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Electromagnetic wave propagation in media consisting of dispersive metamaterials

Propagation d'une onde électromagnétique dans un milieu constitué de métamatériaux dispersifs

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A B S T R A C T

We establish the well-posedness, the finite speed propagation, and a regularity result for Maxwell's equations in media consisting of dispersive (frequency dependent) metamaterials. Two typical examples for such metamaterials are materials obeying Drude's and Lorentz' models. The causality and the passivity are the two main assumptions and play a crucial role in the analysis. It is worth noting that by contrast the well-posedness in the frequency domain is not ensured in general. We also provide some numerical experiments using the Drude's model to illustrate its dispersive behaviour.

R É S U M É

Nous montrons que les équations de Maxwell dans un milieu constitué de métamatériaux dispersifs (dépendant de la fréquence) forment un problème bien posé, à vitesse de propagation finie et satisfaisant un résultat de régularité. Deux exemples typiques de tels métamatériaux sont les matériaux régis par les modèles de Drude et de Lorentz. La causalité et la passivité sont les deux hypothèses principales ; elles jouent un rôle essentiel dans l'analyse. Il vaut la peine de remarquer qu'en revanche, rien n'assure, en général, le caractère bien posé dans le domaine des fréquences. Nous présentons également quelques résultats numériques utilisant le modèle de Drude, afin d'illustrer le comportement dispersif.

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1. Introduction

Metamaterials are smart materials engineered to have properties that have not yet been found in nature. They have recently attracted a lot of attention from the scientific community, not only because of potentially interesting applications, but also because of challenges in understanding their peculiar properties.

An important class of metamaterials is the one of negative index metamaterials (NIMs). The study of NIMs was initiated a few decades ago in the seminal work of Veselago [42], in which the existence of such materials was postulated. The existence of NIMs was confirmed by Shelby, Smith, and Schultz in [39]. New fabrication techniques now allow the construction of NIMs at scales that are interesting for applications, and have made them a very active topic of investigation. One of the interesting properties of NIMs is superlensing, i.e. the possibility to beat the Rayleigh diffraction limit: no constraint between the size of the object and the wavelength is imposed. This was first proposed by Veselago for a slab of index -1 and later studied in various contexts in [21,32,35,36,38]. The rigorous proof of superlensing was given in [25,28] for related lens designs. Another interesting application of NIMs is cloaking objects. Various schemes were suggested in [17,29] and established rigorously in [26,29]. NIMs can be used for cloaking sources, see, e.g., [22,24]. Another attracting class of metamaterials is the one of hyperbolic metamaterials (HMMs). HMMs can be used for superlensing, see [3,15,19]; other promising potential applications of HMMs can be found in [37] and references therein. The peculiar properties and the difficulties in the study of NIMs come from the fact that the modelling equations have sign-changing coefficients. In contrast, the modelling of HMMs involves equations of changing type, elliptic in some regions, hyperbolic in other ones.

The well-posedness of equations modelling metamaterials has been investigated mainly in the frequency domain. Concerning NIMs, it is now known that one needs to impose conditions on the coefficients of the equations near the sign-changing coefficient-interface to insure the well-posedness, see [2,8,23,27,34] and references therein; otherwise, the equations are unstable, see [27]. Concerning HMMs, it is shown in [3] that the stability is very sensitive to the geometry of the hyperbolic region. As far as we know, there are very few works on the stability of metamaterials apart from NIMs in the frequency domain.

This work is on Maxwell's equations in the time domain for media consisting of dispersive metamaterials. These are metamaterials whose material constants are frequency dependent. Two typical examples of such metamaterials are the ones obeying Drude's and Lorentz' models. The study of dispersive metamaterials in the time domain for NIMs was considered by Tip in [41] and Gralak and Tip in [12]. In [12], the authors considered the class of anisotropic media and showed the stability of the energy for smooth solutions. In the two-dimensional space setting, in which NIMs occupy a half-plane and obey Drude's model, Bécache, Joly, and the second author in [1] (see also [43]) showed the instability of the standard PMLs and designed a new one in this context. Again for this setting, the limiting amplitude principle was studied by Cassier in [5] and Cassier, Hazard, and Joly in [6], and confirmed numerically in [43]. In [7], Cassier, Joly, and Kachanovska considered a class of dispersive isotropic media in the spirit of [12] (see also Remark 2.1). For homogeneous media in their class, they established the well-posedness via the auxiliary field approach using Nevanlinna's representation theorem and the Hille–Yosida theory (see Remark 3.2). They also derived the finite-speed propagation for regular solutions for the homogeneous media in the class of materials considered in their paper.

