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Set Functions and Probability Distributions of a Finite Random Sets

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This paper is the investigation of the probability distributions of a finite random set in which the set of random events are considered as a support of the finite random set. These probability distributions can be defined by six equivalent ways (distributions of the I-st — VI-th type). Each of these types of the probability distributions is the set function defined on the corresponding system of events. In this paper the sufficient conditions are formulated and proved. When these conditions are satisfied, then the set function determines the probability distributions of the finite random set of the II-nd and the V-th type. The found conditions supplement the known necessary conditions for the existence of the probability distributions of a finite random set of the II-nd and the V-th type.

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Introduction

The problems of modeling of random objects of non-numeric nature are actual in multivariate data analysis. Such problems arise in medicine, for example, during forecasting the possible complications in patient according to the set of diagnostic symptoms; in Economics — for a research of the consumer demand; in fire safety — for the description of process of accidental spread of a forest fire etc. One possible approach to solving these problems is to use the apparatus of random sets.

The random element with values from set of all subsets of a particular finite set M is called a finite random set. The set M is called a support of a finite random set. In role of the support can be a finite set of objects of numerical or non-numerical nature. Finite random sets with different

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types of supports were studied in works by I. Molchanov [1], H. T. Nguyen [2], O. Yu. Vorobyev [3], A. I. Orlov [4]. In this paper the finite random sets with support in the form of a set of random events are considered. Finite random sets with such support can act as a mathematical model of complex objects and systems that are described by a finite number of attributes. The appearance of any of these attributes is represented as a random event.

Procedures for constructing finite random sets with given distribution laws are demanded at random-set modeling of complex objects and systems. About twenty probability distributions of finite random sets are already known and used in practice. However, the extension of application area of random-set modeling requires the finding out new distributions. One of approaches for constructing new probability distributions of finite random sets is offered in works [5] – [7], [8]. This approach is based on the construction of a set functions and identify the conditions where these functions are probability distributions of the finite random set.

It is known that probability distribution of a finite random set can be defined by six equivalent ways (distributions of the I-st–VI-th type). Each of these types of the probability distributions of a finite random set is the set-function defined on the corresponding system of events. In this paper the sufficient conditions are formulated and proved. When these conditions are satisfied, then the set- functions determines the probability distributions of the finite random set of the II-nd and the V-th type. The found conditions supplement the known necessary conditions for the existence of the probability distributions of a finite random set of the II-nd and the V-th type that were proven in the works [3, 9].

1. Finite random set

The concept of the random element is used for description of finite random set [10].

Definition 1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space, and $(U, \mathcal{A})$ be the measurable space, where U is the arbitrary set, and $\mathcal{A}$ is some σ-algebra of its subsets. We'll say that the function $K = K(\omega)$ defined on Ω and taking values in U is the $\mathcal{F}/\mathcal{A}$–measurable function or random element (with values in U), if $\{\omega : K(\omega) \in A\} \in \mathcal{F}$ is true for any $A \in \mathcal{A}$.*

If we consider some space $(U, \mathcal{A})$ elements of which there are sets, then the measurable mapping K of family of elementary events of arbitrary probability space $(\Omega, \mathcal{F}, \mathbf{P})$ in $(U, \mathcal{A})$ is called a random set [2, 4]. Let M be the finite set and let $(U, \mathcal{A}) = \left(2^M, 2^{2^M}\right)$. Then the measurable function $K : \ \Omega \to \ 2^M$ mapping the probability space into the space of subsets of a finite set M, is called the finite random set and the set M is called a support of K.

This paper investigates the specific random sets. The finite sets of random events $\mathfrak{X} \subset \mathcal{F}$ ($|\mathfrak{X}| < \infty$), selected from the algebra $\mathcal{F}$ of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, are supports of these random sets.

Definition 2. *The finite random set K defined on the support $\mathfrak{X} \subset \mathcal{F}$ represents a mapping $K : \Omega \to 2^{\mathfrak{X}}$ measurable with respect to a pair of algebras $\left(\mathcal{F}, 2^{2^{\mathfrak{X}}}\right)$ that in the sense $K^{-1}(A) \in \mathcal{F}$ is true for every $A \in 2^{2^{\mathfrak{X}}}$.*

The formula $K(\omega) = \{x \in \mathfrak{X} : \omega \in x\}$ can be interpreted as a "random set of the occured events" since for every elementary event of the experiment $\omega \in \Omega$ is corresponded a subset of the events $X \subseteq \mathfrak{X}$ containing all events which occurred in this experiment.

