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Extremal attractors of Liouville copulas

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Abstract

Liouville copulas, which were introduced in [27], are asymmetric generalizations of the ubiquitous Archimedean copula class. They are the dependence structures of scale mixtures of Dirichlet distributions, also called Liouville distributions. In this paper, the limiting extreme-value copulas of Liouville copulas and of their survival counterparts are derived. The limiting max-stable models, termed here the scaled extremal Dirichlet, are new and encompass several existing classes of multivariate max-stable distributions, including the logistic, negative logistic and extremal Dirichlet. As shown herein, the stable tail dependence function and angular density of the scaled extremal Dirichlet model have a tractable form, which in turn leads to a simple de Haan representation. The latter is used to design efficient algorithms for unconditional simulation based on the work of [9] and to derive tractable formulas for maximum-likelihood inference. The scaled extremal Dirichlet model is illustrated on river flow data of the river Isar in southern Germany.

1. Introduction

Copula models play an important role in the analysis of multivariate data and find applications in many areas, including biostatistics, environmental sciences, finance, insurance, and risk management. The popularity of copulas is rooted in the decomposition of Sklar [34], which is at the heart of flexible statistical models and various measures, concepts and orderings of dependence between random variables. According to Sklar's result, the distribution function of any random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ with continuous margins $F_1, \ldots, F_d$ satisfies, for any $x_1, \ldots, x_d \in \mathbb{R}$,

$$\Pr(X_1 \leq x_1, \ldots, X_d \leq x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\},$$

for a unique copula C, i.e., a distribution function on $[0, 1]^d$ whose margins are standard uniform. Alternatively, Sklar's decomposition also holds for survival functions, i.e., for any $x_1, \ldots, x_d \in \mathbb{R}$,

$$\Pr(X_1 > x_1, \ldots, X_d > x_d) = \bar{C}\{\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)\},$$

where $\bar{F}_1, \ldots, \bar{F}_d$ are the marginal survival functions and $\bar{C}$ is the survival copula of $\boldsymbol{X}$, related to the copula of $\boldsymbol{X}$ as follows. If $\boldsymbol{U}$ is a random vector distributed as the copula C of $\boldsymbol{X}$, $\bar{C}$ is the distribution function of $1 - \boldsymbol{U}$.

In risk management applications, the extremal behavior of copulas is of particular interest, as it describes the dependence between extreme events and consequently the value of risk measures at high levels. The purpose of this article is to study the extremal behavior of Liouville copulas. The latter are defined as the survival copulas of Liouville distributions [13, 16, 33], i.e., distributions of random vectors of the form $R\boldsymbol{D}_{\boldsymbol{\alpha}}$, where R is a strictly positive random variable independent of the Dirichlet random vector $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$ with parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. Liouville copulas were proposed by McNeil and Nešlehová [27] in order to extend the widely used class of Archimedean copulas and create dependence structures that are not necessarily exchangeable. The latter property means that for any $u_1, \ldots, u_d \in [0, 1]$ and any permutation π of the integers $1, \ldots, d$, $C(u_1, \ldots, u_d) = C(u_{\pi(1)}, \ldots, u_{\pi(d)})$. When $\boldsymbol{\alpha} = \boldsymbol{1}_d \equiv (1, \ldots, 1)$, $\boldsymbol{D}_{\boldsymbol{\alpha}} = \boldsymbol{D}_{\boldsymbol{1}_d}$ is uniformly distributed on the unit simplex

$$\mathbb{S}_d = \{\boldsymbol{x} \in [0, 1]^d : x_1 + \cdots + x_d = 1\}. \tag{1}$$

In this special case, one recovers the Archimedean copulas. Indeed, according to [26], the latter are the survival copulas of random vectors $R\boldsymbol{D}_{\boldsymbol{1}_d}$, where R is a strictly positive random variable independent of $\boldsymbol{D}_{\boldsymbol{1}_d}$. When $\boldsymbol{\alpha} \neq \boldsymbol{1}_d$, the survival copula of $R\boldsymbol{D}_{\boldsymbol{\alpha}}$ is not Archimedean anymore. It is also no longer exchangeable, unless $\alpha_1 = \cdots = \alpha_d$.

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In this article, we determine the extremal attractor of a Liouville copula and of its survival counterpart. In particular, it will be seen that non-exchangeability of Liouville copulas carries over to their extremal limits. As a by-product, we also obtain the lower and upper tail dependence coefficients of Liouville copulas that quantify the strength of dependence at extreme levels [21]. These results are complementary to [18], where the upper tail order functions of a Liouville copula and its density are derived when $\alpha_1 = \cdots = \alpha_d$.

The extremal attractors of Liouville copulas are interesting in their own right and could be used to model the dependence between extreme risks in the presence of causality relationships [14]. These limiting extreme-value models can be embedded in a single family, termed here the scaled extremal Dirichlet, whose members are new, non-exchangeable generalizations of the logistic, negative logistic, and Coles–Tawn extremal Dirichlet models given in [6]. We examine the scaled extremal Dirichlet model in detail and derive its de Haan spectral representation. The latter is simple and leads to feasible stochastic simulation algorithms and tractable formulas for likelihood-based inference.

The article is organized as follows. The extremal behavior of the margins of Liouville distributions is first studied in Section 2. The extremal attractors of Liouville copulas and their survival counterparts are then derived in Section 3. When $\boldsymbol{\alpha}$ is integer-valued, the results of [23, 27] lead to closed-form expressions for the limiting stable tail dependence functions, as shown in Section 4. Section 5 is devoted to a detailed study of the positive and negative scaled extremal Dirichlet models. In Section 6, their de Haan representation is derived and used for stochastic simulation. Estimation is investigated in Section 7, where expressions for the censored likelihood and the gradient score are also given. An illustrative data analysis of river flow of the river Isar is presented in Section 8, and the paper is concluded by a discussion in Section 9. Lengthy proofs are relegated to the Appendices.

In what follows, vectors in $\mathbb{R}^d$ are denoted by boldface letters, $\boldsymbol{x} = (x_1, \ldots, x_d)$; $\mathbf{0}_d$ and $\mathbf{1}_d$ refer to the vectors $(0, \ldots 0)$ and $(1, \ldots, 1)$ in $\mathbb{R}^d$, respectively. Binary operations such as $\boldsymbol{x} + \boldsymbol{y}$ or $a \cdot \boldsymbol{x}$, $\boldsymbol{x}^a$ are understood as componentwise operations. $\|\cdot\|$ stands for the ℓ_1-norm, viz. $\|\boldsymbol{x}\| = |x_1| + \cdots + |x_d|$, $\perp\!\!\!\perp$ for statistical independence. For any $x, y \in \mathbb{R}$, let $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. The Dirac delta function I_{ij} is 1 if $i = j$ and zero otherwise. Finally, $\mathbb{R}^d_+$ is the positive orthant $[0, \infty)^d$ and for any $x \in \mathbb{R}$, x_+ denotes the positive part of x, $\max(0, x)$.

2. Marginal extremal behavior

A Liouville random vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ is a scale mixture of a Dirichlet random vector $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$ with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) > \mathbf{0}_d$. In what follows, R is referred to as the radial variable of $\boldsymbol{X}$ and $\bar{\alpha}$ denotes the sum of the Dirichlet parameters, viz. $\bar{\alpha} = \|\boldsymbol{\alpha}\| = \alpha_1 + \cdots + \alpha_d$. Recall that $\boldsymbol{D}_{\boldsymbol{\alpha}}$ has the same distribution as $\boldsymbol{Z}/\|\boldsymbol{Z}\|$, where $Z_i \sim \mathsf{Ga}(\alpha_i, 1)$, $i = 1, \ldots, d$ are independent Gamma variables with scaling parameter 1. The univariate margins of $\boldsymbol{X}$ are thus scale mixtures of Beta distributions, i.e., for $i = 1, \ldots, d$, $X_i = RD_i$ with $D_i \sim \mathsf{Beta}(\alpha_i, \bar{\alpha} - \alpha_i)$.

As a first step towards the extremal behavior of Liouville copulas, this section is devoted to the extreme-value properties of the margins of the vectors $\boldsymbol{X}$ and $1/\boldsymbol{X}$, where $\boldsymbol{X}$ is a Liouville random vector with parameters $\boldsymbol{\alpha}$ and a strictly positive radial part R, i.e., such that $\Pr(R \leq 0) = 0$. To this end, recall that a univariate random variable X with distribution function F is in the maximum domain of attraction of a non-degenerate distribution F_0, denoted $F \in \mathcal{M}(F_0)$ or $X \in \mathcal{M}(F_0)$, iff there exist sequences of reals (a_n) and (b_n) with $a_n > 0$, such that, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(a_n x + b_n) = F_0(x).$$

By the Fisher–Tippett Theorem, F_0 must be, up to location and scale, either the Fréchet (Φ_ρ), the Gumbel (Λ) or the Weibull distribution (Ψ_ρ) with parameter $\rho > 0$. Further recall that a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called regularly varying with index $\rho \in (-\infty, \infty)$, denoted $f \in \mathcal{R}_\rho$, if for any $x > 0$, $f(tx)/f(t) \to x^\rho$ as $t \to \infty$. If $\rho = 0$, f is called slowly varying. For more details and conditions for $F \in \mathcal{M}(F_0)$, see, e.g., [11, 30].

Because the margins of $\boldsymbol{X}$ are scale mixtures of Beta distributions, the maximum domain of attraction of the margins of a Liouville vector $\boldsymbol{X}$ can be established using the results of [17]. The following proposition follows directly from Theorems 4.1, 4.4. and 4.5 in the latter paper.

PROPOSITION 1. *Let $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ be a Liouville random vector with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and a strictly positive radial variable R, i.e.,* $\Pr(R \leq 0) = 0$. *Then the following statements hold for any $\rho > 0$:*

(a) *$R \in \mathcal{M}(\Phi_\rho)$ if and only if $X_i \in \mathcal{M}(\Phi_\rho)$ for all $i = 1, \ldots, d$.*

(b) *$R \in \mathcal{M}(\Lambda)$ if and only if $X_i \in \mathcal{M}(\Lambda)$ for all $i = 1, \ldots, d$.*

(c) *$R \in \mathcal{M}(\Psi_\rho)$ if and only if $X_i \in \mathcal{M}(\Psi_{\rho+\bar{\alpha}-\alpha_i})$ for all $i = 1, \ldots, d$.*

Proposition 1 implies that the margins of $\boldsymbol{X}$ are all in the domain of attraction of the same distribution if the latter is Gumbel or Fréchet. This is not the case when the margins are in the Weibull domain of attraction. Note also that there are cases not covered by the above proposition, in which the margins X_i are in the Weibull domain while R is not in the domain of attraction of any extreme-value distribution. For example, when $d = 2$, $\boldsymbol{\alpha} = (1, 1)$ and $R = 1$ almost surely, the margins of $\boldsymbol{X}$ are standard uniform and hence in the maximum domain of attraction of Ψ_1; see Example 3.3.15 in [11]. At the same time, R is clearly neither in the Weibull, nor the Gumbel, nor the Fréchet domain of attraction.

In subsequent sections, we shall also need the extremal behavior of the margins of $1/\boldsymbol{X}$. The proposition below shows that the latter is determined by the properties of $1/R$. In contrast to Proposition 1, however, the margins of $1/\boldsymbol{X}$ are always in the Fréchet domain. The proof may be found in A.

PROPOSITION 2. *Let $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ be a Liouville random vector with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and a strictly positive radial variable R with $\Pr(R \le 0) = 0$. The following statements hold for any $i = 1, \ldots, d$.*

(a) *If $1/R \in \mathcal{M}(\Phi_\rho)$ for $\rho \in (0, \alpha_i]$, then $1/X_i \in \mathcal{M}(\Phi_\rho)$.*

(b) *If $\mathrm{E}(1/R^{\alpha_i+\varepsilon}) < \infty$ for some $\varepsilon > 0$, then $1/X_i \in \mathcal{M}(\Phi_{\alpha_i})$.*

3. Extremal behavior of Liouville copulas

In this section, we will identify the extremal behavior of a Liouville random vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ and of the random vector $1/\boldsymbol{X}$, assuming that $\Pr(R \le 0) = 0$. As a byproduct, we will obtain the extremal attractors of Liouville copulas and their survival counterparts. To this end, recall that a random vector $\boldsymbol{Y}$ with joint distribution function H is in the maximum domain of attraction of a non-degenerate distribution function H_0, in notation $H \in \mathcal{M}(H_0)$ or $\boldsymbol{Y} \in \mathcal{M}(H_0)$, iff there exist sequences of vectors $(\boldsymbol{a}_n)$ in $(0, \infty)^d$ and $(\boldsymbol{b}_n)$ in $\mathbb{R}^d$ such that for all $\boldsymbol{x} \in \mathbb{R}^d$,

$$\lim_{n\to\infty} H^n(\boldsymbol{a}_n\boldsymbol{x} + \boldsymbol{b}_n) = H_0(\boldsymbol{x}).$$

When the margins of H are continuous, $H \in \mathcal{M}(H_0)$ holds if and only if the margins F_i of H are in the maximum domain of attraction of the margins F_{i0} of H_0, i.e., $F_i \in \mathcal{M}(F_{i0})$ for all $i = 1, \ldots, d$, and further if the unique copula C of H is in the domain of attraction of the unique copula C_0 of H_0, denoted $C \in \mathcal{M}(C_0)$, i.e., iff for all $\boldsymbol{u} \in [0, 1]^d$,

$$\lim_{n\to\infty} C^n(\boldsymbol{u}^{1/n}) = C_0(\boldsymbol{u}).$$

In particular, the margins of the max-stable distribution H_0 must each follow a generalized extreme-value distribution, and C_0 must be an extreme-value copula. This means that for all $\boldsymbol{u} \in [0, 1]^d$,

$$C_0(\boldsymbol{u}) = \exp[-\ell\{-\log(u_1), \ldots, -\log(u_d)\}], \tag{2}$$

where $\ell : \mathbb{R}_+^d \to [0, \infty)$ is a stable tail dependence function, linked to the so-called exponent measure ν viz. $\nu\{[\boldsymbol{0}_d, \boldsymbol{x})^{\mathsf{c}}\} = \ell(1/\boldsymbol{x})$, see, e.g., [30]. The latter can be characterized through an angular (or spectral) probability measure σ_d on $\mathbb{S}_d$ given in Equation (1) which satisfies $\int_{\mathbb{S}_d} w_i \, \mathrm{d}\sigma_d(\boldsymbol{w}) = 1/d$ for all $i = 1, \ldots, d$. For all $\boldsymbol{x} \in \mathbb{R}_+^d$, one has

$$\ell(\boldsymbol{x}) = d \int_{\mathbb{S}_d} \max(w_1x_1, \ldots, w_dx_d) \, \mathrm{d}\sigma_d(\boldsymbol{w}). \tag{3}$$

Because ℓ is homogeneous of order 1, i.e., for any $c > 0$ and $\boldsymbol{x} \in \mathbb{R}_+^d$, $\ell(c\boldsymbol{x}) = c\ell(\boldsymbol{x})$, C_0 can also be expressed via the Pickands dependence function $\mathrm{A} : \mathbb{S}_d \to [0, \infty)$ related to ℓ through $\ell(\boldsymbol{x}) = \|\boldsymbol{x}\|\mathrm{A}(\boldsymbol{x}/\|\boldsymbol{x}\|)$. Then at any $\boldsymbol{u} \in [0, 1]^d$,

$$C_0(\boldsymbol{u}) = \exp\left[\log(u_1 \cdots u_d)\mathrm{A}\left\{\frac{\log(u_1)}{\log(u_1 \cdots u_d)}, \ldots, \frac{\log(u_d)}{\log(u_1 \cdots u_d)}\right\}\right].$$

When $d = 2$, it is more common to define the Pickands dependence function $\mathrm{A} : [0, 1] \to [0, 1]$ through $\ell(x_1, x_2) = (x_1 + x_2)\mathrm{A}\{x_2/(x_1 + x_2)\}$ so that, for all $u_1, u_2 \in [0, 1]$,

$$C_0(u_1, u_2) = \exp\left[\log(u_1u_2)\mathrm{A}\left\{\frac{\log(u_2)}{\log(u_1u_2)}\right\}\right]. \tag{4}$$

Now consider a Liouville vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ with a strictly positive radial variable. Theorem 1 below specifies when $\boldsymbol{X} \in \mathcal{M}(H_0)$ and identifies H_0. The proof may be found in B.

THEOREM 1. *Let $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$, $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$, $\boldsymbol{\alpha} > \mathbf{0}_d$, and $\Pr(R \leq 0) = 0$. Then the following statements hold.*

(a) *If $R \in \mathcal{M}(\Phi_\rho)$ for some $\rho > 0$, then $\boldsymbol{X} \in \mathcal{M}(H_0)$, where H_0 is a multivariate extreme-value distribution with Φ_ρ margins and a stable tail dependence function given, for all $\boldsymbol{x} \in \mathbb{R}^d_+$, by*

$$\ell(\boldsymbol{x}) = \frac{\Gamma(\bar{\alpha}+\rho)}{\Gamma(\bar{\alpha})}\mathrm{E}\left[\max\left\{\frac{\Gamma(\alpha_1)x_1D_1^\rho}{\Gamma(\alpha_1+\rho)}, \ldots, \frac{\Gamma(\alpha_d)x_dD_d^\rho}{\Gamma(\alpha_d+\rho)}\right\}\right].$$

(b) *If $R \in \mathcal{M}(\Lambda)$, then $\boldsymbol{X} \in \mathcal{M}(H_0)$, where for all $\boldsymbol{x} \in \mathbb{R}^d$, $H_0(\boldsymbol{x}) = \prod_{i=1}^d \Lambda(x_i)$.*

(c) *If $R \in \mathcal{M}(\Psi_\rho)$ for some $\rho > 0$, then $\boldsymbol{X} \in \mathcal{M}(H_0)$, where for all $\boldsymbol{x} \in \mathbb{R}^d$, $H_0(\boldsymbol{x}) = \prod_{i=1}^d \Psi_{\rho+\bar{\alpha}-\alpha_i}(x_i)$.*

The next result, also proved in B, specifies the conditions under which $1/\boldsymbol{X} \in \mathcal{M}(H_0)$ and gives the form of the limiting extreme-value distribution H_0.

THEOREM 2. *Let $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$, $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$, $\boldsymbol{\alpha} > \mathbf{0}_d$, and assume that $\Pr(R \leq 0) = 0$. Let $\alpha_{\mathrm{M}} = \max(\alpha_1, \ldots, \alpha_d)$. The following cases can be distinguished:*

(a) *If $1/R \in \mathcal{M}(\Phi_\rho)$ for $\rho \in (0, \alpha_{\mathrm{M}}]$, set $\mathbb{I}_1 = \{i : \alpha_i \leq \rho\}$, $\mathbb{I}_2 = \{i : \alpha_i > \rho\}$ and $\bar{\alpha}_2 = \sum_{i\in\mathbb{I}_2}\alpha_i$. Then*

$1/\boldsymbol{X} \in \mathcal{M}(H_0)$, where the margins of H_0 are $H_{i0} = \Phi_{\rho\wedge\alpha_i}$, $i = 1, \ldots, d$, and the stable tail dependence function is given, for all $\boldsymbol{x} \in \mathbb{R}^d_+$, by

$$\ell(\boldsymbol{x}) = \sum_{i\in\mathbb{I}_1} x_i + \frac{\Gamma(\bar{\alpha}-\rho)}{\Gamma(\bar{\alpha})}\mathrm{E}\left[\max_{i\in\mathbb{I}_2}\left\{\frac{\Gamma(\alpha_i)x_iD_i^{-\rho}}{\Gamma(\alpha_i-\rho)}\right\}\right] = \sum_{i\in\mathbb{I}_1} x_i + \frac{\Gamma(\bar{\alpha}_2-\rho)}{\Gamma(\bar{\alpha}_2)}\mathrm{E}\left[\max_{i\in\mathbb{I}_2}\left\{\frac{\Gamma(\alpha_i)x_i\widetilde{D}_i^{-\rho}}{\Gamma(\alpha_i-\rho)}\right\}\right],$$

where $(\widetilde{D}_i, i \in \mathbb{I}_2)$ is a Dirichlet random vector with parameters $(\alpha_i, i \in \mathbb{I}_2)$.

(b) *If $\mathrm{E}\left(1/R^\beta\right) < \infty$ for $\beta > \alpha_{\mathrm{M}}$, then $1/\boldsymbol{X} \in \mathcal{M}(H_0)$, where for all $\boldsymbol{x} \in \mathbb{R}^d$, $H_0(\boldsymbol{x}) = \prod_{i=1}^d \Phi_{\alpha_i}(x_i)$.*

The stable tail dependence functions appearing in Theorems 1 and 2 will be investigated in greater detail in the subsequent sections. Before proceeding, we introduce the following terminology, emphasizing that they can in fact be embedded in one and the same parametric class.