In this paper, we deal with *bi-anisotropic* media, i.e. anisotropic media for which the electric and magnetic induction fields D and B depend on both electric and magnetic fields E and H. This general class of metamaterials covers the usual anisotropic one, for which D (resp. B) depends only on E (resp. H). In particular, the bi-anisotropic class contains NIMs and HMMs. More precisely, we establish the well-posedness for weak solutions associated with this model (Theorem 3.1 in Section 3), the finite-speed propagation of weak solutions associated with these media (Theorem 3.2 in Section 3), and a regularity result for the weak solutions (Theorem 3.3 in Section 3.3). By the dispersivity, the corresponding evolution equations are non-local in time inspired from [30,31]. Two key assumptions in our analysis are the causality (2.13) and the passivity (2.15), which roughly speaking say that the effect cannot precede the cause and that the medium is dissipative rather than producing electromagnetic energy. Causality and passivity are given in [7] through the concept of the Herlglotz functions; in various situations, these definitions of passivity are equivalent (see Remark 2.1). In this paper, we work directly with the non-local equations. This is different from the approaches in [6,7,12,43] (see also [11]) where auxiliary fields are introduced to transform the non-local equations into local ones. The initial data of auxiliary fields are imposed by zero in these works. It is interesting to know whether or not other choices are possible and give the same results.

This paper is organized as follows. In Section 2, we present the dispersive model for Maxwell's equations. We there discuss bi-anisotropic media, but confine ourselves to linear and local-in-space ones. The well-posedness, the finite-speed propagation of electromagnetic fields, and the regularity result are discussed in Section 3. Finally, some numerical experiments are presented in Section 4, in which Drude's model and its particular structure are used for simplicity.

2. Maxwell's equations in dispersive media

In this section, we describe Maxwell's equations in dispersive media. The materials presented here are mainly from [14, chapter 7], [16, chapters 1 and 2], [18, chapter IX], [33] and [20, chapter 1]. The fundamental Maxwell's equations – without source – are

$$\begin{cases} \partial_t D(t,x) = \operatorname{curl} H(t,x), \\ \partial_t B(t,x) = -\operatorname{curl} E(t,x), \end{cases} \quad \text{for } t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{2.1}$$

where $E \in \mathbb{R}^3$ (resp. $H \in \mathbb{R}^3$) is the electric (resp. magnetic) field and $D \in \mathbb{R}^3$ (respectively, $B \in \mathbb{R}^3$) is the electric (respectively, magnetic) induction field. In order to close the system (2.1), one adds constitutive relations that express D and B as functions of E and H. For dispersive media, these relations are more conveniently presented in the frequency domain. In this paper, for a time-dependent field $X(t,x)$, its temporal Fourier transform is given by

$$\widehat{X}(\omega,x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} X(t,x) \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t, \quad \text{for } \omega \in \mathbb{R},\ x \in \mathbb{R}^3 \tag{2.2}$$

(this definition is understood in the distributional sense). In the frequency domain, Maxwell's equations (2.1) are of the form

$$\begin{cases} -\mathrm{i}\omega \widehat{D}(\omega,x) = \operatorname{curl} \widehat{H}(\omega,x), \\ -\mathrm{i}\omega \widehat{B}(\omega,x) = -\operatorname{curl} \widehat{E}(\omega,x), \end{cases} \quad \text{for } \omega \in \mathbb{R},\ x \in \mathbb{R}^3. \tag{2.3}$$

In this paper, we consider linear bi-anisotropic materials, i.e. D and B depend linearly on *both* E and H (see, e.g., [16, Chapters 1 and 2] and [20, Chapter 1]). This class of materials contains the anisotropic ones for which D (resp. B) depends only on E (resp. B), see, e.g., [14, chapter 7] and [18, chapter IX]. We also assume that the media considered are local in space. The constitutive relations in the frequency domain of bi-anisotropic media are then of the form

$$\begin{cases} \widehat{D}(\omega,x) = \big(\varepsilon_{\mathrm{rel}}(x) + \widehat{\chi_{ee}}(\omega,x)\big)\widehat{E}(\omega,x) + \widehat{\chi_{em}}(\omega,x)\widehat{H}(\omega,x), \\ \widehat{B}(\omega,x) = \widehat{\chi_{me}}(\omega,x)\widehat{E}(\omega,x) + \big(\mu_{\mathrm{rel}}(x) + \widehat{\chi_{mm}}(\omega,x)\widehat{H}(\omega,x), \end{cases} \quad \text{for } \omega \in \mathbb{R},\ x \in \mathbb{R}^3. \tag{2.4}$$

Here $\widehat{\chi_{ij}}(\omega,x)$, $(i,j) \in \{e,m\}^2$, are 3×3 matrices called the *susceptibilities* that characterize the dispersive effects of the medium, i.e. its response with respect to the frequency ω at the point x. The permittivity ε and the permeability μ of the medium are given by

$$\widehat{\varepsilon} := \varepsilon_{\mathrm{rel}} + \widehat{\chi}_{ee} \quad \text{and} \quad \widehat{\mu} := \mu_{\mathrm{rel}} + \widehat{\chi}_{mm}. \tag{2.5}$$

We assume that

$$\varepsilon_{\mathrm{rel}} \text{ and } \mu_{\mathrm{rel}} \text{ are two } 3 \times 3 \text{ real symmetric uniformly elliptic matrices defined in } \mathbb{R}^3. \tag{2.6}$$

One can check that $\varepsilon_{\mathrm{rel}}$ and μ_{rel} correspond respectively to $\widehat{\varepsilon}$ and $\widehat{\mu}$ for large frequencies provided that χ_{ee} and χ_{mm} are in $L^1(\mathbb{R}, L^\infty(\mathbb{R}^3)^{3\times 3})$. These constitution relations are Lorentz covariants (see, e.g., [16, chapter 2]).