The $\{\omega : K(\omega) = X, \ X \subseteq \mathfrak{X}\} \in \mathcal{F}$, is a measurable event. This means the finite random set K gets one of its possible values from $X \subseteq \mathfrak{X}$ in other words the occured random events from $\mathfrak{X}$

form the subset $X \subseteq \mathfrak{X}$ and the no-occured random events form the subset $X^c = \mathfrak{X} \setminus X$:

$$\{\omega : K(\omega) = X,\ X \subseteq \mathfrak{X}\} = \left(\bigcap_{x \in X} x\right) \bigcap \left(\bigcap_{x \in X^c} x^c\right) \subseteq \Omega.$$

For simplicity, instead of $\{\omega : K(\omega) = X,\ X \subseteq \mathfrak{X}\}$ will write $\{K_{\mathfrak{X}} = X\}$, where the subscript $\mathfrak{X}$ at K means that the finite random set K defined on the support $\mathfrak{X}$. The set of all these events generated by the set of the events $\mathfrak{X}$ forms a partition of the space of elementary events $\Omega = \sum_{X \subseteq \mathfrak{X}} \{K_{\mathfrak{X}} = X\}$.

2. Types and properties of probability distributions of the finite random set

Probability distributions of finite random sets are the set functions which are defined on various systems of the random events generated by a set of $\mathfrak{X}$.

Definition 3. *The set function $f(X),\ X \in 2^{\mathfrak{X}}$ defined on the $2^{\mathfrak{X}}$ of some arbitrary finite set $\mathfrak{X}$ is the mapping $f : 2^{\mathfrak{X}} \to \mathbb{R}$ with the values on the real axis $\mathbb{R}$.*

Definition 4. *The set function f defined on the $2^{\mathfrak{X}}$ is called*

$$\text{the additive if } f(X \cup Y) + f(X \cap Y) = f(X) + f(Y),\ \forall\, X, Y \subseteq \mathfrak{X}; \tag{1}$$

$$\text{the subadditive if } f(X \cup Y) + f(X \cap Y) \leqslant f(X) + f(Y),\ \forall\, X, Y \subseteq \mathfrak{X}; \tag{2}$$

$$\text{the superadditive if } f(X \cup Y) + f(X \cap Y) \geqslant f(X) + f(Y),\ \forall\, X, Y \subseteq \mathfrak{X}. \tag{3}$$

Let $K_{\mathfrak{X}}$ be the finite random set defined on support $|\mathfrak{X}| = N < \infty$,where $\mathfrak{X} \subset \mathcal{F}$.

Definition 5. *The probability distribution of the finite random set $K_{\mathfrak{X}}$ is called the set of 2^N values of the probability measure $\mathbf{P}$ defined on the system of events $2^{\mathfrak{X}}$.*

Any one of the six types of distributions (I-st–VI-th types) is enough for a complete determination of the probability distribution of the finite random set $K_{\mathfrak{X}}$. The paper deals with the following three types.

PI. The probability distribution of the I-st type of the finite random set $K_{\mathfrak{X}}$ is the set function defined on the system of disjoint events of the following form $\{K_{\mathfrak{X}} = X\} = \left(\left(\bigcap_{x \in X} x\right) \bigcap \left(\bigcap_{x \in X^c} x^c\right)\right)$, where $X^c = \mathfrak{X} \setminus X$, $x^c = \Omega \setminus x$. It is the set of 2^N probabilities of the form $p(X) = \mathbf{P}(\{K_{\mathfrak{X}} = X\})$. The probability distribution of the I-st type satisfies to the condition $\sum_{X \subseteq \mathfrak{X}} p(X) = 1$.

PII. The probability distribution of the II-nd type of the finite random set $K_{\mathfrak{X}}$ is the set function defined on the system of joint events of the following form $\{K_{\mathfrak{X}} \supseteq X\} = \left(\bigcap_{x \in X} x\right)$. It is the set of 2^N probabilities of the form $p_X = \mathbf{P}(\{K_{\mathfrak{X}} \supseteq X\})$. The probability distribution of the II-nd type satisfies to the system of Frechet inequalities for the probabilities of the intersection of events [3]:

$$\max\left\{0, \sum_{x \in X} \mathbf{P}(x) - |X| + 1\right\} \leqslant \mathbf{P}\left(\bigcap_{x \in X} x\right) \leqslant \min_{x \in X} \mathbf{P}(x), \tag{4}$$

where $\mathbf{P}(x) = \mathbf{P}(\{x \in K_{\mathfrak{X}}\})$, $x \in X$, $X \subseteq \mathfrak{X}$, are probabilities of the events themselves.