DEFINITION 1. *For any $\alpha > 0$ and $\rho \in (-\alpha, \infty)$, let $c(\alpha, \rho) = \Gamma(\alpha+\rho)/\Gamma(\alpha)$ denote the rising factorial. For $d \geq 2$ and $\alpha_1, \ldots, \alpha_d > 0$ and let $(D_1, \ldots, D_d)$ denote a Dirichlet random vector with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and set $\bar{\alpha} = \alpha_1 + \cdots + \alpha_d$. For any $-\min(\alpha_1, \ldots, \alpha_d) < \rho < \infty$, the* scaled extremal Dirichlet *stable tail dependence function with parameters ρ and $\boldsymbol{\alpha}$ is given, for all $\boldsymbol{x} \in \mathbb{R}^d_+$, by*

$$\ell^{\mathrm{D}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha}) = c(\bar{\alpha}, \rho)\mathrm{E}\left[\max\left\{\frac{x_1D_1^\rho}{c(\alpha_1, \rho)}, \ldots, \frac{x_dD_d^\rho}{c(\alpha_d, \rho)}\right\}\right], \tag{5}$$

when $\rho \neq 0$ and by $\max(x_1, \ldots, x_d)$ when $\rho = 0$. For any $\rho > 0$, the positive scaled extremal Dirichlet *stable tail dependence function ℓ^{pD} with parameters ρ and $\boldsymbol{\alpha}$ is given, for all $\boldsymbol{x} \in \mathbb{R}^d_+$, by $\ell^{\mathrm{pD}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha}) = \ell^{\mathrm{D}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha})$, while for any $0 < \rho < \min(\alpha_1, \ldots, \alpha_d)$, the* negative scaled extremal Dirichlet *stable tail dependence function ℓ^{nD} is given, for all $\boldsymbol{x} \in \mathbb{R}^d_+$, by $\ell^{\mathrm{nD}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha}) = \ell^{\mathrm{D}}(\boldsymbol{x}; -\rho, \boldsymbol{\alpha})$.*

Remark 1. As will be seen in Section 5, distinguishing between the positive and negative scaled extremal Dirichlet models makes the discussion of their properties slightly easier because the sign of ρ impacts the shape of the corresponding angular measure. When $\rho \to 0$, $\ell^{\mathrm{D}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha})$ becomes $\max(x_1, \ldots, x_d)$, the stable tail dependence function corresponding to comonotonicity, while when $\rho \to \infty$, $\ell^{\mathrm{D}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha})$ becomes $x_1 + \cdots + x_d$, the stable tail dependence function corresponding to independence. Note also that $\rho \in (-\infty, \infty)$ can be allowed, with the convention that all variables whose indices i are such that $\rho \leq -\alpha_i$ are independent, i.e., ℓ^{nD} is then of the form given in Theorem 2 (a).

From Theorems 1 and 2, we can now easily deduce the extremal behavior of Liouville copulas and their survival counterparts. To this end, recall that a Liouville copula C is defined as the survival copula of a Liouville random vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ with $\Pr(R \leq 0) = 0$. The following corollary follows directly from Theorem 2 upon noting that C is also the unique copula of $1/\boldsymbol{X}$.

COROLLARY 1. *Let C be the unique survival copula of a Liouville random vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ with $\Pr(R \leq 0) = 0$. Let $\alpha_{\mathrm{M}} = \max(\alpha_1, \ldots, \alpha_d)$. Then the following statements hold.*

(a) *If $1/R \in \mathcal{M}(\Phi_\rho)$ for $\rho \in (0, \alpha_{\mathrm{M}}]$, then $C \in \mathcal{M}(C_0)$, where C_0 is an extreme-value copula of the form (2) whose stable tail dependence function is given, for all $\boldsymbol{x} \in \mathbb{R}_+^d$, by*

$$\ell(\boldsymbol{x}) = \sum_{i \in \mathbb{I}_1} x_i + \ell^{\mathrm{nD}}(\boldsymbol{x}_2; \rho, \boldsymbol{\alpha}_2),$$

where $\mathbb{I}_1 = \{i : \alpha_i \leq \rho\}$, $\mathbb{I}_2 = \{i : \alpha_i > \rho\}$ and $\boldsymbol{x}_2 = (x_i, i \in \mathbb{I}_2)$, $\boldsymbol{\alpha}_2 = (\alpha_i, i \in \mathbb{I}_2)$.

(b) *If $\mathrm{E}\left(1/R^\beta\right) < \infty$ for $\beta > \alpha_{\mathrm{M}}$, then $C \in \mathcal{M}(\Pi)$, where Π is the independence copula given, for all $\boldsymbol{u} \in [0,1]^d$, by $\Pi(\boldsymbol{u}) = u_1 \cdots u_d$.*

Remark 2. When $\alpha_1 = \cdots = \alpha_d \equiv \alpha$ and $1/R \in \mathcal{M}(\Phi_\rho)$ for $\rho \in (0, \alpha)$, the result in Corollary 1 (a) can be derived from formula (5) in Proposition 3 in [18] by relating the tail order function to the stable tail dependence function when the tail order equals 1.

The survival counterpart $\bar{C}$ of a Liouville copula C is given as the distribution function of $1 - \boldsymbol{U}$, where $\boldsymbol{U}$ is a random vector distributed as C. As C is the unique survival copula of $\boldsymbol{X}$, $\bar{C}$ is the unique copula of $\boldsymbol{X}$. The following result thus follows directly from Theorem 1.

COROLLARY 2. *Let $\bar{C}$ be the unique copula of a Liouville random vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ with $\Pr(R \leq 0) = 0$. Then the following statements hold.*

(a) *If $R \in \mathcal{M}(\Phi_\rho)$ for $\rho > 0$, then $\bar{C} \in \mathcal{M}(C_0)$, where C_0 is an extreme-value copula of the form (2) with the positive scaled extremal Dirichlet stable tail dependence function given, for all $\boldsymbol{x} \in \mathbb{R}_+^d$, by $\ell^{\mathrm{pD}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha})$.*

(b) *If $R \in \mathcal{M}(\Lambda)$ or $R \in \mathcal{M}(\Psi_\rho)$ with $\rho > 0$, then $\bar{C} \in \mathcal{M}(\Pi)$, where Π is the independence copula.*

4. The case of integer-valued Dirichlet parameters

When $\boldsymbol{\alpha}$ is integer-valued, Liouville distributions are particularly tractable because their survival function is explicit. In this section, we will use this fact to derive closed-form expressions for the positive and negative scaled extremal Dirichlet stable tail dependence functions. To this end, first recall the notion of the Williamson transform. The latter is related to Weyl's fractional integral transform and was used to characterize d-monotone functions in [38]; it was adapted to non-negative random variables in [26].

DEFINITION 2. *Let X be a non-negative random variable with distribution function F, and let $k \geq 1$ be an arbitrary integer. The Williamson k-transform of X is given, for all $x > 0$, by*

$$\mathcal{W}_k F(x) = \int_x^\infty \left(1 - \frac{x}{r}\right)^{k-1} dF(r) = \mathrm{E}\left(1 - \frac{x}{X}\right)_+^{k-1}.$$

For any $k \geq 1$, the distribution of a positive random variable X is uniquely determined by its Williamson k-transform, the formula for the inverse transform being explicit [26, 38]. If $\psi = \mathcal{W}_k F$, then, for all $x > 0$,

$$F(x) = \mathcal{W}_k^{-1}\psi(x) = 1 - \sum_{j=0}^{k-2} \frac{(-1)^j x^j \psi^{(j)}(x)}{j!} - \frac{(-1)^{k-1} x^{k-1} \psi_+^{(k-1)}(x)}{(k-1)!},$$

where for $j = 1, \ldots, k-2$, $\psi^{(j)}$ is the jth derivative of ψ and $\psi_+^{(k-1)}$ is the right-hand derivative of $\psi^{(k-2)}$. These derivatives exist because a Williamson k-transform ψ is necessarily k-monotone [38]. This means that ψ is differentiable up to order $k-2$ on $(0,\infty)$ with derivatives satisfying $(-1)^j\psi^{(j)} \geq 0$ for $j=0,\ldots,k-2$ and such that $(-1)^{k-2}\psi^{(k-2)}$ is non-increasing and convex on $(0,\infty)$. Moreover, $\psi(x)\to 0$ as $x\to\infty$ and if $F(0)=0$, $\psi(x)\to 1$ and $x\to 0$.

Now let C be a Liouville copula corresponding to a Liouville random vector $\boldsymbol{X}=R\boldsymbol{D}_{\boldsymbol{\alpha}}$ with integer-valued parameters $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)$ and a strictly positive radial part R, i.e., $\Pr(R\leq 0)=0$. Let ψ be the Williamson $\bar{\alpha}$-transform of R and set $\mathbb{I}_{\boldsymbol{\alpha}}=\{0,\ldots,\alpha_1-1\}\times\cdots\times\{0,\ldots,\alpha_d-1\}$. By Theorem 2 in [27], one then has, for all $\boldsymbol{x}\in\mathbb{R}^d_+$,

$$\Pr(\boldsymbol{X}>\boldsymbol{x})=\bar{H}(\boldsymbol{x})=\sum_{(j_1,\ldots,j_d)\in\mathbb{I}_{\boldsymbol{\alpha}}}(-1)^{j_1+\cdots+j_d}\frac{\psi^{(j_1+\cdots+j_d)}(x_1+\cdots+x_d)}{j_1!\cdots j_d!}\prod_{i=1}^d x_i^{j_i}. \tag{6}$$

In particular, the margins of $\boldsymbol{X}$ have survival functions satisfying, for all $x>0$ and $i=1,\ldots,d$,

$$\Pr(X_i>x)=\bar{H}_i(x)=\sum_{j=0}^{\alpha_i-1}\frac{(-1)^j x^j\psi^{(j)}(x)}{j!}=1-\mathcal{W}^{-1}_{\alpha_i}\psi(x). \tag{7}$$

By Sklar's Theorem for survival functions, the Liouville copula C is given, for all $\boldsymbol{u}\in[0,1]^d$, by

$$C(\boldsymbol{u})=\bar{H}\{\bar{H}_1^{-1}(u_1),\ldots,\bar{H}_d^{-1}(u_d)\}.$$

Although this formula is not explicit, it is clear from Equations (6) and (7) that C depends on the distribution of $\boldsymbol{X}$ only through the Williamson $\bar{\alpha}$-transform ψ of R and the Dirichlet parameters $\boldsymbol{\alpha}$. For this reason, we shall denote the Liouville copula in this section by $C_{\psi,\boldsymbol{\alpha}}$ and refer to ψ as its generator, reiterating that ψ must be an $\bar{\alpha}$-monotone function satisfying $\psi(1)=0$ and $\psi(x)\to 0$ as $x\to\infty$. When $\boldsymbol{\alpha}=\boldsymbol{1}_d$, $C_{\psi,\boldsymbol{1}}$ is the Archimedean copula with generator ψ, given, for all $\boldsymbol{u}\in[0,1]^d$ by $C_{\psi,\boldsymbol{1}}(\boldsymbol{u})=\psi\{\psi^{-1}(u_1)+\cdots+\psi^{-1}(u_d)\}$. Because the relationship between ψ and R is one-to-one [26, Proposition 3.1], we will refer to R as the radial distribution corresponding to ψ.

Now suppose that $1/R\in\mathcal{M}(\Phi_\rho)$ with $\rho\in(0,1)$. By Theorem 2 in [23], this condition is equivalent to $1-\psi(1/\cdot)\in\mathcal{R}_{-\rho}$. It further follows from Corollary 1 (a) that $C_{\psi,\boldsymbol{\alpha}}\in\mathcal{M}(C_0)$ where C_0 is an extreme-value copula with the negative scaled extremal Dirichlet stable tail dependence function $\ell^{\mathrm{nD}}(\cdot;\rho,\boldsymbol{\alpha})$. This is because $\rho<1\leq\min(\alpha_1,\ldots,\alpha_d)$ so that $\mathbb{I}_1=\emptyset$ in Corollary 1 (a). Equation (6) and the results of [23] can now be used to derive the following explicit expression for ℓ^{nD}, as detailed in C.

PROPOSITION 3. *Let $C_{\psi,\boldsymbol{\alpha}}$ be a Liouville copula with integer-valued parameters $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)$ and generator ψ. If $1-\psi(1/\cdot)\in\mathcal{R}_{-\rho}$ for some $\rho\in(0,1)$, then $C_{\psi,\boldsymbol{\alpha}}\in\mathcal{M}(C_0)$, where C_0 is an extreme-value copula with scaled negative extremal Dirichlet stable tail dependence function ℓ^{nD} as given in Definition 1. Furthermore, for all $\boldsymbol{x}\in\mathbb{R}^d_+$,*

$$\ell^{\mathrm{nD}}(\boldsymbol{x};\rho,\boldsymbol{\alpha})=\Gamma(1-\rho)\left[\sum_{j=1}^d\left\{\frac{x_j}{c(\alpha_j,-\rho)}\right\}^{1/\rho}\right]^\rho\times$$
$$\left(1-\rho\sum_{\substack{(j_1,\ldots,j_d)\in\mathbb{I}_{\boldsymbol{\alpha}}\\(j_1,\ldots,j_d)\neq(0,\ldots,0)}}\frac{\Gamma(j_1+\cdots+j_d-\rho)}{\Gamma(1-\rho)}\prod_{i=1}^d\frac{1}{\Gamma(j_i+1)}\left[\frac{\left\{\frac{x_i}{c(\alpha_i,-\rho)}\right\}^{1/\rho}}{\sum_{k=1}^d\left\{\frac{x_k}{c(\alpha_k,-\rho)}\right\}^{1/\rho}}\right]^{j_i}\right).$$

When $\boldsymbol{\alpha}=\boldsymbol{1}_d$, the index set $\mathbb{I}_{\boldsymbol{\alpha}}$ reduces to the singleton $\{\boldsymbol{0}\}$, and the expression for ℓ^{nD} given in Proposition 3 simplifies, for all $\boldsymbol{x}\in\mathbb{R}^d_+$, to the stable tail dependence function of the Gumbel–Hougaard copula, viz.

$$\ell^{\mathrm{nD}}(\boldsymbol{x};\rho,\boldsymbol{1}_d)=\left(x_1^{1/\rho}+\cdots+x_d^{1/\rho}\right)^\rho.$$

The Liouville copula $C_{\psi,\boldsymbol{1}_d}$, which is the Archimedean copula with generator ψ, is thus indeed in the domain of attraction of the Gumbel–Hougaard copula with parameter $1/\rho$, as shown, e.g., in [5, 23].

Remark 3. When $\boldsymbol{\alpha}=\boldsymbol{1}_d$ and $1-\psi(1/\cdot)\in\mathcal{R}_{-1}$, it is shown in Proposition 2 of [23] that $C_{\psi,\boldsymbol{1}}$ is in the domain of attraction of the independence copula. However, when $\boldsymbol{\alpha}$ is integer-valued but such that $\max(\alpha_1,\ldots,\alpha_d)>1$, regular variation of $1-\psi(1/\cdot)$ does not suffice to characterize those cases in Corollary 1 that are not covered by Proposition 3. This is because by Theorem 2 of [23], $1/R\in\mathcal{M}(\Phi_\rho)$ for $\rho\geq 1$, $1/R\in\mathcal{M}(\Lambda)$ and $1/R\in\mathcal{M}(\Psi_\rho)$ for $\rho>0$ all imply that $1-\psi(1/\cdot)\in\mathcal{R}_{-1}$. At the same time, by Corollary 1, $C_{\psi,\boldsymbol{\alpha}}\in\mathcal{M}(\Pi)$ clearly does not hold in all these cases.

Next, let $\bar{C}_{\psi,\boldsymbol{\alpha}}$ be the survival copula of a Liouville copula $C_{\psi,\boldsymbol{\alpha}}$, i.e.i.e., the distribution function of $1-\boldsymbol{U}$, where $\boldsymbol{U}$ is a random vector with distribution function $C_{\psi,\boldsymbol{\alpha}}$. The results of [23] can again be used to restate the conditions under which $\bar{C}_{\psi,\boldsymbol{\alpha}} \in \mathcal{M}(C_0)$ in terms of ψ and to give an explicit expression for the stable tail dependence function of C_0.

PROPOSITION 4. *Let $\bar{C}_{\psi,\boldsymbol{\alpha}}$ be the survival copula of a Liouville copula $C_{\psi,\boldsymbol{\alpha}}$ with integer-valued parameters $\boldsymbol{\alpha}$ and a generator ψ. Then the following statements hold.*

(a) *If $\psi \in \mathcal{R}_{-\rho}$ for some $\rho > 0$, then $\bar{C}_{\psi,\boldsymbol{\alpha}} \in \mathcal{M}(C_0)$, where C_0 has a positive scaled extremal Dirichlet stable tail dependence function ℓ^{pD} as given in Definition 1. The latter can be expressed, for all $\boldsymbol{x} \in \mathbb{R}_+^d$, as*

$$\ell^{\mathrm{pD}}(\boldsymbol{x};\rho,\boldsymbol{\alpha}) = \frac{\Gamma(1+\rho)}{\Gamma(\rho)} \sum_{k=1}^{d} \sum_{1\le i_1<\cdots<i_k\le d} (-1)^{k+1} \left[\left\{ \sum_{j=1}^{k} \frac{x_{i_j}}{c(\alpha_{i_j},\rho)} \right\}^{-1/\rho} \right]^{-\rho} \times \sum_{(j_1,\ldots,j_k)\in \mathbb{I}_{(\alpha_{i_1},\ldots,\alpha_{i_k})}} \frac{\Gamma(j_1+\cdots+j_k+\rho)}{j_1!\cdots j_k!} \prod_{m=1}^{k} \left[\frac{\left\{\frac{x_{i_m}}{c(\alpha_{i_m},\rho)}\right\}^{-1/\rho}}{\sum_{j=1}^{k}\left\{\frac{x_{i_j}}{c(\alpha_{i_j},\rho)}\right\}^{-1/\rho}} \right]^{j_m}.$$

(b) *If $\psi \in \mathcal{M}(\Lambda)$ or $\psi \in \mathcal{M}(\Psi_\rho)$ for some $\rho > 0$, $\bar{C}_{\psi,\boldsymbol{\alpha}} \in \mathcal{M}(\Pi)$, where Π is the independence copula.*

When $\boldsymbol{\alpha} = \mathbf{1}_d$, the expression for ℓ^{pD} in part (a) of Proposition 4 simplifies, for all $\boldsymbol{x} \in \mathbb{R}_+^d$, to

$$\ell^{\mathrm{pD}}(\boldsymbol{x};\rho,\mathbf{1}) = \sum_{A\subseteq\{1,\ldots,d\},\ A\neq\emptyset} (-1)^{|A|+1} \left(\sum_{i\in A} x_i^{-1/\rho} \right)^{-\rho},$$

which is the stable tail dependence function of the Galambos copula [20]. When $\psi \in \mathcal{R}_{-\rho}$ for some $\rho > 0$, $\bar{C}_{\psi,\mathbf{1}_d}$ is thus indeed in the domain of attraction of the Galambos copula, as shown, e.g., in [23].

5. Properties of the scaled extremal Dirichlet models

In this section, the scaled extremal Dirichlet model with stable tail dependence function given in Definition 1 is investigated in greater detail. In Section 5.1 we derive formulas for the so-called angular density and relate the positive and negative scaled extremal Dirichlet models to classical classes of stable tail dependence functions. In Section 5.2 we focus on the bivariate case and derive explicit expressions for the stable tail dependence functions and, as a byproduct, obtain formulas for the tail dependence coefficients of Liouville copulas.

5.1. Angular density

The first property worth noting is that the positive and negative scaled extremal Dirichlet models are closed under marginalization. Indeed, letting $x_i \to 0$ for some arbitrary $1 \le i \le d$, we can easily derive from Lemma 2 that for any $\boldsymbol{\alpha} > \mathbf{0}_d$, $\rho > 0$, and any $\boldsymbol{x} \in \mathbb{R}_+^d$, $\ell^{\mathrm{pD}}(\boldsymbol{x};\rho,\boldsymbol{\alpha}) \to \ell^{\mathrm{pD}}(\boldsymbol{x}_{-i};\rho,\boldsymbol{\alpha}_{-i})$ as $x_i \to 0$, where for any $\boldsymbol{y} \in \mathbb{R}^d$, $\boldsymbol{y}_{-i}$ denotes the vector $(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_d)$. Similarly, for any $1 \le i \le d$, $\boldsymbol{\alpha} > \mathbf{0}_d$, $0 < \rho < \min(\alpha_1,\ldots,\alpha_d)$, and any $\boldsymbol{x} \in \mathbb{R}_+^d$, $\ell^{\mathrm{nD}}(\boldsymbol{x};\rho,\boldsymbol{\alpha}) \to \ell^{\mathrm{nD}}(\boldsymbol{x}_{-i};\rho,\boldsymbol{\alpha}_{-i})$ as $x_i \to 0$.

Since none of the scaled extremal Dirichlet models places mass on the vertices or facets of the simplex $\mathbb{S}_d$ when $\rho \neq 0$, the density of the angular measure σ_d completely characterizes the stable tail dependence function and hence also the associated extreme-value copula. This so-called angular density of the scaled extremal Dirichlet models is given below and derived in D.

