Quantum properties of a parametric four-wave mixing in a Raman-type atomic system

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We present a study of the quantum properties of two light fields used to parametric four-wave mixing in a Raman type atomic system. The system realizes an effective Hamiltonian of beamsplitter type coupling between the light fields, which allows to control squeezing and amplitude distribution of the light fields, as well as realizing their entanglement. The scheme can be feasibly applied to engineer the quantum properties of two single-mode light fields in properly chosen input states.

I. INTRODUCTION

Squeezing [1] and entanglement [2] are two important features of quantum light and have no counterparts in the classical framework. They are the cornerstone of quantum computation and quantum communication with continuous variable light fields, where quantum information is usually encoded into Gaussian states of light [3, 4]. Entanglement and squeezing are normally generated through parametric processes such as parametric down conversion [5] and four-wave mixing (FWM) [6–10].

FWM can realize an effective coupling of beamsplitter type between two light fields [16], which can be applied to generate the important categories of photonic quantum states, such as cat states and symmetric entangled states. In the present work we consider a Raman-type dispersive FWM to realize a similar beamsplitter type coupling for other applications including transferring squeezing between different modes, amplifying the amplitude of one squeezed mode and entangling different quantum modes. There are two degenerate counterpropagating electromagnetic fields as the pump in our concerned system, and they realize an effective standing wave creating the spatial modulation of the nonlinearities for the input quantum light fields. This structure behaves like photonic crystals that are widely used for control light propagation [11, 12]. To avoid the influence of the quantum noises that are critical to the entanglement generation (see, e.g. [13–15]), we use a dispersive parametric interaction, so that the one-photon detuning and two-photon detuning in the process are large enough to prevent the real excitation of the atomic system from its ground states and only the quantum state of the light fields will be changed during the close-loop parametric interaction, which involves the two counter-propagating quantum modes and the standing wave of the pump field.

The rest of the paper is organized as follows. We first present a description of the model in Sec. II. The effective Hamiltonian for the system in the dispersive regime is derived in Sec. III. In Sec. IV we study the properties of quantum light fields, such as the amplitude dynamics, squeezing transferring and entanglement generation, after finding the evolving light field modes. We summarize the results in Sec. V.

II. THE MODEL

Let us consider a three-level atomic system with the ground state $|a\rangle$ and two upper states $|b\rangle$ and $|c\rangle$; see Fig. 1. Through the system the input quantum light fields of single mode, the blue ones in Fig. 1, effectively interact with one another. The classical standing electromagnetic field with the Rabi frequency Ω implements the transition $|a\rangle - |c\rangle$. Such classical standing wave is applied along z - direction and can be decomposed into two running wave with the wave vectors $\pm k$. The two counterpropagating quantum modes that are coupled to the transition $|b\rangle - |c\rangle$ with the coupling parameter g also propagate along z direction.

In the interaction picture the Hamiltonian of the system in Fig. 1 takes the following form ($\hbar \equiv 1$)

$$
\begin{aligned}
H = & -\Delta \left|c\right\rangle \left\langle c\right| - \delta \left|b\right\rangle \left\langle b\right| \\
& W\left(\left|a\right\rangle \left\langle c\right| + \left|c\right\rangle \left\langle a\right|\right) + \widehat{s}^{\dagger} \left|b\right\rangle \left\langle c\right| + \widehat{s} \left|c\right\rangle \left\langle b\right| ,
\end{aligned}
\qquad (1)
$$

where

$$
\begin{aligned}
W &= 2\Omega \cos kz \\
\widehat{s} &= g\left(\widehat{a} e^{ik_0 z} + \widehat{b} e^{-ik_0 z}\right).
\end{aligned}
$$

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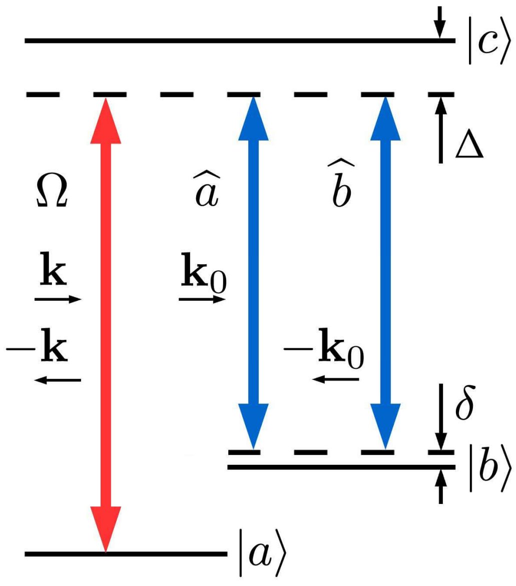