PV. The probability distribution of the V-th type of the finite random set $K_{\mathfrak{X}}$ is the set function defined on the system of joint events of the following form $\{K_{\mathfrak{X}} \nsubseteq X^c\} = \left(\bigcup_{x \in X} x\right)$. It is the set of 2^N probabilities of the form $u_X = \mathbf{P}\big(\{K_{\mathfrak{X}} \nsubseteq X^c\}\big) = 1 - \mathbf{P}\big(\{K_{\mathfrak{X}} \subseteq X^c\}\big)$. The probability distribution of the V-th type satisfies to the system of Frechet inequalities for the probability of the union of events [3]:

$$\max_{x \in X} \mathbf{P}(x) \leqslant \mathbf{P}\left(\bigcup_{x \in X} x\right) \leqslant \min\left\{1, \sum_{x \in X} \mathbf{P}(x)\right\}. \tag{5}$$

For brevity, we'll use the following abbreviations: $p_{\{x\}} = p_x$, $p(\{x\}) = p(x)$, $u_{\{x\}} = u_x$, $p_{\{x,y\}} = p_{xy}$, $p(\{x,y\}) = p(xy)$, $u_{\{x,y\}} = u_{xy}$ etc.

Let's give an explanation about the symbol of the empty set, to avoid terminological confusion in the future. In the Boolean of the finite set $\mathfrak{X}$ designation «$\emptyset$» is used for $\mathfrak{X}^c = \Omega \setminus \mathfrak{X}$, and the designation of an empty set «$\varnothing$» is used for treatment of an impossible event. By the definition, we accept

$$\bigcap_{x \in \emptyset} x = \bigcap_{x \in \mathfrak{X}^c} x = \Omega, \quad \bigcap_{x \in \emptyset} x^c = \bigcap_{x \in \mathfrak{X}^c} x^c = \Omega, \quad \bigcup_{x \in \emptyset} x = \bigcup_{x \in \mathfrak{X}^c} x = \varnothing.$$

Then the event $\{K_{\mathfrak{X}} = \emptyset\} = \left(\left(\bigcap_{x \in \emptyset} x\right) \bigcap \left(\bigcap_{x \in \mathfrak{X}} x^c\right)\right) = \bigcap_{x \in \mathfrak{X}} x^c$ means that no one event from the $\mathfrak{X}$ didn't occur.

For the probability distribution of the II-nd type the next equality

$$p_{\emptyset} = \mathbf{P}(\{K_{\mathfrak{X}} \supseteq \emptyset\}) = \mathbf{P}\left(\left(\bigcap_{x \in \mathfrak{X}^c} x\right) \cap \left(\bigcap_{x \in \mathfrak{X}} \Omega\right)\right) = \mathbf{P}\left(\bigcap_{x \in \mathfrak{X}} \Omega\right) = \mathbf{P}(\Omega) = 1$$

is true always, and for the probability distribution of the V-th type the equality

$$u_{\emptyset} = \mathbf{P}(\{K_{\mathfrak{X}} \nsubseteq \mathfrak{X}\}) = 1 - \mathbf{P}(\{K_{\mathfrak{X}} \subseteq \mathfrak{X}\}) = \mathbf{P}\left(\bigcup_{x \in \mathfrak{X}^c} x\right) = \mathbf{P}(\varnothing) = 0$$

is true always.

In works [3, 9] were obtained the inversion formulas for the probability distributions of the finite random set $K_{\mathfrak{X}}$ which follow from the classical Möbius inversion formulas for functions. The probability distributions of the I-st and the II-nd types for the finite random set $K_{\mathfrak{X}}$ are associated with help of mutually inverse Möbious formulas for all $X \in 2^{\mathfrak{X}}$:

$$p_X = \sum_{Y \in 2^{\mathfrak{X}}:\, X \subseteq Y} p(Y), \tag{6}$$

$$p(X) = \sum_{Y \in 2^{\mathfrak{X}}:\, X \subseteq Y} (-1)^{|Y|-|X|} p_Y. \tag{7}$$

The probability distributions of the I-st and the V-th types for the finite random set $K_{\mathfrak{X}}$ are associated with help of mutually inverse Möbius formulas for all $X \in 2^{\mathfrak{X}}$:

$$u_X = 1 - \sum_{Y \in 2^{\mathfrak{X}}:\, X \subseteq Y} p(Y^c), \tag{8}$$

$$p(X) = \sum_{Y \in 2^{\mathfrak{X}}:\, Y \subseteq X} (-1)^{|X|-|Y|} (1 - u_{Y^c}). \tag{9}$$

Let us prove the properties of the set-functions (1)–(3) for the corresponding probability distributions of the finite random set.