PROPOSITION 5. *Let $d \ge 2$ and set $\alpha_1,\ldots,\alpha_d > 0$ and $\bar{\alpha} = \alpha_1 + \cdots + \alpha_d$. For any $\rho > -\min(\alpha_1,\ldots,\alpha_d)$, let also $\boldsymbol{c}(\boldsymbol{\alpha},\rho) = (c(\alpha_1,\rho),\ldots,c(\alpha_d,\rho))$, where $c(\alpha,\rho)$ is as in Definition 1. Then for any $-\min(\alpha_1,\ldots,\alpha_d) < \rho < \infty$, $\rho \neq 0$, the angular density of the scaled extremal Dirichlet model with parameters $\rho > 0$ and $\boldsymbol{\alpha}$ is given, for all $\boldsymbol{w} \in \mathbb{S}_d$, by*

$$h^{\mathrm{D}}(\boldsymbol{w};\rho,\boldsymbol{\alpha}) = \frac{\Gamma(\bar{\alpha}+\rho)}{d|\rho|^{d-1}\prod_{i=1}^{d}\Gamma(\alpha_i)} \left[\sum_{j=1}^{d} \{c(\boldsymbol{\alpha}_j,\rho)w_j\}^{1/\rho} \right]^{-\rho-\bar{\alpha}} \prod_{i=1}^{d} \{c(\alpha_i,\rho)\}^{\alpha_i/\rho} w_i^{\alpha_i/\rho-1}.$$

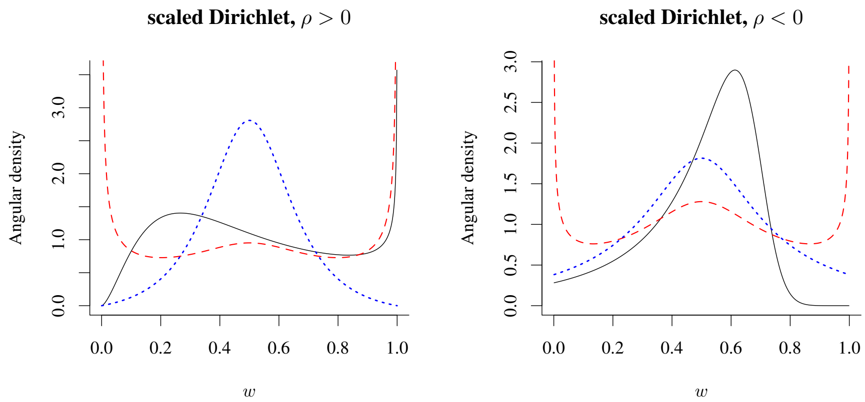

Figure 1: Angular density of the scaled extremal Dirichlet model. Left panel: $\rho = 4/5$ and $\boldsymbol{\alpha} = (2, 1/2)$ (black full), $\rho = 1/4$ and $\boldsymbol{\alpha} = (1/10, 1/10)$ (red dashed), $\rho = 1/4$ and $\boldsymbol{\alpha} = (1/2, 1/2)$ (blue dotted). Right panel: $\rho = -1/4$ and $\boldsymbol{\alpha} = (2, 1/2)$ (black full), $\boldsymbol{\alpha} = (2/5, 2/5)$ (red dashed) and $\boldsymbol{\alpha} = (1/2, 1/2)$ (blue dotted).

The angular density of the positive scaled extremal Dirichlet model with parameters $\rho > 0$ and $\boldsymbol{\alpha}$ is given, for all $\boldsymbol{w} \in \mathbb{S}_d$, by $h^{\mathrm{pD}}(\boldsymbol{w}; \rho, \boldsymbol{\alpha}) = h^{\mathrm{D}}(\boldsymbol{w}; \rho, \boldsymbol{\alpha})$, while the angular density of the negative scaled extremal Dirichlet model with parameters $0 < \rho < \min(\alpha_1, \ldots, \alpha_d)$ and $\boldsymbol{\alpha}$ is given, for all $\boldsymbol{w} \in \mathbb{R}_+^d$, by $h^{\mathrm{nD}}(\boldsymbol{w}; \rho, \boldsymbol{\alpha}) = h^{\mathrm{D}}(\boldsymbol{w}; -\rho, \boldsymbol{\alpha})$.

From Proposition 5, it is easily seen that when $\boldsymbol{\alpha} = \mathbf{1}_d$, the angular density h^{pD} reduces, for any $\rho > 0$ and $\boldsymbol{w} \in \mathbb{S}_d$, to the angular density of the symmetric negative logistic model; see, e.g., Section 4.2 in [6]. In general, the angular density h^{pD} is not symmetric unless $\boldsymbol{\alpha} = \alpha\mathbf{1}_d$. The positive scaled Dirichlet model can be viewed as a new asymmetric generalization of the negative logistic model which does not place any mass on the vertices or facets of $\mathbb{S}_d$, unless at independence or comonotonicity, i.e., when $\rho \to \infty$ and $\rho \to 0$, respectively. Furthermore, h^{pD} can also be interpreted as a generalization of the Coles–Tawn extremal Dirichlet model. Indeed, $h^{\mathrm{pD}}(\boldsymbol{x}; 1, \boldsymbol{\alpha})$ is precisely the angular density of the latter model given, e.g., in Equation (3.6) in [6]. Similarly, the negative scaled extremal Dirichlet model is a new asymmetric generalization of Gumbel's logistic model [15]. Indeed, when $\boldsymbol{\alpha} = \mathbf{1}_d$, h^{nD} simplifies to the logistic angular density, given, e.g., on p. 381 in [6].

Figures 1 and 2 illustrate the various shapes of h^{pD} and h^{nD} that obtain through various choices of $\boldsymbol{\alpha}$ and ρ. The asymmetry when $\boldsymbol{\alpha} \neq \alpha\mathbf{1}_d$ is clearly apparent. For the same value of ρ, the shapes of the angular density can be quite different depending on $\boldsymbol{\alpha}$. In view of the aforementioned closure of both the positive and negative scaled extremal Dirichlet models under marginalization, this means that these models are able to capture strong dependence in some pairs of variables (represented by a mode close to $1/2$ of the angular density) and at the same time weak dependence in others pairs (represented by a bathtub shape).

5.2. The bivariate case

When $d = 2$, the stable tail dependence functions of the positive and negative scaled extremal Dirichlet models have a closed-form expression in terms of the incomplete beta function given, for any $t \in (0, 1)$ and $\alpha_1, \alpha_2 > 0$, by

$$\mathrm{B}(t; \alpha_1, \alpha_2) = \int_0^t x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1}\,\mathrm{d}x.$$

When $t=1$, this integral is the beta function, viz. $\mathrm{B}(\alpha_1,\alpha_2)=\Gamma(\alpha_1)\Gamma(\alpha_2)/\Gamma(\alpha_1+\alpha_2)$. A direct calculation yields the corresponding Pickands dependence function, for any $t\in[0,1]$, $\mathrm{A}^{\mathrm{pD}}(t;\rho,\alpha_1,\alpha_2)=\ell^{\mathrm{pD}}(1-t,t;\rho,\alpha_1,\alpha_2)$, i.e.,

$$\mathrm{A}^{\mathrm{pD}}(t;\rho,\alpha_1,\alpha_2)=\frac{(1-t)}{\mathrm{B}(\alpha_2,\alpha_1+\rho)}\mathrm{B}\left[\frac{\{c(\alpha_2,\rho)(1-t)\}^{1/\rho}}{\{c(\alpha_2,\rho)(1-t)\}^{1/\rho}+\{c(\alpha_1,\rho)t\}^{1/\rho}};\alpha_2,\alpha_1+\rho\right]$$
$$+\frac{t}{\mathrm{B}(\alpha_1,\alpha_2+\rho)}\mathrm{B}\left[\frac{\{c(\alpha_1,\rho)t\}^{1/\rho}}{\{c(\alpha_2,\rho)(1-t)\}^{1/\rho}+\{c(\alpha_1,\rho)t\}^{1/\rho}};\alpha_1,\alpha_2+\rho\right].$$

When $\alpha_1=\alpha_2=1$, A^{pD} becomes the Pickands dependence function of the Galambos copula, viz. $\mathrm{A}(t)=1-\left\{t^{-1/\rho}+(1-t)^{-1/\rho}\right\}^{-\rho}$, as expected given that the positive scaled extremal Dirichlet model becomes the symmetric negative logistic model in this case. Similarly, for any $t\in[0,1]$, the Pickands dependence function $\mathrm{A}^{\mathrm{nD}}(t;\rho,\alpha_1,\alpha_2)=$

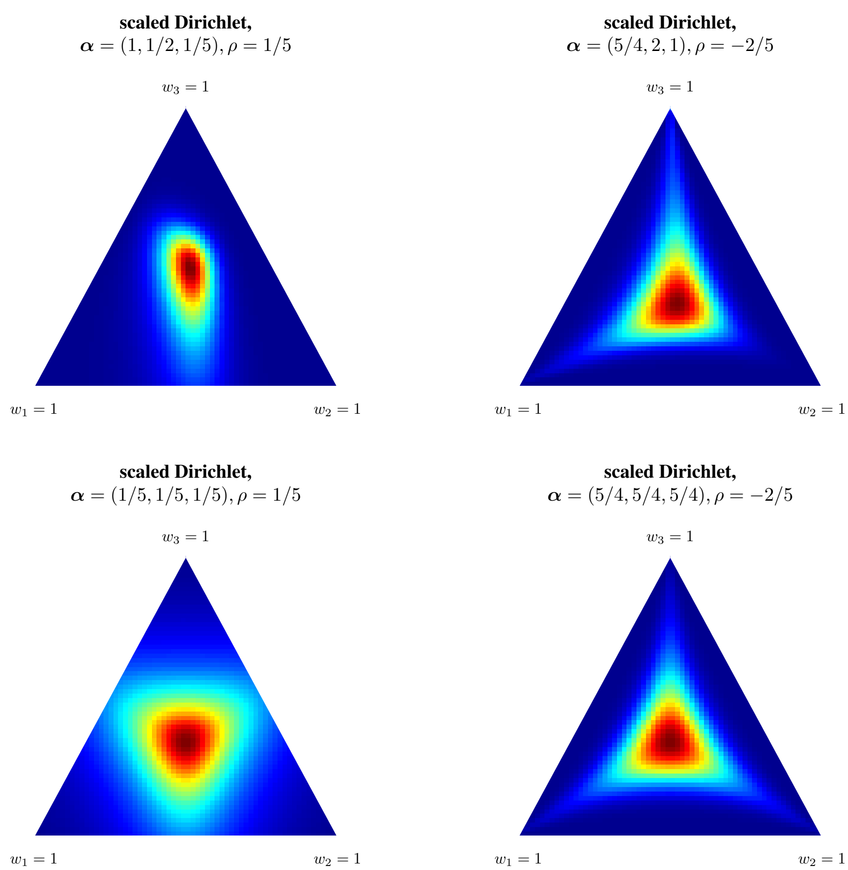

Figure 2: Angular density of the scaled extremal Dirichlet model with $\boldsymbol{\alpha}=(1,1/2,1/5),\rho=1/5$ (top left) and $\boldsymbol{\alpha}=(1/5,1/5,1/5),\rho=1/5$ (bottom left), $\boldsymbol{\alpha}=(5/4,2,1)$, $\rho=-2/5$ (top right) and $\boldsymbol{\alpha}=(5/4,5/4,5/4),\rho=-2/5$ (bottom right). The colors correspond to log density values and range from red (high density) to blue (low density).

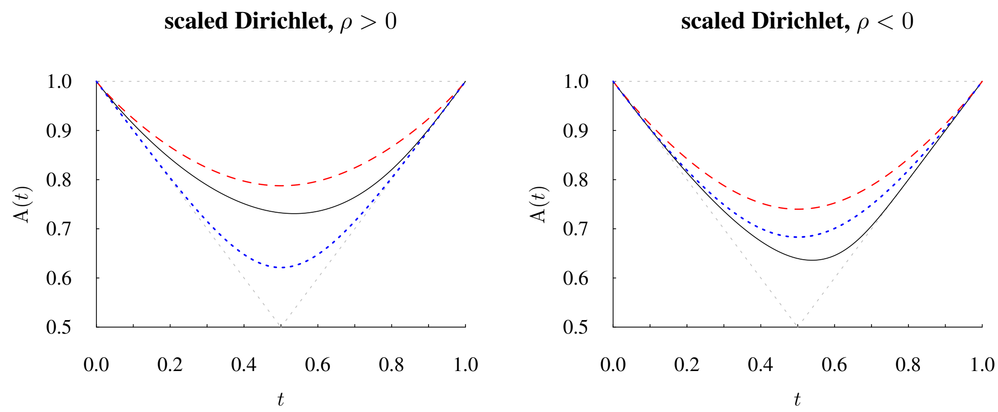

Figure 3: Pickands dependence function of the scaled extremal Dirichlet model.Left panel: $\rho = 4/5$ and $\boldsymbol{\alpha} = (2, 1/2)$ (black full), $\rho = 1/4$ and $\boldsymbol{\alpha} = (1/10, 1/10)$ (red dashed), $\rho = 1/4$ and $\boldsymbol{\alpha} = (1/2, 1/2)$ (blue dotted). Right panel: $\rho = -1/4$ and $\boldsymbol{\alpha} = (2, 1/2)$ (black full), $\boldsymbol{\alpha} = (2/5, 2/5)$ (red dashed) and $\boldsymbol{\alpha} = (1/2, 1/2)$ (blue dotted).

$\ell^{\mathrm{nD}}(1-t, t; \rho, \alpha_1, \alpha_2)$ is

$$\mathrm{A}^{\mathrm{nD}}(t;\rho,\alpha_1,\alpha_2) = \frac{(1-t)}{\mathrm{B}(\alpha_1-\rho,\alpha_2)}\mathrm{B}\left[\frac{\{(1-t)c(\alpha_2,-\rho)\}^{1/\rho}}{\{tc(\alpha_1,-\rho)\}^{1/\rho}+\{(1-t)c(\alpha_2,-\rho)\}^{1/\rho}};\alpha_1-\rho,\alpha_2\right] + \frac{t}{\mathrm{B}(\alpha_2-\rho,\alpha_1)}\mathrm{B}\left[\frac{\{tc(\alpha_1,-\rho)\}^{1/\rho}}{\{tc(\alpha_1,-\rho)\}^{1/\rho}+\{(1-t)c(\alpha_2,-\rho)\}^{1/\rho}};\alpha_2-\rho,\alpha_1\right].$$

When $\alpha_1 = \alpha_2 = 1$, A^{nD} simplifies to the stable tail dependence function of the Gumbel extreme-value copula, viz. $\mathrm{A}(t) = \left\{t^{1/\rho}+(1-t)^{1/\rho}\right\}^{\rho}$. This again confirms that the negative scaled extremal Dirichlet model becomes the symmetric logistic model when $\alpha_1 = \alpha_2 = 1$. The Pickands dependence functions A^{pD} and A^{nD} are illustrated in Figure 3, for the same choices of parameters and the corresponding angular density shown in Figure 1.

The above formulas for A^{pD} and A^{nD} now easily lead to expressions for their upper tail dependence coefficients. Recall that for an arbitrary bivariate copula C, the lower and upper tail dependence coefficients of [21] are given by

$$\lambda_\ell(C) = \lim_{u\to 0}\frac{C(u,u)}{u}, \quad \lambda_u(C) = 2 - \lim_{u\to 1}\frac{C(u,u)-1}{u-1} = \lim_{u\to 0}\frac{\bar{C}(u,u)}{u},$$

where $\bar{C}$ is the survival copula of C, provided these limits exist. When C is bivariate extreme-value with Pickands dependence function A, it follows easily from (4) that $\lambda_\ell(C) = 0$ and $\lambda_u(C) = 2 - 2\mathrm{A}(1/2)$.

Now suppose that $C^{\mathrm{pD}}_{\rho,\boldsymbol{\alpha}}$ is a bivariate extreme-value copula with positive scaled extremal Dirichlet Pickands dependence function A^{pD} and parameters $\rho > 0$ and $\alpha_1, \alpha_2 > 0$. Then

$$\lambda_u(C^{\mathrm{pD}}_{\rho,\boldsymbol{\alpha}}) = 2 - \frac{1}{\mathrm{B}(\alpha_2,\alpha_1+\rho)}\mathrm{B}\left\{\frac{c(\alpha_2,\rho)^{1/\rho}}{c(\alpha_2,\rho)^{1/\rho}+c(\alpha_1,\rho)^{1/\rho}};\alpha_2,\alpha_1+\rho\right\} - \frac{1}{\mathrm{B}(\alpha_1,\alpha_2+\rho)}\mathrm{B}\left\{\frac{c(\alpha_1,\rho)^{1/\rho}}{c(\alpha_2,\rho)^{1/\rho}+c(\alpha_1,\rho)^{1/\rho}};\alpha_1,\alpha_2+\rho\right\}. \quad (8)$$

Similarly, if $C^{\mathrm{nD}}_{\rho,\boldsymbol{\alpha}}$ is a bivariate extreme-value copula with negative scaled extremal Dirichlet Pickands dependence function

A^{nD} and parameters $\alpha_1, \alpha_2 > 0$ and $0 < \rho < \min(\alpha_1, \alpha_2)$, then

$$\lambda_u(C^{\mathrm{nD}}_{\rho,\boldsymbol{\alpha}}) = 2 - \frac{1}{\mathrm{B}(\alpha_1-\rho,\alpha_2)}\mathrm{B}\left\{\frac{c(\alpha_2,-\rho)^{1/\rho}}{c(\alpha_1,-\rho)^{1/\rho}+c(\alpha_2,-\rho)^{1/\rho}};\alpha_1-\rho,\alpha_2\right\} + \frac{1}{\mathrm{B}(\alpha_2-\rho,\alpha_1)}\mathrm{B}\left\{\frac{c(\alpha_1,-\rho)^{1/\rho}}{c(\alpha_1,-\rho)^{1/\rho}+c(\alpha_2,-\rho)^{1/\rho}};\alpha_2-\rho,\alpha_1\right\}. \quad (9)$$

When $\alpha_1 = \alpha_2 \equiv \alpha$, Expressions (8) and (9) simplify to

$$\lambda_u(C^{\mathrm{pD}}_{\rho,\boldsymbol{\alpha}}) = 2 - \frac{2}{\mathrm{B}(\alpha,\alpha+\rho)}\mathrm{B}\left(\frac{1}{2};\alpha,\alpha+\rho\right), \quad \lambda_u(C^{\mathrm{nD}}_{\rho,\boldsymbol{\alpha}}) = 2 - \frac{2}{\mathrm{B}(\alpha-\rho,\alpha)}\mathrm{B}\left(\frac{1}{2};\alpha-\rho,\alpha\right).$$

Formulas (8) and (9) lead directly to expressions for the tail dependence coefficients of Liouville copulas. This is because if $C \in \mathcal{M}(C_0)$, where C_0 is an extreme-value copula with Pickands tail dependence function A_0, $\lambda_u(C) = 2 - 2\mathrm{A}_0(1/2)$ [25, Proposition 7.51]. Similarly, if $\bar{C} \in \mathcal{M}(C_0^*)$, where C_0^* is an extreme-value copula with Pickands tail dependence function A_0^*, $\lambda_\ell(C) = 2 - 2\mathrm{A}_0^*(1/2)$. The following corollary is thus an immediate consequence of Corollaries 1 and 2.

COROLLARY 3. *Suppose that C is the survival copula of a Liouville random vector $R\boldsymbol{D}_{\boldsymbol{\alpha}}$ with parameters $\boldsymbol{\alpha} > 0$ and a radial part R such that $\Pr(R \leq 0) = 0$. Then the following statements hold.*

(a) *If $R \in \mathcal{M}(\Phi_\rho)$ for some $\rho > 0$, $\lambda_\ell(C) = \lambda_u(C^{\mathrm{pD}}_{\rho,\boldsymbol{\alpha}})$ is given by Equation* (8).

(b) *If $R \in \mathcal{M}(\Lambda)$ or $R \in \mathcal{M}(\Psi_\rho)$ for some $\rho > 0$, $\lambda_\ell(C) = 0$.*

(c) *If $1/R \in \mathcal{M}(\Phi_\rho)$ for some $0 < \rho < \alpha_1 \wedge \alpha_2$, $\lambda_u(C) = \lambda_u(C^{\mathrm{nD}}_{\rho,\boldsymbol{\alpha}})$ is given by Equation* (9).

(d) *If $1/R \in \mathcal{M}(\Phi_\rho)$ for $\rho > \alpha_1 \wedge \alpha_2$ or if $\mathrm{E}(1/R^\beta) < \infty$ for $\beta > \alpha_1 \vee \alpha_2$, $\lambda_u(C) = 0$.*

The role of the parameters $\boldsymbol{\alpha}$ and ρ is best explained if we consider the reparametrization $\Delta_\alpha = |\alpha_1 - \alpha_2|$ and $\Sigma_\alpha = \alpha_1 + \alpha_2$. As is the case for the Dirichlet distribution, the level of dependence is higher for large values of Σ_α. Furthermore, λ_u is monotonically decreasing in ρ. Higher levels of extremal asymmetry, as measured by departures from the diagonal on the copula scale, are governed by both Σ_α and Δ_α. The larger Σ_α, the lower the asymmetry. Likewise, the larger Δ_α, the more asymmetry. Contrary to the case of extremal dependence, the behavior in ρ is not monotone. For the negative scaled extremal Dirichlet model, asymmetry is maximal when $\rho \approx \alpha_1 \wedge \alpha_2$. When Σ_α is small, smaller values of ρ induce larger asymmetry, but this is not the case for larger values of Σ_α where the asymmetry profile is convex with a global maximum attained for larger values of ρ.

6. de Haan representation and simulation algorithms

Random samples from the scaled extremal Dirichlet model can be drawn efficiently using the algorithms recently developed in [9]. We first derive the so-called de Haan representation in Section 6.1 and adapt the algorithms from [9] to the present setting in Section 6.2.