If all the $\widehat{\chi_{ij}}$ are independent of ω, the corresponding medium is called a *dielectric medium*; otherwise it is a *dispersive medium*. In the case $\chi_{em} = \chi_{me} = 0$, (2.4) models anisotropic media. In a special case of (2.4), in which χ_{ij} are isotropic, media are called reciprocal chiral and consist of Pasteur and Tellegen ones, see, e.g., [40].

Set

$$\widehat{\lambda_{ij}}(\omega,x) := -\mathrm{i}\omega \widehat{\chi_{ij}}(\omega,x), \quad \text{for } (i,j) \in \{e,m\}^2,\ \omega \in \mathbb{R},\ x \in \mathbb{R}^3. \tag{2.7}$$

Inserting (2.4) in (2.3) gives, for $\omega \in \mathbb{R}$ and $x \in \mathbb{R}^3$,

$$\begin{cases} -\mathrm{i}\omega\varepsilon_{\mathrm{rel}}(x)\widehat{E}(\omega,x) + \widehat{\lambda_{ee}}(\omega,x)\widehat{E}(\omega,x) + \widehat{\lambda_{em}}(\omega,x)\widehat{H}(\omega,x) = \operatorname{curl} \widehat{H}(\omega,x), \\ -\mathrm{i}\omega\mu_{\mathrm{rel}}(x)\widehat{H}(\omega,x) + \widehat{\lambda_{me}}(\omega,x)\widehat{E}(\omega,x) + \widehat{\lambda_{mm}}(\omega,x)\widehat{H}(\omega,x) = -\operatorname{curl} \widehat{E}(\omega,x). \end{cases} \tag{2.8}$$

One can derive that $\hat{\lambda}_{ij}$ is analytic in the upper half ω-plane as long as

$$\lambda_{ij} \in L^1\big(\mathbb{R}, L^\infty(\mathbb{R}^3)^{3\times 3}\big) + L^\infty\big(\mathbb{R}, L^\infty(\mathbb{R}^3)^{3\times 3}\big), \quad \text{for } (i,j) \in \{e,m\}^2. \tag{2.9}$$

This allows us to use Cauchy's theorem and obtain a relation between Re $\widehat{\chi_{ij}}$ and Im $\widehat{\chi_{ij}}$, which is known as the Kramers–Kronig relation (see, e.g., [33] for further information). We will make the following assumptions on λ_{ij}:

$$\widehat{\lambda_{ij}},\ \lambda_{ij} \in L^1_{\mathrm{loc}}\big(\mathbb{R}, L^\infty(\mathbb{R}^3)^{3\times 3}\big) \text{ and } \lambda_{ij} \text{ is real-valued}, \quad \text{for } (i,j) \in \big\{e,m\big\}^2. \tag{2.10}$$

By the inverse Fourier transform [1]

[1] This formula is again understood in the distributional sense.

$$X(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{X}(\omega,x) e^{-i\omega t} \, d\omega, \quad \text{for } t \in \mathbb{R}, \ x \in \mathbb{R}^3, \tag{2.11}$$

the system corresponding to (2.8) in the time domain is

$$\begin{cases} \varepsilon_{\text{rel}}(x)\partial_t E(t,x) + (\lambda_{ee} * E)(t,x) + (\lambda_{em} * H)(t,x) = \operatorname{curl} H(t,x), \\ \mu_{\text{rel}}(x)\partial_t H(t,x) + (\lambda_{me} * E)(t,x) + (\lambda_{mm} * H)(t,x) = -\operatorname{curl} E(t,x), \end{cases} \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3, \tag{2.12}$$

where $*$ stands for the convolution with respect to time t.

Two fundamental assumptions physically relevant on the model, causality and passivity, are imposed.

Causality: the effect cannot precede the cause, i.e. the present states of the system depend only on its states in the past. Mathematically, one requires

$$\lambda_{ij}(t) = 0, \quad \text{for all } t < 0 \text{ and for all } (i,j) \in \{e,m\}^2. \tag{2.13}$$

Under this assumption, we have, for $(i,j) \in \{e,m\}^2$,

$$(\lambda_{ij} * X)(t,\cdot) = \int_{-\infty}^{t} \lambda_{ij}(t-\tau,\cdot)X(\tau,\cdot)\, d\tau = \int_{0}^{\infty} \lambda_{ij}(\tau,\cdot)X(t-\tau,\cdot)\, d\tau, \quad \text{for } t \in \mathbb{R}. \tag{2.14}$$

Passivity: One assumes, for almost every $x \in \mathbb{R}^3$, for almost every $\omega \in \mathbb{R}$ and for all $X \in \mathbb{C}^6$, that[2]

$$\operatorname{Re}\left(\begin{bmatrix} \widehat{\lambda_{ee}}(\omega,x) & \widehat{\lambda_{em}}(\omega,x) \\ \widehat{\lambda_{me}}(\omega,x) & \widehat{\lambda_{mm}}(\omega,x) \end{bmatrix} X \cdot \overline{X} \right) \geq 0, \tag{2.15}$$

