An Axiomatization of Quantum Computational Logic

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In [1; 2; 3] a particular semantics of Quantum Computability Logic (**QCL**) is described in a following way. A sentential language **L** of **QCL** contains the following connectives: the negation ($\neg$), the conjunction ($\curlywedge$) and the square root of the negation ($\sqrt{\neg}$). The notion of sentence (or formula) of L is defined standardly. Let $Form^{\mathbf{L}}$ represent the set of all sentences of **L**. As usual, the metavariables p,q,r,...will range over atomic sentences, while α,β,γ,...will range over sentences. The disjunction ($\curlyvee$) is defined via de Morgan's law:

$\curlyvee := \neg(\neg\curlywedge\neg\ )$.

The basic concept of the semantics is the notion of quantum computational realization which is given by an interpretation of the language L, such that the meaning associated to any sentence is a quregister (qubit-register) – either a qubit or an n-qubit system (any unit vector $|\psi\rangle$ in the product space $\otimes^n\mathbb{C}^2$). This determines that the space of the meanings corresponds not to a unique Hilbert space, but to varying Hilbert spaces, each one of the form $\otimes^n\mathbb{C}^2$. The formal definition is the following.

Definition. A quantum computational realization of L is a function Qub associating to any sentence a quregister in a Hilbert space $\otimes^n\mathbb{C}^2$ (where n depends on the linguistic form of α):

$$\text{Qub}: Form^{\mathbf{L}} \to \bigcup \otimes^n\mathbb{C}^2$$

The mostly intriguing in the situation with **QCL** is that the axiomatizability of **QCL** is still an open problem. Below we will fill up this gap taking into account all peculiarities of the semantics of **QCL**.

Following R. Goldblatt [4], we will conceive a quantum computational logic not as a set of wffs, but as a collection **L** of ordered pairs of wffs that satisfies certain closure conditions, the idea being that the presence of the pair (α,β) in **L** indicates that B can be inferred from A in **L**. Logics of this kind usually are called binary logics, we will write $\alpha \vdash \beta$ in place of $(\alpha,\beta)\in\mathbf{L}$.

Schemes of axioms of **QCL**:

A1. $A \dashv\vdash \neg\neg\alpha$

A2. $\alpha\curlywedge(\beta\curlywedge\gamma) \dashv\vdash (\alpha\curlywedge\beta)\curlywedge\gamma$

A3. $\alpha\curlywedge(\beta\curlyvee\gamma) \vdash (\alpha\curlywedge\beta)\curlyvee(\alpha\curlywedge\gamma)$

A4. $0 \vdash \beta\curlywedge\beta$

A5. $\neg 0 \dashv\vdash 1$

A6. $\alpha \vdash 1 \curlywedge\alpha$

A7. $\alpha\curlywedge\beta \vdash \alpha$

A8. $\alpha\curlywedge\beta \vdash \beta$

A9. $\sqrt{\neg}\sqrt{\neg}\,\neg\alpha \dashv\vdash \neg\alpha$

A10. $\sqrt{\neg}\,\neg\alpha \dashv\vdash \neg\sqrt{\neg}\alpha$

A11. $\sqrt{\neg}\,(\alpha\curlywedge\beta) \dashv\vdash \neg.\ \sqrt{\neg}\,(\alpha\curlywedge\beta)$

Rules of **QCL**:

R1. $\dfrac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$

R2. $\dfrac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}$

R3. $\dfrac{\alpha \vdash \beta \quad \gamma \vdash \delta}{\alpha \wedge \gamma \vdash \beta \wedge \delta}$

The following theorems are proved:

Theorem (**Correctness Theorem for QLC**) $\Gamma \vdash \alpha$ only if $\Gamma \vDash_{Qub} \alpha$

Theorem (**Paraconsistency Theorem for QLC**) $\Gamma \vdash \alpha$ only if $\Gamma \vDash_{Qub*} \alpha$

Theorem (**Completeness Theorem for QLC**) $\Gamma \vdash \alpha$ iff $\Gamma \vDash_{Qub}{}^{c} \alpha$

References

1. Cattaneo G., Dalla Chiara M.L., Giuntini R. An Unsharp Quantum Logic from Quantum Computation // P.Weingartner (ed.), Alternative Logics. Do Sciences Need Them?, Springer Verlag, Berlin-Heidelberg-New York, 2003, pp.323–338
2. Cattaneo G., Dalla Chiara M.L., Giuntini R. and Leporini R. An unsharp logic from quantum computation // International Journal of Theoretical Physics, 2004, Volume 43, Issue 7-8, pp. 1803–1817
3. Dalla Chiara M.L., Giuntini R., Greechie R. Reasoning in Quantum Theory. Sharp and Unsharp Quantum Logic. v. 3, No 1-2, 2004, pp. 240–266
4. Goldblatt, R. Semantic Analysis of Orthologic // Journal of Philosophical Logic, v. 3, No 1-2, 1974, pp. 19–35