6.1. de Haan representation

First, introduce the following family of univariate distributions, which we term the scaled Gamma family and denote $\mathsf{sGa}(a, b, c)$. It has three parameters $a, c > 0$ and $b \neq 0$ and a density given, for all $x > 0$, by

$$f(x; a, b, c) = \frac{|b|}{\Gamma(c)} a^{-bc} x^{bc-1} \exp\left\{-\left(\frac{x}{a}\right)^b\right\}. \quad (10)$$

Observe that when $Z \sim \mathsf{Ga}(c, 1)$ is a Gamma variable with shape parameter $c > 0$ and scaling parameter 1, $Y \stackrel{\mathrm{d}}{=} aZ^{1/b}$ is scaled Gamma $\mathsf{sGa}(a, b, c)$. Consequently, $\mathrm{E}\,(Y) = a\Gamma(c + 1/b)/\Gamma(c) < \infty$ provided that $b < -1/c$. The scaled Gamma family includes several well-known distributions as special cases, notably the Gamma when $b = 1$, the Weibull when $c = 1$ and $b > 0$, the inverse Gamma when $b = -1$, and the Fréchet when $c = 1$ and $b < 0$. When $b > 0$, the scaled Gamma is the generalized Gamma distribution of [36], albeit in a different parametrization.

Now consider the parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $\boldsymbol{\alpha} > \mathbf{0}_d$ and $\rho > -\min(\alpha_1, \ldots, \alpha_d)$, $\rho \neq 0$. Let $\boldsymbol{V}$ be a random vector with independent scaled Gamma margins $V_i \sim \mathsf{sGa}\{1/c(\alpha_i, \rho), 1/\rho, \alpha_i\}$, where for $\alpha > 0$, $c(\alpha, \rho) = \Gamma(\alpha + \rho)/\Gamma(\alpha)$ as in Definition 1. If $\boldsymbol{Z}$ is a random vector with independent Gamma margins $Z_i \sim \mathsf{Ga}(\alpha_i, 1)$ then for all $i = 1, \ldots, d$, $V_i \stackrel{\mathrm{d}}{=} Z_i^{\rho}/c(\alpha_i, \rho)$. Furthermore, recall that $\|\boldsymbol{Z}\| \sim \mathsf{Ga}(\bar{\alpha}, 1)$ is independent of $\boldsymbol{Z}/\|\boldsymbol{Z}\|$, which has the same distribution as the Dirichlet vector $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$. One thus has, for all $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$\mathrm{E}\left\{\max_{1\leq i\leq d}(x_i V_i)\right\} = \mathrm{E}\left[\max_{1\leq i\leq d}\left\{\frac{x_i Z_i^{\rho}}{c(\alpha_i, \rho)}\right\}\right] = \mathrm{E}\left(\|\boldsymbol{Z}\|^{\rho}\right)\mathrm{E}\left[\max_{1\leq i\leq d}\left\{\frac{x_i D_i^{\rho}}{c(\alpha_i, \rho)}\right\}\right] = \ell^{\mathrm{D}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha}), \tag{11}$$

where ℓ^{D} is as in Definition 1, given that $\mathrm{E}\left(\|\boldsymbol{Z}\|^{\rho}\right) = c(\bar{\alpha}, \rho)$.

When $\rho = 1$, the positive scaled Dirichlet extremal model becomes the Coles–Tawn Dirichlet extremal model, $V_i \sim \mathsf{Ga}(\alpha_i, 1)$ and Equation (11) reduces to the representation derived in [32]. When $\boldsymbol{\alpha} = \mathbf{1}_d$, ℓ^{D} becomes the stable tail dependence function of the negative logistic model, V_i is Weibull and Equation (11) is the representation in Appendix A.2.4 of [9]. Similarly, when $\rho < 0$ and $\boldsymbol{\alpha} = \mathbf{1}_d$, the negative scaled Dirichlet extremal model becomes the logistic model, V_i is Fréchet and Equation (11) is the representation in Appendix A.2.4 of [9]. The requirement that $\rho > -\min(\alpha_1, \ldots, \alpha_d)$ ensures that the expectation of V_i is finite for all $i \in \{1, \ldots, d\}$.

Equation (11) implies that the max-stable random vector $\boldsymbol{Y}$ with unit Fréchet margins and extreme-value copula with stable tail dependence function $\ell^{\mathrm{D}}(\cdot; \rho, \boldsymbol{\alpha})$ admits the de Haan [8] spectral representation

$$\boldsymbol{Y} \stackrel{\mathrm{d}}{=} \max_{k\in\mathbb{N}} \zeta_k \boldsymbol{V}_k, \tag{12}$$

where $\mathcal{Z} = \{\zeta_k\}_{k=1}^{\infty}$ is a Poisson point process on $(0, \infty)$ with intensity $\zeta^{-2}\,\mathrm{d}\zeta$ and $\boldsymbol{V}_k$ is an i.i.d. sequence of random vectors independent of $\mathcal{Z}$. Furthermore, the margins of $\boldsymbol{V}_k$ are independent and such that $V_{kj} \sim \mathsf{sGa}\{1/c(\alpha_j, \rho), 1/\rho, \alpha_j\}$ for $j = 1, \ldots, d$ with $\mathrm{E}\left(\boldsymbol{V}_k\right) = \mathbf{1}_d$ for all $k \in \mathbb{N}$.

6.2. Unconditional simulation

The de Haan representation (12) offers, among other things, an easy route to unconditional simulation of max-stable random vectors that follow the scaled Dirichlet extremal model, as laid out in [9] in the more general context of max-stable processes. To see how this work applies in the present setting, fix an arbitrary $j_0 \in \{1, \ldots, d\}$ and recall that the j_0th extremal function $\phi_{j_0}^{+}$ is given, almost surely, as $\zeta_k \boldsymbol{V}_k$ such that $Y_{j_0} = \zeta_k V_{kj_0}$. From eq. (12) and Proposition 1 in [9] it then directly follows that $\phi_{j_0}^{+}/Y_{j_0} \stackrel{\mathrm{d}}{=} (W_{j_0 1}/W_{j_0 j_0}, \ldots, W_{j_0 d}/W_{j_0 j_0})$, where $\boldsymbol{W}_{j_0} = (W_{j_0 1}, \ldots, W_{j_0 d})$ is a random vector with density given, for all $\boldsymbol{x} \in \mathbb{R}_+^d$, by

$$\frac{|1/\rho|}{\Gamma(\alpha_{j_0})} c(\alpha_{j_0}, \rho)^{\alpha_{j_0}/\rho} x_{j_0}^{\alpha_{j_0}/\rho} \exp\left[-\{c(\alpha_{j_0}, \rho) x_{j_0}\}^{1/\rho}\right] \times \prod_{j=1, j\neq j_0}^{d} \frac{|1/\rho|}{\Gamma(\alpha_j)} c(\alpha_j, \rho)^{\alpha_j/\rho} x_j^{\alpha_j/\rho - 1} \exp\left[-\{c(\alpha_j, \rho) x_j\}^{1/\rho}\right].$$

This means that the components of $\boldsymbol{W}_{j_0}$ are independent and such that $W_{j_0 j} \sim \mathsf{sGa}\{1/c(\alpha_j, \rho), 1/\rho, \alpha_j\}$ when $j \neq j_0$ and $W_{j_0 j_0} \sim \mathsf{sGa}\{1/c(\alpha_{j_0}, \rho), 1/\rho, \alpha_{j_0} + \rho\}$. In other words, $W_{j_0 j_0} \sim Z_{j_0}^{\rho}/c(\alpha_{j_0}, \rho)$ where $Z_{j_0} \sim \mathsf{Ga}(\alpha_{j_0} + \rho, 1)$, while for all $j \neq j_0$, $W_{j_0 j} \stackrel{\mathrm{d}}{=} Z_j^{\rho}/c(\alpha_j, \rho)$ where $Z_j \sim \mathsf{Ga}(\alpha_j, 1)$.

The exact distribution of $\phi_{j_0}^{+}/Y_{j_0}$ given above now allows for an easy adaptation of the algorithms in [9]. The first algorithm corresponds to Algorithm 1 in the latter paper and is an adaptation of the procedure in [31]. To draw an observation from the extreme-value copula with the scaled Dirichlet stable tail dependence function ℓ^{D} with parameters $\boldsymbol{\alpha} > \mathbf{0}_d$ and $\rho > -\min(\alpha_1, \ldots, \alpha_d)$, $\rho \neq 0$, follow the steps below.

Alternatively, one can also adapt Algorithm 2 in [9]. This leads to Algorithm 2 below.

Note that $\boldsymbol{S}$ obtained in Step 7 of Algorithm 1 has the angular distribution σ_d of ℓ^{D}; see Theorem 1 in [9]. Similar algorithms for drawing samples from the angular distribution of the extremal logistic and Dirichlet models were obtained in [2]. Algorithm 2 requires a lower number of simulations and is more efficient on average, cf. [9]. Both algorithms are easily implemented using the function `rmev` in the `mev` package within the R Project for Statistical Computing [29], which returns samples of max-stable scaled extremal Dirichlet vectors with unit Fréchet margins, i.e., $\boldsymbol{Y}$ in Algorithms 1 and 2.

Algorithm 1 Exact simulations from the extreme-value copula based on spectral densities.

Simulate $E \sim \mathsf{Exp}(1)$.
Set $\boldsymbol{Y} = \mathbf{0}$.
while $1/E > \min(Y_1, \ldots, Y_d)$ **do**
  Simulate J from the uniform distribution on $\{1, \ldots, d\}$.
  Simulate independent $Z_j \sim \mathsf{Ga}(\alpha_j, 1)$ for $j \in \{1, \ldots, d\} \setminus J$ and $Z_J \sim \mathsf{Ga}(\alpha_J + \rho, 1)$.
  Set $W_j \leftarrow Z_j^\rho / c(\alpha_j, \rho)$, $j = 1, \ldots, d$.
  Set $\boldsymbol{S} \leftarrow \boldsymbol{W} / \|\boldsymbol{W}\|$.
  Update $\boldsymbol{Y} \leftarrow \max\{\boldsymbol{Y}, d\boldsymbol{S}/E\}$.
  Simulate $E^* \sim \mathsf{Exp}(1)$ and update $E \leftarrow E + E^*$.
return $\boldsymbol{U} = \exp(-1/\boldsymbol{Y})$.

Algorithm 2 Exact simulations based on sequential sampling of the extremal functions.

Simulate $Z_1 \sim \mathsf{Ga}(\alpha_1 + \rho, 1)$ and $Z_j \sim \mathsf{Ga}(\alpha_j, 1)$, $j = 2, \ldots, d$.
Compute $\boldsymbol{W}$ where $W_j \leftarrow Z_j^\rho / c(\alpha_j, \rho)$, $j = 1, \ldots, d$.
Simulate $E_1 \sim \mathsf{Exp}(1)$.
Set $\boldsymbol{Y} \leftarrow \boldsymbol{W} / (W_1 E_1)$.
for $k = 2, \ldots, d$ **do**
  Simulate $E_k \sim \mathsf{Exp}(1)$.
  while $1/E_k > Y_k$ **do**
    Simulate independent $Z_k \sim \mathsf{Ga}(\alpha_k + \rho, 1)$ and $Z_j \sim \mathsf{Ga}(\alpha_j, 1)$, $j = 1, \ldots, d, j \neq k$.
    Set $\boldsymbol{W} = (W_1, \ldots, W_d)$ where $W_j \leftarrow Z_j^\rho / c(\alpha_j, \rho)$, $j = 1, \ldots, d$.
    if $W_i/(W_k E_k) < Y_i$ for all $i = 1, \ldots, k-1$ **then**
      Update $\boldsymbol{Y} \leftarrow \max\{\boldsymbol{Y}, \boldsymbol{W}/(W_k E_k)\}$.
    Simulate $E^* \sim \mathsf{Exp}(1)$ and update $E_k \leftarrow E_k + E^*$.
return $\boldsymbol{U} = \exp(-1/\boldsymbol{Y})$.

7. Estimation

The scaled extremal Dirichlet model can be used to model dependence between extreme events. To this end, several schemes can be envisaged. For example, one can consider the block-maxima approach, given that max-stable distributions are the most natural for such data. Another option are peaks-over-threshold models. Yet another alternative, used in [12] for the Brown–Resnick model, is to approximate the conditional distribution of a random vector with unit Fréchet margins given that the j_0th component exceeds a large threshold by the distribution of $\phi^+_{j_0}/Y_{j_0}$ discussed in Section 6.2.

Here, we focus on the multivariate tail model of [24]; see also Section 16.4 in [25]. To this end, let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a random sample from some unknown multivariate distribution H with continuous margins which is assumed to be in the maximum domain of attraction of a multivariate extreme-value distribution H_0. To model the tail of H, its margins F_j, $j = 1, \ldots, d$ can first be approximated using the univariate peaks-over-threshold method. For all x above some high threshold u_j, one then has

$$F_j(x) \approx \tilde{F}_j(x; \eta_j, \xi_j) = 1 - \nu_j \left(1 + \xi_j \frac{(x - u_j)}{\eta_j}\right)_+^{-1/\xi_j}, \tag{13}$$

where $\nu_j = 1 - F_j(u_j)$, and $\eta_j > 0$ and ξ_j are the parameters of the generalized Pareto distribution. Furthermore, for $\boldsymbol{w}$ sufficiently close to $\mathbf{1}_d$, the copula of H can be approximated by the extreme-value copula C_0 of H_0, so that, for $\boldsymbol{x} \geq \boldsymbol{u}$, $H(\boldsymbol{x}) \approx \tilde{H}(\boldsymbol{x}) = C_0\{\tilde{F}_1(x_1), \ldots, \tilde{F}_d(x_d)\}$. The parameters of this multivariate tail model, i.e., the parameters $\boldsymbol{\theta}$ of the stable tail dependence function ℓ_0 of C_0 as well as the marginal parameters $\boldsymbol{\nu}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ can be estimated using likelihood methods; this allows, e.g., for Bayesian inference, generalized additive modeling of the parameters and model selection based on likelihood-ratio tests. We refer the reader to [19] for a comprehensive review of likelihood inference methods for extremes.

The multivariate tail model can be fitted in low-dimensions using the censored likelihood

$$L(\boldsymbol{X}; \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\theta}) = \prod_{i=1}^{n} L_i(\boldsymbol{X}_i; \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\theta}),$$

where for $i = 1, \ldots, n$,

$$L_i(\boldsymbol{X}_i; \boldsymbol{u}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\theta}) = \left.\frac{\partial^{m_i} \tilde{H}(y_1, \ldots, y_d)}{\partial y_{j_1} \cdots \partial y_{j_{m_i}}}\right|_{\boldsymbol{y}=\max(\boldsymbol{X}_i, \boldsymbol{u})} = \left.\frac{\partial^{m_i} \exp\{-\ell_0(1/\boldsymbol{y})\}}{\partial y_{j_1} \cdots \partial y_{j_{m_i}}}\right|_{\boldsymbol{y}=t\{\max(\boldsymbol{X}_i, \boldsymbol{u})\}} \prod_{k=1}^{m_i} J_{j_k}(X_{ij_k}) \tag{14}$$

In this expression, the indices $j_1, \ldots, j_{m_i}$ are those of the components of $\boldsymbol{X}_i$ exceeding the thresholds $\boldsymbol{u}$ and for $\boldsymbol{y} \geq \boldsymbol{u}$, $t(\boldsymbol{y}) = (t_1(y_1), \ldots, t_d(y_d))$ and for $j = 1, \ldots, d$,

$$t_j(x) = -\frac{1}{\log\{\tilde{F}_j(x; \eta_j, \xi_j)\}}, \qquad J_j(x) = \frac{\nu_j}{\eta_j}\left(1 + \xi_j \frac{(x - u_j)}{\eta_j}\right)^{-1/\xi_j - 1} \frac{1}{[\log\{\tilde{F}_j(x; \eta_j, \xi_j)\}]^2 \tilde{F}_j(x; \eta_j, \xi_j)}. \tag{15}$$

The censored likelihood L can be maximized either over all parameters at once, or the marginal parameters $\boldsymbol{\nu}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ can be estimated from each marginalseparately, so that only the estimate of $\boldsymbol{\theta}$ is obtained through maximizing L.

Alternatively, when d is large, one can also maximize the likelihood proposed in [35] that uses the tail approximation $\bar{H}(\boldsymbol{x}) \approx 1 - \ell(1/\boldsymbol{x})$. In any case, ℓ_0 and the higher-order partial derivatives of $\ell_0(1/\boldsymbol{x})$ need to be computed.

When ℓ_0 is the scaled extremal Dirichlet stable tail dependence function $\ell^{\mathrm{D}}(\cdot; \rho, \boldsymbol{\alpha})$ given in Definition 1 with parameters $\boldsymbol{\alpha} > 0$ and $\rho > -\min(\alpha_1, \ldots, \alpha_d)$, $\rho \neq 0$, its expression is not explicit. However, ℓ^{D} can be calculated numerically using adaptive numerical cubature algorithms for integrals of functions defined on the simplex, as implemented in, e.g., the R package `SimplicialCubature`. Given the representation in eq. (5), ℓ^{D} is also easily approximated using Monte Carlo methods. Instead of employing eq. (5) directly and sampling from the Dirichlet vector $\boldsymbol{D}_{\boldsymbol{\alpha}}$, one can use the more efficient importance sampling estimator

$$\widehat{\ell^{\mathrm{D}}}(1/\boldsymbol{u}, \rho, \boldsymbol{\alpha}) = \frac{1}{B} \sum_{i=1}^{B} \frac{\max_{1 \leq j \leq d} \left[\{c(\alpha_j, \rho) u_j\}^{-1} D_{ij}^{\rho}\right]}{\frac{1}{d} \sum_{j=1}^{d} c(\alpha_j, \rho)^{-1} D_{ij}^{\rho}},$$

where $\boldsymbol{D}_i \sim d^{-1} \sum_{j=1}^{d} \mathsf{Dir}(\boldsymbol{\alpha} + \mathrm{I}_j \rho \mathbf{1}_d)$ is sampled from a Dirichlet mixture.

The partial derivatives of ℓ^{D} can be calculated using the following result, shown in E.

PROPOSITION 6. *Let ℓ^{D} be the scaled extremal Dirichlet stable tail dependence function with parameters $\boldsymbol{\alpha} > \mathbf{0}_d$ and $-\min(\alpha_1, \ldots, \alpha_d) < \rho < \infty$, $\rho \neq 0$. Then, for any $\boldsymbol{x} \in \mathbb{R}_+^d$,*

$$\frac{\partial^d \ell^{\mathrm{D}}(1/\boldsymbol{x})}{\partial x_1 \cdots \partial x_d} = -dh^{\mathrm{D}}(\boldsymbol{x}; \rho, \boldsymbol{\alpha}) = -\frac{\Gamma(\bar{\alpha}+\rho)}{|\rho|^{d-1}\prod_{i=1}^d \Gamma(\alpha_i)} \left[\sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho}\right]^{-\rho-\bar{\alpha}} \prod_{i=1}^d \{c(\alpha_i,\rho)\}^{\alpha_i/\rho} x_i^{\alpha_i/\rho - 1}, \quad (16)$$

where h^{D} is as given in Proposition 5. Furthermore, for all $k = 1, \ldots, d-1$ and $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$\frac{\partial^k \ell^{\mathrm{D}}(1/\boldsymbol{x})}{\partial x_1 \cdots \partial x_k} = -\int_0^\infty t^k \prod_{i=1}^k f\left(x_i t; \frac{1}{c(\alpha_i,\rho)}, \frac{1}{\rho}, \alpha_i\right) \prod_{i=k+1}^d F\left(x_i t; \frac{1}{c(\alpha_i,\rho)}, \frac{1}{\rho}, \alpha_i\right) dt,$$

where $f(;a,b,c)$ and $F(;a,b,c)$ denote, respectively the density and distribution function of the scaled Gamma distribution with parameters $a, c > 0$ and $b \neq 0$ given in eq. (10). *Furthermore, if $\gamma(c,x) = \int_0^x t^{c-1}e^{-t}dt$ denotes the lower incomplete gamma function, then for $x > 0$, $F(x;a,b,c) = \gamma\{c,(x/a)^b\}/\Gamma(c)$ when $b > 0$ while $F(x;a,b,c) = 1 - \gamma\{c,(x/a)^b\}/\Gamma(c)$ when $b < 0$.*

Other estimating equations could be used to circumvent the calculation of $\ell^{\mathrm{D}}(1/\boldsymbol{x})$ and its partial derivatives. An interesting alternative to likelihoods in the context of proper scoring functions is proposed in [7]. Specifically, the authors advocate the use of the gradient score, adapted by them for the peaks-over-threshold framework,

$$\delta_w(\boldsymbol{x}) = \sum_{i=1}^d \left(2w_i(\boldsymbol{x})\frac{\partial w_i(\boldsymbol{x})}{\partial x_i}\frac{\partial \log h(\boldsymbol{x})}{\partial x_i} + w_i^2(\boldsymbol{x})\left[\frac{\partial^2 \log h(\boldsymbol{x})}{\partial x_i^2} + \frac{1}{2}\left\{\frac{\partial \log h(\boldsymbol{x})}{\partial x_i}\right\}^2\right]\right)$$

for a differentiable weighting function $w(\boldsymbol{x})$, unit Fréchet observations $\boldsymbol{x}$ and density $h(\boldsymbol{x})$ that would correspond in the setting of the scaled extremal Dirichlet to $dh^{\mathrm{D}}(\boldsymbol{x};\rho,\boldsymbol{\alpha})$. Explicit expressions for the derivatives of $\log dh^{\mathrm{D}}$ may be found in E. The parameter estimates are obtained as the solution to $\mathrm{argmax}_{\boldsymbol{\theta}\in\Theta} \sum_{i=1}^n \delta_w(\boldsymbol{x}_i)\mathrm{I}_{\mathcal{R}(\boldsymbol{x}_i/\boldsymbol{u})>1}$, where $\boldsymbol{\theta} = (\rho, \boldsymbol{\alpha})$ is the vector of parameters of the model and $\mathcal{R}$ is a differentiable risk functional,

usually the ℓ_p norm for some $p \in \mathbb{N}$. Although the gradient score is not asymptotically most efficient, weighting functions can be designed to reproduce approximate censoring, lending the method robustness and tractability.