Lemma 1. The probability distribution of the I-st type of the finite random set $K_{\mathfrak{X}}$ satisfies the property (1), i.e. is an additive set function on $2^{\mathfrak{X}}$.

Proof. The set of 2^N events $\{K_{\mathfrak{X}} = X\}$ generated by the finite set of random events $\mathfrak{X}$ forms the finite partition of Ω. Because the $\{K_{\mathfrak{X}} = X\} \cap \{K_{\mathfrak{X}} = Y\} = \varnothing$ then $\mathbf{P}\big(\{K_{\mathfrak{X}} = X\} \cap \{K_{\mathfrak{X}} = Y\}\big) = 0$. Therefore, the equality $\mathbf{P}\big(\{K_{\mathfrak{X}} = X\} \cup \{K_{\mathfrak{X}} = Y\}\big) + \mathbf{P}\big(\{K_{\mathfrak{X}} = X\} \cap \{K_{\mathfrak{X}} = Y\}\big) = \mathbf{P}\big(\{K_{\mathfrak{X}} = X\}\big) + \mathbf{P}\big(\{K_{\mathfrak{X}} = Y\}\big)$ is true for any $X, Y \subseteq \mathfrak{X}$. □

Lemma 2. *The probability distribution of the II-nd type of the finite random set $K_{\mathfrak{X}}$ satisfies the property (3), i.e. is a superadditive set function on $2^{\mathfrak{X}}$.*

Proof. For the probability distribution of the II-nd type the property (3) takes the form

$$p_{X\cup Y} + p_{X\cap Y} \geqslant p_X + p_Y, \quad X, Y \subseteq \mathfrak{X}. \tag{10}$$

Let us express in (10) the probability of the II-nd type through the probability of the I-st type using the Mobius inversion formula (6). And shall prove that

$$\sum_{X\cap Y\subseteq Z} p(Z) + \sum_{X\cup Y\subseteq Z} p(Z) \geqslant \sum_{X\subseteq Z} p(Z) + \sum_{Y\subseteq Z} p(Z), \ \ Z \in 2^{\mathfrak{X}}. \tag{11}$$

Let's note that for $Z \in 2^{\mathfrak{X}}$

$$\sum_{X\cap Y\subseteq Z} p(Z) = \sum_{X\subseteq Z} p(Z) + \sum_{Y\subseteq Z} p(Z) - \sum_{X\cup Y\subseteq Z} p(Z) + \sum_{\substack{X\cap Y\subseteq Z;\\ X\not\subseteq Z;\ Y\not\subseteq Z}} p(Z), \tag{12}$$

where the summation in the last sum applies for all such intersections $X \cap Y \subseteq Z$, that $X \not\subseteq Z$ and $Y \not\subseteq Z$. Since the sum of (12) is non-negative then we obtain the equality

$$\sum_{X\cap Y\subseteq Z} p(Z) + \sum_{X\cup Y\subseteq Z} p(Z) = \sum_{X\subseteq Z} p(Z) + \sum_{Y\subseteq Z} p(Z) + \sum_{\substack{X\cap Y\subseteq Z;\\ X\not\subseteq Z;\ Y\not\subseteq Z}} p(Z), \ \ Z \in 2^{\mathfrak{X}},$$

from which the proof follows of (11). □

Lemma 3. *The probability distribution of the V-th type for the finite random set $K_{\mathfrak{X}}$ satisfies the property (2), i.e. is a subadditive set function on $2^{\mathfrak{X}}$.*

Proof. For the probability distribution of the V-th type the property (2) takes the form

$$u_{X\cup Y} + u_{X\cap Y} \leqslant u_X + u_Y, \quad X, Y \subseteq \mathfrak{X}. \tag{13}$$

Let us express in (13) the probability of the V-th type through the probability of the I-st type using the Mobius inversion formula (8). And shall prove that

$$\begin{aligned} 1 - \sum_{Z\in 2^{\mathfrak{X}}:\, X\cap Y\subseteq Z} p(Z^c) + 1 - \sum_{Z\in 2^{\mathfrak{X}}:\, X\cup Y\subseteq Z} p(Z^c) \leqslant \\ \leqslant 1 - \sum_{Z\in 2^{\mathfrak{X}}:\, X\subseteq Z} p(Z^c) + 1 - \sum_{Z\in 2^{\mathfrak{X}}:\, Y\subseteq Z} p(Z^c). \end{aligned} \tag{14}$$