Under the terms of χ_{ij} (see (2.7)), condition (2.15) can be written as

$$\omega \operatorname{Im}\left(\begin{bmatrix} \widehat{\chi_{ee}}(\omega,x) & \widehat{\chi_{em}}(\omega,x) \\ \widehat{\chi_{me}}(\omega,x) & \widehat{\chi_{mm}}(\omega,x) \end{bmatrix} X \cdot \overline{X} \right) \geq 0. \tag{2.16}$$

Assumption (2.15) means that the medium is dissipative, i.e. it does not produce electromagnetic energy by itself. We emphasize that no assumption on the sign of the real part of the χ_{ij} in (2.16) is required (or equivalently on the imaginary part of the λ_{ij} in (2.15)). Moreover, no symmetry on the χ_{ij} (or equivalently on the λ_{ij}) is assumed.

Some comments on these assumptions are in order in the anisotropic case ($\lambda_{em} = \lambda_{me} = 0$) and in the frequency domain. It is possible, for some frequencies, that $\widehat{\varepsilon}$ and $\widehat{\mu}$ are both negative in some regions. This corresponds to NIMs (see Lorentz' and Drude's models below). It is also possible that $\widehat{\varepsilon}$ and $\widehat{\mu}$ have both positive and negative eigenvalues in some region. In this case, one deals with HMMs. In the anisotropic case, condition (2.16) is equivalent to[3]

$$\omega \operatorname{Im} \widehat{\varepsilon}(\omega), \ \omega \operatorname{Im} \widehat{\mu}(\omega) \geq 0, \quad \text{for almost all } \omega \in \mathbb{R}. \tag{2.17}$$

Condition (2.17) ensures that when small loss is added, the problem associated with the outgoing (Silver–Müller) condition at infinity is well-posed (see, e.g., [28]). Adding a small loss is the standard mechanism to study phenomena related to metamaterials in the frequency domain. Nevertheless, condition (2.17) does not exclude the ill-posedness in the frequency domain (see [27, Proposition 2]). As one sees later, even if the problem is ill-posed in the frequency domain for some frequency, the well-posedness is ensured for the problem in the time domain under, roughly speaking, the causality and passivity conditions mentioned above (see Theorem 3.1).

Remark 2.1. The causality and passivity used in [7], where the authors dealt with isotropic media, i.e. $\lambda_{em} = \lambda_{me} = 0$ and λ_{ee} and λ_{mm} are functions, are defined as follows: the causality means that (see [7, page 2795])

$$\omega \mapsto \hat{\lambda}_{ee}(\omega) \text{ and } \omega \mapsto \hat{\lambda}_{mm}(\omega) \text{ are analytic in } \{\omega \in \mathbb{C}; \Im(\omega) > \alpha\}, \text{ for some } \alpha \geq 0, \tag{2.18}$$

and the passivity means that (see [7, Definition 2.5])

$$\omega \to \omega\hat{\lambda}_{ee}(\omega,x) \text{ and } \omega \to \omega\hat{\lambda}_{mm}(\omega,x) \text{ are Herglotz functions.} \tag{2.19}$$

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[2] Here $\cdot$ stands for the Euclidean scalar product in $\mathbb{C}^6$.

[3] Here for a 3×3 matrix A, we denote $A \leq 0$ if $Ax \cdot x \leq 0$ for all $x \in \mathbb{R}^3$.

Recall that (see, e.g., [7, Definition 2.3]) a Herglotz function φ is a function $\varphi : \{\omega \in \mathbb{C}; \Im(\omega) > 0\} \to \mathbb{C}$ that is analytic and satisfies the condition $\Im\varphi(\omega) \geq 0$ (compare (2.19) with (2.9) and (2.17)). Note that several notions concerning passivity are discussed in [7]. In various situations (see [7, Theorem 2.11, Remark 2.12, and Definitions 2.5 and 2.9]), (2.19) is equivalent to (2.17).

We next recall two typical examples of dispersive anisotropic media ($\chi_{me} = \chi_{em} = 0$) satisfying condition (2.10), the causality (2.13), and the passivity (2.15). The first one is that of media obeying *Lorentz' model*. For a homogeneous isotropic medium, the susceptibilities χ_{ee} and χ_{mm} are of the form (see, e.g., [14, (7.51)])

$$\widehat{\chi}(\omega) = \sum_{\ell=1}^{n} \frac{\omega_{p,\ell}^2}{\omega_{0,\ell}^2 - \omega^2 - 2\mathrm{i}\gamma_\ell \omega} I_3, \quad \text{for } \omega \in \mathbb{R}, \tag{2.20}$$

where $\omega_{p,\ell}$ (resp. $\omega_{0,\ell}$ and γ_ℓ) are positive (resp. non negative) material constants. Here and in what follows, I_3 denotes the 3×3 identity matrix. Using the residue theorem, one can show (see, e.g., [14, (7.110)]) that for $t \in \mathbb{R}$, one has