FIG. 1: Scheme of a Raman-type dispersive four-wave mixing of two counterpropogating quantum modes $\widehat{a}$ and $\widehat{b}$ with the wave vectors $\pm k_0$ and the classical standing wave with the Rabi frequency Ω in a media of a three-level quantum systems.

In Eq. (1) we introduce the one-photon detuning $\Delta = \omega - \omega_{ac}$ and the two-photon detuning $\delta = \omega - \omega_0 - \omega_{ab}$, where ω, ω_0 are the optical frequencies of the classical field and quantum modes correspondingly, while ω_{ac} and ω_{ab} are the frequencies of the corresponding transitions.

III. EFFECTIVE HAMILTONIAN

In what follows we will consider a regime of the dispersive interaction, where the large detunings of the electromagnetic fields prevent the real transitions of the three-level system from its ground state. So, the atomic system remains the same initial state $|a\rangle$ during the interaction, and only the quantum states of the optical fields can be changed. The classical field is assumed to be much stronger than the quantum ones. In order to derive the effective Hamiltonian in the dispersive regime we use an adiabatic elimination technique [16–18]. We start from the Schrödinger equation $i\hbar\frac{d\Psi(t)}{dt} = H\Psi(t)$ and project it onto atomic states $|a\rangle$, $|b\rangle$ and $|c\rangle$:

$$i\frac{d}{dt}\langle a|\Psi(t)\rangle = W\langle c|\Psi(t)\rangle \tag{2}$$

$$i\frac{d}{dt}\langle b|\Psi(t)\rangle = -\delta\langle b|\Psi(t)\rangle + \widehat{s}^{\dagger}\langle c|\Psi(t)\rangle \tag{3}$$

$$i\frac{d}{dt}\langle c|\Psi(t)\rangle = -\Delta\langle c|\Psi(t)\rangle + W\langle a|\Psi(t)\rangle + \widehat{s}\langle b|\Psi(t)\rangle, \tag{4}$$

following the derivations detailed in [16, 18] under the

conditions (n is the maximal photon number in the quantum modes)

$$\left|\frac{W}{\Delta}\right| << 1 \text{ and}$$
$$\frac{g\sqrt{n}W}{|\Delta\delta|} << 1,$$

which allows to avoid one-photon and two-photon transition so that the system remains in its ground state $|a\rangle$ and only the state of the light fields will be changed. Then we can eliminate the states $|c\rangle$ and $|b\rangle$ from the dynamical equations, to obtain the effective dynamical evolution

$$i\frac{d}{dt}\langle a|\Psi(t)\rangle = \widetilde{H}_{eff}\langle a|\Psi(t)\rangle, \tag{5}$$

where

$$\widetilde{H}_{eff} = \frac{\widehat{s}^{\dagger}\widehat{s}W^2}{\Delta^2\delta}$$

is the effective Hamiltonian.

Furthermore we can simplify the present Hamiltonian and eliminate $z-$ dependence. By omitting the fast oscillation terms and making tranformation $T_\delta = \exp i\Delta k\left(a^{\dagger}a + b^{\dagger}b\right)z$, we obtain a beamsplitter type Hamiltonian

$$H_{eff} = \chi_0\left(\widehat{a}^{\dagger}\widehat{a} + \widehat{b}^{\dagger}\widehat{b}\right) + \sigma_0\left(\widehat{a}\widehat{b}^{\dagger} + \widehat{a}^{\dagger}\widehat{b}\right) \tag{6}$$

where

$$\chi_0 = \frac{2\Omega^2 g^2}{\Delta^2\delta} - \Delta kc, \tag{7}$$

with $\Delta k = k - k_0$, is due to a self-phase modulation and

$$\sigma_0 = \frac{\Omega^2 g^2}{\Delta^2\delta} \tag{8}$$

is the cross-coupling between the quantum modes.