Here

$$\sum_{Z\in 2^{\mathfrak{X}}:\, X\cap Y\subseteq Z} p(Z^c) + \sum_{Z\in 2^{\mathfrak{X}}:\, X\cup Y\subseteq Z} p(Z^c) \geqslant \sum_{Z\in 2^{\mathfrak{X}}:\, X\subseteq Z} p(Z^c) + \sum_{Z\in 2^{\mathfrak{X}}:\, Y\subseteq Z} p(Z^c).$$

Because $\sum\limits_{Z\in 2^{\mathfrak{X}}:\, X\subseteq Z} p(Z^c) = \sum\limits_{Z\in 2^{\mathfrak{X}}:\, Z\subseteq X^c} p(Z)$, then the last inequality has the form

$$\sum_{Z\subseteq (X\cap Y)^c} p(Z) + \sum_{Z\subseteq (X\cup Y)^c} p(Z) \geqslant \sum_{Z\subseteq X^c} p(Z) + \sum_{Z\subseteq Y^c} p(Z), \quad Z\in 2^{\mathfrak{X}}. \tag{15}$$

It is not difficult to prove that set function $f(X^c) = \sum\limits_{Z\in 2^{\mathfrak{X}}:\, Z\subseteq X^c} p(Z)$ is the superadditive if the $f(X) = \sum\limits_{Z\in 2^{\mathfrak{X}}:\, Z\subseteq X} p(Z)$ is the superadditive. Let us show that

$$\sum_{Z\subseteq X\cap Y} p(Z) + \sum_{Z\subseteq X\cup Y} p(Z) \geqslant \sum_{Z\subseteq X} p(Z) + \sum_{Z\subseteq Y} p(Z), \quad Z\in 2^{\mathfrak{X}}. \tag{16}$$

Let's note that for $Z\in 2^{\mathfrak{X}}$

$$\sum_{Z\subseteq X\cap Y} p(Z) = \sum_{Z\subseteq X} p(Z) + \sum_{Z\subseteq Y} p(Z) - \sum_{Z\subseteq X\cup Y} p(Z) + \sum_{\substack{Z\subseteq X\cap Y;\\ Z\nsubseteq X;\, Z\nsubseteq Y}} p(Z), \tag{17}$$

where the summation in the last sum applies to all such intersections $Z\subseteq X\cap Y$, that $Z\nsubseteq X$ and $Z\nsubseteq Y$. Since the sum of (17) is a non-negative then we obtain

$$\sum_{Z\subseteq X\cap Y} p(Z) + \sum_{Z\subseteq X\cup Y} p(Z) = \sum_{Z\subseteq X} p(Z) + \sum_{Z\subseteq Y} p(Z) + \sum_{\substack{Z\subseteq X\cap Y;\\ Z\nsubseteq X;\, Z\nsubseteq Y}} p(Z), \quad Z\in 2^{\mathfrak{X}},$$

from which implies the validity of (16). hence it proves (15). □

From Lemmas 1–3 implies that the choice of this or that system of events determines properties of additivity (1)–(3) of considered probability distributions of the finite random sets. Note that only the probability distribution of the I-st type satisfies the probabilistic normalization because it is defined on the complete group of disjoint events $\{K_{\mathfrak{X}} = X\}$. Probabilistic normalization of the probability distributions of the II-nd and tthe V-th types defined on systems of a joint events $\{K_{\mathfrak{X}} \supseteq X\}$ and $\{K_{\mathfrak{X}} \nsubseteq X^c\}$ respectively isn't realized.

3. Sufficient conditions for the existence of probability distributions of a finite random set of the II-nd and the V-th type

The following obvious lemma allows to consider the set function $f(X)$ defined on system of events $\{K_{\mathfrak{X}} = X\}$ as distribution of the I-st type.

Lemma 4 (Sufficient conditions for the existence of probability distribution of the I-st type). *The set function $f : 2^{\mathfrak{X}} \to [0,\ 1]$ defined on the system of events $\{K_{\mathfrak{X}} = X\}$ is the probability distribution of the I-st type of the finite random set $K_{\mathfrak{X}}$, if the set function satisfy to the condition* $\sum\limits_{X\subseteq\mathfrak{X}} f(X) = 1$.

The set function $f(X)$ defined on the system of events $\{K_{\mathfrak{X}} \supseteq X\}$ which values satisfy to the Frechet bounds (4) does not always determine the probability distribution of the II-nd type of the finite random set. Since the transformation $f(X)$ by formulas (7) can lead to a set function $p(X)$ defined on the system of events $\{K_{\mathfrak{X}} = X\}$ with values not necessarily from $[0,\ 1]$.

