Is the principle of contradiction a consequence of $x^2 = x$?

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1. A strange proposition

PROPOSITION IV of Chapter III of the Laws of Thought of Boole states the following:

> … the principle of contradiction … is a consequence of the fundamental law of thought, whose expression is $x^2 = x$.

This is strange for two main reasons. Firstly, before Boole nobody considered that $x^2 = x$ was a fundamental law of thought. Secondly, it is not clear how we can derive the principle of contradiction from this fundamental law.
This proposition can be seen as establishing connections between two heterogeneous fields, on the one hand metaphysics, on the other hand mathematics. We can draw a parallel with the famous controversy in St Petersburg in the 1770s when Euler told Diderot:

Sir! $(a + b^n)/n = x$, therefore God exists, respond!

Since algebra sounds for Diderot like Hebrew he was ridiculed and ran back to Paris with the tails between its legs.
If we put the above proof of PROPOSITION IV of the Laws of Thought as an exercise for a student, or even a professor, of the University of Oxbridge he will probably not be able to present such a proof. (S)he may even claim that there is no such a proof because it is false.
The examination of the proof of PROPOSITION IV is interesting for various reasons. It is related to the question of notation. Does this proof depend on the concept of notation? Does any proof depend on notation?
And what is the relation between notation and conceptual framework? Two notations may differ only due to superficial aspects. We can change the fonts of our text, choosing bigger fonts or another type of fonts. We can also transcribe a Russian text using Latin alphabet; this does not mean we are translating Russian into Latin.
But sometimes change of notation goes hand to hand with change of conceptual framework. A pictogramatic language does not work in the same way as an alphabetic language – we can still however claim that a Chinese basically think in the same way as an Russian.
When we go to mathematical notation, things are not the same. It is much more than a change of language; it is a change of way of thinking.
Using mathematics do deal with logic, Boole changed the theory of reasoning. He also changed the way of reasoning and changed mathematics (introducing non numerical algebra, going out of the sphere of quantities).

2. Analysis of the original three-step derivation of Boole

(1*)	$x^2 = x$
(2*)	$x - x^2 = 0$
(3*)	$x\,(1 - x) = 0$

TABLE 1 – BOOLE ORIGINAL DERIVATION

Each line of the proof is as in the original work of Boole. The three equations on the right column are written exactly in this way in the first edition of The Laws of Thought on page 49 (III.15).
The original version of Boole's Laws of thought has been reproduced in the Dover edition with the same pagination. A scan of the original book is available on the internet by Internet

Archive, a non-profit digital library founded by Brewster Khale. Note also that Boole uses Roman numbers for Chapter and to each paragraph is attributed an Arabic number. So it is quite easy to make precise references to the Bible of modern logic.
However Boole does not present this in a table and does not use the word "proof" to qualify what we call here a "3-step derivation". We are using a table to make easier the reading and to compare with other derivations we will present.
The word "derivation" means here a development of a theorem. This is item 5a of the definition of "derivation" in Dictionary.com It is commonly used in this sense both by specialist of proof-theory and by non-specialists, lay mathematicians. It is more neutral than "proof" and leaves open the degree of analysis and formalism.
This is why we will use this word and, as it is also commonly done, we can use it both to speak of the whole process, or to speak of a specific part of it: going from step 1 to step 2 is a derivation.
Boole successively presents these three equations in the order presented in TABLE 1. (1*) is presented at the end of PROPOSITION IV and he does not repeat it in a separate line. (2*) is presented at a middle of a line ending with a comma:

$$x - x^2 = 0,$$

and so is (3*) but with an semicolon and "(1)" at the very end on the right side:

$$x(1 - x) = 0; \qquad (1)$$

To write these three equations successively in separate lines to express the fact that we go from the first to the third through the second and that all are true is a procedure which still common in contemporary books of algebra as well as the details of the fonts: italic for the variable, no italics for the numbers, the parentheses, the exponent and the identity sign. It is also common to write something like "(1)" at the beginning or the end of the line, using it as a label for the asserted equation. Centering the equation is also common as well as the use of punctuation marks. The whole thing is like a sentence, the idea to divide the sentence starting new lines is to emphasize the assertion of each equations, their truth.
As it is known Frege introduced the sign "⊢" to make the distinction between a proposition and its assertion, before him this distinction was operated by this "lining-device". Frege's sign what adopted by Whitehead and Russell in Principia Mathematica but not by Hilbert who didn't like it and kept using the traditional lining-device.
This way of writing (lining-device, italic/non-italic fonts, label) derivations of equations was developed before Boole during the 18th century and has been preserved up to now independently of the development of mathematical logic (showing the weak, not say null, impact on logic upon mathematics, that for many people turns things more complicated without any further advantage).
In a standard contemporary book of algebra it is not necessarily explained what justifies the passage to the first equation from the second and the second from the third. Students generally learned this as kind of informal rules.
Boole goes from step 1 to step 2, saying "let us write this equation in the form" and then presenting (2B), but after going form step 2 to step 3 on the basis of a "whence", he qualifies both derivations by using the word "transformation", saying "these transformations being justified by the axiomatic laws of combination and transposition (II.13)".

3. Boolean algebra

If we present this 3-step derivation to a sample student, let say Natasha, she will think that it is part of algebra, not part of logic; in particular not part of propositional logic where there are not equations, no identity sign, except if we consider idiosyncratic constructions by a guy like Roman Suszko.

Which algebra? At first sight Natasha may think that it is about an algebra with 1 and 0 and other numbers on which x is ranging: natural numbers, rational numbers or reals numbers.
But looking more closely Natasha will change her mind because $x^2 = x$ is false if we consider that this free way to use the variable x corresponds to universal quantification, which is the standard interpretation.
What is also standard for a lay mathematician is to consider that the line-device corresponds to truth. So Natasha will only think of this derivation in the context where $x^2 = x$ is true. She will not consider the validity of the derivation independently of the truth of $x^2 = x$, only logicians do that or postmodern mathematicians…
If we consider that x is ranging other natural numbers $x^2 = x$ is false, just consider that the value of x is Natasha's age (for sake of privacy we will not reveal it here). This equation is true among natural numbers exactly for two of them: 0 and 1. That is why Natasha will claim "This is (a derivation of) Boolean algebra".
If her teacher is a bit tricky she will correct her adding: "Boolean algebra on {0,1} !".
Let us emphasize that Boole's 3-step derivation is performed/written exactly in the same way nowadays in standard mathematics. But people have totally forgotten the meaning originally given by Boole to it: derivation of the principle of contradiction from the fundamental law of thought. They generally don't interpret $x^2 = x$ as a fundamental law of thought and $x(1 - x) = 0$ as the principle of contradiction.

4. Boolean algebra from the point of view of model theory

Boolean algebra can be considered today from the point of view of classical first-order logic. First-order logic can itself be considered from two perspectives: model theory and proof theory. Sometimes this distinction is presented with a linguistic flavor as a distinction between semantics and syntax. This is rather ambiguous.
If we develop a strong link between syntax and computation, then many things that are called semantics can be called syntax. This is in fact what Chang and Keisler do in their famous book calling truth-tables syntax.
Moreover the two guys define model theory with the following equation

universal algebra + logic = model theory

Although we can enjoy their sense of humor, structures do not reduce to algebras. Model theory is a relation between syntax and semantics, syntax not in the sense of proof-theory but in the sense of formulas and formulations. The study of syntax in this sense has been developed in details by logicians. Mathematicians are rather fuzzy about that.
Birkhoff's HSP theorem, one of the main theorems in the pre-history of model theory, states that a class of structures is closed under homomorphism, subalgebra and product iff it can be defined by a set of equations. It establishes a relation between syntax and semantics. Semantics here in the sense of algebraic structures.

(1F)	$\forall x\, x \times x = x$
(2F)	$\forall x\, x - (x \times x) = \mathbf{0}$
(3F)	$\forall x\, x \times (\mathbf{1} - x) = \mathbf{0}$

TABLE 2 – FIRST-ORDER-LOGIC

The question is not to eliminate abbreviation for multiplication but to consider that multiplication and subtraction are not part of the syntax. What about =, **1** , **0** ?
"×" can be interpreted in different ways. Its meaning is fixed by some axioms. A sign similar to "x" is used, because it is the intended-interpretation.

Now than we have put the correct syntax, can we say that the three lines form a correct derivation?

Consequence in model theory (Tarski):
All models of (1F) are models of (2F) are models of (3F).

