responses on various occasions, among them "I wanted to become human" and "For the sake of our children."

On October 8, 1979, I addressed my first letter to Hassler Whitney, asking him if he were willing to discuss some of his pedagogical views in connection with a course (Mathematical Thinking) we had designed at NYU for incoming students. He must have responded by telephone; he was always glad to be consulted about mathematics education. He came to New York and had lunch with Peter and me. This was the first of a long series of occasional visits Hassler paid to NYU, often inserting them between other business he had in New York on the same day—see somebody in The Little Red School House, attend a meeting at Columbia, talk to Debbie Meyer, look something up in a big library, renew his passport, or attend a reunion of alpinists. Yet, he never seemed to be in a hurry. He would sit in my office, listen carefully to our accounts of what happened in a Mathink class, to our questions and doubts, and respond with examples; for instance of how one group of children derived a long division algorithm by sharing a given amount of play money equally, or how different children used a string of colored beads to get from 18 to 42 and invented various subtraction schemes to find 42-18 without having to use a new "rule" for taking the units' digit 8 away from the smaller digit 2, or other "taught" schemes. We exchanged examples of how a group of learners might "act out" a familiar situation, observe, reflect and eventually pursue various mathematical activities to sort out its sundry aspects and take it further. We exchanged articles we had noticed, disagreed with or liked; he, it turned out, knew almost all the people I was just discovering: David Hawkins (whom I had met in Boulder and whose work at the Mountain View Center intrigued me); Stella Baruk, the French advocate for children defeated by school mathematics; and a host of others.

Hass would telephone, tell me when he planned to be in New York, we would arrange a time, and he would arrive, usually with a small back pack containing papers and his lunch. He was unperturbed by goings on in my office—students who came and went, telephone calls, etc.—content to read something while our conversations were interrupted. He met several of my collaborators and colleagues, snared Erika Duncan's and my interest in the role of language in learning mathematics and visited our workshop. He had immediate personal rapport with each individual he met, said little, listened intently, even "between the lines."

His pedagogical views might best be expressed by quotes from his papers:

> A primary aim of studying mathematics must be to grow in your own natural reasoning powers, especially in domains where precise reasoning is valid. The growth must include creative and critical thinking, increasing control over one's work, seeing connections with related matters, and raising communication skills. Simply stated, the individual must take increasing responsibility over the work being done..."(1)

He almost always illustrates such statements with little mathematical examples serving as implicit advice to any teacher who might want to try out similar ideas. He obviously did not believe in separating pedagogy from "content", nor intellectual from emotional aspects of learning. He did, however, separate school mathematics from the spontaneous mathematical explorations children pursue out of school and as pre-schoolers. He wrote, for example,

> "...the 'mathematical' experience of most children these days, especially in the inner city, is one of trying to learn the rules of the day; they give up seeing meanings somewhere in the early grades. The great need for the children is to return to their wonderful preschool learning, when they were full of vitality and curiosity, exploring their environment, observing a myriad of inter-connections, and learning complex concepts and skills like communicating verbally and nonverbally, beyond what any of us adults do, and without any formal teaching." (2, p.1)

Later, in the same paper (2, p.8) he follows an example of a simple "subtraction through addition" scheme that mirrors a child's thoughts with the following:

> "Warning: DO NOT try to teach this! for they must understand, i.e. get it into their thoughts, in their way. The superiority in other countries is largely because they teach less, knowing the children are intelligent. We say we know this, but act otherwise, through . . . teaching each little step, destroying the kids' thought processes in the school situation ..."

His opposition to the formal teaching of arithmetic in the early grades was strengthened by his discovery of a 1935 report by L.P. Benezet (3) on an educational experiment carried out in New Hampshire. Hassler distributed copies and called attention to it at every opportunity. In case you have not seen it, let me just say that the experiment, as I recall it, was run in the public schools of some New Hampshire mill towns with large immigrant populations and consisted in selecting classes in which formal arithmetic instruction was suspended during grades 1-3. These children and a "control" group of their traditionally taught contemporaries were later examined and interviewed. Those who had not received formal arithmetic instruction performed better than their counterparts not only in solving mathematical problems; they also reasoned more maturely and responsibly, were more involved with their studies and more "alive."

Hassler's curiosity about this experiment and the reasons for its abandonment led him to search for people who remembered Benezet, and learn about political repercussions of the experiment. (All this will be reported elsewhere (4).) In any case, I asked Hassler what, in the experiment, had replaced formal arithmetic instruction. He said more time was devoted to communication skills. I thought Hassler's condemnation of teaching was a bit extreme and questioned him about what role he thought teachers sould play. I found some of my questions in a letter I had written to him on September 13, 1984, after he had visited us in our Adirondack retreat. This and some other letters in my file refer to "unfinished conversations", evidence of seeds of thoughts—sometimes mathematical, sometimes educational—that Hassler planted in people with whom he chatted, and which they then mulled over and often developed further.

About his warning against teaching, I asked how teachers might initiate the kinds of fruitful explorations he advocates. I have since learned a bit about that, partly by following Hassler's example, and partly by listening to students and pointing out connections between their concerns and some piece of mathematics that they might find helpful. I also pointed out to Hassler that his ideal non-teaching facilitator would need to ask the right questions to help the exploring novices get unstuck or take a next step before they get too frustrated or discouraged in their investigations. I reminded Hassler of the questions he had asked me when I tried to figure out why sailboats, when you let go of the mainsheet and rudder, turn into the wind—questions that were rooted in a deep understanding of scientific principles which most teachers/facilitators rarely have.

The sailboat conversation was, as the reader will discover in the next page a very hands-on, real world problem, right up Hassler's pedagogical alley.

In the summer of 1934, Peter and I invited Hass to spend a few days with us at our Adirondack retreat on Loon Lake. He had planned to be at the Bennington music camp (he played violin) and decided to combine visiting us with finding a place called Hurricane, where he had spent some time as a boy on a family vacation. Characteristically, he arranged his drive from Vermont so as to allocate a prescribed time interval to his search and still reach his destination as planned; and sure enough, although Hurricane Village no longer existed, he found the area by orienting himself with the help of Hurricane Mountain, which he had climbed more than six decades earlier.

We have a Sailfish (a tiny sailboat). He had not sailed before and was eager to try. After sailing with me for a while, he wanted to handle the boat alone. I got off and swam to our dock, remembering how, a few years earlier, I had capsized repeatedly in the process of learning how to sail, and how tiring it was to capsize, right the boat, get back on, capsize again, several times in a row. I watched Hass from the dock, saw him capsize, right the boat and get back on only to capsize again. Hass was then about 77, and I became concerned that he, too, might find the struggle tiring. I then noticed that he was holding the mainsheet at a wrong place and shouted instructions to him. He revised his grip and had no further trouble. However, as he brought the boat back to the dock, his eyeglasses dropped into the water, out of his sight. We looked for them, took turns diving for them in the hope of seeing them or feeling them in the sand, but to no avail. Hass had a square pair and was, usual, not at all upset by the mishap.

The next morning was sunny and clear, not windy, so I decided to make my 6:30 am swim near the dock and make another attempt at retrieving the glasses. When I got to the dock, there was Hass already in the water, evidently with the same plan. He clearly enjoyed the challenge though the object to be retrieved was of little important to him. The glasses were not recovered during that season, nor to this day.

From the Adirondacks, Hassler returned to Princeton, I to New York. I took him up on his offer to drive me to New York and drop me at my office. The conversation about sailboats occurred during that trip. He asked me also if I had ever wondered in what ways tightrope walkers were helped in maintaining their balance by the long rods they hold in their hands. I had no idea, and he again asked some incisive questions that started my thinking. Months later, when I mentioned the tightrope problem in another context, Hass asked if I had had a chance to think about it and take it further. I sheepishly admitted that I had not. In retrospect, I wonder if his Harvard graduate students similarly got distracted or were insufficiently involved with their dissertation problems to take them further, and if this had made Hassler feel that his lecturing and teaching at Harvard had been too remote from his students, and that he had to find ways of helping them gain the necessary confidence and control to get the kind of joy out of doing mathematics that he was experiencing in his research.

Hassler deposited me and my duffle bag at my office before going on to Princeton. Just as I was settling down to work, he appeared at my door asking if he might use my telephone. He explained that he had locked his car keys into his trunk and was investigating ways of getting himself out of this dilemma. These included telephoning to see if his second set of keys might be brought from his house to New York; if not, telephoning his Princeton garage; and as last resort, calling a local locksmith to open his trunk. His equanimity, even slight amusement at having done such a silly thing, immediately stopped me from thinking "this would not have happened if he had gone directly to Princeton, nor if I had helped get my duffle bag out of his trunk" and other guilty musings I am ordinarily prone to. Hassler solved his problem via a local locksmith who arrived sooner and charged less than expected.

One day, when Hassler was to drop by before going to a meeting at Columbia, he displayed the same amazing equanimity. He telephoned, saying he would be a little late because his wallet had been stolen by a subway pick pocket, and he needed to do something about important stolen papers. When he arrived, only a little late, he showed no signs of annoyance, let alone anger, or any kind of stress. He seemed to take things as they came, to stay on top of all situations.

During the period when Erika and I led workshops for teachers, we worked in two high schools and one elementary school in Brooklyn, under a Ford Foundation grant. We told Hassler how much we enjoyed the hospitable atmosphere created by a supportive principal and an enthusiastic bunch of teachers of grades K-4; and that this was the least pressured, most receptive school we regularly visited. Hassler said he was interested in Brooklyn elementary schools and decided to join us at the next workshop there. As usual, he listened hard and said little. However, he left some sheets of paper (see below and p. 5b) for any child or teacher who might want to play with them. I noticed them only by chance at the end of the session and stuffed them into my briefcase. We learned in subsequent weeks that a couple of teachers had also seen them, had fiddled around with them, used them in classes, and told us what their kids had done with them.

Before leaving the elementary school that day, Hassler had a chat with the principal. I learned later that she had put him in touch with another, extremely troubled Brooklyn elementary school. Hassler was grateful to her and enthusiastic about a new challenge. It was quite usual for him to be pleased when things were going well,

EXPLORATIONS

Small and large numbers

We read in the papers about the federal government paying millions for this, billions for that. It seems really too much to pay for such things. Millions, billions, is there any difference? We have a very little sense of this. On the other hand, no househusband would mix up ten dollars with a hundred dollars. We just are not used to larger numbers.

Is there some way we can see all the way from the small to the large?

So, let us put one and a million both on the same line. What are some other numbers you would like to put between them? Where would you put those numbers? That is what I ask you to explore.

And the beautiful thing is that the further you explore, the further you may wish to go. For instance, have you put in 2, and 3? Different people will surely make different choices. But some amy be more satisfying than others. Here is a chance for sharing of views! One group might come to somewhat of a consensus, with another group having different ideas.

Certainly an arbitrary, wiggling, choice, will not be very useful. So might there be some meaning to "useful," which could guide us in placing numbers?

A word about several people or groups working on a question or situation: If it is intriguing, and you want to share, it might be better to share some vague generality than specifics; the latter might "gives things away" and take away some basic joy of discovery. Of course brilliant thoughts might come out and prove keys to answers, without even the realization that this is happening. Spreading success can also be illuminating and pleasurable to all or most concerned.

To end, I do mention that some channels of this exploration can lead to very basic and beautiful parts of mathematics: and you don't at all need to be brilliant mathematically to get into these directions.

Where's your pencil?

					71
		42		56	63
		37	43		55
			37		
		27		35	39
16	19	22		28	

but to feel that his presence was not needed in such places. I suspect he planned to use the occasion to gain entry into a "difficult" school where he would be needed and that he just ascertained, *en passant*, that things were going well enough at our school, well enough to leave behind these little gems—not your ordinary educational "materials", but more seeds for thoughts that might fall on fertile ground.

I saw Hassler at the January meetings in Phoenix talking to a group of people including U. D'Ambrosio in the lobby one evening. As I stopped to greet this little group, one member (I cannot recall who) asked if any of us had sen V.G. Paley's article "On Listening to What the Children Say" (5). I quickly said "yes, in fact I sent a copy of it to Hassler," and I heard Hassler say "This is the most important article I have seen since Benezet's!" I found this rather refreshing, because at many previous meetings Hass had seemed so preoccupied with Benezet's experiment that most people who wanted to hear about some of his other ideas and concerns began to feel he was a bit hung-up on Benezet. But his most recent activities and writings show that he was forging ahead on many educational fronts, was gaining breadth and balance in is expression, and kept growing.

During our mathematical or educational conversations, Hassler would often make some personal remarks. At some point he began to speak fondly and admiringly of Barbara, a painter. Some time later he told us that he and Barbara had decided to get married. Eventually we met Barbara, a warm, delightful person, who seemed to support and supplement his educational interests and to add an artistic, aesthetic component, helping him towards his goal of becoming even more "human".

On April 28, 1989, Shirley Hill was scheduled to speak at NYU on "The Future of Education in Urban America", one of several regional talks she was giving about the report *Everybody Counts*. I had called it to Hassler's attention. He phoned to say that he planned to come to New York, also to take care of some other matters in preparation for a trip to Europe he and Barbara were planning to take in the summer. They were to spend the night at our house and go back to Princeton on April 29, the next day. On April 27 Barbara called to let me know that Hassler had had a severe stroke. He did not recover.

Barbara was in touch with me during her sad time since April 27. Hassler's family gathered in Princeton; he died on May 10. I visited the following Sunday. His daughter Moll, who had come from New Zealand, had called me, wanting to talk about her father's work in education. She had developed some explorations of pentominoes in her father's spirit. She also thought she had alerted him to the importance of keeping track of units (dimensional analysis) when setting up and solving equations that describe scientific phenomena or, for that matter, any situation. (Hassler had published a beautiful little article on the motion of a simple pendulum using only the units to derive the fundamental relations, see (6). I have never understood why high school and college teachers so rarely use this essential tool in the classroom to keep the meaning of formulas in mind and/or to check that algebraic manipulations and results make sense, i.e., are dimensionally correct.)

Molly told about her own scientific training and her present work in New Zealand. She recited the words of a round her father had written about a butterfly struggling out of its cocoon, and I hope to see a written version also of the melody. The words symbolize his educational philosophy.

I briefly met some of Hassler's other children, their spouses and his grandchildren; they would all disperse the next day. Barbara, still a bit overwhelmed, would stay in Princeton.

After talking with Molly, I thought Hassler's "for the sake of our children" undoubtedly included his own. He was one of the rare people who, in discussing a mathematical problem with others, did not let his own agenda get in the way of listening, thus allowing himself to learn from others, including young students. He would often report about such experiences with delight. Molly may well have initiated his emphasis on units in elementary mathematics instruction.