IV. ENGINEERING OF THE QUANTUM LIGHT FIELDS

The field operators' evolution is described by the Heisenberg equation as the two coupled propagation ones

$$\frac{d}{dz}\widehat{a} = i\chi\widehat{a} + i\sigma\widehat{b}$$
$$\frac{d}{dz}\widehat{b} = -i\sigma\widehat{a} - i\chi\widehat{b},$$

where the parameters $z = \alpha_0 z_0$ (the replacement $t \to z_0/c$ has been used), $\chi = -\chi_0 c/\alpha_0$ and $\sigma = -\sigma_0 c/\alpha_0$ are renormalized with the unpertubed absorption coefficient α_0, and the counterpropagation geometry of the quantum modes is taken into account. Since the two modes have a counterpropagating geometry, the boundary conditions become $\widehat{a}(z=0) = \widehat{a}_0$ and $\widehat{b}(z=L) = \widehat{b}_L$, where L is the length of the medium. The solution for the output modes $\widehat{a}(z=L) = \widehat{a}_L$ and $\widehat{b}(z=0) = \widehat{b}_0$ can be written in the following form:

$$\widehat{a}_L = S_1(L)\,\widehat{a}_0 + S_2(L)\,\widehat{b}_L \quad (9)$$
$$\widehat{b}_0 = S_2(L)\,\widehat{a}_0 + S_1(L)\,\widehat{b}_L \quad (10)$$

where

$$S_1 = \left[\cos sL - i\frac{\chi}{s}\sin sL\right]^{-1} \quad (11)$$
$$S_2 = iS_1\frac{\sigma}{s}\sin sL \quad (12)$$
$$s = \sqrt{\chi^2 - \sigma^2} \quad (13)$$

This result is in agreement with that obtained by treating the currently concerned system as a classical one [19]. During the process the commutation relations, $\left\langle\widehat{a}_L\widehat{a}_L^\dagger\right\rangle - \left\langle\widehat{a}_L^\dagger\widehat{a}_L\right\rangle = \left\langle\widehat{b}_0\widehat{b}_0^\dagger\right\rangle - \left\langle\widehat{b}_0^\dagger\widehat{b}_0\right\rangle = |S_1|^2 + |S_2|^2 = 1$, for the photon modes are always preserved as it should be.

An interesting property of the system is that the parameter s can be imaginary when $\chi^2 < \sigma^2$. This situation can happen in the range of the parameters

$$\frac{1}{3} < P < 1, \quad (14)$$

where we have introduced a new notation for the dimensionless parameter

$$P \equiv \frac{\Omega^2 g^2}{\Delta^2 \delta(\Delta k)\, c}. \quad (15)$$

This situation is analogous to the presence of the band gap in photonic crystal [11, 12], as predicted in [20].

In the following, we will apply the above-mentioned dynamics to study the properties of propagation such as the conversion, and squeezing transferring between the light fields, and entanglement generation under the different input parameters.

A. field mode swapping

We start with the discussion of how the amplitude of the quantum modes can be changed in our concerned process. We can find the photon number in the modes $A_a = \left\langle\widehat{a}^\dagger\widehat{a}\right\rangle$ and $A_b = \left\langle\widehat{b}^\dagger\widehat{b}\right\rangle$ from Eqs.(9) and (10):

$$A_a^L = |S_1|^2 A_a^0 + |S_2|^2 A_b^L$$
$$A_b^0 = |S_1|^2 A_b^L + |S_2|^2 A_a^0,$$

where it is assumed that initially the two modes $\widehat{a}$ and $\widehat{b}$ are not correlated $\left\langle\widehat{a}_0^\dagger\widehat{b}_L\right\rangle = \left\langle\widehat{a}_0\widehat{b}_L^\dagger\right\rangle = 0$. In the numerical calculation we consider the case when input state for the mode $\widehat{a}$ is in the coherent state $|\alpha\rangle$ and the second mode $\widehat{b}$ is in the vacuum state $|0\rangle$. In this case the amplitudes of photon modes take a simple form

$$A_a^L = |S_1|^2 |\alpha|^2$$
$$A_b^0 = |S_2|^2 |\alpha|^2 .$$

The possibility of the squeezing transferring by coherent process involving electromagnetically induced transparency in atomic system was studied in [21], but principally it is impossible to obtain more than 25% of the initial squeezing in that way. Here we demonstrate more than 90% of the squeezing transferring between the modes. Amplitude of the squeezed mode and amount of squeezing is easy to control by the intensity of the pump field.