8. Data illustration

In this section, we illustrate the use of the scaled extremal Dirichlet model on a trivariate sample of daily river flow data of the river Isar in southern Germany; this dataset is a subset of the one analyzed in [1]. All the code can be downloaded from `https://github.com/lbelzile/ealc`. For this analysis, we selected data measured at Lenggries (upstream), Pappinger Au (in the middle) and Munich (downstream). To ensure stationarity of the series and given that the most extreme events occur during the summer, we restricted our attention to the measurements for the months of June, July and August. Since the sites are measuring the flow of the same river, dependence at extreme levels is likely to be present, as is indeed apparent from Figure 4. Directionality of the river may further lead to asymmetry in the asymptotic dependence structure, suggesting that the scaled extremal Dirichlet model may be well suited for these data. Furthermore, given that other well-known models like the extremal Dirichlet, logistic and negative logistic are nested within this family, their adequacy can be conveniently assessed through likelihood ratio tests.

To remove dependence between extremes over time, we decluster each series and retain only the cluster maxima based on three-day runs. Rounding of the measurements has no impact on parameter estimates and is henceforth neglected. The multivariate tail model outlined in Section 7 is next fitted to the cluster maxima. The thresholds $\boldsymbol{u} = (u_1, u_2, u_3)$ were selected to be the 92% quantiles using the parameter stability plot of [37] (not shown here). Next, set $\boldsymbol{\theta} = (\boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{\alpha}, \rho)$, where $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are the marginal parameters of the generalized Pareto distribution in eq. (13) and ρ and $\boldsymbol{\alpha}$ are the parameters of the scaled Dirichlet model. To estimate $\boldsymbol{\theta}$, the trivariate censored likelihood (14) could be used. To avoid numerical integration and because of the relative robustness to misspecification, we employed the pairwise composite log-likelihood l_C of [24] instead; the loss of efficiency in this trivariate example is likely small. Specifically, we maximized

$$l_C(\boldsymbol{\theta}) = \sum_{i=1}^n \sum_{j=1}^{d-1} \sum_{k=j+1}^d \left[\log g\{t_j(x_{ij}), t_k(x_{ik}); \boldsymbol{\theta}, t_j(u_j), t_k(u_k)\} + \mathrm{I}_{x_{ij}>u_j} \log J_j(x_{ij}) + \mathrm{I}_{x_{ik}>u_k} \log J_k(x_{ik})\right],$$

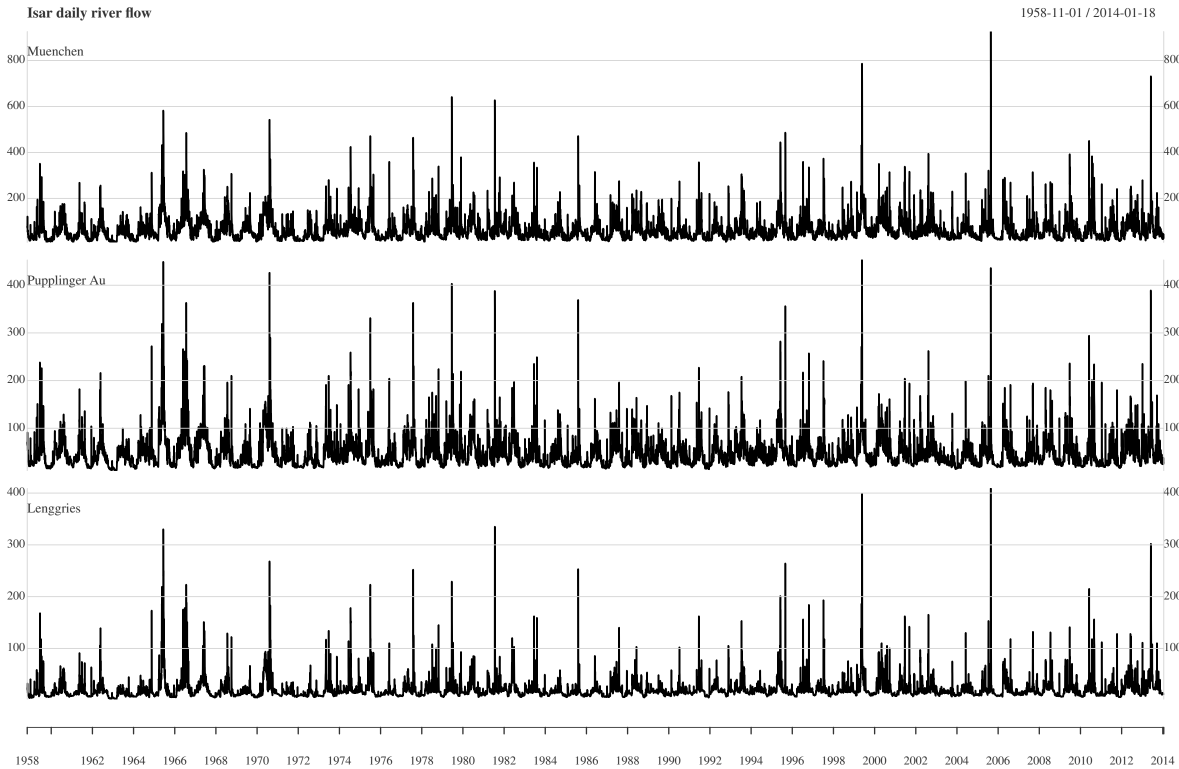

Figure 4: Daily river flow of the Isar river at three sites

where

$$
g(y_j, y_k; \boldsymbol{\theta}, u_j, u_k) = \begin{cases} \exp\{-\ell(1/u_j, 1/u_k)\}, & y_j \leq u_j, y_k \leq u_k \\ -\partial\ell(1/y_j, 1/u_k)/\partial y_j \exp\{-\ell(1/y_j, 1/u_k)\}, & y_j > u_j, y_k \leq u_k \\ -\partial\ell(1/u_j, 1/y_k)/\partial y_k \exp\{-\ell(1/u_j, 1/y_k)\}, & y_j \leq u_j, y_k > u_k \\ \left[\{\partial\ell(1/y_j, 1/y_k)/\partial y_j\}\{\partial\ell(1/y_j, 1/y_k)/\partial y_k\} - dh^{\mathrm{D}}(y_j, y_k)\right] \exp\{-\ell(1/y_j, 1/y_k)\}, & y_j > u_j, y_k > u_k \end{cases}
$$

where $\ell = \ell^{\mathrm{D}}$ and for all $j = 1, \ldots, d$, t_j and J_j are as in Equation (15). Uncertainty assessment can be done in the same way as for general estimating equations. Specifically, let $g(\boldsymbol{\theta})$ denote an unbiased estimating function and define the variability matrix $\mathbf{J}$, the sensitivity matrix $\mathbf{H}$ and the Godambe information matrix $\mathbf{G}$ as

$$
\mathbf{J} = \mathrm{E}\left(\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^\top\right), \qquad \mathbf{H} = -\mathrm{E}\left(\frac{\partial^2 g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top}\right), \qquad \mathbf{G} = \mathbf{H}\mathbf{J}^{-1}\mathbf{H}. \tag{17}
$$

The maximum composite likelihood estimator is strongly consistent and asymptotically normal, centered at the true parameter $\boldsymbol{\theta}$ with covariance matrix given by the inverse Godambe matrix $\mathbf{G}^{-1}$.

Using the pairwise composite log-likelihood l_C, we fitted the scaled extremal Dirichlet model as well as the logistic and negative logistic models that correspond to the negative and positive scaled extremal Dirichlet models, respectively, and the parameter restriction $\boldsymbol{\alpha} = \mathbf{1}_d$. The estimates of the marginal generalized Pareto parameters $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are given in Table 1. As the estimates were obtained by maximizing ℓ_C, their values depend on the fitted model; the line labeled “Marginal" corresponds to fitting the generalized Pareto distribution to threshold exceedances of each one of the three series separately. The marginal QQ-plots displayed in Figure 5 indicate a good fit of the model as well.

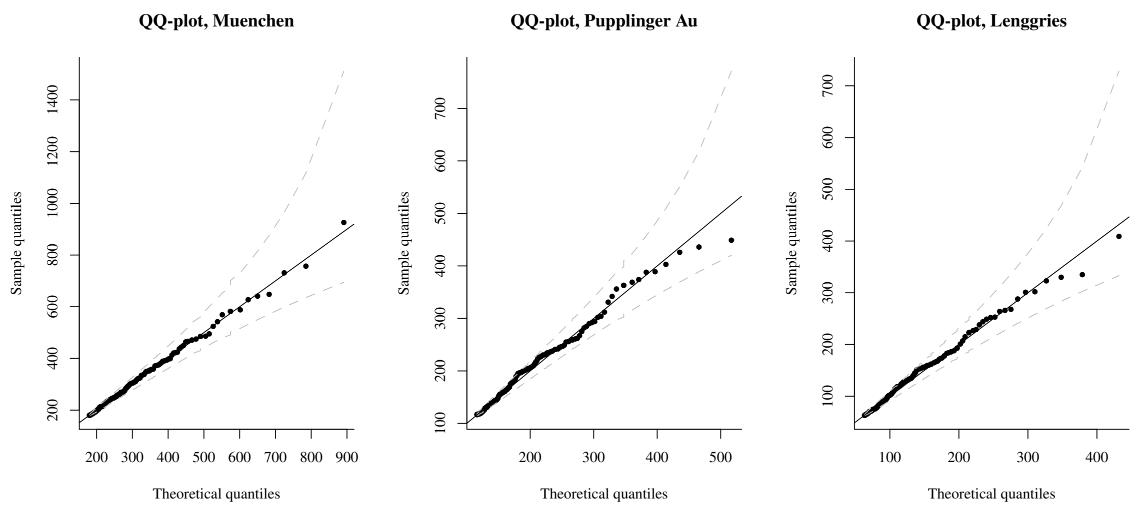

Figure 5: Marginal QQ-plots for the three sites based on pairwise composite likelihood estimates for the scale and shape parameters obtained from the scaled Dirichlet model, retaining marginal exceedances of the 92% quantiles. The pointwise confidence intervals were obtained from the transformed Beta quantiles of the order statistics.

	η_1	η_2	η_3	ξ_1	ξ_2	ξ_3
Scaled Dirichlet	123.2 (7.5)	84.4 (5)	68.1 (4.2)	0.05 (0.04)	-0.03 (0.04)	0.02 (0.04)
Neg. logistic	117.1 (6.8)	86.2 (5.1)	70 (4.3)	0.08 (0.04)	-0.05 (0.04)	0 (0.04)
Logistic	117.3 (6.8)	86.6 (5.1)	70.4 (4.3)	0.08 (0.04)	-0.05 (0.04)	0 (0.04)
ext. Dirichlet	114.4 (6.8)	84.3 (4.9)	68.2 (4.1)	0.12 (0.04)	-0.02 (0.04)	0.04 (0.04)
Marginal	129.1 (14.5)	95.1 (10.6)	76 (8.7)	-0.01 (0.08)	-0.15 (0.08)	-0.08 (0.08)

Table 1: Generalized Pareto parameter estimates and standard errors (in parenthesis) for the trivariate river example for four different models.

The estimates of the dependence parameters $\boldsymbol{\alpha}$ and ρ are given in Table 2. The last line displays the maximum gradient score estimates were obtained from the raw data, i.e., ignoring the clustering, after transforming the observations to the standard Fréchet scale using the probability integral transform. We retained only the 10% largest values based on the ℓ_p norm with $p = 20$; this risk functional is essentially a differentiable approximation of ℓ_∞. We selected the weight function $w(\boldsymbol{x}, u) = \boldsymbol{x}[1 - \exp\{-(\|\boldsymbol{x}\|_p/u - 1)\}]$ based on [7] to reproduce approximate censoring. The estimates are similar to the composite maximum likelihood estimators, though not efficient.

The angular densities of the fitted logistic, negative logistic and scaled extremal Dirichlet models are displayed in Figure 6. The right panel of this figure shows asymmetry caused by a few extreme events that only happened downstream. Whether this asymmetry is significant can be assessed through composite likelihood ratio tests; recall that the logistic, negative logistic and the extremal Dirichlet model of [6] are all nested within the scaled extremal Dirichlet model. To this end, consider a partition of $\boldsymbol{\theta} = (\boldsymbol{\psi}, \boldsymbol{\lambda})$ into a q dimensional parameter of interest $\boldsymbol{\psi}$ and a $3d + 1 - q$ dimensional nuisance parameter $\boldsymbol{\lambda}$, and the corresponding partitions of the matrices $\mathbf{H}$, $\mathbf{J}$ and $\mathbf{G}$. Let $\widehat{\boldsymbol{\theta}}_C = (\widehat{\boldsymbol{\psi}}_C, \widehat{\boldsymbol{\lambda}}_C)$ denote the maximum composite likelihood parameter estimates and $\widehat{\boldsymbol{\theta}}_0 = (\boldsymbol{\psi}_0, \widehat{\boldsymbol{\lambda}}_0)$ the restricted parameter estimates under the null hypothesis that the simpler model is adequate. The asymptotic distribution of the composite likelihood ratio test statistic $2\{\log l_C(\widehat{\boldsymbol{\theta}_C}) - \log l_C(\widehat{\boldsymbol{\theta}_0})\}$ is equal to $\sum_{i=1}^q c_i Z_i$ where Z_i are independent χ^2_1 variables and c_i are the eigenvalues of the $q \times q$ matrix $(\mathbf{H}_{\boldsymbol{\psi}\boldsymbol{\psi}} - \mathbf{H}_{\boldsymbol{\psi}\boldsymbol{\lambda}}\mathbf{H}^{-1}_{\boldsymbol{\lambda}\boldsymbol{\lambda}}\mathbf{H}_{\boldsymbol{\lambda}\boldsymbol{\psi}})\mathbf{G}^{-1}_{\boldsymbol{\psi}\boldsymbol{\psi}}$; see [22]. We estimated the inverse Godambe information matrix, $\mathbf{G}^{-1}$, by the empirical covariance of B nonparametric boostrap replicates. The sensitivity matrix $\mathbf{H}$ was obtained from the Hessian matrix at the maximum composite likelihood estimate and the variability matrix $\mathbf{J}$ from eq. (17). Since the Coles–

	α_1	α_2	α_3	ρ
Scaled Dirichlet	0.76 (0.3)	1.65 (0.82)	2.03 (1.15)	−0.32 (0.1)
Neg. logistic	1	1	1	0.36 (0.02)
Logistic	1	1	1	0.28 (0.01)
ext. Dirichlet	3.34 (0.52)	10.2 (2.84)	12.78 (3.93)	1
Gradient score	1	2.72	2.66	−0.39

Table 2: Dependence parameters estimates and standard errors (in parenthesis) for the trivariate river example.

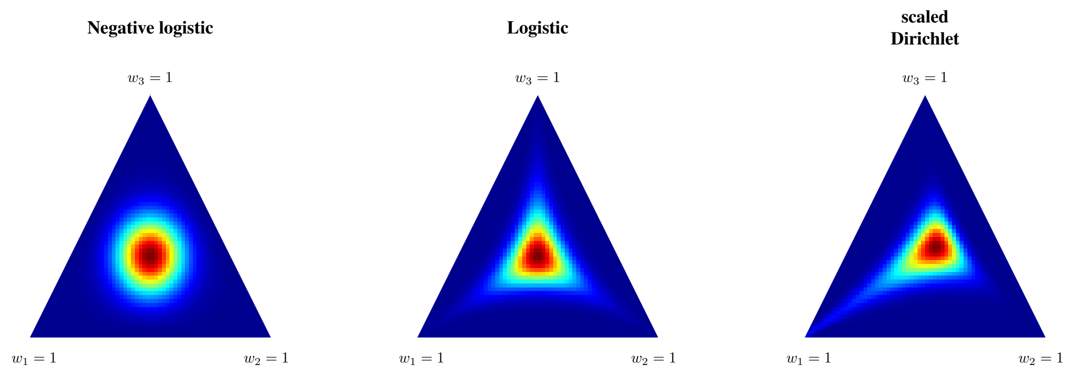

Figure 6: Angular density plots for the three models, the negative logistic (left), logistic (middle) and negative scale Dirichlet (right). The colours correspond to log density values and range from red (high density) to blue (low density).

Tawn extremal Dirichlet, negative logistic and logistic models are nested within the scaled Dirichlet family, we test for a restriction to these simpler models; the respective approximate P-values were 0.003, 0.74 and 0.78. These values suggest that while the Coles–Tawn extremal Dirichlet model is clearly not suitable, there is not sufficient evidence to discard the logistic and negative logistic models. The effects of possible model misspecification are also visible for the Coles–Tawn extremal Dirichlet model, as the parameter values of α_1, α_2 and α_3 are very large (viz. Table 2) and this induces negative bias in the shape parameter estimates, as can be seen from Table 1.

9. Discussion

In this article, we have identified extremal attractors of copulas and survival copulas of Liouville random vectors $R\boldsymbol{D}_{\boldsymbol{\alpha}}$, where $\boldsymbol{D}_{\boldsymbol{\alpha}}$ has a Dirichlet distribution on the unit simplex with parameters $\boldsymbol{\alpha}$, and R is a strictly positive random variable independent of $\boldsymbol{D}_{\boldsymbol{\alpha}}$. The limiting stable tail dependence functions can be embedded in a single family, which can capture asymmetry and is a valid model in dimension d. The latter is novel and termed here the scaled extremal Dirichlet; it includes the well-known logistic, negative logistic as well as the Coles–Tawn extremal Dirichlet models as special cases. In particular, therefore, this paper is first to provide an example of a random vector attracted to the Coles–Tawn extremal Dirichlet model, which was derived by enforcing moment constraints on a simplex distribution rather than as the limiting distribution of a random vector.

A stable extremal Dirichlet stable tail dependence function ℓ^{D} has $d+1$ parameters, ρ and $\boldsymbol{\alpha}$. The parameters $\boldsymbol{\alpha}$ are inherited from $\boldsymbol{D}_{\boldsymbol{\alpha}}$ and induce asymmetry in ℓ^{D}. The parameter ρ comes from the regular variation of R at zero and infinity, respectively; this is reminiscent of the extremal attractors of elliptical distributions [28]. The magnitude of ρ has impact on the strength of dependence while its sign changes the overall shape of ℓ^{D}. Having $d+1$ parameters, the scaled extremal Dirichlet model may not be sufficiently rich to account for spatial dependence, unlike the Hüsler–Reiss or the extremal Student-t models, which have one parameter for each pair of variables and are thus easily combined with distances. Also, it is less flexible than Dirichlet mixtures [3], which are however hard to estimate in high dimensions and require sophisticated machinery. Nonetheless, the scaled extremal Dirichlet model may naturally find applications whenever asymmetric extremal dependence is suspected; the latter may be caused, e.g., by causal relationships between the variables [14]. The stochastic structure of the scaled extremal Dirichlet model has several major advantages, that make the model easy to interpret, estimate and simulate from. Its angular density has a simple form; in contrast to the asymmetric generalizations of the logistic and negative logistic models, this model does not place any mass on the vertices and lower-dimensional facets of the unit simplex. Another plus are the tractable de Haan representation and extremal functions, both expressible in terms of independent scaled Gamma variables; this allows for feasible inference and stochastic simulation. While the scaled extremal Dirichlet stable tail dependence function ℓ^{D} does not have a closed form in general, closed-form algebraic expressions exist when $\boldsymbol{\alpha}$ is integer-valued and in the bivariate case. Model selection for well-known families of extreme-value distributions can be performed through likelihood ratio tests. Another potentially useful feature is that $\rho \in (-\infty, \infty)$ can be allowed, with the convention that all variables whose indices i are such that $-\rho \leq -\alpha_i$ are independent.

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Appendix

A. Proofs from Section 2

Proof of Proposition 2. To prove parts (a) and (b), recall that $1/X_i$ is distributed as $1/(RD_i)$, where $D_i \sim \mathsf{Beta}(\alpha_i, \bar{\alpha}-\alpha_i)$ is independent of R. Furthermore, it is easy to show that $1/D_i \in \mathcal{M}(\Phi_{\alpha_i})$, which implies that $\mathrm{E}(1/D_i^{\beta}) < \infty$ for any $\beta < \alpha_i$. The extremal behavior of $1/X_i$ will thus be determined by the extremal behavior of either $1/R$ or $1/D_i$, depending on which one has a heavier tail. Indeed, Breiman's Lemma [4] implies that $1/X_i \in \mathcal{M}(\Phi_\rho)$ if $1/R \in \mathcal{M}(\Phi_\rho)$ for some $\rho < \alpha_i$ and that $1/X_i \in \mathcal{M}(\Phi_{\alpha_i})$ if $\mathrm{E}(1/R^{\alpha_i+\varepsilon}) < \infty$ for some $\varepsilon > 0$. Finally, the fact that $1/X_i \in \mathcal{M}(\Phi_{\alpha_i})$ when $1/R \in \mathcal{M}(\Phi_{\alpha_i})$ follows directly from the Corollary to Theorem 3 in [10].