In the present climate of educational reform, the question "What is the impact (of this project or that)?" is often raised, especially by program officers of funding agencies who want to maximize the effects of the grants they award. "What has been the impact of Hassler's work in mathematics education?" will undoubtedly be asked by many people. As others tell their reminiscences, very different Hasslers may appear; he touched different individuals in different ways. He did not teach or preach. I conjecture that his impact will grow as the seeds of thought and reflection he planted subtly continue to transform people he touched, and who, in turn, will have their educational ripple effects.

Hassler enjoyed solving problems—in mathematics, in mountain climbing, in education. He took in the whole situation, formulated simple goals and enjoyed the process—the means of attaining them—at least as much as getting there. I have heard him comment on how good it feels to negotiate a path through thick woods, even in the dark, to be physically fit; and though I did not hear him say so, I am sure he felt that same delight in blazing his mathematical trails and in feeling intellectually fit and in control. He did not see why these struggles and pleasures of growth should belong only to a privileged few. Everybody, he thought, should have a chance to grow, gain control, become responsible, and that schools should help rather than hinder this process. After attending some mathematics education policy committee of a school system, he said "the goals are so simple, but in the discussions people lose sight of them and clutter up the path wit short term 'objectives'".

Hassler attacked problems with his eyes wide open, tolerant and patient with impediments to be overcome, deeply involved. He said to Louise Raphael, when she was a rotator at the N.S.F. "people think I am naive; I am not naive." Indeed, he was not naive, but neither was he *blasse*.

Hassler was not a martyr who, from a sense of duty, sacrificed his time and mathematical talent for the sake of education. On the contrary, he did precisely what he wanted to do in both his careers, his mountaineering, his music, and in his interactions with others. May his example convince the mathematics research and education communities that there is, in fact, no real schism between them. What he advocated in education—using children's curiosity to help them explore, individually and in groups; have them communicate and revise their findings, learn to justify and defend their reasoning methods; struggle and feel good when they gain ground and control; become so involved that they no longer need either approval from authorities or external incentives—is, after all, what good researchers do, albeit at a different level. This is what human learning is all about, not just for mathematicians and scientists, but for all who want to understand, help modify, and find their places in our complex world.

Notes and references

1. H. Whitney. *Coming Alive in School Math and Beyond* , June, 1985

2. H. Whitney. *Mathematical Reasoning, Early Grades*. Growth through involvement, curriculum outline. May, 1988

3. L. P. Benezet. The Story of an Experiment, *Journal of the National Education Association*, 1935, 24(8) pp. 241-244, 24(9) pp. 301-305, and 1936, 25(1) pp. 7-8.

4. Whitney's educational papers and correspondence is being collected with the help of his widow, Barbara. She was established a Fund for a Mathematical Education, c/o W. Guthrie, Rider College School of Education and Human Services, Lawrenceville, N.J. 08648, so that Whitney's educational works can be catalogued and made available for study and biographical information. W. Guthrie and other colleagues are helping with this task; further details will be announced probably in Focus (MAA), The Mathematics Teacher (NCTM) and other publications.

5. V.G. Paley On Listening to What the Children Say, *Harvard Educational Review*, 1986, 56(2) pp. 121-131.

6. H. Whitney The Mathematics of Physical Quantities, Part II, Quantity Structures and Dimensional Analysis, *American Mathematical Monthly*, March 1968, p.254.

The Visits of Hassler Whitney to Brazil
Hassler Whitney, In Memoriam

by Ubiratan D'Ambrosio

Beginning in 1975, Hassler Whitney visited Brazil almost every year until 1985. Regularly, in April or May we would count on him for a months visit to Campinas.

In 1975 I was the Director of the Mathematics Institute of the State University of Campinas (UNICAMP) in Brazil. We had received a research grant which allowed us to invite foreign visitors. I was working in Geometric Measure Theory and was particularly interested in the late mathematical work of Hassler Whitney. But he was such a renowned mathematician that it was unlikely that he would accept an invitation to visit a new Mathematics Institute in a relatively small town in Brazil, invited by someone whom he did not know personally. With some hesitation, I dared write to him. The reply came promptly, very simple and sincere saying, to my surprise, that he was afraid he had not much to offer in Mathematics because since a few years back he had fully committed himself to Mathematics Education. Even though, he was ready to come if we would have interest in his kind of work.

By then we were beginning a major project in Math and Science Education, sponsored by the Organization of American States and I was also committing myself to Math Education. Whitney's letter was a pleasant surprise and I wrote back, inviting him as an OAS consultant. The visit of Hassler Whitney to the Math Education group would bring to our group, which was struggling for survival among hard core mathematicians, some respectability.

Indeed, he arrived and in his affable style got involved with our projects in Math Education, visited and lectured in elementary and secondary schools and to groups of school teachers and absorbed just about everything we were trying to do, always offering acute remarks and ideas for improvement. At the same time he agreed to offer a few lectures to the Mathematics Institute, more in the nature of mathematics, about his processes of creating mathematics and about history. Of Course, his presence in the Mathematics Institute was a source of excitement, but he was really more at ease among Math Educators, school teachers and children.

Of course, we reserved for him the best hotel in town. A few days later he moved to downtown, to a very simple hotel, and every year since he would return to the same hotel. He became known to the people in town, since everyday, he was jogging in downtown Campinas. We would pick him up in the morning and he worked all day in his office. He was very excited about his "part B" as he used to call a set of notes he brought in. He became very excited about Benezet's paper. But he always preferred to visit schools. He was eager to see the schools, talk to teachers and children and to revise our work. He devoted himself to us. He would require practically nothing of the formalities we always have with visitors and insisted on being called Hass. It was very easy to please Hass with a regular family dinner and he loved to bring his violin and play a duo or a trio with my children before they would go to bed. Then amid good and varied conversation we would learn superb lessons of life, of love and of care for mankind. Several colleagues had the same experience and the return of Hass was always requested. He very easily accepted invitations to spend a few days in other small towns, lecturing and always visiting schools and talking with children and teachers.

To a number of Mathematics Educators in Brazil Hassler Whitney was a most influential figure. I personally owe him much of my views on Math Education. His first visit to Brazil and my first contact with Hass was when I had finished my first version of the Chapter on "Objectives and Goals for Mathematics Education", for ICME 3, in Karlsruhe. His reaction to my controversial position was highly positive. Our friendship and professional ties grew and a convergence of views on Math Education was for me reassuring and surely the best encouragement I received to develop what I later called "ethnomathematics". I was reassured of his support to these controversial ideas when in Helsinki he invited me to be a Vice-President of ICMI while he was the President. Our relations in the four years in the EC of ICMI were excellent, always very close to children and to school teachers. Besides his regular visits to Brazil we would meet in three or four occasions every year in different places. Those four years were for me a most rewarding and beautiful human and professional experience.

With Hassler Whitney I learned much about education, kindness and, above all, I experienced the company of someone that revealed, while rejecting pride and arrogance, the highest moral and intellectual standing.

EDUCATION IS FOR THE STUDENTS' FUTURE

(Draft)
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The problems of education are still on us. The public wants something done now, so we can get back to other pressing matters. Professionals say "The schools are improving; the test scores show a little upturn. The worst is over." District boards of education and politicians are calling for stronger superintendents and principals, who will "not let them get away with it." And state departments of education are warning schools to "Raise those scores, or we will have to take over."

In the center of all this chaos lie the students and teachers. They are getting the real blame; yet they are powerless to do anything about it. The increasing pressures force them into routines (drill and practice, mostly) that have been there the whole century (see Cuban, 3), and the resulting rote learning is making impossible the meaningful learning of skills, let alone using them outside the school.

The real tragedy is that the result is increasing anxiety and desperation on the part of the students, resulting in increased dropping out, delinquency, drugs, crime and suicide. The long-range problem of shifting the whole process is basic; but the immediate need is to lessen the pressures and allow (and aid) the schools to change their ways. There is increasing desire and understanding which can get the process started, if we can only help rather than attack them. But the negative forces are strong and on high levels. It is essential that we make concerted efforts to get communication and cooperation towards a truce so understanding and reform can take place. This is the reason for my writing these lines.

The continued failure is proof that we are missing something basic. It is clear that we must apply real problem solving, decision making, to make a valid upturn. The whole situation is very complex, with many features of varied sorts playing important roles. We cannot expect to change the workings of such a situation for the better without a deep understanding of the whole dynamics lying underneath; so we have to find the features of most importance, choose basic and workable goals, and use those goals to check on what we are doing. Only in this way can we make either immediate improvement or long-range success; so I am asking us to get at this job, along with any other activities we feel pressed to do.

We know that rote learning is useless for applications; we know that pressures and mass testing holds teachers down to teaching by rote; and we know that, in spite of our struggles, nothing has changed. But also, we have no real faith in children's being "able to think" (I speak especially of we who study them in the classroom), and we are just too busy to give the needed attention to this massive ferment; so we let it go. But what we can do is make it a *basic commitment*, and aid others in the task.

If the working goals contain the most essential elements, that will be sufficient to keep us on a good track. So I consider first where we want young adults to be, and then the schooling needed to help them get there.

Basic goal for the future: Adults will have the power and commitment to help mankind towards keeping us and the planet as a well functioning whole.

There are several parts to this, from the deep to the outwardly clear.

(1) The commitment involves a real sense of responsibility. This is moral, spiritual, ethical, rather than intellectual.

(2) We need a deep understanding of the world and society in all its complex interrelations; this must allow us to carry out sane decision making.

(3) Power of reasoning is used to sort out and comprehend our complex world.

(4) The actual knowledge is at the basis of the understanding and decision making.

Schooling is a process, so we state its goals in terms of the process.

Basic goal for education: Students will be aided in growing toward the above goals.

Growing is *integrating* oneself in all one's functions, essentially through one's life experiences. As an adult, there will be no teacher to watch over you; you have to control everything you do. So *experiencing* such activities, controlled by you, is a fundamental part of your learning.

We look more at the individual goals. Responsibility and commitment cannot be taught. They come from experiencing and appreciating the common good when you have worked toward that end. For instance (from Glenn and Warner, 5) at breakfast on the farm, all the extended family there, the three-year old is admonished: "how do you expect us to do our work when you have not brought in the eggs which nourish us?" The child senses the warm and nourishing tone behind this, and is moved inside to greater commitment.

Decision making is an intellectual conclusion; in difficult situations it must be based on both your careful thinking and on your deeper life experiences. (How to bring these experiences to your attention has been studied by Gendlin 4.) It is particularly the humanities, including history, which are the source of growth in this area in school and college. But outside experiences may be still more important.

Goal 3 will be served in part by all school work, especially through discussion, and partly by one's continued need for growth and control. And goal 4 is essentially on factual content (and easily describable processes); it is this part which is tested on the usual standardized tests, hence concentrated on by teachers.

Note that language skills, and communciation more generally, is a basic part of the above. So discussions should be, in part, about such communication.

Are all students inherently capable of such goals? As preschoolers, they certainly grew in all ways, learning complicated and subtle things like communciation, faster than they ever will again; and all without formal teaching. Like all animals, they lose none of these capabilities while growing. So their apparent loss in school is due to our pressures, not to them. The fact that there are essentially no "disadvantaged" children is shown for example by Benezet's experiences; see (1). Those of us who have worked carefully with children in difficulties are in general well aware of this.

We are now at the basic part of our work, to pick out the essential features of schooling and see what can be done about them to work toward the goals. So let us look at that clear issue, rote learning. First of all, what sort of thing is learning? Has it degrees of intellect? I see four distinct levels:

Level 0. You do as you are supposed to, mechanically, without thought. Drill goes directly to this. It is though you were becoming the hardware of a computer.

Level 1. Like level 0, except that you must do some *translating* from words to task. Thus this is like software.

Level 2. A situation is described, from which you must pick out the particular task required. This is like being a computer operator. It is like our usual solving of life's little problems, without trying to push into new directions.

Level 3. The situation is complex, perhaps with subtleties; you attempt to find meanings, questions, and come up with answers that may contain breakthroughs. This is like being a designer of computers.

A final level would consist of creative inspirations, leading to new points of view.

The difference between the first two levels is shown by the National Assessment (NAEP) questions: Do 21 + 54 (written in vertical form), correctly answered by 90 percent of the 9-year olds, and "What is the sum of 21 and 54?", with 69 percent correct. Apparently, the children were well drilled in the simplest addition; but when mere translating "sum" into the command (these were difficult children), the could not be "software," having been trained to be "hardware."

This makes evident also the difficulty in trying to teach children problem solving: Teaching it turns it into a lower level; but in real life it is not on that level (and the teacher is not there for you).

Is the US falling behind other nations? We certainly mean this question on a high level of thinking. Yet the "international comparisons" are carried out mostly through standardized tests, which are almost completely on the lowest levels. These tests may as well be disregarded. They tell us about schooling, not about high thinking.

What is rote learning in school, and why does it happen? We mostly believe that in school math, the children are simply learning what you should do on given tasks, for instance in two-digit subtraction, without trying to get the meaning behind it. I find reality quite different. The children are mostly trying to put down *correct marks for that day*, not correct answers to *questions* (or tasks). They don't look at the questions; they guess at what sort of marks (numbers) to put down.

I have described in (10) how children are confounded by changing rules for subtraction, given a *pattern* (as they see it) of four digits making a square box. And this is typical of their attitudes which are formed early and never change (until college perhaps): Learn the rules of that day. And guess what to do; meanings don't count.

But then how can they be sure of correct answers in *using* arithmetic outside school? Of course, they don't try; the know very well that school math is for school, not for the outside. (compare Carraher et al., (2).)

Let us look at the attitude about "math" of the textbook writers, the testers, and probably the teacher trainers; for these of necessity become the attitudes of the teachers.

For them, math is a set of skills and subskills, with right ways to do everything. The students are supposed to *see relations* among skills, which make it easier to remember them; but for them, under "rules for the day" the complexities merely get all mixed up with each other, so they can only guess what to do.

The general professional attitude is: "know all the skills; in a particular case see which skill is needed." This is similar to the attitude of mathematicians, directed at students: "Learn more math; you will need it later." It is *experience in mathematical reasoning* in various situations that is needed; the bare mathematical facts can be learned *through* that reasoning, and on the job.

The basic difference can be seen through levels of thought: Level 0 (and 1) learning is almost useless; it can only help speed you up. It is level 2 that can be practiced routinely; with level 3 to challenge students and get them used to carrying out their work, especially through explorations, organizing, and the like.

Thus the math *curriculum*, in terms of topics to be taught, misses the students; and the *instructional methods* should certainly not be a bare presenting of material.

I believe I am expressing what has been the normal attitude about math among professionals, and among those trying to make immediate improvement in schools. But I also see an increasing move toward deeper understanding, leading to the view I am presenting here. I have certainly seen a far greater commitment to true learning in the last year or two than I could have expected before that time, so I have real hopes for basic improvement.