Theorem 1 (Sufficient conditions for the existence of the probability distribution of the II-nd type). *The set function $f : 2^{\mathfrak{X}} \to [0,\ 1]$ defined on the event system $\{K_{\mathfrak{X}} \supseteq X\}$ is the probability distribution of the II-nd type of the finite random set $K_{\mathfrak{X}}$, if simultaneously following conditions are satisfied:*

(i) $f(\emptyset) = \mathbf{P}(\{K_{\mathfrak{X}} \supseteq \emptyset\}) = 1$;

(ii) $f(x) = \mathbf{P}(\{x \in K_{\mathfrak{X}}\})$ *are the values of the set function on events $x \in \mathfrak{X}$ coincides with the probabilities of these events*;

(iii) *values of the set function satisfy to the system of the Frechet inequalities for the probabilities the intersection of events for all $X \in 2^{\mathfrak{X}}$, $|X| \geqslant 2$,*

$$\max\left\{0, 1 - \sum_{x\in X}(1 - f(x))\right\} \leqslant f(X) \leqslant \min_{x\in X} f(x);$$

(iv) *values of the set function*

$$p(X) = \sum_{Y\in 2^{\mathfrak{X}}:X\subseteq Y} (-1)^{|Y|-|X|} f(Y), \quad X \in 2^{\mathfrak{X}}, \tag{18}$$

derived from the $f(X)$ by the inversion Möbius formula are non-negative.

Proof. Terms (i) – (iii) are necessary conditions for the existence probability distribution of the II-nd type of finite random set $K_{\mathfrak{X}}$. However, they are not sufficient. In the works [3, 9] it is shown that the formula (18) transforms the set function defined on the system of events $\{K_{\mathfrak{X}} \supseteq X\}$ in the set function defined on the system of events $\{K_{\mathfrak{X}} = X\}$. Let's prove that the result of this transformation is the distribution of the I-st type.

Nonnegativity of $p(X)$ follows from the condition (iv). Let's replace in (18) the summation over all $X \subseteq Y$, $Y \in 2^{\mathfrak{X}}$ on a double sum: the inner sum over $X \subseteq Y_n$, where Y_n are subsets of fixed power $|Y_n| = n$, and the outer summation over the possible values of power $n = |X|, \dots, |\mathfrak{X}|$. Then (18) takes the form

$$p(X) = \sum_{n=|X|}^{|\mathfrak{X}|} \sum_{X\subseteq Y_n} (-1)^{n-|X|} f(Y_n) = \sum_{n=|X|}^{|\mathfrak{X}|} (-1)^{n-|X|} \sum_{X\subseteq Y_n} f(Y_n).$$

Let's prove that the sum of all values of $p(X)$ is equal to unity:

$$\begin{aligned} \sum_{X\subseteq\mathfrak{X}} p(X) &= \sum_{X\subseteq\mathfrak{X}} \left(\sum_{n=|X|}^{|\mathfrak{X}|} (-1)^{n-|X|} \sum_{X\subseteq Y_n} f(Y_n) \right) = \\ &= f(\emptyset) + \sum_{n=1}^{|\mathfrak{X}|} \left(\sum_{m=0}^{n} C_n^m \, (-1)^{n-m} \sum_{Y\in\mathfrak{C}_{\mathfrak{X}}^n} f(Y) \right) = f(\emptyset) = 1, \end{aligned} \tag{19}$$

where

$\mathfrak{C}_{\mathfrak{X}}^n = \{Y \subseteq \mathfrak{X} : \ |Y| = n\}$ is the n-layer of set $\mathfrak{X}$ [3, стр. 50];

$$\sum_{m=0}^{n} C_n^m (-1)^{n-m} = \begin{cases} 1, & n = 0; \\ 0, & n > 0. \end{cases}$$

According to Lemma 4 the set function $p(X)$ determines the probability distribution of the I-st type of the finite random set $K_{\mathfrak{X}}$. Set functions which are probability distributions of the I-st and the II-nd types for the finite random set $K_{\mathfrak{X}}$ are coupled mutually by inversion Möbius formulas (6) and (7). Thus if the set function $p(X)$ determines the probability distribution of the I-st type, then $f(X)$ is a probability distribution of the II-nd type. □

The set function $f(X)$ defined on the system of events $\{K_{\mathfrak{X}} \not\subseteq X^c\}$, which values satisfy to the Frechet bounds (5), also does not always determine the probability distribution of the V-th type of the finite random set.