The following lemma is a side result of Proposition 2, which is needed in the subsequent proofs.

LEMMA 1. *Suppose that $\boldsymbol{X} = R\boldsymbol{D}_\alpha$. If $1/R \in \mathcal{M}(\Phi_{\alpha_i})$ for some $i \in \{1, \ldots, d\}$, then*

$$\lim_{x\to\infty} \frac{\Pr(1/R > x)}{\Pr(1/X_i > x)} = 0.$$

Proof of Lemma 1. First write

$$\Pr\left(\frac{1}{X_i} > x\right) = \Pr\left(\frac{1}{RD_i} > x\right) = \int_0^1 \Pr\left(\frac{1}{R} > xb\right) \frac{b^{\alpha_i-1}(1-b)^{\bar{\alpha}-\alpha_i-1}}{\mathrm{B}(\alpha_i, \bar{\alpha}-\alpha_i)}\,\mathrm{d}b.$$

Because $1/R \in \mathcal{M}(\Phi_{\alpha_i})$, it holds that, for all $x > 0$, $\Pr(1/R > x) = x^{-\alpha_i}L(x)$ for some slowly varying function L. Making the change of variable $y = xb$ in the above integral, one thus has

$$\Pr\left(\frac{1}{X_i} > x\right) = \frac{x^{-\alpha_i}}{\mathrm{B}(\alpha_i, \bar{\alpha}-\alpha_i)} \int_0^x y^{-1}L(y)\left(1-\frac{y}{x}\right)^{\bar{\alpha}-\alpha_i-1}\mathrm{d}y.$$

To show the claim of the lemma, it suffices to show that, as $x \to \infty$,

$$\frac{1}{L(x)}\int_0^x y^{-1}L(y)\left(1-\frac{y}{x}\right)^{\bar{\alpha}-\alpha_i-1}\mathrm{d}y = \int_0^1 \frac{L(xb)}{L(x)}\frac{1}{b}(1-b)^{\bar{\alpha}-\alpha_i-1}\,\mathrm{d}b \to \infty.$$

Because L is slowly varying, one has, as $x \to \infty$,

$$\frac{L(xb)}{L(x)}\frac{1}{b}(1-b)^{\bar{\alpha}-\alpha_i-1} \to \frac{1}{b}(1-b)^{\bar{\alpha}-\alpha_i-1}.$$

An application of Fatou's lemma thus gives that

$$\infty = \int_0^1 \frac{1}{b}(1-b)^{\bar{\alpha}-\alpha_i-1}\,\mathrm{d}b \le \liminf_{x\to\infty}\int_0^1 \frac{L(xb)}{L(x)}\frac{1}{b}(1-b)^{\bar{\alpha}-\alpha_i-1}\,\mathrm{d}b,$$

and hence the result.

B. Proofs from Section 3

First recall the following property of the Dirichlet distribution, which is easily shown using the transformation formula for Lebesgue densities.

LEMMA 2. *Let $\boldsymbol{D}_\alpha$ be a Dirichlet random vector with parameters $\boldsymbol{\alpha}$. Then for any $2 \le k \le d$ and any collection of distinct indices $1 \le i_1 < \cdots < i_k \le d$,*

$$(D_{i_1}, \ldots, D_{i_k}) \stackrel{d}{=} B_{i_1,\ldots,i_k} \times \boldsymbol{D}_{(\alpha_{i_1},\ldots,\alpha_{i_k})},$$

where $B_{i_1,\ldots,i_k} \sim \mathsf{Beta}(\alpha_{i_1}+\cdots+\alpha_{i_k}, \bar{\alpha}-(\alpha_{i_1}+\cdots+\alpha_{i_k}))$ is independent of the k-variate Dirichlet vector $\boldsymbol{D}_{(\alpha_{i_1},\ldots,\alpha_{i_k})}$ with parameters $(\alpha_{i_1}, \ldots, \alpha_{i_k})$.

Proof of Theorem 1. In order to prove part (a), recall that $\|\boldsymbol{X}\| \stackrel{\mathrm{d}}{=} R$ is independent of $\boldsymbol{X}/\|\boldsymbol{X}\| \stackrel{\mathrm{d}}{=} \boldsymbol{D}_{\boldsymbol{\alpha}}$. Because $R \in \mathcal{M}(\Phi_\rho)$, there exists a sequence (b_n) of constants in $(0,\infty)$ such that, for any Borel set $B \subseteq \mathbb{S}_d$ and any $r > 0$,

$$\lim_{n\to\infty} n \Pr\left(\|\boldsymbol{X}\| > b_n r, \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|} \in B\right) = \lim_{n\to\infty} n \Pr(R > b_n r) \Pr(\boldsymbol{D}_{\boldsymbol{\alpha}} \in B) = r^{-\rho} \Pr(\boldsymbol{D}_{\boldsymbol{\alpha}} \in B).$$

By Corollary 5.18 in [30], $\boldsymbol{X} \in \mathcal{M}(H_0)$ where for all $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$H_0(\boldsymbol{x}) = \exp\left[-\mathrm{E}\left\{\max\left(\frac{D_1^\rho}{x_1^\rho}, \ldots, \frac{D_d^\rho}{x_d^\rho}\right)\right\}\right].$$

Let $\mathrm{B}(\cdot,\cdot)$ denote the Beta function. The margins of H_0 are given, for all $i = 1,\ldots,d$ and $x > 0$, by

$$H_{i0}(x) = \exp\left\{-x^{-\rho}\mathrm{E}(D_i^\rho)\right\} = \exp\left\{-x^{-\rho}\frac{\mathrm{B}(\rho+\alpha_i, \bar\alpha-\alpha_i)}{\mathrm{B}(\alpha_i, \bar\alpha-\alpha_i)}\right\} = \exp\left\{-x^{-\rho}\frac{\Gamma(\alpha_i+\rho)\Gamma(\bar\alpha)}{\Gamma(\bar\alpha+\rho)\Gamma(\alpha_i)}\right\}$$

for $i \in \{1,\ldots,d\}$. The copula of H_0 then satisfies, for all $\boldsymbol{u} \in [0,1]^d$,

$$C_0(\boldsymbol{u}) = H_0\{H_{01}^{-1}(u_1), \ldots, H_{0d}^{-1}(u_d)\} = \exp\left(-\frac{\Gamma(\bar\alpha+\rho)}{\Gamma(\bar\alpha)}\mathrm{E}\left[\max_{1\le i\le d}\left\{\frac{(-\log u_i)\Gamma(\alpha_i)D_i^\rho}{\Gamma(\alpha_i+\rho)}\right\}\right]\right).$$

By Equation (2), the stable tail dependence function of C_0 thus indeed equals, for all $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$\ell(\boldsymbol{x}) = \frac{\Gamma(\bar\alpha+\rho)}{\Gamma(\bar\alpha)}\mathrm{E}\left[\max\left\{\frac{x_1\Gamma(\alpha_1)D_1^\rho}{\Gamma(\alpha_1+\rho)}, \ldots, \frac{x_d\Gamma(\alpha_d)D_d^\rho}{\Gamma(\alpha_d+\rho)}\right\}\right].$$

The parts (b) and (c) can be proved jointly. To this end, set, for all $i = 1,\ldots,d$, $H_{0i} = \Lambda$ when $R \in \mathcal{M}(\Lambda)$ and $H_{0i} = \Psi_{\rho+\bar\alpha-\alpha_i}$ when $R \in \mathcal{M}(\Psi_\rho)$ for some $\rho > 0$. Recall that from Proposition 1, one then has that for $i = 1,\ldots,d$, $X_i \in \mathcal{M}(H_{0i})$ and hence there exist sequences $(a_{ni}) \in (0,\infty)$, $(b_{ni}) \in \mathbb{R}$, such that for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n \Pr(X_i > a_{ni}x + b_{ni}) = -\log\{H_{0i}(x)\}.$$

Next, observe that as in the proof of Proposition 5.27 in [30], $\boldsymbol{X} \in \mathcal{M}(H_0)$ follows if for all $1 \le i < j \le d$ and x_k such that $H_{0k}(x_k) > 0$ for $k = i,j$,

$$\lim_{n\to\infty} n \Pr(X_i > a_{ni}x_i + b_{ni}, X_j > a_{nj}x_j + b_{nj}) = 0. \tag{B.1}$$

To prove that (B.1) indeed holds, it suffices to assume that $d = 2$. This is because for arbitrary indices $1 \le i < j \le d$, Lemma 2 implies that $(X_i, X_j) \stackrel{\mathrm{d}}{=} R^*(B, 1-B)$, where $B \sim \mathrm{Beta}(\alpha_i, \alpha_j)$, $R^* \stackrel{\mathrm{d}}{=} RY$ is independent of B and $Y \sim \mathrm{Beta}(\alpha_i+\alpha_j, \bar\alpha-\alpha_i-\alpha_j)$ is independent of B and R. Because $\Pr(R^* \le 0) = 0$, Theorems 4.1 and 4.5 in [17] imply that $R^* \in \mathcal{M}(\Lambda)$ and $R^* \in \mathcal{M}(\Psi_{\rho+\bar\alpha-\alpha_i-\alpha_j})$ when $R \in \mathcal{M}(\Lambda)$ and $R \in \mathcal{M}(\Psi_\rho)$ for some $\rho > 0$, respectively.

Hence, suppose that $d = 2$ and write $(D_1, D_2) \equiv (B, 1-B)$, where $B \sim \mathsf{Beta}(\alpha_1, \alpha_2)$. Fix arbitrary $x_1, x_2 \in \mathbb{R}$ are such that $H_{0i}(x_i) > 0$ for $i = 1, 2$. Then because for any $a, c > 0$ and $b \in (0,1)$, $\max\{a/b, c/(1-b)\} \ge a + c$, one has

$$\begin{aligned} 0 \le \Pr(X_1 > a_{n1}x_1 + b_{n1}, X_2 > a_{n2}x_2 + b_{n2}) &= \Pr\left\{R > \max\left(\frac{a_{n1}x_1 + b_{n1}}{B}, \frac{a_{n2}x_2 + b_{n2}}{1-B}\right)\right\} \\ &\le \Pr(R > a_{n1}x_1 + b_{n1} + a_{n2}x_2 + b_{n2}). \end{aligned}$$

In order to prove Equation (B.1), it thus suffices to show that

$$\lim_{n\to\infty} n \Pr(R > a_{n1}x_1 + b_{n1} + a_{n2}x_2 + b_{n2}) = 0. \tag{B.2}$$

To accomplish this, the following two cases have to be distinguished.

Case I. The upper endpoint $r = \sup\{x : \Pr(R \le x) < 1\}$ of R is finite. Then for $i = 1, 2$, r is also the upper endpoint of X_i and as $n \to \infty$, $a_{ni}x_i + b_{ni} \to r$. This means that there exists $n_0 \in \mathbb{N}$ so that for all $n \ge n_0$, $a_{n1}x_1 + b_{n1} + a_{n2}x_2 + b_{n2} > r$. Hence Equation (B.2) indeed holds. This proves Equation (B.1) and hence also Theorem 1 (c), as well as (b) when the upper endpoint of R is finite.

Case II. The upper endpoint of R is infinite. This case can only occur in part (b) when $R \in \mathcal{M}(\Lambda)$. Hence, for $i = 1, 2$, without loss of generality, b_{ni} is such that $\Pr(X_i > b_{ni}) = 1/n$ and $a_{ni} = 1/w_i(b_{ni})$,

where w_i is the so-called scaling function of X_i. By Theorem 4.1 in [17], $w_1 = w_2 = w$, where w is the scaling function of R. From Lemma 4.2 and Theorem 4.1 in [17], it further holds that, for any $c > 1$,

$$\lim_{x\to\infty} \frac{\Pr(R > cx)}{\Pr(RB > x)} = \lim_{x\to\infty} \frac{\Pr(R > cx)}{\Pr\{R(1-B) > x\}} = 0. \tag{B.3}$$

To establish the validity of Equation (B.2), three further sub-cases have to be distinguished.

Sub-case IIa. $\alpha_1 = \alpha_2$. Here, $RB \stackrel{\mathrm{d}}{=} R(1-B)$ and for all $n \in \mathbb{N}$, $a_{n1} = a_{n2} \equiv a_n$ and $b_{n1} = b_{n2} \equiv b_n$. The limit in Equation (B.2) thus equals $\lim_{n\to\infty} n \Pr\{R > a_n(x_1 + x_2) + 2b_n)\}$ and Equation (B.3) implies

$$\begin{aligned} 0 \le \lim_{n\to\infty} n \Pr\{R > a_n(x_1 + x_2) + 2b_n\} &\le \lim_{n\to\infty} n \Pr[R > 2\{a_n(x_1 \wedge x_2) + b_n\}] \\ &= \lim_{n\to\infty} n \Pr\{RB > a_n(x_1 \wedge x_2) + b_n\} \frac{\Pr[R > 2\{a_n(x_1 \wedge x_2) + b_n\}]}{\Pr\{RB > a_n(x_1 \wedge x_2) + b_n\}} = 0, \end{aligned}$$

given that as $n \to \infty$, $n \Pr\{RB > a_n(x_1 \wedge x_2) + b_n\} \to e^{-x_1 \wedge x_2}$ and $a_n(x_1 \wedge x_2) + b_n \to \infty$.

Sub-case IIb. $\alpha_1 < \alpha_2$. Then by Equations (4.5) and (4.9) in [17],

$$\lim_{x\to\infty} \frac{\Pr\{R(1-B) > x\}}{\Pr(RB > x)} = \lim_{x\to\infty} \frac{\{1 + o(1)\}\Gamma(\alpha_1)}{\{1 + o(1)\}\Gamma(\alpha_2)} \{xw(x)\}^{\alpha_2 - \alpha_1} = \infty.$$

This means that for all x sufficiently large, $\Pr\{R(1-B) > x\} > \Pr(RB > x)$ so that there exists $n_0 \in \mathbb{N}$ such that $b_{n1} < b_{n2}$ for all $n \ge n_0$. Indeed, if $b_{n1} \ge b_{n2}$ were true, then $1/n = \Pr(RB > b_{n1}) < \Pr\{R(1-B) > b_{n1}\} \le \Pr\{R(1-B) > b_{n2}\} = 1/n$, which is a contradiction. Thus, for $n \ge n_0$,

$$\frac{a_{n2}x_2 + b_{n2}}{a_{n1}x_1 + b_{n1}} = \frac{b_{n2}}{b_{n1}} \frac{\frac{x_2}{b_{n2}w(b_{n2})} + 1}{\frac{x_1}{b_{n1}w(b_{n1})} + 1} > \frac{\frac{x_2}{b_{n2}w(b_{n2})} + 1}{\frac{x_1}{b_{n1}w(b_{n1})} + 1}.$$

By Equation (4.5) in [17], the right-most expression converges to 1 as $n \to \infty$. Consequently, for some fixed $\varepsilon \in (0, 1)$, there exists $n_0^* \ge n_0$ so that for all $n \ge n_0^*$, $a_{n2}x_2 + b_{n2} \ge (a_{n1}x_1 + b_{n1})(1 - \varepsilon)$. This in turn implies that for all $n \ge n_0^*$,

$$n \Pr(R > a_{n1}x_1 + b_{n1} + a_{n2}x_2 + b_{n2}) \le n \Pr(RB > a_{n1}x_1 + b_{n1}) \frac{\Pr\{R > (a_{n1}x_1 + b_{n1})(2 - \varepsilon)\}}{\Pr(RB > a_{n1}x_1 + b_{n1})}.$$

By Equation (B.3) and since $n \Pr(RB > a_{n1}x_1 + b_{n1}) \to e^{-x_1}$, the right-hand side indeed tends to 0 as $n \to \infty$.

Sub-case IIc. $\alpha_1 > \alpha_2$. Proceed analogously as in the previous sub-case: Conclude that for a fixed $\varepsilon \in (0, 1)$ there exists $n_0^* \in \mathbb{N}$ such that for all $n \ge n_0^*$, $a_{n1}x_1 + b_{n1} \ge (a_{n2}x_2 + b_{n2})(1 - \varepsilon)$ and hence

$$n \Pr(R > a_{n1}x_1 + b_{n1} + a_{n2}x_2 + b_{n2}) \le n \Pr\{R(1-B) > a_{n2}x_2 + b_{n2}\} \frac{\Pr\{R > (a_{n2}x_2 + b_{n2})(2 - \varepsilon)\}}{\Pr\{R(1-B) > a_{n2}x_2 + b_{n2}\}}.$$

The last expression again converges to 0 by Equation (B.3) and the fact that $n \Pr\{R(1-B) > a_{n2}x_2 + b_{n2}\} \to e^{-x_2}$ as $n \to \infty$. Consequently, Equation (B.2) holds in this case as well, as claimed.

The proof of Theorem 2 requires the following technical lemma.

LEMMA 3. *Suppose that* $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$ *is a Dirichlet random vector with parameters* $\boldsymbol{\alpha}$*. Further let* R *be a positive random variable independent of* $\boldsymbol{D}_{\boldsymbol{\alpha}}$ *such that* $\Pr(R \le 0) = 0$*, and let* $\boldsymbol{X} = R\boldsymbol{D}_{\alpha}$*. Then for any* $1 \le i < j \le d$ *and any* $x_i, x_j \in (0, \infty)$*,*

$$\lim_{n\to\infty} n \Pr\left(\frac{1}{X_i} > a_{ni}x_i, \frac{1}{X_j} > a_{nj}x_j\right) = 0$$

if either:

(i) $1/R \in \mathcal{M}(\Phi_\rho)$ *with* $\rho \in [\alpha_i \wedge \alpha_j, \alpha_1 \vee \alpha_2]$, *and for* $k = i, j$, (a_{nk}) *is a sequence of positive constants such that* $n \Pr(1/X_k > a_{nk}x_k) \to x_k^{-(\alpha_k \wedge \rho)}$ *as* $n \to \infty$*;*

(ii) $\mathrm{E}(1/R^\beta) < \infty$ *for some* $\beta > \alpha_i \vee \alpha_j$ *and for* $k = i, j$, (a_{nk}) *is a sequence of positive constants such that* $n \Pr(1/X_k > a_{nk}x_k) \to x_k^{-\alpha_k}$ *as* $n \to \infty$.

Proof of Lemma 3. Observe first that when $d > 2$, Lemma 2 implies that $(X_i, X_j) \stackrel{d}{=} R^*(B, 1-B)$, where $R^* \perp\!\!\!\perp B$, $B \sim \mathsf{Beta}(\alpha_i, \alpha_j)$ and $R^* = RY$, with $Y \perp\!\!\!\perp R$ and $Y \sim \mathsf{Beta}(\alpha_i + \alpha_j, \bar{\alpha} - \alpha_i - \alpha_j)$. Now note that $1/Y \in \mathcal{M}(\Phi_{\alpha_i+\alpha_j})$. Thus if $1/R \in \mathcal{M}(\Phi_\rho)$ for $\rho \in [\alpha_i \wedge \alpha_j, \alpha_1 \vee \alpha_2]$, $\rho < \alpha_i + \alpha_j$ and Breiman's Lemma implies that $1/R^* \in \mathcal{M}(\Phi_\rho)$. Further, if $\mathrm{E}(1/R^\beta) < \infty$ for some $\beta \in (\alpha_i \vee \alpha_j, \alpha_i + \alpha_j)$, $\mathrm{E}\{1/(R^*)^\beta\} < \infty$ given that $\mathrm{E}(1/Y^\beta) < \infty$. We can thus assume without loss of generality that $d = 2$ and $\alpha_1 \leq \alpha_2$; we shall also write $(D_1, D_2) \equiv (B, 1-B)$, where $B \sim \mathsf{Beta}(\alpha_1, \alpha_2)$.

To prove part (i), note first that the existence of the sequences (a_{nk}), $k = i, j$, follows from Proposition 2, by which $1/X_k \in \mathcal{M}(\Phi_{\rho \wedge \alpha_k})$ for $k = i, j$, and the Poisson approximation [11, Proposition 3.1.1]. Next, observe that for any constants $a, c > 0$ and any $b \in (0, 1)$,

$$\frac{ac}{a+c} \leq ab \vee c(1-b) < a \vee c. \tag{B.4}$$

Indeed, when $b < c/(a+c)$, $ab \vee c(1-b) = c(1-b)$ and $c(1-b) \in (ac/(a+c), c)$, while when $b \geq c/(a+c)$, $ab \vee c(1-b) = ab$ and $ab \in [ac/(a+c), a)$. To show the claim in part (i), distinguish the cases below:

Case I. $\alpha_1 = \alpha_2$. Here, $\rho = \alpha_1 = \alpha_2$ and $X_1 \stackrel{d}{=} X_2$, so that $a_{n1}/a_{n2} \to 1$ by the Convergence to Types Theorem [30]. By Equation (B.4),

$$0 \leq n \Pr\left(\frac{1}{X_1} > a_{n1}x_1, \frac{1}{X_2} > a_{n2}x_2\right) = n \Pr\left[\frac{1}{R} > \max\{a_{n1}x_1 B, a_{n2}x_2(1-B)\}\right] \leq n \Pr\left(\frac{1}{R} > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1 + a_{n2}x_2}\right). \tag{B.5}$$

Because $(x_1x_2)/\{(a_{n1}/a_{n2})x_1 + x_2\} \to (x_1x_2)/(x_1+x_2)$ as $n \to \infty$,

$$\lim_{n\to\infty} n \Pr\left(\frac{1}{X_1} > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1 + a_{n2}x_2}\right) = \left(\frac{x_1x_2}{x_1+x_2}\right)^{-\alpha_1}.$$

Furthermore, by Lemma 1, given that $(a_{n1}a_{n2}x_1x_2)/(a_{n1}x_1 + a_{n2}x_2) \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\Pr\left(1/R > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1+a_{n2}x_2}\right)}{\Pr\left(1/X_1 > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1+a_{n2}x_2}\right)} = 0,$$

so that the right-hand side in Equation (B.5) tends to 0 as $n \to \infty$, and this implies the claim.