We all want very much to help others, and try to do it through advising them. But this *tells* them what we think they should do, reducing them to level 0 or 1: do what you should. And we really do not want that to happen. So we must also realize the difficulties of school teachers shifting from presentation of material to helping the students start exploring situations. (And of course it applies on the college level also.)

I now look at the problem of getting real improvement started. The *fundamental* problem is that of getting *changed attitudes*.

On the one hand, it is easy. Chatting with a student who "can't think," if I speak of things I am interested in, say going shopping, the student is likely to think of similar thins also and shortly she or he is doing real mathematical thinking.

The teacher attitude "they cannot" (place value, long division, percent, whatever) is very strong. For one example, a group of six teachers (in Brazil) said "It is impossible to teach long division!" (with a one-digit divisor). After some discussion, I "paid them for writing that article together": Plain popsicle sticks were ones, reds were tens, blues were hundreds, those two yellows were thousands. "Why don't you share it?" Then remained passive. In time, they decided to use the money bag, and exchanged a yellow for ten blues (tied together with a rubber band). Soon the sharing was accomplished, with some ones left over. I then asked them to repeat, a bit structured (big money first), and record. Looking at the recording and seeing the algorithm, they exclaimed "*Now* I know that *my children can!*" They knew their children could carry out that experience as well as they did. In ( ) I describe how to continue this process to a solidity in the topic, which many say should be thrown out of the teaching schedule (of course it should not be *taught*).

Turning to these students again, what might we look for in them to show real progress being made? Of course the first thing is, they must *be there*. This means, not just in body but in spirit also. There is no way to grow toward the goals without this. So let us walk into a typical classroom and look at the students. We know from Goodlad (6) (and from our own experience) that we are most apt to see boredom, passivity, lethargy; they are half alive. We would not choose any of these students to come and help us. So the **climate** of classrooms is basic; *involvement* of the students in *real study* is absolutely essential, for at least a good part of their time and energy.

And how can we get involvement? Is it there automatically with a good climate? Unfortunately, no; the students, having had those poor attitudes for years, have *no experience with involvement*. "So we have to motivate them!" We let them play games, getting points for right answers. This is blowing on that bit of spark inside, hoping a flame will be lit. But such a spark will burn out at once without inner nourishment. The *spark of life* must be *nourished from inside*. And this comes from inner involvement, that preschool type of curiosity and desire for exploration, that must be revived. And just as attitudes can jump forward, then go forward later more easily, the same is true for the spark of life, the true involvement.

Finally, I look more directly at what we can do. On the local level, this involves communication and cooperation with the groups in school, so we work together. We are just beginning to see cooperation among the education groups involved; compare Meier and Shanker. Helping get such cooperation started can work wonders.

On the higher level, for instance with state boards of education, getting into real communication may be difficult. But it is extremely important, because of the power exercised over the schools. Getting in touch, to get a real level of communication, may not be easy. They do not understand, seem not to hear what we say, and vice versa. *Continued contact*, finding some bits of topics with similar interests, can lead to better communication. I find my attitude of acceptance and respect for anyone I am with to allow closer communication rapidly; but continuing may still be hard.

In a group of a few people, desiring to get into the heart of matters, there are some simple principles that help. Making short, carefully thought out, statements, and then *pausing*, promotes real listening and consideration. The others *and* you gain from this. Neither pushing nor pulling may help; but accepting the *process* underway and looking for a time to continue it, is likely to at least give a few people happy to go further.

I have said a lot without being able to hear responses, so I stop here. After more contacts, I expect to write further, and I hope better still. In particular, I am starting a series of exploration topics which can be played with (without help, as far as possible); these, in part, combine science with math. And a particular purpose is to have them used in groups, which will help with language skills and communication in general.

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PDP/Academic Excellence Workshops in Mathematics

Talk delivered at the Southern California MAA meeting November 11, 1988.

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Let us fantasize for a bit about the characteristics we would like to see in our "ideal" student of mathematics: curious, logically precise, persistent, understands concepts and their applications, communicates effectively in the language of mathematics, and so on. How can we as mathematicians develop these traits in our students?

If at this time we are not satisfied with our students' performance, we must realize that our educational challenge will become even greater as our classrooms reflect the growing cultural diversity of our country as we move to the Twenty-first Century. By the year 2010 in California, for instance, the white non-Hispanic students will be in the minority; for some of our campuses that is already the reality. Thus we mathematicians must not only learn how to teach more effectively the traditional 18 to 21 year-old, middle-class white student, but we also must develop pedagogy that is effective with those from other cultural and educational backgrounds.

Now imagine a group of Black, Hispanic and American Indian students meeting voluntarily 4 hours each week to discuss mathematics. They usually are working in self-selected groups of 3 or 4. As the quarter progresses, they have become quite comfortable with each other and have no hesitancy to move around the room to check on how another group is approaching a problem. Friendly rivalries develop, and they will good-humoredly challenge each other's solutions. To break the routine, the Facilitators will sometimes divide the group into two to four teams. Each team will then compete to solve a "challenge" problem—one that requires a higher degree of sophistication. They will work intensely and with great enthusiasm in hopes of becoming the first team to complete the problem correctly. Many times when the workshop period is over, students remain to complete the solution of a problem or to conclude a discussion of some technical point. Frequently, they will arrange to study together at additional times, especially to review for an examination. Thus the students not only master the material with a higher level of understanding, and learn how to communicate technical material, they also experience the rewards of membership in an academic community. What creates this enthusiasm for learning that we too seldom see? If we can develop strategies that increase the academic performance of minority students, then we will have gained an insight in to how to teach *all* students more effectively.

The workshop model as developed by Prof. Treisman at UC Berkeley and implemented at Cal Poly Pomona is designed to provide a means to develop that academic community for the Black and Hispanic student. These workshops therefore are based on the following premises:

> Students that we see in our freshman calculus are the best from their communities. This is especially true for the Black, Hispanic, and American Indian student since only 60% of Hispanics and 75% of the Blacks that enter high school graduate as compared with 83% for the non-Hispanic white. Of the 18-24 year olds, 28% of the non-Hispanic whites are enrolled in college as compared with 18% of the Hispanics and 20% of the Blacks.
>
> These students are highly motivated; the minority student, especially, is under great pressure to be successful: both from within and from family and community who see this student's success as a reflection of the capabilities of that culture.
>
> The "brightest" minority students (that is, those with higher SAT's) historically have all too often been those least successful in traditional courses.
>
> One of the primary factors that precludes success for such students is the intellectual isolation within which they operate. The Asian and the fraternity/sorority networks are very effective; however, the bulk of our students

have no means by which they may develop their own intellectual community.

These premises may be startling to faculty who have assumed that students come to us either with many of the characteristics of the "ideal" mathematics student or that they do not deserve our time and our resources: that is, our students can "shape up." Some faculty may feel that to assume another posture is to lower our academic standards and let "weak" students through who will be unable to perform in the future.

In the less sophisticated student, this attitude is hostile to the development of those traits we desire in our "ideal" student. We can have an effect on the qualities that we expect and demand. It is not student apathy or perversity that causes the difficulty. Dr. Clarence Stephens' Mathematics Department at SUNY Potsdam, the UC Berkeley PDP workshops, and developing Academic Excellence workshop program at Cal Poly Pomona demonstrate that, first, when we can create an academic community among our students to support their development, and second, when we encourage them to practice learning mathematics in that community, we enable them to develop the ability to synthesize the fundamental principles we so wish them to learn.

Thus, there are two levels of teaching for which we are responsible: the first, which we all recognize is the mathematical content, the second is the process by which students learn mathematics. We give homework for the students to practice their mastery of the content, and we judge this progress through quizzes and tests. We ordinarily, however, provide no structure to guide them to develop their learning strategies, and we test their mastery only indirectly in so far as we test the application of these strategies to the content.

We can more consciously model in detail our problem-solving strategies in our lectures, and we can create a structured opportunity for the students to develop their learning strategies through cooperative learning.

By structuring discussion among students about mathematics, we can help them develop a network of peers and a mode of communication through which they may continue to mature mathematically. In order to thoroughly understand a concept, one must be willing to test that understanding by applying it in a variety of settings and to articulate the distinctions and similarities among them. By sharing insights, by learning whether errors were errors of mechanics or of understanding, by sharing different approaches to the material, all students not only master the content, but they teach each other how to learn mathematics.

The greatest increase in understanding occurs when we explore new approaches, employ different techniques, and reflect on the results. That is, in order to learn the most, we must increase the risk of being wrong, then analyze the outcome. A woman or minority student may not be willing to take those risks if he/she does not feel the support of a community of learners or have the audience within which to refine his/her thinking. The women, Hispanic, and Black students in engineering or science, may view themselves as standard bearers for their group. Many feel that their performance is the basis upon which their sex/ethnic group will be judged. No student will risk appearing incompetent in a group to which he/she feels excluded.

The isolation of minority and women students is further compounded: not only are they likely NOT to feel a sense of belonging on our campuses, they may feel isolated from the cultural community from which they come because of their goals. Therefore, we need diverse ways to nurture and mould an effective academic community for those who are highly motivated yet who in the past have not had such an opportunity. Thus we strive to foster cooperative learning among Workshop students so that they may learn in the same way that we continue to learn—from our peers.

Specifically, then, we assume that the traits of an "ideal" mathematics student can be developed in those less experienced, and further, we assume it is our responsibility to do so. The professor is the one who establishes the atmosphere of inclusion or exclusion for the students.

Let us now examine one way to create that community in which students "learn to learn." In the fall of 1986, Cal Poly Pomona's Minority Engineering Program adapted Berkeley's PDP model and began its Academic Excellence Workshops in mathematics. The Workshops are now jointly sponsored by the Minority Engineering Program and The Science Educational Enhancement Services (SEES), and encompass 11 courses in college algebra, calculus, chemistry, physics, statistics and dynamics. Each quarter about 5 workshops have a total of approximately 75 enrollees.

The students who have participated have earned on the average at least 0.5 grade point above the remainder of the class. Frequently it is a full grade point higher. The norm is that 60% of the participants earn A's and B's; the usual expectation for these Black, Hispanics, and American Indian youth is that 60% would be earning D's and F's. Several faculty who have taught the lecture for the Workshop students have noted a sharp change in their classroom: more students participate, the questions are more sophisticated, and test performance is better—not only for Workshop participants, but for the class as a whole. In particular, one professor (who supports, but has been naturally cautious about the workshops) was surprised to find that a subsequent class without Workshop students was a much weaker class overall. He found that the performance of this non-workshop section was a full letter grade below that of one with workshop participants. Not only had Workshop students earned higher grades, but they had brought the entire group to a higher level of understanding.

What is the process by which a Workshop enlivens learning so that students are more able to understand the basic concepts and their applications? Students who elect to participate in a Workshop enroll in one of the designated lectures where they constitute from 10% to 30% of the enrollment. This group of 8 to 25 students agree to regularly attend two 2-hour workshops per week where they will work problems above and beyond homework. They are expected to work on their homework and to read assignments before the workshop session. These sessions are NOT homework sessions, nor tutorials, nor reviews of the lecture.

The Facilitators, upper-division undergraduates, prepare a worksheet of problems in consultation with the lecture professor, and facilitates the discussion and solution of the problems among the students. Since the sessions are designed to coach the students in learning how to learn mathematics, the Facilitator, when ever possible, does not directly answer a student's question; either the student is asked another question to guide him/her to greater insight or the student is referred to another student. The Facilitator models the behavior of our "ideal" student, by asking those questions which a superior students would ask of him/herself. Thus the Facilitator needs not only to be a strong student of mathematics, but needs to understand the conceptual challenges of the material from the participants' perspective. Only when several students are unable to resolve the question does the Facilitator step in. The following questions characterize the Facilitators' primary involvement:

"Why did you do that?"

"Is this problem similar to any others you have worked? How?"

"What do you have in your class notes that might relate to this problem?"

"What makes this problem different?"

"How do you know your answer/procedure is correct?"

"What do you think?"

"Is there another way to do this?"

"How are these problems related, or are they?"

"What other versions are there of this type of problem?"

The environment that the Facilitator strives to create is one of mutual support and friendly competitiveness. The students move from problems similar to the homework to those much more challenging—more difficult that they are likely to encounter on tests. The problems selected for the Worksheet are deliberately chosen to require the student to synthesize from homework and class and to apply that knowledge in a new setting. Through this graded structure of the worksheet, the best student is challenged while those less quick have the support of others to clarify concepts and with whom they may test their understanding. Thus the difficulty of the problems force students to collaborate. For some students this is the first time that cooperative learning has been encouraged and rewarded.

The students are challenged to articulate exactly WHAT the underlying structure is and how to apply it. The students thus are forced to engage in ACTIVE learning, rather than memorizing an algorithm to apply by rote. The students are encouraged to debate among themselves about tactics, procedures, and results. They learn from each other when there are several methods available and discuss how they know when each is appropriate. No student is permitted, no matter how strong (or weak) to avoid this dialogue with others. The student who finishes a problem quickly is encouraged to explain his/her approach to those with questions. All must engage in discussions about mathematics. They learn to use the technical vocabulary and to correct each other's errors. When they examine each other's work, they learn that the process of working out a problem on paper is a form of communication: that there *is* a standard grammar for mathematics.

The title "Academic Excellence Workshops" conveys the level of activity expected. Too frequently the student who has been among the top of his/her high-school class finds that the pace in college is much faster, that the course more rigorous, and that the support of faculty and peers is sparse. Such students, particularly if they are minorities, will avoid at all costs any tutoring or other assistance that may be perceived of as "remedial." If they go at all, it is after the situation if hopeless. For this reason, the commitments, the expectations of the workshop, and the rewards (greater likelihood of A's and B's) are clearly stated. Thus participation of those who would ordinarily shun support is gained. The workshops are all but billed as "honors."

There are several critical elements necessary for a workshop to produce the desired results:

> The students must be challenged with novel, inventive problems that require a synthesis of concepts taught.
>
> The structure must reinforce all students' active participation; specifically it should preclude one or two doing the work for the rest.
>
> The evaluation of student work must focus on the positive results and provide guidance on how to eliminate the unproductive strategies so that all aspects of the students' efforts lead to a more full understanding of how to approach and solve problems.

The Workshops *continue* to affect the students' academic performance in subsequent courses. They have learned to value the peer network so that they schedule their future coursework with peers in order to form their own independent study sessions. In these groups they continue to employ the strategies that they learned in the workshop: to question results, to clarify concepts, to encourage each other to a higher level of mastery of the material. They have also discovered that most faculty welcome questions and student involvement so they are more assertive in their classes. More importantly, however, they have experienced the excitement of quality academic performance and know how to work with others to create that same level of intellectual involvement in their other courses. The Workshop, as Dr. Clarence Stephens states, "teaches the students HOW to learn" making them more independent of us.