This is indeed no true for all models, only for a certain class of models: rings.
Ring is a sufficient condition, but not a necessary condition.
From the point of view of model theory, a Boolean algebra is any structure obeying axioms of ring + (1F), i.e. an idempotent ring, also called a Boolean ring. It can also be defined as a complemented distributive lattice (equivalence of structure).

5. Algebra of sets

The most famous model of Boolean algebra is the Boolean algebra on {0,1}. Another famous model is algebra of sets.
It is possible to prove that the power set of any set forms a Boolean algebra and there is the famous Stone representation theorem stating that every Boolean algebra is isomorphic to a field of sets.

(1FAS)	$\forall x\ x \cap x = \emptyset$
(2FAS)	$\forall x\ x \setminus (x \cap x)\ = \emptyset$
(3FAS)	$\forall x x \cap (\cup \setminus x)\ = \emptyset$

TABLE 3 – FIRST-ORDER ALGEBRA OF SETS

(1AS)	A∩A = A
(2AS)	A \ A∩A = ∅
(3AS)	A∩(**U** \ A) = ∅
(4AS)	A∩$\bar{A}$ = ∅

TABLE 4 – INFORMAL ALGEBRA OF SETS

∅ is the empty set.
U is the Universe, i.e. the set of all objects (not the set of all sets, because elements of sets are not specified, we are not in set theory!)
The derivation from (1AS) to (4AS) is performed using intuitive properties of set-theoretical operations.

6. Propositional classical logic

Let us finally examine the relation between Boole's original derivation and classical propositional logic (hereafter CPL), not CPL as conceived by Boole, but as it is nowadays understood.
It is common to claim that CPL is a Boolean algebra. What does it mean exactly?
Model theory permits a clear formulation of the problem: Is CPL a first-order structure which is a model of the axioms of Boolean ring?
Note that it is not necessarily absurd to consider a propositional logic as a first-order logical structure. A move made by Tarski.
We have written the formulas on TABLE 2 following the intended-interpretation's spirit. Having in mind CPL as intended-interpretation we can rewrite TABLE 2 as follows:

(1P)	$\forall p\ p \wedge p \leftrightarrow p$
(2P)	$\forall p\ p - (p \wedge p) \leftrightarrow \bot$
(3P)	$\forall p\ p \wedge (\top - p) \leftrightarrow \bot$

TABLE 5 – FIRST-ORDER-CLASSICAL PROPOSITIONAL LOGIC

Similarly as we have simplified TABLE 3 into TABLE 4, we can simplify TABLE 5:

(1P)	$p \wedge p \leftrightarrow p$
(2P)	$p - (p \wedge p) \leftrightarrow \bot$
(3P)	$p \wedge (\top - p) \leftrightarrow \bot$

TABLE 6 – CLASSICAL PROPOSITIONAL LOGIC

(1P)	$p^2 \leftrightarrow p$
(2P)	$p - p^2 \leftrightarrow \bot$
(3P)	$p\ (\top - p) \leftrightarrow \bot$
(4P)	$p \wedge\neg p \leftrightarrow \bot$

TABLE 7 – CLASSICAL PROPOSITIONAL LOGIC WITH EXPONENTIATION AND NEGATION

$\top$ is verum and $\bot$ is $falsum$. But What is $-$?

p	q	p⟶q	¬(p⟶q)	p ⟵q	¬(p⟵q)
0	0	1	0	1	0
0	1	1	0	0	1
1	0	0	1	1	0
1	1	1	0	1	0

This derivation can be performed in a subsystem of CPL. Which one exactly?

7. Formalization of the principle of non-contradiction – PNC

$$p \wedge\neg p \leftrightarrow \bot$$

Ex-contradictione sequitur quod libet

$$p, \neg p \vdash q$$

different from

$$\vdash \neg(p\wedge\neg p)$$

The two are independent.

p	¬p	p∧¬p	¬(p∧¬p)
0	1	0	1
½	½	½	½
1	0	0	1

If ½ is considered as designated, ⊢ ¬(p∧¬p) is valid but not p, ¬p ⊢ q.
If ½ is considered as non designated, p, ¬p ⊢ q is valid but not ⊢ ¬(p∧¬p).
Standard definition of paraconsistent logics: rejection of p, ¬p ⊢ q.
Genuine paraconsistent logic: a logic in which none of the two forms of PNC is valid.