Case II. $\alpha_1 < \alpha_2$ and $\rho = \alpha_2$. Then for $i = 1, 2$, there exists a slowly varying function L_i such that $a_{ni} = n^{1/\alpha_i} L_i(n)$. Hence $a_{n2}/a_{n1} \to 0$ and $(x_1x_2)/\{x_1 + x_2(a_{n2}/a_{n1})\} \to x_2$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} n \Pr\left(\frac{1}{X_2} > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1 + a_{n2}x_2}\right) = x_2^{-\alpha_2}.$$

Moreover, by Lemma 1, given that $(a_{n1}a_{n2}x_1x_2)/(a_{n1}x_1 + a_{n2}x_2) \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\Pr\left(1/R > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1+a_{n2}x_2}\right)}{\Pr\left(1/X_2 > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1+a_{n2}x_2}\right)} = 0,$$

so that again the right-hand side in Equation (B.5) tends to 0 as $n \to \infty$.

Case III. $\alpha_1 < \alpha_2$ and $\rho \in [\alpha_1, \alpha_2)$. In this case, $1/X_1 \in \mathcal{M}(\Phi_{\alpha_1})$ and $1/X_2 \in \mathcal{M}(\Phi_\rho)$. Therefore, either directly when $\rho > \alpha_1$ or by Lemma 1, one can easily deduce that

$$\lim_{x\to\infty} \frac{\Pr(1/R > x)}{\Pr(1/X_1 > x)} = 0. \tag{B.6}$$

At the same time, Breiman's Lemma [4] implies that

$$\lim_{x\to\infty} \frac{\Pr(1/R > x)}{\Pr(1/X_2 > x)} = \frac{1}{\mathrm{E}\{1/(1-B)^\rho\}} = \frac{\mathrm{B}(\alpha_1,\alpha_2)}{\mathrm{B}(\alpha_1,\alpha_2-\rho)}. \tag{B.7}$$

Hence, for any $b \in (0,1)$,

$$\begin{aligned}\lim_{n\to\infty} n\Pr\{1/R > a_{n2}x_2(1-b)\} &= \lim_{n\to\infty} n\Pr\{1/X_2 > a_{n2}x_2(1-b)\}\frac{\Pr\{1/R > a_{n2}x_2(1-b)\}}{\Pr\{1/X_2 > a_{n2}x_2(1-b)\}}\\ &= \{x_2(1-b)\}^{-\rho}\frac{\mathrm{B}(\alpha_1,\alpha_2)}{\mathrm{B}(\alpha_1,\alpha_2-\rho)}\end{aligned}$$

so that

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 n\Pr\{1/R > a_{n2}x_2(1-b)\}\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b = n\Pr(1/X_2 > a_{n2}x_2) = x_2^{-\rho}\\ = x_2^{-\rho}\int_0^1 \frac{b^{\alpha_1-1}(1-b)^{\alpha_2-\rho-1}}{\mathrm{B}(\alpha_1,\alpha_2-\rho)}\,\mathrm{d}b = \int_0^1 \lim_{n\to\infty} n\Pr\{1/R > a_{n2}x_2(1-b)\}\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b.\end{aligned} \tag{B.8}$$

Given that for any $b \in (0,1)$, $\Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\} \leq \Pr\{1/R > a_{n2}x_2(1-b)\}$,

$$\begin{aligned}&\int_0^1 \liminf_{n\to\infty}\Big(n\big[\Pr\{1/R > a_{n2}x_2(1-b)\} - \Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\}\big]\Big)\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b\\ &\leq \liminf_{n\to\infty}\int_0^1 n\big[\Pr\{1/R > a_{n2}x_2(1-b)\} - \Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\}\big]\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b\end{aligned}$$

by Fatou's Lemma. Because of Equation (B.8), this inequality simplifies to

$$\begin{aligned}&x_2^{-\rho} - \int_0^1 \limsup_{n\to\infty}\big[n\Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\}\big]\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b\\ &\leq x_2^{-\rho} - \limsup_{n\to\infty}\int_0^1 n\Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\}\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b\end{aligned}$$

and hence

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}\big\{n\Pr(1/X_1 > a_{n1}x_1, 1/X_2 > a_{n2}x_2)\big\}\\ &\leq \int_0^1 \limsup_{n\to\infty}\big[n\Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\}\big]\frac{b^{\alpha_1-1}(1-b)^{\alpha_2-1}}{\mathrm{B}(\alpha_1,\alpha_2)}\,\mathrm{d}b.\end{aligned}$$

To show the desired claim, it thus suffices to show that for arbitrary $b \in (0,1)$,

$$\lim_{n\to\infty} n\Pr\{1/R > a_{n1}x_1b, 1/R > a_{n2}x_2(1-b)\} = 0. \tag{B.9}$$

To this end, fix $b \in (0,1)$ and observe that $a_{n1}/a_{n2} \to \infty$. Indeed, if $\rho > \alpha_1$, this follows directly from the fact that $a_{n1} = n^{1/\alpha_1}L_1(n)$ and $a_{n2} = n^{1/\rho}L_2(n)$ for some slowly varying functions L_1, L_2. When $\rho = \alpha_1$, suppose that $\liminf_{n\to\infty} a_{n1}/a_{n2}$ were finite. Then there exists a subsequence a_{n_k1}/a_{n_k2} such that $a_{n_k1}/a_{n_k2} \to a$ as $k \to \infty$ for some $a \in [0,\infty)$. Hence, for a fixed $\varepsilon > 0$ and all $k \geq k_0$, $a_{n_k1}/a_{n_k2} \leq a + \varepsilon$. Using the latter observation and Equation (B.7),

$$\begin{aligned}\lim_{k\to\infty} n_k\Pr(1/R > a_{n_k1}) &\geq \lim_{k\to\infty} n_k\Pr\{1/R > a_{n_k2}(a+\varepsilon)\}\\ &= \lim_{k\to\infty} n_k\Pr\{1/X_2 > a_{n_k2}(a+\varepsilon)\}\frac{\Pr\{1/R > a_{n_k2}(a+\varepsilon)\}}{\Pr\{1/X_2 > a_{n_k2}(a+\varepsilon)\}} = (a+\varepsilon)^{-\rho}\frac{\mathrm{B}(\alpha_1,\alpha_2)}{\mathrm{B}(\alpha_1,\alpha_2-\rho)} > 0.\end{aligned}$$

At the same time, by Equation (B.6),

$$\lim_{k\to\infty} n_k \Pr(1/R > a_{n_k 1}) = \lim_{k\to\infty} n_k \Pr(1/X_1 > a_{n_k 1}) \frac{\Pr(1/R > a_{n_k 1})}{\Pr(1/X_1 > a_{n_k 1})} = 0$$

and hence a contradiction. Therefore, $\liminf_{n\to\infty} a_{n1}/a_{n2} = \infty$ and hence $a_{n1}/a_{n2} \to \infty$ as $n \to \infty$. Because $a_{n1}b > a_{n2}(1-b)$ if and only if $b > a_{n2}/(a_{n1}+a_{n2})$ and $a_{n2}/(a_{n1}+a_{n2}) \to 0$ as $n \to \infty$, there exists n_0 such that for all $n \geq n_0$,

$$\begin{aligned} n \Pr\{1/R > a_{n1}x_1 b, 1/R > a_{n2}x_2(1-b)\} &= n \Pr(1/R > a_{n1}x_1 b) \\ &= n \Pr(1/X_1 > a_{n1}x_1 b) \frac{\Pr(1/R > a_{n1}x_1 b)}{\Pr(1/X_1 > a_{n1}x_1 b)}. \end{aligned}$$

The last expression tends to 0 as $n \to \infty$ by Equation (B.6) and hence Equation (B.9) indeed holds.

To prove part (iv), first recall that by Proposition 2 (b), $1/X_i \in \mathcal{M}(\Phi_{\alpha_i})$, $i = 1, 2$, and hence there exists scaling sequences (a_{n1}) and (a_{n2}). Recall that for $i = 1, 2$, $a_{ni} = n^{1/\alpha_i} L_i(n)$ for some slowly varying function L_i. As in the proof of part (i), $n \Pr(1/X_1 > a_{n1}x_1, 1/X_2 > a_{n2}x_2)$ can be bounded above by the right-hand side in Equation (B.5). Markov's inequality further implies that for $\beta \in (\alpha_2, \alpha_1 + \alpha_2)$ such that $\mathrm{E}(1/R^\beta) < \infty$,

$$n \Pr\left(\frac{1}{R} > \frac{a_{n1}a_{n2}x_1x_2}{a_{n1}x_1 + a_{n2}x_2}\right) \leq n\mathrm{E}\left(1/R^\beta\right) \frac{(a_{n1}x_1 + a_{n2}x_2)^\beta}{(a_{n1}a_{n2}x_1x_2)^\beta} = \frac{\mathrm{E}\left(1/R^\beta\right)}{(x_1x_2)^\beta} \left\{\frac{x_1}{n^{1/\alpha_2 - 1/\beta}L_2(n)} + \frac{x_2}{n^{1/\alpha_1 - 1/\beta}L_1(n)}\right\}^\beta$$

The right-most expression tends to 0 as $n \to \infty$ because for any $i = 1, 2$ and $\rho > 0$, $n^\rho L_i(n) \to \infty$.

Proof of Theorem 2. First note that a positive random vector $\boldsymbol{Y}$ is in the maximum domain of attraction of a multivariate extreme-value distribution H_0 with Fréchet margins if and only if there exist sequences of positive constants $(a_{ni}) \in (0, \infty)$, $i = 1, \ldots, d$, so that for all $\boldsymbol{y} \in \mathbb{R}_+^d$,

$$\begin{aligned} &\lim_{n\to\infty} n\{1 - \Pr(Y_1 \leq a_{n1}y_1, \ldots, Y_d \leq a_{nd}y_d)\} = \\ &\qquad \lim_{n\to\infty} n \left\{\sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} (-1)^{k+1} \Pr(Y_{i_1} > a_{ni_1}y_{i_1}, \ldots, Y_{i_k} > a_{ni_k}y_{i_k})\right\} = -\log H_0(\boldsymbol{y}). \end{aligned} \tag{B.10}$$

This multivariate version of the Poisson approximation holds by the same argument as in the univariate case [11, Proposition 3.1.1].

To prove part (a), suppose that $1/R \in \mathcal{M}(\Phi_\rho)$ for some $\rho \in (0, \alpha_M]$. By Proposition 2, one then has that for any $i \in \mathbb{I}_1$, $1/(RD_i) \in \mathcal{M}(\Phi_{\alpha_i})$. For any $i \in \mathbb{I}_1$, let (a_{ni}) be a sequence of positive constants such that, for all $x > 0$, $n \Pr\{1/(RD_i) > a_{ni}x\} \to x^{-\alpha_i}$ as $n \to \infty$; such a sequence exists by the univariate Poisson approximation [11, Proposition 3.1.1]. The same result also guarantees the existence of a sequence (a_n) of positive constants such that, for all $x > 0$, $n \Pr(1/R > a_n x) \to x^{-\rho}$ as $n \to \infty$. Now set, for any $i \in \mathbb{I}_2$,

$$b_i = \mathrm{E}\left(D_i^{-\rho}\right) = \frac{\Gamma(\alpha_i - \rho)\Gamma(\bar{\alpha} - \alpha_i)}{\Gamma(\bar{\alpha} - \rho)} \times \frac{\Gamma(\bar{\alpha})}{\Gamma(\alpha_i)\Gamma(\bar{\alpha} - \alpha_i)} = \frac{\Gamma(\bar{\alpha})/\Gamma(\bar{\alpha} - \rho)}{\Gamma(\alpha_i)/\Gamma(\alpha_i - \rho)}, \tag{B.11}$$

and define, for any $i \in \mathbb{I}_2$ and $n \in \mathbb{N}$, $a_{ni} = b_i^{1/\rho} a_n$. As detailed in the proof of Proposition 2 (a), Breiman's Lemma then implies that, for all $i \in \mathbb{I}_2$ and $x > 0$,

$$\lim_{n\to\infty} n \Pr\left\{\frac{1}{RD_i} > a_{ni}x\right\} = \lim_{n\to\infty} n \Pr\left\{\frac{1}{R} > a_n(b_i^{1/\rho}x)\right\} \frac{\Pr\left\{\frac{1}{RD_i} > a_n(b_i^{1/\rho}x)\right\}}{\Pr\left\{\frac{1}{R} > a_n(b_i^{1/\rho}x)\right\}} = x^{-\rho} b_i^{-1} b_i = x^{-\rho},$$

given that for all $i \in \mathbb{I}_2$, $D_i \sim \mathsf{Beta}(\alpha_i, \bar{\alpha} - \alpha_i)$.

Next, fix an arbitrary $\boldsymbol{x} \in (0,\infty)^d$, $k \in \{2,\ldots,d\}$ and indices $1 \le i_1 < \cdots < i_k \le d$. To calculate the limit of $n\Pr(1/(RD_{i_1}) > a_{ni_1}x_{i_1},\ldots,1/(RD_{i_k}) > a_{ni_k}x_{i_k})$, two cases must be distinguished:

Case I. $\{i_1,\ldots,i_k\} \cap \mathbb{I}_1 \neq \emptyset$. In this case, suppose, without loss of generality, that $i_1 \in \mathbb{I}_1$. Then

$$0 \le n\Pr\left(\frac{1}{RD_{i_1}} > a_{ni_1}x_{i_1},\ldots,\frac{1}{RD_{i_k}} > a_{ni_k}x_{i_k}\right) \le n\Pr\left(\frac{1}{RD_{i_1}} > a_{ni_1}x_{i_1},\frac{1}{RD_{i_2}} > a_{ni_2}x_{i_2}\right).$$

Now either $i_2 \in \mathbb{I}_1$, in which case $\rho \ge \alpha_{i_1} \vee \alpha_{i_2}$, or $i_2 \in \mathbb{I}_2$, so that

$\alpha_{i_1} \le \rho < \alpha_{i_2}$. Either way, Lemma 3 implies that

$$\lim_{n\to\infty} n\Pr\left(\frac{1}{RD_{i_1}} > a_{ni_1}x_{i_1},\frac{1}{RD_{i_2}} > a_{ni_2}x_{i_2}\right) = 0$$

and consequently $n\Pr\{1/(RD_{i_1}) > a_{ni_1}x_{i_1},\ldots,1/(RD_{i_k}) > a_{ni_k}x_{i_k}\} \to 0$ as $n\to\infty$.

Case II. $\{i_1,\ldots,i_k\} \cap \mathbb{I}_1 = \emptyset$. In this case, let $Z_{i_1,\ldots,i_k} = \max(x_{i_1}(b_{i_1})^{1/\rho}D_{i_1},\ldots,x_{i_k}(b_{i_k})^{1/\rho}D_{i_k})$ and observe that for any $\varepsilon > 0$ such that $\rho+\varepsilon < \min(\alpha_1,\ldots,\alpha_d)$,

$$\mathrm{E}\left(\frac{1}{Z^{\rho+\varepsilon}_{i_1,\ldots,i_k}}\right) \le x_{i_1}^{-\rho-\varepsilon} b_{i_1}^{-(\rho+\varepsilon)/\rho}\mathrm{E}\left(\frac{1}{D^{\rho+\varepsilon}_{i_1}}\right) < \infty.$$

Therefore, by Breiman's Lemma,

$$\begin{aligned}\lim_{n\to\infty} n\Pr\left(\frac{1}{RD_{i_1}} > a_{ni_1}x_{i_1},\ldots,\frac{1}{RD_{i_k}} > a_{ni_k}x_{i_k}\right) &= \lim_{n\to\infty} n\Pr\left(\frac{1}{RZ_{i_1,\ldots,i_k}} > a_n\right) = \mathrm{E}\left(Z^{-\rho}_{i_1,\ldots,i_k}\right)\\ &= \mathrm{E}\left[\left\{\max_{1\le j\le k}\left(x_{i_j}b_{i_j}^{1/\rho}D_{i_j}\right)\right\}^{-\rho}\right] = \mathrm{E}\left[\min_{1\le j\le k}\left\{\frac{\left(x_{i_j}D_{i_j}\right)^{-\rho}}{b_{i_j}}\right\}\right].\end{aligned}$$

Putting the above calculations together, one then has, for any $\boldsymbol{x} \in \mathbb{R}^d_+$,

$$\lim_{n\to\infty} n\left\{1-\Pr\left(\frac{1}{RD_1} \le a_{n1}x_1,\ldots,\frac{1}{RD_d} \le a_{nd}x_d\right)\right\} = \sum_{i\in\mathbb{I}_1} x_i^{-\alpha_i} + \sum_{k=1}^{|\mathbb{I}_2|}\sum_{\substack{\{i_1,\ldots,i_k\}\subseteq\mathbb{I}_2\\ i_1<\cdots<i_k}} (-1)^{k+1}\mathrm{E}\left[\min_{1\le j\le k}\left\{\frac{\left(x_{i_j}D_{i_j}\right)^{-\rho}}{b_{i_j}}\right\}\right]$$

Furthermore, one can readily establish by induction that for any $\boldsymbol{t} \in \mathbb{R}^d$,

$$\sum_{k=1}^{|\mathbb{I}_2|}\sum_{\substack{\{i_1,\ldots,i_k\}\subseteq\mathbb{I}_2\\ i_1<\cdots<i_k}} (-1)^{k+1}\min(t_{i_1},\ldots,t_{i_k}) = \max_{i\in\mathbb{I}_2}(t_i).$$

Hence, for any $\boldsymbol{x} \in \mathbb{R}^d_+$,

$$\lim_{n\to\infty} n\left\{1-\Pr\left(\frac{1}{RD_1} \le a_{n1}x_1,\ldots,\frac{1}{RD_d} \le a_{nd}x_d\right)\right\} = \sum_{i\in\mathbb{I}_1} x_i^{-\alpha_i} + \mathrm{E}\left[\max_{i\in\mathbb{I}_2}\left\{\frac{(x_iD_i)^{-\rho}}{b_i}\right\}\right].$$

By the multivariate Poisson approximation (B.10), $1/\boldsymbol{X} \in \mathcal{M}(H_0)$, where for all $\boldsymbol{x} \in \mathbb{R}^d_+$,

$$H_0(\boldsymbol{x}) = \exp\left(-\sum_{i\in\mathbb{I}_1} x_i^{-\alpha_i} - \mathrm{E}\left[\max_{i\in\mathbb{I}_2}\left(\frac{(x_iD_i)^{-\rho}}{b_i}\right\}\right]\right).$$

The margins of H_0 are given, for all $i \in \mathbb{I}_1$, by $H_{0i}(x) = x^{-\alpha_i}$ and for all $i \in \mathbb{I}_2$, $H_{0i}(x) = \exp(-x^{-\rho})$. By Sklar's Theorem, the unique copula of H_0 is given, for all $\boldsymbol{u} \in [0,1]^d$, by (2), where for all $\boldsymbol{x} \in \mathbb{R}^d_+$,

$$\ell(\boldsymbol{x}) = \sum_{i\in\mathbb{I}_1} x_i + \mathrm{E}\left\{\max_{i\in\mathbb{I}_2}\left(\frac{x_iD_i^{-\rho}}{b_i}\right)\right\}.$$

The first expression for ℓ follows immediately from Equation (B.11). The second expression is readily verified using Lemma 2, given the fact that if $B \sim \mathrm{Beta}(\bar\alpha_2, \bar\alpha - \bar\alpha_2)$, $\mathrm{E}(B^{-\rho}) = \Gamma(\bar\alpha_2 - \rho)\Gamma(\bar\alpha)/\Gamma(\bar\alpha - \rho)\Gamma(\bar\alpha_2)$.

To prove part (b), recall that by Proposition 2 (b), $1/X_i \in \mathcal{M}(\Phi_{\alpha_i})$. Hence, there exist sequences of positive constants (a_{ni}), $i = 1, \dots, d$, such that for all $i = 1, \dots, d$ and all $x > 0$, $n \Pr(1/(RD_i) > a_{ni}x) \to x^{-\alpha_i}$ as $n \to \infty$. By Lemma 2 (ii), it also follows that for arbitrary $\boldsymbol{x} \in (0,\infty)^d$, $k \in \{2, \dots, d\}$ and indices $1 \le i_1 < \cdots < i_k \le d$,

$$\begin{aligned} 0 &\le \lim_{n\to\infty} n \Pr\left(\frac{1}{RD_{i_1}} > a_{ni_1}x_{i_1}, \dots, \frac{1}{RD_{i_k}} > a_{ni_k}x_{i_k}\right) \\ &\le \lim_{n\to\infty} n \Pr\left(\frac{1}{RD_{i_1}} > a_{ni_1}x_{i_1}, \frac{1}{RD_{i_2}} > a_{ni_2}x_{i_2}\right) = 0. \end{aligned}$$

Thus, by Equation (B.10), $1/\boldsymbol{X}$ is in the domain of attraction of the multivariate extreme-value distribution given, for all $\boldsymbol{x} \in \mathbb{R}_+^d$, by $H_0(\boldsymbol{x}) = \exp(-x_1^{-\alpha_1} - \cdots - x_d^{-\alpha_d})$, as was to be showed.