B. squeezing transferring

We define the quadratures of the quantum modes

$$X_{a,b} = \left[\widehat{a}_L\left(\widehat{b}_0\right) + \widehat{a}_L^\dagger\left(\widehat{b}_0^\dagger\right)\right]/2 \quad (16)$$
$$Y_{a,b} = \left[\widehat{a}_L\left(\widehat{b}_0\right) - \widehat{a}_L^\dagger\left(\widehat{b}_0^\dagger\right)\right]/2i \quad (17)$$

and their fluctuations $\langle\Delta X_{a,b}\rangle^2 = \left\langle X_{a,b}^2\right\rangle - \langle X_{a,b}\rangle^2$ and $\langle\Delta Y_{a,b}\rangle^2 = \left\langle Y_{a,b}^2\right\rangle - \langle Y_{a,b}\rangle^2$.

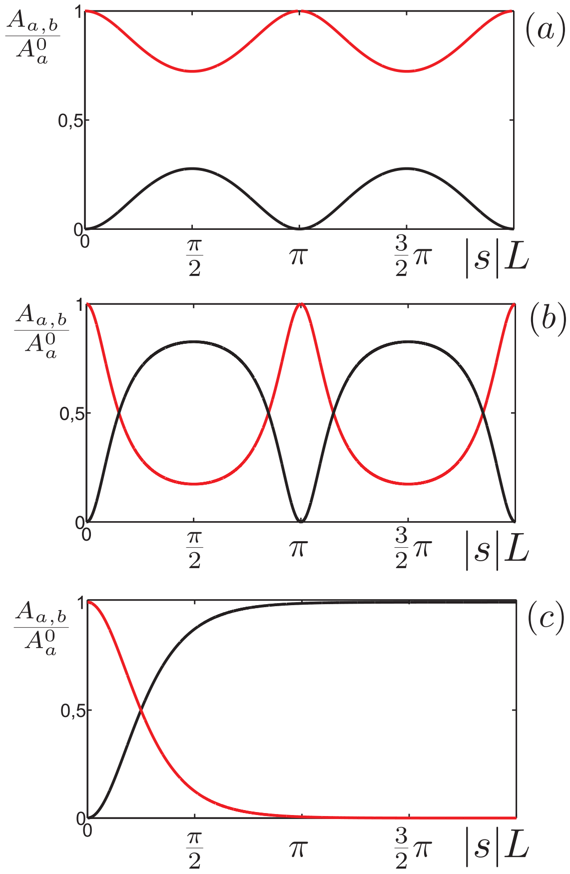

FIG. 2: Dynamics of the normalized over $A_0 = |\alpha|^2$ quantum field's amplitude as a function of the dimensionless parameter $|s|\,L$ where s is given in Eq.13 and L is the interaction length. The red line is the function A_a/A_a^0 and the black one is the A_b/A_a^0. a) $P = 10$ b) $P = 1.1$ c) $P = 0.4$

If the fluctuation in one of the quadratures is less than 1/4, the field will be in squeezed state. For the input coherent states in the considered system, the output are always in coherent states and there is no way to get a squeezed state

$$\langle \Delta X_{a,b}\rangle^2 = \langle \Delta Y_{a,b}\rangle^2 = \frac{1}{4}\left(|S_1|^2 + |S_2|^2\right) = \frac{1}{4}$$

The situation will be different in the case when one of the modes is initially in a squeezed state. As an example we assume that mode $\widehat{a}$ is in the coherent state $|\alpha\rangle$ and mode $\widehat{b}$ in the squeezed state $|\xi\rangle$ with squeezing parameter r. The expressions for the four quadratures fluctuations are

$$\langle \Delta X_{a,b}\rangle^2 = \frac{1}{4} + \frac{1}{4}\left[2\,|S_{2,1}|^2 \sinh^2 r - \left(S_{2,1}^2 + \left(S_{2,1}^*\right)^2\right)\sinh r \cosh r\right]$$

$$\langle \Delta Y_{a,b}\rangle^2 = \frac{1}{4} + \frac{1}{4}\left[2\,|S_{2,1}|^2 \sinh^2 r + \left(S_{2,1}^2 + \left(S_{2,1}^*\right)^2\right)\sinh r \cosh r\right]$$