As an aside, a secondary benefit of the Workshops is the faculty mentoring of the Facilitators: some are now planning graduate study and some are considering a teaching career. With the growing need for American-educated students to enter graduate school in technical fields we need to be alert to means by which we can encourage more of our students to consider graduate study. Further, by guiding the Facilitators through their work, we are giving them the opportunity to see the personal rewards to teaching.

While Cal Poly's program is for a targeted group in the calculus and is structured to be independent of the course, there are other ways to encourage this type of group activity for all students. Some campuses build the study group into the course structure as a lab. Others, where there is strong faculty commitment, model the class itself after workshops as was done at SUNY Potsdam. With some reflection we can find ways to build in a structure through which we can guide students to develop their own problem-solving strategies and become independent learners. If we can create this atmosphere, I believe we will increase the possibility that our students will more nearly approximate our "ideal" mathematics student.

Pumps, Filters, and Lenses; Humanistic Issues in Calculus Reform

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Today's slogans describe calculus as a pump or filter. According to this metaphor, mathematics is just a treatment plant for students in the pipeline from high school to technical or professional employment. But I like to think of calculus as a lens through which students may catch a glimpse of the humanistic side of mathematics, and I am confident that many other teachers share this view. With this attitude, it is a natural reaction for us to view the various proposals for calculus reform with alarm. If the syllabus is made more relevant, more accessible, more applicable to the subject areas that have made calculus a requirement, what will become of the liberal arts component of calculus instruction? The advocates of reform have my sympathy when they argue that mathematics departments should get out of the business of filtering students for other disciplines. But this may mean reducing calculus to a course that emphasizes the skills and intuitive understanding needed for applying calculus. Can reform of this type be consistent with the liberal arts goals of education? I intend to argue that the answer is *yes*, Provided we adopt a broader viewpoint: Curriculum reform.

Before making my case, Let me discuss the liberal arts component of calculus. An educated member of our society should have an understanding of the historical evolution of the culture, of the forces that shaped it, and that continue to shape it. Without question one of the significant forces has been mathematics. this is not only a reflection of the role that mathematics has played in the development of science and technology. Mathematics has also played a significant role in philosophy.

Calculus offers many opportunities for discussing the impact of mathematics on our culture. On one level, as the foundation for Newtonian mechanics, the contribution of mathematics to classical physics can be made clear. On a second level, calculus provides a stage for exhibiting the epistemological issues that lead from naive Platonism to formalism. We describe the constructs of calculus in intuitive terms, but then present formal definitions. The intuitive descriptions depend on a shared perception of an ideal platonic reality; the formal definitions recognize the need to create an abstract universe for the precise description of that perception. The historical context for the development of the formal definitions is inseparable from a broader movement in philosophy. The arithmetization of analysis at once provides a microcosm for and exemplifies a contributing force to this philosophical movement. It is appropriate to discuss these issues in calculus courses, both to provide a historical background, and to give a clearer understanding of the methodology and epistemology of mathematics. Thus, to me, the point of including limits in calculus has never been to further the students' understanding of calculus, but rather to further the understanding of calculus, but rather to further the understanding of mathematics.

Calculus can be a vehicle for addressing other aspects of liberal arts instruction, as well. It provides opportunities for exposing students to aesthetic considerations in the formulation and derivation of mathematical knowledge. It is a subject whose development is widely recognized as one of the intellectual triumphs of our culture, and we may attempt to show our students why it is so considered.

As the foregoing makes clear, I have a number of instructional goals for calculus that go far beyond imparting a conceptual and operational mastery of the content. There seems to be little room for these goals in the calculus that is to come. Yet, after an initial negative reaction, I find that I can agree with many of the reform proposals. How can this be? In part, my agreement with the proposals stems from reconsidering whether my goals for instruction are appropriate for the student audience, particularly that segment of the audience conscripted by course work requirements from other disciplines. Additionally, my agreement is contingent upon the adoption of a wider scope for the reform. I can accept the deletion from calculus of liberal arts instructional goals, provided that these goals are provided a suitable platform *somewhere* in the curriculum. These points are elaborated below.

The first point has to do with the fit between instructional goals and student audience. One of the criticisms of the calculus status quo is that we attempt to make every calculus student a mathematician. Indeed, I fear I *have* been guilty of exploiting the captive audience of calculus students to further my own goals for liberal education. But as a participant in many faculty senate debates on universal requirements for a liberal arts degree, I recognize that there are more subjects that an educated citizen out to be comfortable with than can be accommodated in a set of universal requirements for a four year degree. Education must include some selection, and we understand that the tre goal is to make the student self educating. Accordingly, I refrain from insisting that every educated citizen must be acquainted with the discipline of mathematics. It would be inconsistent to insist on this same kind of exposure for the subset of students who happen to be required to master the techniques of calculus. They may be right who advise us to focus on the concepts and applications of calculus.

As I said, I don't insist that every undergraduate student be exposed to mathematics in the liberal arts sense. What I do insist is that the universities and colleges provide every student an *opportunity* for this kind of exposure. We have an obligation to offfer and to promote courses that will address the issue I have already mentioned: the historical evolution of mathematics, its impact on our culture, a sense of its methodology and epistemology, and the role of aesthetics. We all recognize that college algebra or trigonometry (as usually taught) do not contribute to these goals of liberal education, and we should not be content to offer these alone to students that lack the background to tackle calculus. Meaningful courses taught by dedicated faculty are needed, not just for the mathematically unsophisticated, but for our majors, as well. These should be courses in which we take pride; courses we are pleased to recommend to our talented majors, and which we use to showcase our discipline to the rest of the academic community. There should be courses of this type that is the logical offering for students who currently enroll in calculus because they desire collegiate mathematics experience and find that calculus is the *next course*. Then, *let* them turn calculus into a skills course. *Let* them offer it in high school for equivalent college credit. And let us get out of the filtering business.

Calculus reform has grown out of a deep discontent with a narrow part of the mathematics curriculum. Heightened awareness of curricular issues and participation in discussion of these issues is one of the results. But we should not address only that part that has become the focus of criticism. Rather, let us take this opportunity to address the curriculum as a whole. The liberal arts component of mathematics instruction can thus be a beneficiary, and not a victim, of calculus reform.

Lessons From Cognitive Theory For Teaching Mathematical Modeling to Freshmen

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Three years ago Pomona College embarked on a program of Freshman Seminars entitled "Critical Inquiry". In these seminars at most fifteen entering freshmen think, discuss and write extensively. Participating faculty choose the subject of their own seminars. My topic of "Mathematical Modeling and Exposition" provokes special difficulties for the students which may best be understood through a model for their "Cognitive And Ethical Growth" developed by William G. Perry, Jr.

Contrary to the expectations of some, these seminars have been just as popular in the sciences and mathematics as they have in the humanities and social sciences. Mathematically oriented seminars have been offered on the philosophy and art of pure mathematics, on societal uses and abuses of statistics, on symmetry, and on mathematical modeling. Topics in the sciences vary from plate tectonics to nuclear war to biological determinism.

My original hope for my seminar was to emphasize mathematical modeling as part of and as a means of mathematical exposition. It was my underlying thesis that a kind of simple, naive, mathematical modeling underlies the way many of us understand the social and physical world around us. If so, then that modeling constitutes an integral part of the way we communicate our ideas.

Fundamental to our exposition of mathematics is our complete acceptance of working from assumptions. At the pure end of the mathematics spectrum we emphasize axiom systems, while at the applied end we value the conceptual and computational simplifications resulting from well-considered assumptions. Even in ordinary everyday discussions we mathematicians tend to be fairly conscious of the assumptions, or axiomatic base, which we bring to bear in understanding social and physical phenomena. (Of course in everyday discourse we tend not to worry too much about consistency of our axioms.)

Not everyone shares our love for axiomatically connected discourse. In our own classrooms, how often are we accused of having our heads in the clouds, just for wanting to think carefully concerning "all those obvious facts about real numbers?" Even our colleagues in the physical an social sciences frequently blur the distinction between "reality" and consequences obtained from assumptions. They naturally influence the way our students think when they come to the mathematics classroom.

So it shouldn't come as a surprise when our students believe they can understand everything as it "really is". No system is too complex, no detail, too fine to deter them. They are filled to overflowing with the confidence that they can understand anything completely. You can easily imagine them saying, "If things seem a little complicated today, then we'll pull an all-nighter and straighten this out for tomorrow." Their optimism is nothing, if not charming.

But they have little use for simplifying assumptions. They are certain that making an assumption is tantamount to the admission that they cannot know everything to be either fact or false. This they find to be far less acceptable and a far more bitter pill than I do. Moreover, at this point they would consider me to be slightly irrelevant if I suggested there might be a logical problem with the concept of knowing everything to be fact or false.

Probably many of you are less surprised than I am, at the lack of sophistication of my freshmen. After all, they are just beginning to mature mentally. Indeed, they are just beginning that mental maturation process which is central to developing their world views. Therefore it would be helpful for us to understand that maturation process, in order to teach them effectively about the relevance of modeling for their worldviews.

William G. Perry, Jr., has suggested a model for understanding the "Cognitive and Ethical Growth" of our students. The model contains a scheme of development consisting of nine literary ordered "positions" and prototypical transitions between them. The positions begin with the simplistic and dualistic attitudes wherein students categorize everything as to "good vs. bad," "right vs. wrong," "true vs. false," and so forth. It then proceeds in discovering relativistic standards, according to Perry, wherein truth becomes relative to context. For example for the student in Perry's earliest relativist positions (which are just beyond the dualistic positions), good writing in mathematics can be different than good writing in literature classes, because the authorities, the professors, are different. Further growth, if it occurs, moves in this model toward a commitment to a more mature relativism. This position Perry characterizes by the statement of attitude:

> I must be wholehearted while tentative,
> fight for my values yet respect others, believe
> my deepest values right yet be ready to learn.

It seems that Perry's model has much to suggest about the attitudes of students toward mathematical modeling. The attitudes by which I have characterized my students earlier are consistent with Perry's dualistic position at the beginning of the development ladder. They are saying that a description of the "real" world is either right or wrong. In this viewpoint, simplifying assumptions might be seen to make the description wrong. There is very little room for meaningful approximation. Our expectation that they accept a model which only approximates experiential evidence in only a limited set of scenarios should be understood as an expectation that these students make significant strides in growing through Perry's positions. We therefore recognize the implications of those expectations in terms of fundamental personal growth.

Consider my original hopes for my students: that they understand mathematical modeling as a part of, and as a means of mathematical exposition; that they come to use modeling approach for casual understanding of the social and physical world around them. That is, I was hoping that they would come to be aware of how the conclusions and even values they form about the world around them depend on the assumptions they bring to their analyses. I further hoped they would be self-consciously aware of the tentative nature of their assumptions.

Now compare these hopes for my students' development with Perry's position of highest "Cognitive and Ethical Growth" characterized by the statement of attitude above. For me, the correspondence between my hopes, and Perry's position of highest development was amazing and dismaying. Clearly if Perry is right about the positions through which we must progress in our development, and if that progress is as slow as he indicates, then we are forced to realize that my hopes were wildly unrealistic and desperately need modification.

I believe we can develop a freshman pedagogy for mathematical modeling which is comfortable for students in the earliest of the Perry positions. After all, they are well accustomed to accepting some other kinds of models as correct and useful. Toys and dolls are used by all children to model a more complicated reality. Many high school students are fairly sophisticated users of maps and models. They recognize that topographic maps may not be good indicators of economic activity. They also know that a refined map might include economy with topography. Students value these models as aids to studying the world. However, they would probably disagree with my suggestions that their concept of geographical reality depends to a large extent upon such models.

We cannot expect freshmen to accept models as tentative replacements for their reality. That would be tantamount to the expectation of immediate progression to more sophisticated positions in Perry's model. A model as a separate entity can be useful for displaying information about a separately conceived reality. But in order for the model to retain its legitimacy, it must not be held up as a replacement for that reality. For if it is, it will be discarded as being incorrect in some respects, and therefore false in the dualist perspective.

Just as children perform musically long before they acquire an interpretative maturity, so can our freshmen model proficiently independently of their progress toward cognitive maturity. Fortunately they are already familiar with many powerful mathematical concepts and tools. Even regression models and dynamic systems are viable for some of them. Their powers of deduction are frequently equal to the task of finding a conclusions. Subsequent comparisons with data from the real world fit all too well into their dualist's perspective. Thus modeling as a craft, if not as a world view, can be practiced by students in any of the positions of Perry's model.

Realizing this, we can better introduce our students to mathematical modeling. If they can achieve an intellectual understanding of the modeling process early in their cognitive development, then perhaps they can incorporate a modeling attitude in their later development to a tentative relativism. In fact, I hope they can thereby grow more easily in their cognitive and ethical senses, according to Perry's model, toward a more personalized, relativist stance in their worldviews.

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REFLECTIONS

on attending three contributed paper sessions on humanistic mathematics in Phoenix 1989

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The term "Humanistic Mathematics" is a redundancy, isn't it, just like "Humanistic Literature" or "Humanistic Philosophy" would be. So why do we need a special interest group devoted to "Humanistic Mathematics"? We have dehumanized mathematics, and some of us are trying to reverse that trend. It is high time that we did so.

How have we dehumanized mathematics? Let me count some ways:

- We present mathematics to our students as a rigid set of rules.
- We tell our students that mathematics stands here before us, a set of eternal, changeless truths that must be memorized and regurgitated on examinations.
- We tell our students that all mathematical problems have exactly one answer which can always be found by following a mechanical system of rules.
- We show our students only those problems that are of no interest to them whatsoever.
- We insist that mathematicians never get emotionally involved with their work.
- We tell our students that mathematics can only be done by highly talented, wise and clever people such as we are and such as they can never hope to be.
- We tell our students that mathematics does not appeal to the senses.
- The "average educated man in the street" and "average" public school teacher believes all these things we tell our students, and they teach these things to their children and their students who believe these things even before we tell it to them.
- And so most people dislike mathematics, because who would not dislike a subject about which all these things are true?

All speakers seem to agree: Mathematics is important in society. I have no quarrel with that statement if you modify "society" to read "our society". Does it they follow that mathematics is important to all societies?

Some speakers say "Yes!" They point to tally sticks and other elementary computing devices found in even the most primitive societies as evidence that mathematical activities are universal. But other speakers point out that mathematics is not merely symbol manipulation. Is the presence of a tally stick evidence of the presence of genuine mathematical activitymerely evidence of symbol manipulation of a kind more primitive than that engaged in by a student doing long division with paper and pencil?

To continue: Is the Rhind Papyrus a mathematics book or merely a set of algorithms? Was A'h-mose', or the person whose original work A'h-mose' copied, a true mathematician or one of the first dehumanizers of mathematics? Or perhaps mathematics was not a humanistic discipline in those days; perhaps mathematics became a humanistic discipline only in the 19th Century after the discovery of non-Euclidean Geometry, non-commutative algebras and the discovery of paradoxes in the very foundation of mathematics forced us to approach our discipline in a different manner.