C. Proofs from Section 4

Proof of Proposition 3. In view of Corollary 1 and Theorem 2 in [23], it only remains to derive the explicit expression for ℓ^{nD}. Because $1 - \psi(1/\cdot) \in \mathcal{R}_{-\rho}$, there exists a slowly varying function L such that for all $x > 0$, $1 - \psi(1/x) = x^{-\rho}L(x)$. Given that the distribution function $\psi(1/\cdot)$ is in the domain of attraction of Φ_ρ, the Poisson approximation implies that there exists a sequence (a_n) of positive constants such that, for all $x > 0$,

$$\lim_{n\to\infty} n\left[1 - \psi\{1/(a_n x)\}\right] = \lim_{n\to\infty} n(a_n x)^{-\rho} L(a_n x) = x^{-\rho}. \tag{C.1}$$

Furthermore, by Equation (A6) in the proof of Theorem 2 (a) in [23], one has, for any $j = 1, \dots, \bar\alpha - 2$,

$$\lim_{x\to\infty} \frac{(-1)^j x^{-j} \psi^{(j)}(1/x)}{\kappa_j x^{-\rho} L(x)} = 1, \tag{C.2}$$

where $\kappa_j = \rho\Gamma(j - \rho)/\Gamma(1 - \rho)$. Now for all $i = 1, \dots, d$, Equation (7) yields, for any $x > 0$,

$$n \Pr\left(\frac{1}{X_i} > a_n x\right) = n\left\{1 - \bar H_i\left(\frac{1}{a_n x}\right)\right\} = n(a_n x)^{-\rho} L(a_n x)\left\{1 - \sum_{j=1}^{\alpha_i - 1} \frac{(-1)^j (a_n x)^{-j} \psi^{(j)}(1/a_n x)}{j!(a_n x)^{-\rho} L(a_n x)}\right\}.$$

Given that $a_n \to \infty$ as $n \to \infty$, the last expression converges by Equations (C.1) and (C.2) as $n \to \infty$ to

$$x^{-\rho}\left\{1 - \sum_{j=1}^{\alpha_i - 1} \frac{\kappa_j}{j!}\right\} = x^{-\rho}\left\{1 - \rho \sum_{j=1}^{\alpha_i - 1} \frac{\Gamma(j - \rho)}{\Gamma(j+1)\Gamma(1-\rho)}\right\} = x^{-\rho} \frac{c(\alpha_i, -\rho)}{\Gamma(1-\rho)}.$$

The Poisson approximation thus implies that, as $n \to \infty$, for all $i = 1, \dots, d$ and $x > 0$,

$$\bar H_i^n\left(\frac{1}{a_n x}\right) \to \exp\left\{-x^{-\rho} \frac{c(\alpha_i, -\rho)}{\Gamma(1-\rho)}\right\}. \tag{C.3}$$

For any $\boldsymbol{x} \in (0,\infty)^d$, let $1/(a_n\boldsymbol{x}) = \{1/(a_n x_1), \dots, 1/(a_n x_d)\}$ and denote by $\bar x_H$ the harmonic mean of $\boldsymbol{x}$, viz. $\bar x_H = d/(1/x_1 + \cdots + 1/x_d)$. From Equation (6) one then has

$$\begin{aligned} n\left\{1 - \bar H\left(\frac{1}{a_n \boldsymbol{x}}\right)\right\} &= n\left(\frac{a_n \bar x_H}{d}\right)^{-\rho} L\left(\frac{a_n \bar x_H}{d}\right) \\ &\times \left\{1 - \sum_{\substack{(j_1,\dots,j_d)\in\mathbb{I}_{\boldsymbol{\alpha}} \\ (j_1,\dots,j_d)\neq \mathbf{0}_d}} \frac{(-1)^{j_1+\cdots+j_d} \left(\frac{a_n \bar x_H}{d}\right)^{-j_1-\cdots-j_d} \psi^{(j_1+\cdots+j_d)}\left(\frac{d}{a_n \bar x_H}\right)}{j_1!\cdots j_d! \left(\frac{a_n \bar x_H}{d}\right)^{-\rho} L\left(\frac{a_n \bar x_H}{d}\right)} \prod_{i=1}^d \left(\frac{\bar x_H}{d x_i}\right)^{j_i}\right\} \end{aligned}$$

By Equation (C.2), the right most expression in the curly brackets converges, as $n \to \infty$, to

$$\left\{1-\rho \sum_{\substack{(j_1,\dots,j_d)\in\mathbb{I}_{\boldsymbol{\alpha}}\\(j_1,\dots,j_d)\neq\mathbf{0}_d}} \frac{\Gamma(j_1+\cdots+j_d-\rho)}{\Gamma(1-\rho)j_1!\cdots j_d!}\prod_{i=1}^{d}\left(\frac{\bar{x}_H}{dx_i}\right)^{j_i}\right\}$$

$$=\left\{1-\rho \sum_{\substack{(j_1,\dots,j_d)\in\mathbb{I}_{\boldsymbol{\alpha}}\\(j_1,\dots,j_d)\neq\mathbf{0}_d}} \frac{\Gamma(j_1+\cdots+j_d-\rho)}{\Gamma(1-\rho)}\prod_{i=1}^{d}\frac{1}{\Gamma(j_i+1)}\left(\frac{1/x_i}{1/x_1+\cdots+1/x_d}\right)^{j_i}\right\}.$$

Furthermore, Equation (C.1) implies that, as $n \to \infty$,

$$n\left(\frac{a_n\bar{x}_H}{d}\right)^{-\rho} L\left(\frac{a_n\bar{x}_H}{d}\right) \to \left(\frac{1}{x_1}+\cdots+\frac{1}{x_d}\right)^{\rho}.$$

Consequently, as $n \to \infty$, $n\{1-\bar{H}(1/a_n\boldsymbol{x})\} \to -\log H_0(\boldsymbol{x})$, where

$$-\log H_0(\boldsymbol{x}) = \left(\frac{1}{x_1}+\cdots+\frac{1}{x_d}\right)^{\rho}\left\{1-\rho \sum_{\substack{(j_1,\dots,j_d)\in\mathbb{I}_{\boldsymbol{\alpha}}\\(j_1,\dots,j_d)\neq\mathbf{0}_d}} \frac{\Gamma(j_1+\cdots+j_d-\rho)}{\Gamma(1-\rho)}\prod_{i=1}^{d}\frac{1}{\Gamma(j_i+1)}\left(\frac{1/x_i}{\sum_{j=1}^{d}\frac{1}{x_j}}\right)^{j_i}\right\}.$$

By Equation (B.10), $1/\boldsymbol{X} \in \mathcal{M}(H_0)$. From Equation (C.3), the margins of H_0 are scaled Fréchet, and Sklar's theorem implies that the unique copula of H_0 is of the form (2) with stable tail dependence function as in Proposition 3.

Proof of Proposition 4. In view of Corollary 2 and Theorem 1 in [23], it only remains to compute the expression for ℓ^{pD} given in part (a). Suppose that $\psi \in \mathcal{R}_{-\rho}$ for some $\rho > 0$. This means that there exists a slowly varying function such that for all $x > 0$, $\psi(x) = x^{-\rho}L(x)$. Because ψ is itself a survival function, $\psi \in \mathcal{M}(\Phi_\rho)$ and by the univariate Poisson approximation, there exists a sequence (a_n) of strictly positive constants such that, for all $x > 0$,

$$\lim_{n\to\infty} n\psi(a_n x) = x^{-\rho}. \tag{C.4}$$

Furthermore, by Equation (A1) in the proof of Theorem 1 (a) in [23], one has, for any $j = 1,\dots,\bar{\alpha}-1$,

$$\lim_{x\to\infty} \frac{(-1)^j x^j \psi^{(j)}(x)}{\psi(x)} = c(j,\rho). \tag{C.5}$$

Now let $\boldsymbol{X}$ be the Dirichlet random vector with parameters $\boldsymbol{\alpha}$ and radial part R whose Williamson $\bar{\alpha}$-transform is ψ. Denote the distribution function of $\boldsymbol{X}$ by H and its margins by H_i, $i = 1,\dots,n$. Then for all $i = 1,\dots,d$, Equations (C.4) and (C.5) imply that

$$\lim_{n\to\infty} n\bar{H}_i(a_n x) = \lim_{n\to\infty} n\sum_{j=0}^{\alpha_i-1}\frac{(-1)^j(a_n x)^j\psi^{(j)}(a_n x)}{j!} = x^{-\rho}\sum_{j=0}^{\alpha_i-1}\frac{\Gamma(j+\rho)}{\Gamma(\rho)\Gamma(j+1)} = \frac{x^{-\rho}c(\alpha_i,\rho)}{\Gamma(\rho+1)} \tag{C.6}$$

and hence, by the Poisson approximation, $H_i^n(x) \to \exp\{-x^{-\rho}c(\alpha_i,\rho)/\Gamma(\rho+1)\}$ as $n\to\infty$.

Next, for arbitrary $k = 1,\dots,d$ and $1 \le i_1 < \cdots < i_k \le d$, let $\mathbb{I}_{(\alpha_{i_1},\dots,\alpha_{i_k})} = \{0,\dots,\alpha_{i_1}-1\}\times\cdots\times\{0,\dots,\alpha_{i_k}-1\}$. For any $\boldsymbol{x} \in (0,\infty)^d$, Equations (6), (C.4) and (C.5) imply that

$$\lim_{n\to\infty} n\Pr(X_{i_1} > x_{i_1},\dots,X_{i_k} > x_{i_k})$$

$$= \lim_{n\to\infty} n \sum_{(j_1,\dots,j_k)\in\mathbb{I}_{(\alpha_{i_1},\dots,\alpha_{i_k})}} (-1)^{j_1+\cdots+j_k}\frac{\psi^{(j_1+\cdots+j_k)}\{a_n(x_{i_1}+\cdots+x_{i_k})\}}{j_1!\cdots j_k!}\prod_{m=1}^{k}(a_n x_{i_m})^{j_m}$$

$$= (x_{i_1}+\cdots+x_{i_k})^{-\rho}\sum_{(j_1,\dots,j_k)\in\mathbb{I}_{(\alpha_{i_1},\dots,\alpha_{i_k})}}\frac{\Gamma(j_1+\cdots+j_k+\rho)}{\Gamma(\rho)j_1!\cdots j_k!}\prod_{m=1}^{k}\left(\frac{x_{i_m}}{x_{i_1}+\cdots+x_{i_k}}\right)^{j_m}.$$

Therefore, for any $\boldsymbol{x} \in (0,\infty)^d$,

$$\lim_{n\to\infty} n \left\{ \sum_{k=1}^{d} \sum_{1\le i_1<\cdots<i_k\le d} (-1)^{k+1} \Pr(X_{i_1} > a_n x_{i_1}, \ldots, X_{i_k} > a_n x_{i_k}) \right\} = -\log H_0(\boldsymbol{x}),$$

where

$$-\log H_0(\boldsymbol{x}) = \sum_{k=1}^{d} \sum_{1\le i_1<\cdots<i_k\le d} (-1)^{k+1} (x_{i_1} + \cdots + x_{i_k})^{-\rho} \times \sum_{(j_1,\ldots,j_k)\in \mathbb{I}_{(\alpha_{i_1},\ldots,\alpha_{i_k})}} \frac{\Gamma(j_1+\cdots+j_k+\rho)}{\Gamma(\rho) j_1!\cdots j_k!} \prod_{m=1}^{k} \left(\frac{x_{i_m}}{x_{i_1}+\cdots+x_{i_k}} \right)^{j_m}.$$

By Equation (B.10), $\boldsymbol{X} \in \mathcal{M}(H_0)$. As argued above, the margins of H_0 are given, for all $i = 1,\ldots,d$ and $x > 0$, by $\exp\{-x^{-\rho} c(\alpha_i,\rho)/\Gamma(\rho+1)\}$. Sklar's theorem thus implies that the unique copula of H_0 is of the form (2) with stable tail dependence function indeed as given by the expression in part (a).

D. Proofs from Section 5

Proof of Proposition 5. First, we show that for any $\rho > -\min(\alpha_1,\ldots,\alpha_d)$, $\rho \neq 0$,

$$\mathrm{E}\left[\max_{1\le i\le d} \left\{ \frac{x_i D_i^\rho}{c(\alpha_i,\rho)} \right\} \right] = \frac{\Gamma(\bar\alpha)}{|\rho|^{d-1} \prod_{i=1}^d \Gamma(\alpha_i)} \int_{\mathbb{S}_d} \max(x_i t_i) \left[\sum_{i=1}^d \{c(\alpha_i,\rho) t_i\}^{1/\rho} \right]^{-\rho-\bar\alpha} \prod_{i=1}^d \{c(\alpha_i,\rho)\}^{\alpha_i/\rho} (t_i)^{\alpha_i/\rho-1} \,\mathrm{d}\boldsymbol{t}. \tag{D.1}$$

Indeed, using the fact that $(D_1,\ldots,D_d) \stackrel{\mathrm{d}}{=} \boldsymbol{Z}/\|\boldsymbol{Z}\|$, where $Z_i \sim \mathsf{Ga}(\alpha_i,1)$, $i = 1,\ldots,d$ are independent,

$$\mathrm{E}\left[\max_{1\le i\le d} \left\{ \frac{x_i D_i^\rho}{c(\alpha_i,\rho)} \right\} \right] = \int_{\mathbb{R}_+^d} \max_{1\le i\le d} \left\{ \frac{x_i z_i^\rho}{c(\alpha_i,\rho)} \right\} (z_1+\cdots+z_d)^{-\rho} \prod_{i=1}^d \frac{e^{-z_i} z_i^{\alpha_i-1}}{\Gamma(\alpha_i)} \,\mathrm{d}\boldsymbol{z}.$$

Make a change of variable $t_i = \{z_i^\rho / c(\alpha_i,\rho)\} / \sum_{j=1}^d z_j^\rho / c(\alpha_j,\rho)$ for $i = 1,\ldots,d-1$ and $w = \sum_{j=1}^d z_j^\rho / c(\alpha_j,\rho)$. For ease of notation, set also $t_d = 1 - \sum_{i=1}^{d-1} t_i$. Then, for $i = 1,\ldots,d$, $z_i = \{c(\alpha_i,\rho) t_i w\}^{1/\rho}$ and the absolute value of the Jacobian is

$$|\mathbf{J}| = \frac{1}{|\rho|^d} w^{d/\rho-1} \prod_{i=1}^d c(\alpha_i,\rho)^{1/\rho} t_i^{1/\rho-1}.$$

Therefore,

$$\mathrm{E}\left[\max_{1\le i\le d} \left\{ \frac{x_i D_i^\rho}{c(\alpha_i,\rho)} \right\} \right] = \frac{1}{|\rho|^d \prod_{i=1}^d \Gamma(\alpha_i)} \int_{\mathbb{S}_d} \max_{1\le i\le d} (x_i t_i) \left[\sum_{i=1}^d \{c(\alpha_i,\rho) t_i\}^{1/\rho} \right]^{-\rho} \prod_{i=1}^d c(\alpha_i,\rho)^{\alpha_i/\rho} t_i^{\alpha_i/\rho-1} \times \int_0^\infty w^{\bar\alpha/\rho-1} e^{-w^{1/\rho} \sum_{i=1}^d \{c(\alpha_i,\rho) t_i\}^{1/\rho}} \,\mathrm{d}w \,\mathrm{d}\boldsymbol{t}.$$

Equation (D.1) now follows from the fact that

$$\int_0^\infty w^{\bar\alpha/\rho-1} e^{-w^{1/\rho} \sum_{i=1}^d \{c(\alpha_i,\rho) t_i\}^{1/\rho}} \,\mathrm{d}w = |\rho|\, \Gamma(\bar\alpha) \left[\sum_{i=1}^d \{c(\alpha_i,\rho) t_i\}^{1/\rho} \right]^{-\bar\alpha}.$$

The expression for h^{D} now follows directly from Eqs. (3) and (D.1), while the formulas for h^{pD} and h^{nD} obtain upon setting $\rho = \rho$ and $\rho = -\rho$, respectively.

E. Proofs from Section 7

Proof of Proposition 6. For $k = 1, \ldots, d$, the formula for the kth order mixed partial derivatives of $\ell^{\mathrm{D}}(1/\boldsymbol{x})$ can be established from eq. (12). Indeed, if $\boldsymbol{V}$ denotes a random vector with independent scaled Gamma components $V_i \sim \mathsf{sGa}\{1/c(\alpha_i,\rho), 1/\rho, \alpha_i\}$, then the point process representation eq. (12) implies that, for all $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$\ell^{\mathrm{D}}(1/\boldsymbol{x}) = \int_0^\infty \Pr\left(\frac{V_i}{t} > x_i \text{ for at least one } i \in \{1,\ldots,d\}\right) \mathrm{d}t = \int_0^\infty \left[1 - \prod_{i=1}^d F\left\{x_i t; \frac{1}{c(\alpha_i,\rho)}, \frac{1}{\rho}, \alpha_i\right\}\right] \mathrm{d}t. \quad (18)$$

For any $k = 1, \ldots, d$, the expression on the right-hand side of eq. (18) can be differentiated with respect to $x_1, \ldots, x_k$ under the integral sign. This gives the formulas for $\partial \ell^{\mathrm{D}}(1/\boldsymbol{x})/\partial x_1 \ldots \partial x_k$. When $k = d$, eq. (18) implies that

$$\begin{aligned}
\frac{\partial^d \ell^{\mathrm{D}}(1/\boldsymbol{x})}{\partial x_1 \cdots \partial x_d} &= -\int_0^\infty t^d \prod_{i=1}^d f\left\{x_i t; \frac{1}{c(\alpha_i,\rho)}, \frac{1}{\rho}, \alpha_i\right\} \mathrm{d}t \\
&= \frac{1}{\rho^d} \prod_{j=1}^d \frac{c(\alpha_j,\rho)\{c(\alpha_j,\rho)x_j\}^{\alpha_j/\rho-1}}{\Gamma(\alpha_j)} \int_0^\infty t^{\alpha_j/\rho} \exp\left[-t^{1/\rho} \sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho}\right] \mathrm{d}t \\
&= \frac{\Gamma(\bar{\alpha}+\rho)}{\rho^{d-1}\left[\sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho}\right]^{\bar{\alpha}+\rho}} \prod_{j=1}^d \frac{c(\alpha_j,\rho)\{c(\alpha_j,\rho)x_j\}^{\alpha_j/\rho-1}}{\Gamma(\alpha_j)},
\end{aligned}$$

where the last equality follows upon making the change of variable $u = \sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho} t^{1/\rho}$. Alternatively, Theorem 1 in [6] implies that that the dth order mixed partial derivative of $\ell^{\mathrm{D}}(1/\boldsymbol{x})$ equals $-d\|\boldsymbol{x}\|^{-d-1} h^{\mathrm{D}}(\boldsymbol{x}/\|\boldsymbol{x}\|; \rho, \boldsymbol{\alpha})$, which indeed simplifies to $-dh^{\mathrm{D}}(\boldsymbol{x};\rho,\boldsymbol{\alpha})$ given that $h^{\mathrm{D}}(\boldsymbol{x}/\|\boldsymbol{x}\|;\rho,\boldsymbol{\alpha}) = \|\boldsymbol{x}\|^{d+1} h^{\mathrm{D}}(\boldsymbol{x};\rho,\boldsymbol{\alpha})$.

Finally, the formulas for $F(x;a,b,c)$ follow immediately from the fact that the scaled Gamma distribution is also the distribution of the random variable $aZ^{1/b}$, where Z is Gamma with shape α_i and unit scaling.

Derivation of the gradient score. Straightforward calculations show that

$$\begin{aligned}
\frac{\partial \log dh^{\mathrm{D}}(\boldsymbol{x})}{\partial x_i} &= -\frac{(\bar{\alpha}+\rho)c(\alpha_i,\rho)^{1/\rho} x_i^{1/\rho-1}}{\rho \sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho}} + \left(\frac{\alpha_i}{\rho} - 1\right)\frac{1}{x_i} \\
\frac{\partial^2 \log dh^{\mathrm{D}}(\boldsymbol{x})}{\partial x_i \partial x_k} &= -\frac{(\bar{\alpha}+\rho)c(\alpha_i,\rho)^{1/\rho} x_i^{1/\rho-1}}{\rho \sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho}} \left[\left(\frac{1}{\rho}-1\right)\frac{\mathrm{I}_{ik}}{x_i} - \frac{c(\alpha_k,\rho)^{1/\rho} x_k^{1/\rho-1}}{\rho \sum_{j=1}^d \{c(\alpha_j,\rho)x_j\}^{1/\rho}}\right] - \left(\frac{\alpha_i}{\rho}-1\right)\frac{\mathrm{I}_{ik}}{x_i^2}.
\end{aligned}$$