Theorem 2 (Sufficient conditions for the existence of the probability distribution of the V-th type). *The set function $f : 2^{\mathfrak{X}} \to [0,\ 1]$ defined on the event system $\{K_{\mathfrak{X}} \not\subseteq X^c\}$ is the probability distribution of the V-th type of the finite random set $K_{\mathfrak{X}}$, if simultaneously following conditions are satisfied:*

(i) $f(\emptyset) = \mathbf{P}(\{K_{\mathfrak{X}} \not\subseteq X^c\}) = 0$;

(ii) $f(x) = \mathbf{P}(\{x \in K_{\mathfrak{X}}\})$ *what means that values of the set function on events at $x \in \mathfrak{X}$ coincide with probabilities of these events*;

(iii) *values of the set function satisfy to the system of Frechet inequalities for probability of union of events for all $X \in 2^{\mathfrak{X}}$, $|X| \geqslant 2$*

$$\max_{x \in X} f(x) \leqslant f(X) \leqslant \min\left\{1, \sum_{x \in X} f(x)\right\};$$

(iv) *values of the set function*

$$p(X) = \sum_{Y \in 2^{\mathfrak{X}} : Y \subseteq X} (-1)^{|X|-|Y|} \left(1 - f(Y^c)\right), \quad X \in 2^{\mathfrak{X}}, \tag{20}$$

derived from $f(X)$ by the inversion Möbius formula are non-negative.

Proof. Terms (i) – (iii) are necessary conditions for the existence probability distribution of the V-th type of finite random set $K_{\mathfrak{X}}$. However, they are not sufficient. In the works [3, 9] it is shown that the formula (20) transforms the set function defined on the system of events $\{K_{\mathfrak{X}} \not\subseteq X^c\}$ in the set function defined on the system of events $\{K_{\mathfrak{X}} = X\}$. Let's prove that the result of this transformation is the distribution of the I-st type.

Nonnegativity of $p(X)$ follows from the condition (iv). Let's replace in (20) the summation over all $Y \subseteq X$, $Y \in 2^{\mathfrak{X}}$ on a double sum: the inner sum over $Y_n \subseteq X$, where Y_n are subsets of fixed power $|Y_n| = n$, and the outer summation over the possible values of power $n = 0, \ldots, |X|$. Then (20) takes the form

$$p(X) = \sum_{n=0}^{|X|} \sum_{Y_n \subseteq X} (-1)^{|X|-n} \big(1 - f(Y_n^c)\big) = \sum_{n=0}^{|X|} (-1)^{|X|-n} \sum_{Y_n \subseteq X} \big(1 - f(Y_n^c)\big).$$

Let's prove that the sum of all values of $p(X)$ is equal to unity:

$$\sum_{X \subseteq \mathfrak{X}} p(X) = \sum_{X \subseteq \mathfrak{X}} \left(\sum_{n=0}^{|X|} (-1)^{|X|-n} \sum_{Y_n \subseteq X} \big(1 - f(Y_n^c)\big) \right) = 1. \tag{21}$$

Indeed, for a finite random set defined on the support of the unit capacity $\mathfrak{X} = \{x\}$ the formula (21) takes the form

$$\sum_{X\subseteq\{x\}} p(X) = p(\emptyset) + p(x) =$$

$$= (-1)^{0-0}(1 - f(x)) + \left((-1)^{1-0}(1 - f(x)) + (-1)^{1-1}(1 - f(\emptyset))\right) =$$

$$= \left(C_1^0(-1)^{0-0} + C_1^1(-1)^{1-0}\right)(1 - f(x)) + C_0^0(-1)^{1-1}(1 - f(\emptyset)) = 1.$$

All summands are mutually removed, except the last, which gives as a result unit.

For $\mathfrak{X} = \{x, y\}$ the formula (21) takes the form

$$\sum_{X\subseteq\mathfrak{X}} p(X) = p(\emptyset) + p(x) + p(y) + p(xy) = \sum_{m=0}^{2} C_{2-0}^{m-0}(-1)^{m-0}(1 - f(xy)) +$$

$$+ \sum_{m=1}^{2} C_{2-1}^{m-1}(-1)^{m-1} \sum_{Y\in\mathfrak{C}_{\mathfrak{X}}^1} (1 - f(Y^c)) + C_{2-2}^{2-2}(-1)^{2-2}(1 - f(\emptyset)) =$$

$$= \sum_{n=0}^{2}\left(\sum_{m=n}^{2} C_{2-n}^{m-n}(-1)^{m-n} \sum_{Y\in\mathfrak{C}_{\mathfrak{X}}^n}\left(1 - f(Y^c)\right)\right) = 1.$$