If mathematics is so important to all societies why did the Chinese abandon mathematical research after making such an auspicious beginning in number theory as the discovery of the Chinese Remainder Theorem?

If mathematics is not important in all societies but only in some of them, could we then characterize each type of society and distinguish between the characteristics of each type? Could we then, perhaps, make valid and meaningful statements about the nature of the human condition in each of these different types of societies?

A final question: Is mathematics a humanistic discipline only in a small subset of those societies in which it is important? In other words, are those bad things about mathematics which we still teach our students true in certain parts of the world, and were they true in the western world at some time in the recent past?

If that were true we would not be trying to reverse a bad trend of our own making. It would be correct to say that we are trying to introduce a new and exciting element into our discipline, one which we have long felt was there but which the rest of the world has yet to discover.

Mathematics Appreciation: A Humanities Course

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In 1983 the CUPM Panel on Mathematics Appreciation Courses issued a report which articulated the goal of such courses. "Students must come to understand the historical and contemporary role of mathematics, and to place the discipline properly in the context of other human achievements." [4]. The statement of the CUPM Panel is an assertion of the place of mathematics in a liberal arts education. Liberally educated people have a broad base of knowledge in a variety of aspects of human culture and society. They are able to assess their own experience and opinions in the context of human cultural tradition. The mathematical component of a liberal education should help students see mathematics as an integral part of human culture, not as an isolated discipline.

In this paper I describe a course developed at Villanova University, which offers three perspectives on the connection of mathematics to other aspects of human culture: the fact that mathematics has always been part of our culture, the Pythagorean view that number is the basis of all creation, and the tradition of using geometry to structure our visual perceptions of the world.

Synopsis of the Course

The course begins by recognizing that mathematics has been an integral part of human society from the beginning. Bones and sticks containing tally marks can be found among the artifacts of human species from the very beginning of the species [2]. Examples of such bones are shown and the speculation about more advanced mathematical ideas implicit in the Ishango Bone is discussed [1,3]. We point out that every human society has had some concept of number and some language for speaking of numbers. The numeral systems and computational techniques of ancient Egypt and Mesopotamia are discussed. This included the duplication method of multiplication as well as the use of unit fractions in Egypt and sexigesimal fractions in Mesopotamia. Note is taken of the fact that astronomers used the sexigesimal system until the Renaissance.

A discussion of place value systems in general naturally follows. Base 60 has been seen. Base 10 should be familiar. One or two other bases are discussed to emphasize the society's choice of the decimal system is arbitrary, at least from the mathematical standpoint. Students sometimes have difficulty with other bases than ten because our language for numbers as well as our symbolism is based on ten. The words one thousand, three hundred eighty-five mimic the numeral 1385. If we write this in base eight, 2551 cannot be pronounced. The problem can be addressed by working in base twelve, in which there is a language. Our number is 975, and is read as nine gross, seven dozen five. It may also be helpful to remind students of phrases such as, "four score and seven years ago", which has a base of twenty and of the representation of fractional parts of angles in minutes and seconds, using the Babylonian base, sixty.

It is important to discuss the binary system and its relevance to modern computing.

The discussion of positional numeration culminates with the theorem that the radix representation of every rational number either terminates or repeats; for a reduced fraction the representation terminates iff each prime factor of the denominator is a factor of the base.

The mathematics at this stage is easy. The material is important for helping students appreciate that math is an inseparable component of human culture not only in modern technological society but throughout the history of culture. The question is raised, why is this the case and how did it come to be so. The customary answer is that as society grew in complexity, the practical requirements of society demanded ever more sophisticated mathematics. The evidence put forward in support of this view is first that it seems reasonable. Secondly one can cite problems of an applied nature in ancient mathematical writings such as the Rhind Mathematical Papyrus [5]. A completely different opinion is found in the very interesting theory of A. Seidenberg that mathematics grew up with ritual practice [6].

One criterion for knowing if we understand a natural phenomenon is being able to express our understanding in mathematical terms. In the words of Galileo, "Philosophy is written in that book which lies ever before our eyes—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in the mathematical language . . ." [2, pp. 328, 329]. This view originated early in the development of Greek rational philosophy when Pythagoras is supposed to have considered that, "all is number," by which he meant that we can understand all of creation in terms of relationships among whole numbers. In the next part of the course we explore these ideas. Some aspects of number theory are studied on the assumption that Pythagoras initiated number theory to lay a foundation for understanding the universe. The subject is treated frankly as an intellectual exercise in furtherance of a philosophical position.

Triangular, square, and oblong numbers are discussed. It is shown by partitioning the figures that a triangular number is a sum of consecutive integers, a square is a sum of odds and an oblong is a sum of evens. Obviously a square number is a product of a number with itself, an oblong is a product of two consecutive numbers. One sees by drawing a diagonal that a triangular number is half an oblong number from which we get the traditional formula for a sum of integers

$$1 + 2 + 3 + \ldots + n = 1/2\, n(n + 1).$$

The Fundamental Theorem of Arithmetic is stated with emphasis on the uniqueness of the representation of a number as a product of primes. Because the factorization is unique one can extract from the product of primes a complete list of divisors of a number. This enables one to check if a number is perfect.

Moreover the uniqueness of factorization can be used as a basis for proving that the square root of every natural number is either integral or irrational. This is related to the Pythagorean discovery of incommensurable quantities and the consequent collapse of the Pythagorean philosophical program.

Although the Pythagorean formulation of the mathematical foundation of all things could not be maintained, its residue survives in the attitude common in the sciences that the best explanation is one which can be expressed in mathematical terms. This view was one of the main themes of the Scientific Revolution and is illustrated by Galileo's mathematical theory of motion.

Galileo begins with uniform motion, motion at constant speed, which exists only as an ideal but is mathematically simple. He then discusses the motion of free fall for which he takes $v = at$ as the basic formula for uniform motion, $d = rt$. From $v = at$ he uses mathematics to deduce $d = 1/2\ vt$ and $d = 1/2\ at^2$. Then projectile motion is analyzed as a combination of uniform motion and free fall.

Many believe that the Greeks made such significant contributions to geometry because Pythagoras' emphasis on number proved untenable. Plato is said to have insisted on a knowledge of geometry as a prerequisite to entry to his academy. Over the centuries Euclidean geometry came to be the dominant model of the physical world. Therefore Euclidean geometry is reviewed with an emphasis on parallelism. Then it is pointed out what Euclidean geometry is not logically necessary and a brief introduction to non-Euclidean geometry is presented.

An intuitive discussion of spherical geometry brings out the points that a line is a great circle, parallel lines do not exist, and the sum of the angles in a triangle is greater that 180 degrees and varies with the area of the triangle. There is a brief and superficial discussion of Lobachevskian geometry and its potential for modeling the universe.

Course Objectives

The primary goal of this course is to help students understand how mathematics has always been a primary tool in human efforts to understand our world. Mathematics has been with us since the beginning of our species. At least since the time of Pythagoras mathematics has been seen as a language for expressing abstract models of natural phenomena. Because the models are abstract they gloss over some details but they may also yield deeper insights. The models may be developed out of experience, observation or intuition of nature. But it often happens that mathematics developed for its own sake is seen later to model some phenomenon (for example, the parabola as the path of a projectile). The insistence in mathematics on precise formulation and the deductive proof provides a discipline for checking a scientific theory.

Another goal of the course is to give students insight into the nature of mathematics as one of the many areas of human activity. Mathematics has an integrity of its own and provides impetus for its own development apart from any utility. There is a beauty in a body of mathematical theory that can be seen for example in Pythagoras' work on figurate numbers, the theory of conic sections, or Galileo's theory of motion. Mathematics has its own standards of truth, the criteria for what constitutes a valid proof. We point out how these standards are different from standards in other realms of human activity - law, science, everyday affairs, religion, etc. A few selected results in the course are proved (irrationality of some square roots, some theorems in geometry) and the opportunity is taken to discuss the nature of proof in mathematics and how it is different from other systems of evidence.

General Comments

This is a one semester course. There is more than enough material for one semester and decisions have to be made about what to omit. On the other hand there is not enough material for two semesters. In any case a course of this kind extending over two semesters would probably tax the patience of students who take the course to satisfy a requirement rather that out of genuine interest. A second semester course should concentrate on the uses of mathematics in the twentieth century, treating such topics as statistical thinking, linear, polynomial and exponential models, and modern computing.

An obvious difficulty with this course is the choice of a textbook. Nothing suitable is available and I have written notes for my course.

Some people with whom I've discussed these ideas have raised the objection that mathematicians are accustomed to teaching strictly mathematical material and either can't teach non-technical material or do not want to. It is asserted that we don't know how to lead a classroom discussion of Pythagorean philosophy or that we do not know how to formulate exercises and test questions on the cultural implications of non-Euclidean geometry. This objection seems to me a luxury we can ill afford. If it is true we must seek resources that will help us. The center section of [4] contains an excellent list of resources. We might seek assistance from our colleagues in the humanities. A common view of mathematics is that it is an essential subject for modern society, but one which few can appreciate; ignorance of mathematics is not only permissible but something of a distinction. The popularity of this view indicates not only the cynicism of our academic colleagues but the failure of our mathematics appreciation courses. We must learn to discuss mathematics in a broad cultural contest.

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Tapping Creativity and Ingenuity of Liberal Arts Majors in a Math Course

(For Humanistics Mathematics Session at Phoenix Joint Meetings)

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As a component of a liberal arts education, Mathematics should provide "Equal opportunity for all" students, regardless of major, to develop skill in mathematical thinking. But most mathematics students (or non-students as the case may be) do not enjoy merely rehashing their prior mathematical coursework: if they did well with the material previously, they will be bored because they know it already and can anticipate what is to come; if they did poorly with the material previously, they may well be fearful that here they are facing one more failure in the pursuit of their mathematical fortunes. I believe that one successful strategy for enlivening liberal arts mathematics courses is to incorporate into them material that is totally new to at least ninety per cent of the students enrolled in the course.

An effective and accessible tool for achieving this goal is graph theory, which has the potential for linking mathematical modeling with the everyday experiences of all students, even those who may not yet recognize the usefulness of mathematical thinking in everyday life.

Although the term has come into vogue only in recent decades, the concept of mathematical modeling has been used for centuries in the so-called "hard sciences" for the solution of myriad problems, the majority of which involve functions and/or equations, because the nature of the variables involved was continuous. Within the past half-century or so, however, the management sciences, the social sciences, and even the humanities have become aware of the applicability of discrete mathematical models to a broad spectrum of problems in their fields.

Mathematical modeling, particularly using graphs and directed graphs as models, is a very appropriate vehicle for developing skill at thinking mathematically in non-mathematical settings. A course using graph theory can provide liberal arts and other non-science majors with terms, tools and techniques for learning to apply mathematical thinking in situations with which they never dreamed mathematics could be associated.

All students may expect to encounter in their daily personal and/or professional experience, for example, a variety of situations that could require such skills as: 1) scheduling activities without time conflicts; 2) arranging several people or objects that have compatibility restrictions; 3) matching qualified people with available positions; 4) determining the minimal cost required for a project that has multiple option packages available; 5) computing the minimal time required to complete a project composed of several activities, some of which may be accomplished simultaneously and some of which must precede others immediately or intermediately; 6) ranking a variety of choices expressed by several people to determine a valid composite order of preference; or 7) ascertaining the maximal amount of resources that can be transported along existing channels, where portions of those channels may have different capacities.

Resolution of situations such as the aforementioned can be accomplished very directly using applicable graph and directed graph theory techniques: map-coloring, matching theory, minimal spanning tree algorithms, activity analysis digraph methods, tournament theory, flow in network procedures. If this theory and these techniques are presented geometrically, with an emphasis on diagrams rather than on set theoretic notation, liberal arts majors, social science majors and management science majors are soon able to use graphs and directed graphs to model situations from their own fields of study as well as from their living experiences.

In developing such an elementary course in math modeling over the past ten years, as an option for our humanities majors who need a non-specified mathematics core requirement, I have incorporated an extremely successful pedagogical device for enabling students to recognize the relevance of graph, theoretic techniques in their future experiences beyond the mathematics classroom: an independent project in which each student composes then solves a variety of original problems based on the student's major and/or personal experience, actual or potential, with each problem lending itself to solution using a different graph or digraph theoretic technique.

It is in the execution of the individual projects that the ingenuity and creativity of the different members of the class are evidenced. One project requirement is that each of the various original problems must lend itself to different solution technique with some problems needing graph theory and others, directed graph theory. Some students conjure up problems that have no relationship to each other. However, a trend that began to develop about the third or fourth semester that I included the project in the course was that of having a theme to unify the various problems which they composed for their projects: one used work situations in his father's business; a second used her experiences during an internship at a local television station; another fantasized about how graph theory and digraph theory could have alleviated some of the problems encountered in certain Mother Goose Rhymes; and still another became Diane Graph, authoress of a "Dear Di" advice column in the MATRIX, the school newspaper of Modern Math University. English majors in particular have had a field day with creativity in their projects!

Reactions from students over the years have been generally quite positive, in part because many of the students have performed considerably better than was usual in their previous math courses. A significant number of these students have indicated to me verbally and/or in written evaluations, "This is the first mathematics course I have ever really enjoyed." Included among my most enthusiastic students have been several married women who have returned to college after a time lapse and who have found the course interesting and non-intimidating.

My original purpose for the project was to increase my supply of applications in the cross-section of majors generally represented in a class section. My discovery of its pedagogical value was accidental: student's performance on word problem sections of examinations improved significantly after their completion of the projects. Moreover, many students have returned after taking the course (often after graduation) to relate further applications the have been able to make in their jobs or living experiences. To me this says that they have learned to apply mathematical modeling to their life experiences, that they have developed their ability to think mathematically in a non-mathematical setting. To derive such results is for me a crowning achievement for the course!

Introducing Undergraduates to Mathematics Information Resources

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Trinity University

Undergraduate students of mathematics are traditionally not heavy users of library materials for their coursework in mathematics. Likewise, many instructors of mathematics are inclined to limit the scope of their undergraduate students' work to "straight" mathematics—working problems and doing homework assignments. Why should students be introduced to the information resources in their field? First of all, those students who plan to go to graduate school will need to know how to get information on basic research in mathematics. Likewise, those students who are planning a career in mathematics in an industrial or applied situation need to know how to gain rapid, timely access to mathematics information. These are the most obvious and practical benefits; however, by using mathematics resources, students will both develop a sense of the richness and variety of mathematical research, and understand the processes and development of the discipline. This can be true for both math majors and non-majors. By taking mathematics out of the textbook so to speak, and into a larger scheme of things, this can become part of a student's complete educational experience in mathematics.