$$\chi(t) = \sqrt{2\pi}\theta(t) \sum_{\ell=1}^{n} \omega_{p,\ell}^2 \frac{\sin(\nu_\ell t)}{\nu_\ell} \mathrm{e}^{-\gamma_\ell t} I_3 \quad \text{and} \quad \lambda(t) = \sqrt{2\pi}\theta(t) \sum_{\ell=1}^{n} \omega_{p,\ell}^2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\sin(\nu_\ell t)}{\nu_\ell} \mathrm{e}^{-\gamma_\ell t} \right) I_3, \tag{2.21}$$

where $\nu_\ell^2 = \omega_{0,\ell}^2 - \gamma_\ell^2$ (if $\omega_{0,\ell} > \gamma_\ell$) and θ is the Heaviside function, i.e. $\theta(t) = 1$ if $t \geq 0$ and $\theta(t) = 0$ otherwise. Here λ is defined in such a way that $\widehat{\lambda}(\omega) = -\mathrm{i}\omega\widehat{\chi}(\omega)$ for $\omega \in \mathbb{R}$.

One can easily check that Lorentz' model satisfies conditions (2.10), (2.13), and (2.15) (which implies (2.16)).

The second example is *Drude's model*. It is a particular case of the Lorentz model (2.20) with $n = 1$ and $\omega_{0,1} = 0$:

$$\widehat{\chi}(\omega) = \frac{\omega_p^2}{-\omega^2 - 2\mathrm{i}\gamma\omega} I_3, \quad \text{for } \omega \in \mathbb{R}. \tag{2.22}$$

One thus has

$$\chi(t) = \sqrt{2\pi}\omega_p^2 \gamma^{-1}(1 - \mathrm{e}^{-\gamma t})\theta(t) I_3 \quad \text{and} \quad \lambda(t) = \sqrt{2\pi}\omega_p^2 \mathrm{e}^{-\gamma t}\theta(t) I_3, \quad \text{for } t \in \mathbb{R}. \tag{2.23}$$

Remark 2.2. Using homogeneization theory, one can obtain HMMs from positive-index materials and NIMs (see, e.g., [3]).

3. Electromagnetic wave propagation in dispersive media

In this paper, we study (2.12) under the form of the initial problem at time $t = 0$, assuming that the data are known in the past $t < 0$. Set

$$(\lambda_{ij} \star X)(t, \cdot) := \int_0^t \lambda(t - \tau, \cdot) X(\tau, \cdot) \, \mathrm{d}\tau, \quad \text{for } t > 0. \tag{3.1}$$

For $X = E$ or H, under the causality assumption (2.13)–(2.14), one has for $t > 0$ that

$$\begin{aligned} (\lambda_{ij} * X)(t, \cdot) &= \int_0^t \lambda_{ij}(t - \tau, \cdot) X(\tau, \cdot) \, \mathrm{d}\tau + \int_{-\infty}^0 \lambda_{ij}(t - \tau, \cdot) X(\tau, \cdot) \, \mathrm{d}\tau \\ &= (\lambda_{ij} \star X)(t, \cdot) + \int_{-\infty}^0 \lambda_{ij}(t - \tau, \cdot) X(\tau, \cdot) \, \mathrm{d}\tau. \end{aligned} \tag{3.2}$$

Hence if the data are known for the past $t < 0$, then the last term is known at time $t > 0$. With the presence of sources, one can then reformulate system (2.12) under the form

$$\begin{cases} \varepsilon_{\mathrm{rel}}(x)\partial_t E(t, x) + (\lambda_{ee} \star E)(t, x) + (\lambda_{em} \star H)(t, x) = \operatorname{curl} H(t, x) + f_e(t, x), \\ \mu_{\mathrm{rel}}(x)\partial_t H(t, x) + (\lambda_{me} \star E)(t, x) + (\lambda_{mm} \star H)(t, x) = -\operatorname{curl} E(t, x) + f_m(t, x), \\ E(0, x) = E_0(x), \ H(0, x) = H_0(x), \end{cases} \tag{3.3}$$

for $t > 0$ and $x \in \mathbb{R}^3$. Here E_0, H_0 are the initial data at time $t = 0$ and f_e, f_m are given fields that can be considered as "effective" sources since they also take into account the last terms in (3.2). Note that if sources are 0 for $t < 0$, then the initial problem considered here with $E_0 = H_0 = 0$ gives exactly the solutions to (2.12), admitting that $E = H = 0$ for $t < 0$ since there is no source for $t < 0$ (see Remark 3.5).