Similarly, for any finite set $\mathfrak{X}$ the formula (21) takes the form

$$\sum_{X\subseteq\mathfrak{X}} p(X) = \sum_{X\subseteq\mathfrak{X}}\left(\sum_{n=0}^{|X|}(-1)^{|X|-n}\sum_{Y_n\subseteq X}\left(1 - f(Y_n^c)\right)\right) =$$

$$= \sum_{n=0}^{|\mathfrak{X}|}\left(\sum_{m=n}^{|\mathfrak{X}|} C_{|\mathfrak{X}|-n}^{m-n}(-1)^{m-n}\sum_{Y\in\mathfrak{C}_{\mathfrak{X}}^n}\left(1 - f(Y^c)\right)\right) = C_0^0(-1)^{|\mathfrak{X}|-|\mathfrak{X}|}(1 - f(\emptyset)) = 1,$$

where

$\mathfrak{C}_{\mathfrak{X}}^n = \{Y \subseteq \mathfrak{X} : |Y| = n\}$ – n is the layer of the set $\mathfrak{X}$;

$$\sum_{n=0}^{|\mathfrak{X}|}\sum_{m=n}^{|\mathfrak{X}|} C_{|\mathfrak{X}|-n}^{m-n}(-1)^{m-n} = \begin{cases} 1, & |\mathfrak{X}| = n; \\ 0, & \text{otherwise.} \end{cases}$$

According to Lemma 4 the set function $p(X)$ determines the probability distribution of the I-st type of a finite random set $K_{\mathfrak{X}}$. The probability distributions of the I-st and the V-th types of the finite random set $K_{\mathfrak{X}}$ are the set functions. These functions are linked each other by the mutually inversion Möbius formulas (8) and (9). Thus if the set function $p(X)$ determines the probability distribution of the I-st type, then $f(X)$ is a probability distribution of the V-th type. The theorem is proved. □

Conclusion

Sufficient conditions are formulated and proved in this paper. These conditions allow to consider the set function as the probability distribution of the finite random set of the II-nd and the V-th type. The found conditions supplement the known necessary conditions for the existence of probability distributions of the finite random set of the II-nd and the V-th type. These results extend the area of application of the random-set modeling of socio-economic, environmental and medical systems.

References

[1] I.Molchanov, The Theory of Random Sets, Springer, New York, 2011.

[2] H.T.Nguyen, An Introduction to Random Sets, Taylor & Francis Group, LLC, 2006.

[3] O.Yu.Vorobyev, Eventology, Siberian Federal University, Krasnoyarsk, 2007 (in Russian).

[4] A.I.Orlov, Stability in the socio-economic models, Moscow, Nauka, (1979) (in Russian).

[5] N.A.Lukyanova, D.V.Semenova, Associative Frank functions in constructing families of discrete probability distributions of random sets of events, *Zh. prikladnoi diskretnoi matematiki*, **32**(2016), no. 2, 5–19 (in Russian).

[6] D.V.Semenova, N.A.Lukyanova, Recurrent formation of discrete probabilistic distributions of random sets of events, *Zh. prikladnoi diskretnoi matematiki*, **26**(2014), no. 4, 47–58 (in Russian).

[7] N.A.Lukyanova, D.V.Semenova, The study of discrete probabilistic distributions of random sets of events using associative function, *J. of Siberian Federal University. Math. & Phys.*, **7**(2014), no. 4, 500–514.

[8] D.V.Semenova, N.A.Lukyanova, Formation of Probabilistic Distributions of RSE by Associative Functions, *Communications in Computer and Information Science*, Springer International Publishing, Switzerland, **487**(2014), 377–386.

[9] O.Yu.Vorobyev, A.O.Vorobyev, Summation of the set-additive functions and Mobius inversion formula. *Doklady Ross. Akad. Nauk*, **336**(1994), no. 4, 417–420 (in Russian).

[10] A.N.Shiryaev, Probability-1, Moscow, 2004 (in Russian).

Функции множества и распределения вероятностей конечных случайных множеств

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В работе исследуются распределения вероятностей конечного случайного множества, носителем которых выступает множество случайных событий. Данные распределения вероятностей можно задать шестью эквивалентными способами (распределениями I–VI рода). Каждый из этих типов распределений вероятностей является функцией множества, заданной на соответствующей системе событий. В работе сформулированы и доказаны достаточные условия, при выполнении которых функция множества определяет распределение вероятностей конечного случайного множества II и V рода. Найденные условия дополняют известные необходимые условия существования распределения вероятностей конечного случайного множества II и V рода.

Ключевые слова: конечное случайное множество, функция множества, распределение вероятностей.