Probably the best way to introduce students to mathematics information sources, and achieve the aims stated above, is through library-based assignments. Following are three suggestions for assignments that could help students achieve these ends. These assignments have been developed as alternatives to actually writing papers. It has been my experience that unless such an assignment is approached very carefully, having students write a term paper is not the most effective way for students to become acquainted with resources in a particular area, and a more directed assignment, like these, is much more successful.

The first of these assignments is designed to allow students to gain facility in using mathematics information resources, before going on to work on a project or assignment in which they will have to use these resources. Students have an introduction to their library and its materials in mathematics, and are given a handout describing these. Then they must complete a worksheet to demonstrate their facility with these materials. The worksheet consists from eight to fifteen question that must be answered by using library resources. This assignment is particularly useful for students because it not only gives them a thorough preparation for their course assignment, but gives them a general introduction to mathematics resources that they will find essential in either graduate school or in the working world. I have used it the past two years in a course in Mathematical Modelling, where the students are required to develop an actual model of a real world situation. They are expected to do a literature search as part of their project. The response from the students has been very encouraging. In spite of the course instructor's and my own worries that the students might find this too much busy work, they have considered the experience very useful for both this course and others.

The second assignment addresses the third and fourth reasons that I gave for introducing students to mathematics information resources, namely that such assignments allow students to gain appreciation of the richness and variety of mathematics, and a better understanding of the processes and development of the discipline. In this assignment, students would develop an annotated bibliography on a topic currently being investigated by mathematicians. They would find between ten and fifteen articles written about this topic over the last ten years, and write short descriptions of the articles and why they are important. They would also write a one page description of their search process. The assignment would culminate with a five to seven minute oral report by the student in which he or she would summarize the current state of research on the chosen topic. This is a particularly useful assignment because of its flexibility. The number of articles to be found can be increased or decreased, or the pitch of the articles can be changed, from fairly simple, straight-forward descriptive works for an introductory class, to more advanced, scholarly works for an upper-level class. The aim of the assignment is for students to begin to find out about some of the areas of modern mathematical research, and tell how they found out about it, i.e. through systematically using and evaluating library resources. I have not used this assignment in a mathematics class, yet. However, it has been used successfully in a class in microanatomy for several years.

The third assignment is designed to help promote students' understanding of the development of the discipline of mathematics. In contrast to the second assignment, which emphasized students' exploration of areas of current research in math, this assignment takes a more historical approach. Students assess the scholarly contributions of a single mathematician, based on his or her published research and its evaluation by their peers. They would read about their subject, and either read or read about their publications, depending on the level of the class, and the particular area of mathematics in which their subject worked. They would evaluate these publications, using citation analysis, secondary materials, and other sources, to determine the impact this particular mathematician had on their area of research and on the development of the field as a whole. The students would then write a short report (1-2 pages) describing their findings and research process. This assignment was recently used for the first time in a class in genetics at Trinity. The instructor and I developed a list of geneticists who had made fairly important contributions to the field, and let the students each pick a different name on the list on which to do their report. The students were particularly encouraged to relate the work of their subject to what they had discussed in class.

These three assignments can be used in a variety of mathematics courses, and with students with a variety of levels of expertise. The assignments themselves can be easily modified to suit the particular needs of different groups of students or different institutions. Ideally, the three assignments would be part of a continuum, beginning with the use of the first assignment in an intermediate math majors course, then using the second and third assignments in upper level courses. The aim would be to have all mathematics majors do all three kinds of assignments before they complete their program. However, the assignments are equally changeable to be used in lower-level, non-majors courses. Whatever the class, though, students' use of information resources in mathematics can give them a better sense of perspective on the entire field of mathematics, and increase their appreciation of its richness and diversity.

Note: Examples illustrating the three assignments discussed here are available from the author. Please send requests to: Sallie H. Barringer, Maddux Library, Trinity University, 715 Stadium Drive, San Antonio, TX 78284, (512) 736-7343.

An Empowering, Participatory Research Model for Humanistic Mathematics Pedagogy

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Introduction

Within the community of mathematicians and mathematics educators who identify with the term, 'humanistic mathematics,' an agreement on its meaning is still under negotiation. However, discussions in the community have been skewed toward improving the teaching and learning of mathematics, a member of the "hard sciences," by attending to its decidedly human dimensions. Therefore, a primary concern of the community is the possibility of "teaching humanistically." Abstracting from White's description of the concerns of humanistic mathematics, we have distinguished four processes involved in teaching humanistically: (1) placing students more centrally in the position of the inquirer, (2) acknowledging the emotional climate of learning mathematics, (3) having students learn from each other, and (4) making mathematics meaningful rather than arbitrary(White, 1987, p1)

Though these four processes represent a significant departure from typical concerns of the prevailing "chalk-and-chalk" pedagogy, they, nevertheless, are somewhat limited. What we propose is a broader, more inclusive vision of the third process distinguished above by including the notion of interdependent learning between students and instructors. Further, we propose that 'teaching humanistically' ought to involve an additional, or fifth, process which attends to the more general, human process of empowerment. Our concern is for the empowerment of all actors in various settings of mathematics education.

To facilitate interdependent learning and empowerment and to apply the by products of these processes to improving mathematics pedagogy is a principal function of participatory research activities. Research activities into pedagogy are participatory, and potentially empowering, when they give authority to the voices of students. For they generally feel, and are often considered, to be without power in many instructional settings. To give authority to their voices and to incorporate their perspectives in transforming mathematics pedagogy, instructors can start most profitably by listening to students.

There exists much anecdotal evidence to show that listening to students is important in improving their attitudes toward mathematics along with their performance. Reflecting on the effectiveness of instructional programs targeted at special groups of students, Lax (1988) suggested that, independent of the population of students, a common underlying spirit contributed to the success of these programs. Lax concluded that the authors of these programs "had gone to considerable lengths to find out who the students in these programs were, where they came from, what went on in their heads and hearts when they worked on math and how well and by what means they could cope with problems in their out-of-school life." This conclusion supports the research of Rosamond (1982) who, in the context of mathematics-assistance laboratory at a large, prestigious university, documented that listening to students can positively effect their learning.

In a course, one effective and efficient vehicle for "listening" to all students is journals. Certain types of journals writing activities have been shown (Countryman, 1985; Gopen and Smith, in press; Hoffman and Powell, 1989; Lopez and Powell, 1989[1]; Mett,1987; Powell, 1986) to be efficacious vehicles for a number of pedagogical imperatives. Among these imperatives, Lopez and Powell (1989) described some of what can be "heard" from students through their journals. In their case study, they identified affective and cognitive items. Affective ones included preoccupations, dispositions, and feelings; and some cognitive items were what students know, what they have yet to know, misconceptions, and discrepancies between conceptual understandings and computational and algebraic manipulations.

Their case study was a participatory investigation since Lopez was both the student whose journal writings were analyzed and a co-investigator analyzing those writings. In addition to discovering that journal writing improved Lopez's affective and cognitive functionings in mathematics, the results of the study indicated that the dynamics of the student-instructor collaboration con-

[1] This paper was originally presented at the writing and mathematics session of the 1988 annual joint mathematics meeting, Atlanta, Georgia.

tributed to the overall empowering effect of the case study (Lopez and Powell, 1989). This effect raises some important questions concerning components of the student-instructor collaboration that might have contributed to these findings.

That study also generated a number of questions about writing as a vehicle to learn mathematics. The present investigation examined one of these questions: In what ways can personal, reflective journal writings best support the enhancement of mathematical thinking? Two related sub-questions also guided the investigation.

1. Do students' writings display their attempts to specialize and generalize as well as to make conjectures and to provide justifications for them?
2. How does writing help students to construct or negotiate meaning?

To investigate these questions, a team was assembled that included students and the instructor. In the process of the investigation, we conjectured that it would be worthwhile to examine the nature of our collaboration and its qualitative and transformative effects on the substance of both learning and teaching. In this report, we present the results of the investigation into the empowering effects of our participatory research model and suggest its relationship to a humanistic mathematics perspective.

Setting

This study was conducted, in the fall semester of 1988, in one section of a computation course, Developmental Mathematics I, at the Newark College of Arts and Sciences. The college, whose students are primarily commuters, is an urban campus of Rutgers, the State University of New Jersey. The course includes the study of some concepts of number theory, fractions, decimals, percents, and word problems as well as an introduction to elementary algebra. These topics, more or less common to such courses, were taught through a not so common pedagogical approach. It is based on an approach and course material developed by Hoffman and Powell (1988, 1987), both of which depart fundamentally from those within a "chalk-and-chalk" paradigm.

The course met three times a week for fourteen weeks and had an initial enrollment of twenty-six students out of which seventeen completed the course. Most were first-year students, and all were placed in the course on the basis of their performance on the New Jersey Test of Basic Skills or, on an in-house instrument, the Mathematics Placement Test. The content of both instruments is arithmetic computation and elementary algebra.

Based on previous scholastic experiences, many students in Developmental Mathematics I have fostered negative feelings and beliefs about mathematics and themselves as mathematics learners. A student expressed one such view as, "Mathematics is something you do, not something you understand." Like students in similar settings (Buerk, 1982) and generally (McKnight et al., 1987, pp. 42-49), most students in this course consider mathematics not only as an abstruse symbol system but also an arcane and fixed body of knowledge whose secrets almost never reveal themselves though they are expected to demonstrate a degree of mastery. They have developed an estranged relationship with academic mathematics which manifests itself in their relative high level of mathematics anxiety and phobia. This estrangement is also manifested in students' developed strategies of avoidance which include their learning passivity, inappropriate study routines, and reluctance to participate actively in class. In essence, these behaviors are manifestations of interacting sets of low expectations that students have for themselves and that most remedial programs have of them. For many students, the force of these debilitating expectations effectively have silenced and marginalized them in mathematics and related disciplines.

Method

To counter and reverse the disempowering effects of these expectations, a participatory research model was selected as the methodological process of this investigation. The process of journal writing complemented this method since writing requires an active rather than a passive involvement of learners. Focusing on these processes, this investigation aimed to empower students in the following ways:

- to promote students' awareness of and facility in the use of writing as a vehicle for learning,
- to put students at the center and in control of their own learning by engaging them in reflection and critical reflection on mathematical experiences,
- to provide opportunities for students to reflect on and transform the affective and cognitive effects of silence and marginalization, and

- to give space interdependent learning between the instructor and students by valuing their voices and so that they could affect instruction and learning.

The investigation attempted to realize these aims by involving students as co-investigators and through journal writing. Students were asked to write journals daily, or at least for each class or assignment, on any topic or issue related to their learning of or feelings about the mathematics of the course or the course itself. To help remove the chore-like conception some have of writing and to relieve anxieties many associate with the quantity to be produced, students were advised that five minutes of writing was sufficient for a journal entry. After adjusting to the idea of writing journals for a mathematics class, many found themselves spending quality time expressing their thoughts. Only to stimulate thought and reflection, a list of topics was offered (see Appendix A).

Journals were collected weekly and returned with comments on the substance of what was written. The comments were intended to be non-judgmental and, most often, took the form of questions about or suggestions on issues, ideas, and so on that students discussed to encourage them to explore further. The objective was to use journal writing as a tool for learning mathematics. Therefore, it was emphasized to students that neither their grammar nor syntax were of concern, only what they had to say. Aside from moral and other intrinsic incentives, neither penalties nor rewards, in the form of grades or otherwise, were given.

Chronologically, the participatory research model consisted of five stage: information, selection of research collaborators, background meetings, weekly meetings, and post-semester meetings. The information stage occurred during the second week of the semester. At the time, the nature and objectives of this study were discussed with the class verbally and in a letter, and research collaborators were solicited (see Appendix B). Students were asked to respond in writing, explaining whether they wished to be a research collaborator and why.

In the selection of research collaborators stage, students were chosen from among those who responded affirmatively to the letter. Three students (Selby, Sheridan, and Walker) did so and were accepted as collaborators. In the fifth week of the semester, another student (Jeffries) was encouraged to and did join the research team. Each student either held a part-time job or was involved in a College-sponsored sports team. Before the end of the semester, one student withdrew from the College to accommodate his need to work full-time, while the demands of work and other course work lead another to drop out of the project. Along with the instructor, the two student collaborators who remained are the co-authors of this paper and whose work and interactions in the team are the bases of this investigation.

Before the weekly meeting stage, Powell met to discuss with each student their history with mathematics. By the third week of the semester, weekly meetings of the research team were held. At these meetings, which were approximately an hour and twenty minutes each, collaborators distributed among themselves copies of their journals, instructor's comments, from the previous week. During these meetings, student collaborators reflected on written evidence of mathematical thinking and any other striking feature of another's journal and wrote their reflections. Afterward, Powell and the students read and discussed their comments as well as raised questions concerning the course. Finally, from the journal writing and class discussions, the team identified twenty-eight processes that it found and determined were involved in thinking mathematically (see Appendix C).

During post-semester meetings, the research reviewed, discussed, analyzed its data. These consisted of the following:

- the weekly journals of the student collaborators,
- their and the instructor's comments and analysis of each journal entry, and
- tape recording of discussions among team members on the nature of the participatory research activity and its effects on both students' learning and the instructor's teaching.

Results

The two student authors became research participants in different ways and for different reasons. Selby responded to Powell's letter immediately and perceived the project as an opportunity to confront her fears of mathematics. The following are excerpts from Selby's reflections on why she accepted the invitation to become a research collaborator.

> I found the goal of the research project intriguing because it presented me with new way of learning. The goal of the project was also interesting. Because I have had negative experiences learning mathematics, I immedi-

ately jumped at the opportunity to collaborate on this project.

> Another thing that attracted me to this project was the idea of working with a professor as well as with other students. This was appealing. I was never offered the opportunity to work closely with a professor. I believe that working with a group can have its strong points. In the past, I found that working with a group was rewarding and allowed me to benefit from the opinions and views of others.
>
> Most importantly, I decided to accept the invitation to eliminate the fear I had for mathematics. I hope to learn how to think mathematically. Being able to think mathematically seems to be essential in learning mathematical concepts.

The needs that Selby recognized motivated her to join the project. In addition to overcoming her fears of mathematics, she wanted to improve her ability to think mathematically and to collaborate with instructor and other students. The latter motivation indicates that Selby wished to have a voice and to be heard as well as to gain the benefits of perspectives other than her own.