Set

$$u := \begin{bmatrix} E \\ H \end{bmatrix}, \quad u_0 := \begin{bmatrix} E_0 \\ H_0 \end{bmatrix}, \quad f := \begin{bmatrix} f_e \\ f_m \end{bmatrix}, \quad \mathbb{A}u := \begin{bmatrix} \operatorname{curl} H \\ -\operatorname{curl} E \end{bmatrix}, \tag{3.4}$$

$$\Lambda := \begin{bmatrix} \lambda_{ee} & \lambda_{em} \\ \lambda_{me} & \lambda_{mm} \end{bmatrix} \quad \text{and} \quad M := \begin{bmatrix} \varepsilon_{\text{rel}} & 0 \\ 0 & \mu_{\text{rel}} \end{bmatrix}. \tag{3.5}$$

System (3.3) can then be rewritten in the following compact form:

$$\begin{cases} M(x)\partial_t u(t,x) + (\Lambda \star u)(t,x) = \mathbb{A}u(t,x) + f(t,x), \\ u(0,x) = u_0(x), \end{cases} \quad \text{for } t > 0,\ x \in \mathbb{R}^3. \tag{3.6}$$

The goal of this paper is to establish the well-posedness, the finite speed propagation, and to present a regularity result for (3.6).

Define

$$\mathcal{H} := L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3 \quad \text{and} \quad \mathcal{V} := H_{\text{curl}}(\mathbb{R}^3) \times H_{\text{curl}}(\mathbb{R}^3), \tag{3.7}$$

equipped with the standard inner products induced from $L^2(\mathbb{R}^3)^3$ and $H_{\text{curl}}(\mathbb{R}^3)$. One can verify that $\mathcal{H}$ and $\mathcal{V}$ are Hilbert spaces. We also denote

$$\mathcal{M}_6(L^\infty(\mathbb{R}^3)) \text{ the space of } 6 \times 6 \text{ real matrices whose entries are } L^\infty(\mathbb{R}^3) \text{ functions.} \tag{3.8}$$

In what follows, in the time domain, we only consider *real* quantities.

The first result of this paper is the well-posedness of (3.6), whose proof is given in Section 3.1.

Theorem 3.1. *Let $T \in (0, +\infty)$, $u_0 \in \mathcal{H}$, $f \in L^1(0, T; \mathcal{H})$, and $\Lambda \in L^1\big(0, T; \mathcal{M}_6(L^\infty(\mathbb{R}^3))\big)$. Assume that (2.6), (2.10), (2.13) and (2.15) hold. There exists a unique weak solution $u \in L^\infty(0, T; \mathcal{H})$ to (3.6) on $(0, T)$. Moreover, the following estimate holds*

$$\langle Mu(t,\cdot), u(t,\cdot)\rangle_{\mathcal{H}} \le \left(\langle Mu_0, u_0\rangle_{\mathcal{H}}^{1/2} + C \int_0^t \|f(s,\cdot)\|_{\mathcal{H}}\, \mathrm{d}s \right)^2 \quad \text{in } (0, T), \tag{3.9}$$

where C is a positive constant depending only on the coercivity of M.

The notion of weak solutions to (3.6) is as follows.

Definition 3.1. Let $T \in (0, +\infty)$, $u_0 \in \mathcal{H}$ and $f \in L^1(0, T; \mathcal{H})$. A function $u \in L^\infty(0, T; \mathcal{H})$ is called a *weak solution* to (3.6) on $[0, T]$ if

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Mu(t,\cdot), v\rangle_{\mathcal{H}} + \langle (\Lambda \star u)(t,\cdot), v\rangle_{\mathcal{H}} = \langle u(t,\cdot), \mathbb{A}v\rangle_{\mathcal{H}} + \langle f(t,\cdot), v\rangle_{\mathcal{H}} \text{ in } (0,T) \text{ for all } v \in \mathcal{V}, \tag{3.10}$$

and

$$u(0,\cdot) = u_0. \tag{3.11}$$

Remark 3.1. One can easily check that if u is a smooth solution and decays enough at infinity, then u is a weak solution by integration by parts, and that if u is a weak solution and smooth, then u is a classical solution.

Some comments on Definition (3.1) are in order. Equation (3.10) is understood in the distributional sense. Initial condition (3.11) is understood as

$$\langle Mu(0,\cdot), v\rangle_{\mathcal{H}} = \langle Mu_0, v\rangle_{\mathcal{H}}, \quad \text{for all } v \in \mathcal{V}. \tag{3.12}$$

Under the assumptions $u \in L^\infty(0, T; \mathcal{H})$, $v \in \mathcal{V}$, $f \in L^1(0, T; \mathcal{H})$ and $\Lambda \in L^1\big(0, T; \mathcal{M}_6(L^\infty(\mathbb{R}))\big)$, one can check that $\langle (\Lambda \star u)(t), v\rangle_{\mathcal{H}}$, $\langle u(t), \mathbb{A}v\rangle_{\mathcal{H}}$, $\langle f(t), v\rangle_{\mathcal{H}}$ are in $L^1(0, T)$. It follows from (3.10) that

$$\langle Mu(t), v\rangle_{\mathcal{H}} \in W^{1,1}(0, T). \tag{3.13}$$

This in turn ensures the trace sense of $\langle Mu(0,\cdot), v\rangle_{\mathcal{H}}$ in (3.12).