In this case the amplitude of the squeezed mode can be sufficiently amplified due to the parametric energy transferring from the coherent mode. In Fig. 3(a) about $P = 10$ [see Eq. (15)] there is the effective squeezing transferring between two quadratures of the same mode. At the same time in Fig. 2(a) we can see that at the point $L = \frac{\pi}{2}$ the mode, which is initially weak and squeezed, gets sufficiently amplified due to the parametric interaction with another quantum mode and preserves almost the initial level of the squeezing. Next we consider the case when the squeezing transfers from one mode to another [$P = 1.1$ as in Fig. 3 (b)]. The quantum mode which is initially coherent state and has a large amplitude quantum mode around the point $L = \frac{\pi}{2}$ will become squeezed [see Fig. 3(b)] and it's amplitude will be still sufficiently large [see Fig. 2(b)]. For the case when the parameters lie in the region where the band gap exists $P = 0.4$ [see Eq. (15)], there will be no sufficient squeezing in any quadrature as in Fig. 3(c).

The possibility of the squeezing transferring by coherent process involving electromagnetically induced transparency in atomic system was studied in [21], but principally it is impossible to obtain more than 25% of the initial squeezing in that way. Here we demonstrate more than 90% of the squeezing transferring between the modes. Amplitude of the squeezed mode and amount of squeezing is easy to control by the intensity of the pump field.

C. entanglement generation

In this section we demonstrate the possibility of generating the entanglement between quantum modes. As the criteria of entanglement we use the inequality $Q \equiv (\Delta u)^2 + (\Delta \upsilon)^2 < 1$ [22], where $u = X_a + X_b$ and $\upsilon = Y_a - Y_b$. The quadratures $X_{a,b}$ and $Y_{a,b}$ are given in Eqs. (16) and (17). When both modes are in the coherent state, there will be no possibility to generate entanglement and function Q is simply larger than 1. When mode $\widehat{a}$ is in a coherent state $|\alpha\rangle$ and mode $\widehat{b}$ in a squeezed state $|\xi\rangle$, the system will be able to generate an entanglement between the modes in a certain range of system parameters. For the concerned system we have

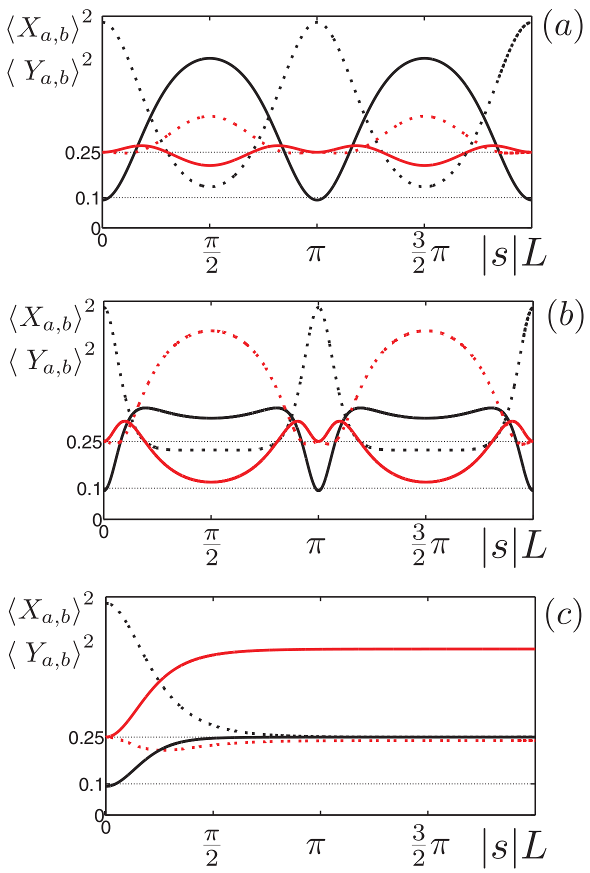

FIG. 3: Dynamics of the fields quadratures fluctuations as a function of the dimentionless parameter $|s|\,L$ where s is given in Eq.13 and L is the interaction length. $\langle\Delta X_a\rangle^2$ - red solid line, $\langle\Delta X_b\rangle^2$ - black solid line, $\langle\Delta Y_a\rangle^2$ - red doted line, $\langle\Delta Y_b\rangle^2$ - black doted line. a) $P=10$ b) $P=1.1$ c) $P=0.4$