Unlike Selby, Jeffries did not volunteer first. He contended that his involvement on the College's fencing team precluded his participation in the project. Though Powell felt that potentially Jeffries and the project could benefit from his involvement, Powell did not attempt to persuade Jeffries to reconsider his initial decision. It was not until the fifth week of the semester that Powell urged Jeffries to join the project. Some time later, Jeffries disclosed that there were reasons other than sports that prevented him from volunteering, even though he was encouraged to join after reading Powell's letter:

> When the invitation was extended to me, I initially rejected the idea. I felt that I was not competent enough. I also was afraid that I would be successful. I knew that if I was successful in this endeavor I would be expected to repeat that success. I didn't know if I was ready to fulfill those expectations, because all my life I had been a poor math student. Why change? I had been labeled a poor math student and I had long ago since accepted the label, and what's worse is that I believed it. I was a lazy student when it came to math. I had no confidence in my mathematical ability because I was never given the opportunity to take risks in math, it was always a subject that I loathed and feared, and I was happy with poor grades as long as I passed the course.
>
> After the semester started, after my confidence grew, after my professor pushed, I was finally persuaded to join the research project.

Jeffries' performance had been predicted by his primary and secondary school authorities, and he had accepted their low expectations of his mathematical abilities. In any case, he argued, his poor grades bore out these expectations. As one can well understand, he developed both a fear and a loathing for the subject as a way of justifying it all. This accounts for his initial reluctance to join the project.

During the weekly meetings, he and other research collaborators commented on features of each other's journal entries that they considered striking. In most cases, entries were considered striking if they revealed the presence or absence of one's affective or cognitive struggles with some aspect of the course. In particular, evidence of mathematical thinking was especially looked for. At first, borrowing from Mason, Burton, and Stacey (1985), we distinguished four processes, or habits of the mind, involved in thinking mathematically: generalizing, specializing, conjecturing, and justifying. Later, through the course of the semester, we identified twenty-four other processes of mathematical thinking (see Appendix C). These were abstracted both from considering journal entries and from analyzing what the students involved in the research team did as they worked on mathematical problems.

The extent to which the writing that students do supports and reveals their mathematical thinking depends on attributes of their writing. Hoffman and Powell (1989) conjectured that journal writing is more useful for learning and best supports mathematical thinking when it is personal and reflective. Journal entries are personal to the extent that they represent the subjective understanding and feeling of the writer as opposed to the writer's perception the viewpoints or feelings of others. Reflective writing goes beyond the mere description and approaches of analysis. In reflective writing, the writer is inquisitive and contemplative and searches for meaning.

Attributes of the writing that the student researchers produced were not immediately personal and reflective. These attributes were encouraged through comments that Powell made on the substance of the journal entries. For example, during the third week of the semester, Selby wrote the following journal entry:

> I have to finally admit to myself that for once in my life I truly enjoy doing math. I feel good inside when I can take something learned in class one day and apply to something new on a different day. The homework assignment in Chapter 2/Section 2, was a combination of what I learned in class two or three days ago. When or while I was completing the assignment, I was surprised that I was able to do each problem without some kind of struggle. It was very unusual for me. One of the reasons why I am able to understand the class & homework is because first, in class it is explained to me in a very simple & understandable fashion. Another reason is that the worksheets also break up each step in a simple way which is easy to understand & follow. I have never had math taught to me in the manner & methods that I am now learning from. I love it!!

In the above entry, Selby wrote a personal, nonreflective, and general summary. She neither stated specifically what she learned nor what she did not learn. It appears, as she later verified, that she wrote down what she thought the instructor wanted to hear.

Through the process of weekly meetings the student authors became aware of the attributes of personal and reflective writing and included an evaluation of these attributes in their commentary on each other's journal entries. After Jeffries joined the research team, he read and commented on above Selby's entry.

> This journal appears to me to just fill the page. I think that was Aleshia's goal. She doesn't give specific examples of her problems, she instead gives blanket statements concerning her work. I think this is so because Aleshia didn't know what to write so she simply filled the page.

Jeffries recognized that the blanket statements, given without examples or context, were attempts to fill the page. He too produced a similar entry the week he joined the research team.

> In my problem solving course this summer, I got to use to signed numbers but I found that I confused myself. It's one thing to see something as an equation, but it is another thing when that equation is embedded in a word problem. Why is that? I thought that if I mastered an equation, I could do it if I saw it in a word problem. To my surprise, I found I couldn't. Why? Maybe you know.

This journal entry was written during the sixth week of class, a week after Jeffries joined the research team. He shows little attachment to the writing and gives the reader little context in which to interpret his questions. Reflecting on this entry some weeks later, he states that his motives were simply to fill up a page while hoping that the instructor would not read the entry. He also stated that he did not fully understand the purpose of the journals or what he was expected to do. This was true, Jeffries claimed, although Powell had written comments on previous journals suggesting ways that he might use them more profitably.

Suggestions on profitable ways to use journal writing were discussed during each team meeting. In fact, the participatory nature of the research project affected teaching as well as affective and cognitive features of learning. During each meeting, students read and commented on each others journal entries. The comments that students made were similar to those made by the instructor and, at times, were in a language that they could easily comprehend. As the semester progressed, the interactions between student investigators grew more substantive and lively; their observations about learning became increasingly more insightful and elaborate. In addition, as we will show in the journal excerpts below, the movement toward personal, reflective writing was facilitated by the interactions that occurred among the students.

For instance, Jeffries transformed the nature of his journal writing with the help of the substantive comments he received from the other student investigators during the team meetings. Consider the following journal entry written during the ninth week of the semester.

> On page 44 of Chapter 4, Section 5 problem number 2 gave me some difficulties. It reads as follows:
>
> $$\left\{\frac{\left[5(x+1)^{\frac{1}{3}}-1\right]}{3}+7\right\}^{-2}=\frac{1}{1000}$$
>
> Now when I went to solve this as a circle equation[2], a problem occurred.
>
> When I saw the fraction 1/1000, I made some sort of mental error. I felt that 1/1000 meant that I had to divide something in the equation. Instead of taking the reciprocal, I attempted to incorporate division into the equation.
>
> My question is why does the fraction bar in some cases mean division and in other cases the fraction bar does not? More importantly, what is a fraction? The only thing that I am sure of if that when the number underneath the fraction bar is 1, you accept the fraction to be an integer such as 10/1 = 10. In what I refer to as regular fractions, such as 2/3, what does this expression mean? Does it mean that 2 parts of 3 are being spoken for. Perhaps it means 2 divided by 3 or vice versa.

The above carefully written entry is characteristic of others that Jeffries wrote that week and, more or less, throughout the rest of the semester. Like this one, they were both personal and reflective and reveal his ability to identify what confuses him.

In the above entry, Jeffries states an example and, thereby, provides the reader with a context for the questions he later poses. He understands that to solve the equation he must begin by reversing the action of the given exponent. He also demonstrates awareness of two interpretations of the division bar. His question is which interpretation should he act on. Jeffries is puzzled by the choices before him. Should he divide 100 into 1? If he chooses this operational interpretation, then he would have a representation of the number, a decimal, which would make it difficult for *him* to reverse the action of the given exponent. It appears that Jeffries is comfortable with raising a fraction to a negative exponent; as such, one senses that he would prefer to interpret the division bar of the fraction, 1/1000, according to its non-operational meaning. However, through the process of writing, it appears that Jeffries stumbles upon another question: What meaning should he attach to those fractions he calls regular?

In the latter part of the semester, this process of discovery and negotiation of meaning, illustrated above in Jeffries' journal entry, was evidenced more frequently in his writing and the writings of other team members, as well. Selby, for example, wrote the following entry during the seventh week.

> I have found a way to solve the problems that seems easiest to me. I have no problems adding integers, however, I had problems subtracting. Now, I found that by changing all of my subtraction problems to addition problems that they are easier to solve.
>
> Ex: $3 - {}^{-}2 = $ $\quad {}^{-}5 - {}^{-}2 =$
> $3 + 2 = 5$ $\quad {}^{-}5 + 2 = -3$
>
> Also, I was confused about making connections to problems, transforming them into other problems, and about how to link them to a problem that would give me the same result. I believe what confused me was for example, making 5 - 3 look differently, yet having the same result. After or should I say during class, I realized how simple it was to convert or transform 5 - 3 to make it look like ${}^{-}3 - {}^{-}5$. What helped me understand the procedure of transforming the two was the commutative property and the concept of additive opposite. The concept of additive opposites seems like the same thing I did when I changed subtraction problems to find the result

This journal entry is personal and reflective and gives evidence of mathematical thinking. In the above entry, Selby describes and analyzes insights that lead her to create a generalized procedure, one which she finds easier for subtracting signed numbers. In the first part of the entry, she articulates two concepts that she synthesized to devise her procedure. The procedure involves transforming subtraction expressions into equivalent ad-

[2]Circle equations are a technique for solving a certain class of equations. For an elaboration of this technique see, Hoffman and Powell (1988).

ditions. Moreover, in the second part of the entry, using the technical language meaningfully, she discusses her struggle to see and create links between subtraction and equivalent addition problems. Making connections between equivalent expressions and using these specialized equivalences to devise and conjecture a generalized procedure for transforming a given problem into an easier, equivalent one, these are complex processes in which Selby engaged her mind and are powerful manifestations of mathematical thinking.

Conclusions

As we have defined it, a humanistic mathematic perspective includes the notion that students and instructors can learn together. Such interdependent learning is unlikely to occur through a "chalk-and-chalk" instructional method; for it presupposes the instructor as the only authority on matters of content and form and monologue as the discursive mode. Students and instructor infrequently engage in dialogue about either the nature of mathematics or approaches to learning and teaching mathematics. When dialogue does occur, rarely is its purpose to transform, more than in a superficial manner, the nature of instruction and learning. Within the perspective of humanistic mathematics, to realize interdependent learning and to transform instruction require new pedagogical and research methods.

The methodological approach of our study is offered as a first attempt to develop a new research model consistent with and facilitative of the following five processes which we have suggested are involved in teaching mathematics humanistically:

- placing students more centrally in the position of inquirer,
- acknowledging the emotional climate of learning mathematics,
- interdependent learning among students as well as between instructors and students,
- making mathematics meaningful rather than arbitrary, and
- empowering instructors and students.

These processes are best catalyzed by participatory investigations. We recognize that all investigative initiatives manipulate and transform reality and, therefore, posit that the structure of a participatory model should skew change in the direction of improved teaching and learning. Furthermore, we posit that the structure of research model should contribute to empowering both instructors and students. This imperative implies that all actors participate in the research as investigators. Students are transformed from objects of educational research into active subjects or co-investigators. That is, students participate in and are integral to the interpretation of data collected from their work and the analysis of pedagogical techniques and approaches under which they are taught. There are three important reasons for including students as co-investigators. They are (a) to ensure the ethical quality, (b) to include multiple perspectives so as to ensure the validity of research findings, and (c) to empower students intellectually.

We observed that ways in which our participatory research project affected the learning and teaching as well as contributed to empowering students and the instructor can be located in one of the following ten categories:

1. becoming an independent mathematics learner
2. learning how to learning mathematics
3. gaining insights into teaching
4. expressing ideas using mathematical terms
5. becoming an mathematics autonomous learner
6. quality and quantity of involvement lead to
   a) enjoyment
   b) diffusion of fears
   c) finding mathematics interesting, and
   d) vicarious learning
7. gaining confidence
8. gaining a sense of responsibility
9. communicating clearly
10. gaining authority

Space does not allow us to elaborate on each of these categories. Here we will discuss aspects of how our project influenced the ability of students to communicate mathematics clearly and the instructor to listen to students and have that affect his teaching. For the student researchers, participating in the project promoted a sense of community and increased their quantitatively and qualitatively writing and thinking about the mathematics of the course. In turn, these features of their involvement led to a number of by products. First, each collaborator felt committed to writing and had a sense that others depended on her or his written contributions. This commitment encouraged more writing, more often. Second, reading, analyzing, and discussing their journal entries during project meetings simply increased the number of reflections students made on the mathematics of the course. In addition to more writing, project meetings also increased the opportunities for students to do and talk about mathematics. Over time, we observed a corresponding increase in the range, depth, and clarity of the mathematical talk and writing.

Finally, in addition to contributing to the empowerment and learning of students, the participatory nature of this research project ensured that the instructor listened to students. Opportunities to listen occurred in project meetings when students read and commented on each other's journals. Their verbal and written commentary were insightful, rich, and honest. The comment that a student made about another's journal entries in one project meeting positively affected that student's subsequent writings. The powerful and efficacious nature of these interactions stimulated Powell to think of ways to incorporate aspects of the project meeting as regular features of instruction. Through the course of the semester, it became clear that the verbal and written critiques that students made of each other's journals contributed significantly to promoting personal, reflective writing. To reproduce this type interactions among students requires that instruction be transformed to give value to group work. Since cooperative, small group work is already a feature of the course, students within a group could become an interacting community reflecting and commenting on each other's journal writings. This would make widespread the empowering intellectual experience that the student authors had.

Appendix A

Professor Arthur Powell
Developmental Mathematics I

ABOUT JOURNALS

You are asked to keep a journal on 8 1/2" x 11" sheets of loose-leaf paper. Generally, one or two sheets will be sufficient for a week's worth of journal writing. Neither your syntax nor grammar will be a concern or checked; my only concern and interest is what you say, not how you say it. You are asked to make, at least, one journal entry for each meeting that we have, and, as a rule of thumb, you need not spend more than five to ten minutes writing each entry. Each week, the latest journal entries will be collected and returned with comments.

The focus of your journal entries should be on your *learning* of mathematics or on the *mathematics* of the course. That is, your reflections should be on what you do, feel, discover, or invent. Within this context, you may write on any topic or issue you choose. To stimulate your thoughts and reflections, here are some questions and suggestions.

- What did you learn from the class activity and discussion or the assignment?
- What questions do you have about the work you are doing or not able to do?
- Describe any discoveries you make about mathematics (patterns, relationships, procedures, and so on) or yourself.
- Describe the process you undertook to solve a problem.
- What attributes, patterns, or relationships have you found?
- How do you feel about your work, discoveries, the class or the assignment?
- What confused you today? What did you especially like? What did you not especially like?
- Describe any computational procedure you invent?

APPENDIX B

19 September 1988

Dear Developmental Mathematics I Student:

This semester, I will conduct a research project for which I am looking for student collaborators. The goal of the research project is to discover whether writing about the mathematics that one is learning and doing can be helpful in learning mathematics. Let me tell what the project is about.

In this course, I am asking each of you to keep a journal about your learning and to do other types of short writing assignments related to the course. Most of the writings that you do I will collect and analyze, and to some writings I will respond. Those who collaborate with me may be asked to do a bit more writing than others. Each week, collaborators and I will meet as a research team to analyze their writings.

The central research questions that I hope to answer by the end of this research project is: In what ways can personal, reflective journal writings best support the enhancement of mathematical thinking? In addition, there are also two sub-questions that I will be asking about the writing that you do.

- Do students' writings display their attempts to specialize and generalize as well as to make conjectures and to provide justifications for them?
- How does writing help students to construct or negotiate meaning?