Remark 3.2. In [7, Theorem 4.7 and Corollary 4.11], the authors establish the well-posedness of electromagnetic waves in homogeneous isotropic dispersive media, i.e. $\lambda_{em} = \lambda_{me} = 0$ and λ_{ee} and λ_{mm} are constant functions (with respect to space variables) that satisfy (2.18), (2.19), and the following condition (see [7, (HP) on page 2795])

$$\forall \eta > 0 \text{ if } \Im(\omega) > \eta, \lim_{|\omega|\to+\infty} \hat{\varepsilon}(x,\omega) = \varepsilon_0 \text{ and } \lim_{|\omega|\to+\infty} \hat{\varepsilon}(x,\omega) = \mu_0, \tag{3.14}$$

for some positive constants ε_0, μ_0. They use the auxiliary field method, and apply Nevanlinna's representation theorem and the Hille–Yosida semi-group theory.

We next discuss the finite speed propagation for (3.6). In what follows, $B(a, R)$ stands for the ball of $\mathbb{R}^3$ of radius $R > 0$ centred at $a \in \mathbb{R}^3$ and $\partial B(a, R)$ denotes its boundary. In the case $a = 0$ – the origin –, we simply denote $B(0, R)$ by B_R. Set

$$c(x) := \gamma_e(x)\gamma_m(x), \quad \text{for } x \in \mathbb{R}^3, \tag{3.15}$$

where $\gamma_e(x)$ and $\gamma_m(x)$ are respectively the largest eigenvalues of $\varepsilon_{\text{rel}}(x)^{-1/2}$ and $\mu_{\text{rel}}(x)^{-1/2}$. According to assumptions (2.6), $c(x)$ is bounded below and above by a positive constant. For $a \in \mathbb{R}^3$ and $R > 0$, we denote

$$c_{a,R} := \operatorname*{ess\,sup}_{x\in B(a,R)} c(x). \tag{3.16}$$

The second result of this paper is on the finite speed propagation of (3.6), whose proof is given in Section 3.2.

Theorem 3.2. *Let $R > 0$, $a \in \mathbb{R}^3$ and $u_0 \in \mathcal{H}$. For $T > R/c_{a,R}$, let $f \in L^1(0, T; \mathcal{H})$ and $\Lambda \in L^1(0, T; \mathcal{M}_6(L^\infty(\mathbb{R}^3)))$. Assume that (2.6), (2.10), (2.13) and (2.15) hold,*

$$\operatorname{supp} u_0 \cap B(a, R) = \emptyset, \tag{3.17}$$

and

$$\operatorname{supp} f(t,\cdot) \cap B(a, R - c_{a,R}t) = \emptyset, \quad \textit{for almost every } t \in (0, R/c_{a,R}). \tag{3.18}$$

Let $u \in L^\infty(0, T; \mathcal{H})$ be the unique weak solution to (3.6) on $(0, T)$. Then

$$\operatorname{supp} u(t,\cdot) \cap B(a, R - c_{a,R}t) = \emptyset, \quad \textit{for almost every } t \in (0, R/c_{a,R}). \tag{3.19}$$

We finally discuss the regularity of the weak solutions to (3.6). To motivate the regularity result stated below, let us first assume that u is a weak solution to (3.6) and that u, Λ, and f are regular in $[0, T] \times \mathbb{R}^3$. Set

$$v(t,x) := \partial_t u(t,x), \quad \text{for } t \in (0,T),\ x \in \mathbb{R}^3. \tag{3.20}$$

Differentiating (3.6) with respect to t, we have

$$M(x)\partial_t v(t,x) + (\Lambda \star v)(t,x) = \mathbb{A}v(t,x) + g(t,x), \quad \text{for } t \in (0,T),\ x \in \mathbb{R}^3, \tag{3.21}$$

where

$$g(t,x) := \partial_t f(t,x) - \Lambda(t,x)u_0(x) \quad \text{in } [0,T) \times \mathbb{R}^3. \tag{3.22}$$

Applying Theorem 3.1 to v and noting that $Mv(0,\cdot) = \mathbb{A}u_0 + f(0,\cdot)$, we obtain

$$\|v(t,\cdot)\|_{\mathcal{H}} \le C\Big(\|u_0\|_{\mathcal{V}} + \|f(0,\cdot)\|_{\mathcal{H}} + \int_0^t \|\partial_s f(s,\cdot)\|_{\mathcal{H}} + \|\Lambda(s,\cdot)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\Big), \quad \text{in } (0,T), \tag{3.23}$$

for some positive constant C depending only on the ellipticity of M.

In fact, we can prove the following result.

Theorem 3.3. *Let $T \in (0, +\infty)$, $u_0 \in \mathcal{V}$, $f \in L^1(0, T; \mathcal{H})$, and $\Lambda \in L^1(0, T; \mathcal{M}_6(L^\infty(\mathbb{R}^3)))$. Assume that (2.6), (2.10), (2.13) and (2.15) hold and $\partial_t f \in L^1(0, T; \mathcal{H})$. Let $u \in L^\infty(0, T; \mathcal{H})$ be the unique weak solution to (3.6) on $(0, T)$. Then $\partial_t u \in L^\infty(0, T; \mathcal{H})$ and, for $t \in (0, T)$,*

$$\|\partial_t u(t,\cdot)\|_{\mathcal{H}}^2 \le C\left(\|u_0\|_{\mathcal{V}} + \|f(0,\cdot)\|_{\mathcal{H}} + \int_0^t \|\partial_s f(s,\cdot)\|_{\mathcal{H}} + \|\Lambda(s,\cdot)\|_{L^\infty(\mathbb{R}^3)}\|u(s,\cdot)\|_{\mathcal{H}}\,\mathrm{d}s\right)^2, \tag{3.24}$$

for some positive constant C depending only on the coercivity of M.