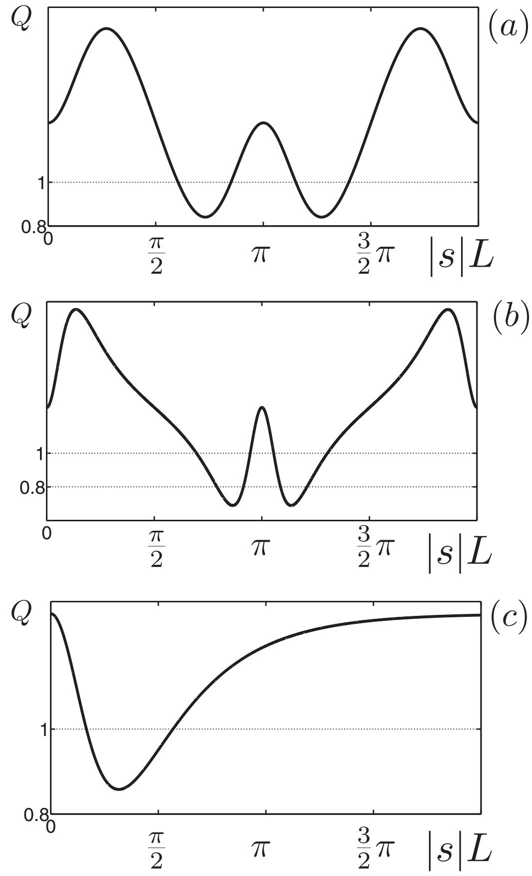

FIG. 4: The function Q that is an entanglement creteria as a function of dimentionless parameter $|s|\,L$ where s is given in Eq.13 and L is the interaction length. a) $P=10$ b) $P=1.1$ c) $P=0.4$

$$
\begin{aligned}
Q &= (\Delta u)^2 + (\Delta v)^2 \\
&= \left(1+\sinh^2 r\right)\left(|S_1|^2 + |S_2|^2\right) \\
&- \frac{1}{2}\sinh r\cosh r\left[S_1^2 + S_2^2 + (S_1^*)^2 + (S_2^*)^2\right]
\end{aligned}
$$

Through numerical analysis we find a few regimes where two quantum modes can be entangled.

In Fig. 4 we plot the function Q as compared with the entanglement criteria [22]. We see that, for $P=10$ and $P=1.1$, there are two dips around $|s|\,L=\pi+\pi l$ for $l=0,1,2...$, to have the two modes entangled; see Figs. 4 (a) and 4 (b). And when $P=0.4$, we have only one dip where these modes are entangled; see Fig. 4 (c). This parametrically induced beam splitter is capable of entangling two light fields as the usual beamsplitter [23].

Before ending the discussion on these applications, we give an estimation of the experimental requirement to observe the phenomena. An ensemble of atomic density $N\sim 10^{12}$ cm^{-3} and with the size $L\sim 10$ cm suffices to realize the parameters used in the above figures. One could use sodium vapor on the $D1$ spectral line as the medium, while choosing $\Delta\sim 3000$ MHz, $\delta\sim 50$ MHz, $\Omega\sim 60$ MHz for the light fields.

V. CONCLUSION

In summary, we have studied the engineering of two quantum modes counterpropagating in a medium of three-level atomic system also in the presence of a strong classical standing electromagnetic field. The interaction between atomic system and the electromagnetical modes is via dispersive FWM, to have all fields well detuned from one-photon and two-photon resonances, so that the FWM process is purely dispersive and the atomic system itself will be always in the ground state $|a\rangle$. Under such condition there will be no loss in the system in the presence of decays and Langevin noises. The derived effective Hamiltonian has the similar form to a beamsplitter

Hamiltonian [5]. We demonstrate that there are possibilities for the parametric amplitude amplification of a squeezed state, as well as the squeezing transferring and the entanglement generation, with such type of interaction. Also we have found that, in a certain range of the parameters as in Eq. (14), the features similar to those in a photonic crystal band gap appear.

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