Why do I ask students to write in a mathematics class? Last year, a Developmental Mathematics I student and I collaborated on a research project to determine whether journal and free writing were useful vehicles to learn mathematics. Based on that study, which will be published soon, we have concluded that writing can be a powerful tool in learning mathematics. Now I wish to examine more closely how writing can support the development of mathematical thinking.

This close examination of your writing will, I believe, benefit you in two ways. First, the writing that you do will improve your learning. Second, what you choose to write about will inform my teaching and, thereby, improve the lessons I conduct.

I intend to co-author a paper, with those who collaborated with me, on the finding of this project. Let me know by letter whether you would like to work with me on this project. If you would like to collaborate with me and have the time,in your letter, discuss why you are interested and what you wish discover about yourself as a learner of mathematics. I will collect these letters on Wednesday 21, September.

Sincerely,

Arthur B. Powell

Appendix C
Processes Involved in Thinking Mathematically (or Habits of the Mind)

- ☐ posing problems and questions
- ☐ exploring a question systematically
- ☐ generating examples
- ☐ specializing
- ☐ generalizing
- ☐ devising symbols and notations
- ☐ making observations
- ☐ recording observations
- ☐ identifying patterns, relationships, and attributes
- ☐ formulating conjectures (inductively and deductively)
- ☐ testing conjectures
- ☐ justifying conjectures
- ☐ communicating with an audience
- ☐ writing to explore one's thoughts
- ☐ writing to inform an audience
- ☐ using appropriate techniques to solve a problem
- ☐ using technical language meaningfully
- ☐ devising methods, ways of solving problems
- ☐ struggling to be clear
- ☐ revising one's views
- ☐ making connections between equivalent statements or expressions, transformations
- ☐ making comparisons
- ☐ being skeptical, searching for counter examples
- ☐ reflecting on experiences
- ☐ suspending judgement
- ☐ sleeping on a problem
- ☐ suspending temporarily work on a problem and returning to it later
- ☐ listening actively to peers

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Abstracts ofAll Papers Presented at the Humanistic Mathematics Sessions in Phoenix 1989

Humanistic Mathematics, Part A

The role of the knower in mathematics, CATHERINE GORINI, Maharishi International University. Knowledge has three components: the knower, the object of knowledge, and the process of gaining knowledge. Modern education focuses on the object of knowledge although the knower is certainly the most fundamental component of knowledge. An understanding of the nature of the knower, the mathematician or student of mathematics, has direct applications in mathematics education. Experience with this approach will be discussed.

Mathematical thinking and heuristics, JOHN LUCAS, University of Wisconsin-Oshkosh. Mathematics is a product of human thinking and problem solving. In the teaching and learning process, those reasoning techniques (heuristics) which help us produce mathematics and solve new problems are even more important than mathematics itself, because they are the essence of the human ingredient--the "production mechanisms" by which mathematics evolves and becomes an extension of our thinking. Conveying mathematics by emphasizing heuristics motivates and challenges students by appealing to their creativity and involving them actively in the process of discovery. This talk explores some common myths about the learning of mathematics, examines the nature of heuristic, and offers some examples of how emphasizing human reasoning in the classroom can enhance learning.

Using the history of mathematics to generate interest in forging its future, EMELIE KENNEY, Wilkes College. Many secondary and beginning college students think of mathematics as a stagnant body of long-before discovered facts--all mathematics has already been discovered, they seem to think. Unfortunately, mathematics is sometimes presented this way. We should change students' perceptions not only for accuracy's sake, but also to promote curiosity, inventiveness, and desire for discovery. Introduction of historical elements actually helps to effect the change. Here, several ways of introducing mathematics history in the curriculum are discussed, and some examples are presented to illustrate how this may be accomplished.

Introducing undergraduates to mathematics information resources, SALLIE BARRINGER, Trinity University Library. Undergraduate students of mathematics have traditionally not been heavy users of library materials in mathematics. In most of their studies, they concentrate on learning the "basics" of mathematics, often to the exclusion of developing any sense of the history of mathematics, range of mathematical literature, or methods of mathematical research and inquiry. This talk describes three mathematical alternative, library-based assignments requiring students to use and evaluate a variety of materials that could be integrated into intermediate and advanced level undergraduate mathematics courses, and would promote students' understanding and appreciation of mathematics as a rich and diverse discipline.

Presenting problem solving as a universal activity, GEORGE DAY, Allegheny college. Problem solving is the process of combining intuition, knowledge, and skills, toward the satisfaction of a need or desire. John Mason's paradigm of mathematical problem solving can be used to demonstrate the similarity of the mathematical process to other creative activity. Students who, in the same course, examine the relevance of his model to non-mathematical situations and also use it to analyze and improve their own mathematical techniques, may see more clearly that doing mathematics is a natural human behavior.

Let us teach philosophy of mathematics, REUBEN HERSH, University of New Mexico. A historically oriented humanistic course in the philosophy of mathematics will be described. Such a course is advocated as a more exciting alternative to the traditional foundations course.

Mathematics and the existence of God, PAUL R. MANNING, Oratory Preparatory School. The general public still does not seem to realize that there is a close connection between philosophy and mathematics. Pascal, Descartes, Newton, Russel and Whitehead, to name a few, have worked in both fields successfully. The modern Canadian Philosopher-theologian, Bernard Lonergan, has continued in this tradition. In particular, mathematics permeates his classical work, *Insight* (1957). This paper will focus on Lonergan's use of mathematics in the development of his "transcendental method". Finally, Lonergan's approach to the question of the existence of God will be compared/contrasted with the approaches of Aquinas and Pascal.

Mathematics and music, JOEL K. HAACK, Oklahoma State University. Several modern composers have returned to mathematical theories of harmony to develop systems of tunings that they employ in their work. Terry Riley, in particular, has developed a system of just intonation based on the first five pitches in the overtone series. The theoretical background of tunings and a mathematical analysis of this system provides the beginning of an explanation of the appeal of his music.

Humanistic mathematics, Part B.

Mathematics: a significant force in our culture, HARALD M. NESS, University of Wisconsin Center--Fond du Lac. A discussion of mathematics as a significant force in the development of our culture, the effects of the culture on the development of mathematics, and implications for mathematics courses for liberal arts students.

Mathematics appreciation: a humanistic course, THOMAS L. BARTLOW, Villanova University. The course described in this paper attempts to present mathematics in relation to other currents of the western intellectual tradition rather than as a subject separate from other trends in society. The course connects with intellectual history in three ways; by examining mathematical practices of ancient societies, by exploring the pythagorean philosophical view that number, and by extension mathematics, is the foundation of reality, and by studying the use of geometry to understand the nature of the universe. Many of the mathematical topics covered are common in a Mathematics Appreciation course; the distinctive feature of this course is examining them as part of a larger culture.

Topping the creativity and ingenuity of liberal arts majors, HELEN CHRISTENSEN, Loyola College in Maryland. A major challenge to the teacher of liberal arts majors who have a mathematics requirement to fulfill is that of preventing simultaneously the two extremes of frustration and boredom, with one being related to the background of the individual student. A course in graph theory applied at elementary level and centered around a "real-life" project that elicits student creativity and ingenuity often leads to the final evaluative statement: "This is the first mathematics course I've really enjoyed."

Mathematics for liberal arts, FREDERICK SOLOMON, Warren Wilson College. I have taught the course "Mathematics for Liberal Arts" for three years. It is an alternative to the Precalculus-Calculus stream and is also an alternative to Finite Math and Statistics. The course consists one-third of traditional mathematics, one-third history and philosophy, and one-third art and design projects. The latter are very popular with the students. The approach I take is entirely aesthetic. There are several goals: to connect mathematics with other areas of liberal arts, to involve students in aesthetic aspects of mathematics through projects, to relate to students who aren't interested in the mathematics subjects as taught in high school. Students are required to write papers and construct projects in addition to the normal types of problems sets.

Cosmologies, FREDERICK SOLOMON, Warren Wilson college. I teach a freshman seminar entitled Cosmologies. It covers different ways of knowing the world--in particular, physical cosmology, depth psychology, and the spiritual traditions. By combining these radically distinct approaches to understand the world in one course, the differences and similarities are apparent. The connection with mathematics is this: Training in mathematical thinking uniquely enables one to entertain imaginative constructions and to see their interrelations in the realm of abstract thought. The different ways of looking at the world--cosmologies--can be seen as distinct logical structures. To deal with them requires an axiomatic approach similar to that used in dealing with mathematical axiomatic systems.

Where are your eyebrows?, SHARON M. STENGLEIN, The College of St. Catherine. Most beginning calculus students have been successful at algebraic manipulation and have found satisfaction in completing mathematical problems correctly. Few have any sense of mathematics as a beautiful part of the history of human learning and culture. A variety of lectures, discussions and assignments are used to deepen students' understanding and appreciation of mathematics while "raising their eyebrows".

A course in the history and philosophy of mathematics, ROBERT W. OWENS, Lewis and Clark College. We studied epistemological and ontological issues concerning the nature of mathematical knowledge from the Platonist, conceptualist-intuitionist, realist, and empiricist perspective. Arguments for and against the notion of "a priori" knowledge were investigated, as were warrants for mathematical truths and justifications for changes in mathematical practice. Finally, we tested these philosophical arguments by considering the history and evolution of the calculus.

The rediscovery of hyperbolic geometry, DICK A. WOOD, Seattle Pacific University. Students will explore new ideas, develop novel constructions, and create new proofs by changing the Euclidean plane slightly. Just omit a region which contains some key point(s) used in the typical Euclidean construction. The students find it challenging and fun to form their own constructions. They hone writing skills when describing the idea and verifying its correctness. This can be modified to cover a wide range of skill levels.

Humanistic Mathematics, Part C

A "famous equation" seminar course, RICHARD G. MONTGOMERY, Southern Oregon State College. A history seminar-course is described wherein weekly public talks on "famous equations" were given by students enrolled in a supporting course. This format provided natural opportunities for dedicated individual research, talk preparation and expository mathematical writing within a supportive and instructive group environment. Strategies to make this a practical and humanistic approach while achieving quality results are detailed. Course mementos, including the anthology of student papers, are displayed.

Student initiated, team taught history of mathematics course, EDWIN F. BAUMGARTNER, Le Moyne College. At Le Moyne College, a liberal arts college in the Jesuit tradition, a senior studies course is required of students and its catalog listing is: ". . . to help them integrate their educational experiences and improve their ability to express their ideas." Besides certain specified courses, there is a "student initiatives" option under which "students are encouraged to seek out an instructor, as a group, design and pursue a program of study which meets the design and purpose of the senior studies requirement." In responses to student requests, we are offering such a course this term. None of us have taught such a course before, so three of us decided to team teach it as an overload. We wanted to make library research and report writing an integral part of the course. We also wanted to emphasize that both students and faculty were to be learning together. In addition to having regular textbook readings, students and faculty select and present reports and problems at our meetings. The topics to study were chosen by both students and faculty from a faculty generated list that was augmented by student suggestions. More emphasis is placed upon researching topics, and writing reports for the benefit of all participants, than upon oral presentations by students. Students are graded based upon individual learning contracts negotiated between them and faculty members. While only a month into the course, we're very hopeful of it being successful and of repeating such an experiment.

A mathematics seminar from the National Endowment for the Humanities, WILLIAM DUNHAM, Ohio State University. This past summer, the National Endowment for the Humanities sponsored a five-week seminar for school teachers entitled "The Great Theorems of Mathematics in Historical Context" and held on the campus of The Ohio State University. The seminar's director will discuss this unusual offering at the first mathematics seminar funded by NEH as part of its "Summer Seminars for School Teachers" program--with emphasis on the seminar's format and content, the response rate from across the country, and the general nature of an enterprise explicitly designed to examine mathematical masterpieces as landmarks of human creativity.

Applications, sources and research, RAYMOND F. COUGHLIN, Temple University. A difficult pedagogical challenge is to convince students that mathematics is applied in a wide variety of disciplines. Most textbook applications are clearly manufactured by the author and do nothing to convince the student of the applicability of mathematics. We have developed from journals and books a list of over 300 applications that present case studies of mathematics being used to solve problems in business, economics, the social sciences and the biological sciences. This talk describes several of these applications. In addition, we show how the applications can be used effectively in the classroom, how the student can find the referenced article or book in the library, and how further research on the topic can lead to a term paper or a class presentation.

Discussing and debating conjectures, ANNELI P. LAX, NYU-Courant. Students' preconceptions about the nature of mathematics and instructors' preconceptions about the nature of students often combine to hinder mathematical progress in our mathematics classes. Class conversations based on observations which lead students to formulate conjectures and to test, prove, or disprove them allow us to identify both sets of preconceptions and to tackle the misconceptions among them. Class discussions provide opportunities for explorations of both mathematical topics and individual learning styles. Such inquiries demonstrate the spirit of what we usually call "research" and confirm its crucial role in learning and teaching.

The communication of mathematics, a rational and irrational process, PHILIP D. EVANSTOCK, Park College and the Phoenix Union High School District. The teaching of mathematics is a process of communication, the purpose of which is to sustain and extend the knowledge of mathematics. It fulfills our technical and intellectual needs. However, the person communicating this knowledge is not always well defined. Indeed he/she may be a hindrance and/or a catalyst for the proper assimilation of this knowledge. This talk will focus on the mathematical teacher as a communicator of concepts, stress the nature of the teacher-student communication, consider some of the problems involved in this communication and suggest some solutions to the obstacles inherent in this human intercourse.

An empowering, participatory research model for humanistic mathematics pedagogy, ARTHUR B. POWELL, Rutgers University at Newark. A defining feature of a humanistic mathematics perspective is the notion that students and instructors can learn together. To ensure interdependent learning among actors in the classroom, a research paradigm which differs from conventional ones is required when investigating the effectiveness of instructional and learning techniques. Furthermore, since all investigative initiatives manipulate and transform reality, the methodology of this new research paradigm must skew change in the direction of improved teaching and learning and empowerment. This implies that all actors participate in the research which aims to meet the above criteria. For in actual practice, I will provide an example of a participatory research project conducted in a developmental mathematics course. The research concerned the ways in which personal, reflective journal writings best support the enhancement of mathematical thinking.

How women have been and are encouraged to pursue mathematical knowledge, SYLVIA SVITAK and MONA FABRICANT, Queensborough Community college. Mathematics as a humanistic endeavor must seek ways to enhance women's chances for successful pursuit of mathematical knowledge. Contemporary educational research points to a number of factors that affect a woman's decision to study mathematics and the history of mathematics strongly demonstrates those factors to be operative from the time of Pythagoras to the present. A supportive family environment, early successful exposure to significant mathematics and emphathetic role models are, among others, keys to fostering the development of women's mathematical abilities and expertise. This paper shows how the history of women in mathematics coupled with observations from recent educational studies can guide us to provide a nurturing environment in which women can study mathematics in today's culture.

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