Remark 3.3. One can bound $\|u(s,\cdot)\|_{\mathcal{H}}$ using (3.9).

The next three sections are respectively devoted to the proof of Theorems 3.1, 3.2, and 3.3.

3.1. Proof of Theorem 3.1

The proof is based on the standard Galerkin approach, see, e.g., [9,10]. We first establish the existence of a weak solution. Let $(\phi_k)_{k\in\mathbb{N}}$ be a (real) orthogonal basis of $\mathcal{V}$. For $n \in \mathbb{N}$, consider u_n of the form

$$u_n(t,x) = \sum_{k=1}^{n} d_{n,k}(t)\phi_k(x), \qquad \text{for } t \in (0,T),\ x \in \mathbb{R}^3, \tag{3.25}$$

such that for all $k \in \{1,\dots,n\}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Mu_n(t),\phi_k\rangle_{\mathcal{H}} + \langle(\Lambda \star u_n)(t),\phi_k\rangle_{\mathcal{H}} = \langle u_n(t),\mathbb{A}\phi_k\rangle_{\mathcal{H}} + \langle f(t),\phi_k\rangle_{\mathcal{H}}, \quad \text{in } (0,T), \tag{3.26}$$

and

$$u_n(0) = u_{0,n},\ \text{the projection of } u_0 \text{ to the space spanned by } \{\phi_1,\cdots,\phi_n\} \text{ in } \mathcal{H}. \tag{3.27}$$

Since $(\phi_k)_{k\in\mathbb{N}}$ is linearly independent in $\mathcal{V}$, it is also linearly independent in $\mathcal{H}$. This implies that the $n\times n$ matrix whose (i,j)-entry is given by $\langle\phi_i,\phi_j\rangle_{\mathcal{H}}$ is invertible. Since

$$\|\Lambda\star u\|_{L^\infty(0,T;\mathcal{H})} \le \|\Lambda\|_{L^1\left(0,T;\mathcal{M}_6(L^\infty(\mathbb{R}))\right)}\|u\|_{L^\infty(0,T;\mathcal{H})}, \tag{3.28}$$

the existence and uniqueness of $d_{n,k} \in W^{1,1}(0,T)$ follow by a standard point-fixed argument (see, e.g., [4, Theorem 2.1.1]). This implies the existence and uniqueness of u_n.

We now derive an estimate for u_n. The key point of the analysis is the following two observations:

$$\int_0^t \langle(\Lambda\star v)(s,\cdot), v(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s \ge 0 \quad \text{for } v \in L^\infty(0,T;\mathcal{H}),\ t\in(0,T), \tag{3.29}$$

and

$$\langle v,\mathbb{A}v\rangle_{\mathcal{H}} = 0, \qquad \text{for } v\in\mathcal{V}. \tag{3.30}$$

Note that (3.30) follows by an integration by parts and the density of $\mathcal{C}^1_c(\mathbb{R}^3)^6$ in $\mathcal{V}$. We now verify (3.29). Let $\mathbf{v}$ be the extension of v in $\mathbb{R}$ by 0 for $t\in\mathbb{R}\setminus[0,T]$. It follows from (2.13) and (3.1) that

$$(\Lambda\star v)(s,\cdot) = (\Lambda * \mathbf{v})(s,\cdot), \qquad \text{for } s\in[0,t]. \tag{3.31}$$

By Parseval's identity, one has, for $t\in(0,T)$,

$$\begin{aligned}\int_0^t \langle(\Lambda\star v)(s,\cdot), v(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s &= \int_{\mathbb{R}} \langle(\Lambda * \mathbf{v})(s,\cdot),\mathbf{v}(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s \\ &= \mathrm{Re}\int_{\mathbb{R}} \left\langle \mathcal{F}(\Lambda*\mathbf{v})(\omega,\cdot),\overline{\widehat{\mathbf{v}}(\omega,\cdot)}\right\rangle_{\mathcal{H}}\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \mathrm{Re}\left\langle \widehat{\Lambda}(\omega,\cdot)\widehat{\mathbf{v}}(\omega,\cdot),\overline{\widehat{\mathbf{v}}(\omega,\cdot)}\right\rangle_{\mathcal{H}}\mathrm{d}\omega \ge 0,\end{aligned} \tag{3.32}$$

thanks to the passivity (2.15). Assertions (3.29) and (3.30) are proved.

Multiplying (3.26) by $d_{n,k}(t)$ and summing with respect to k yields that, in $(0,T)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Mu_n(t,\cdot),\ u_n(t,\cdot)\rangle_{\mathcal{H}} + \langle(\Lambda\star u_n)(t,\cdot),u_n(t,\cdot)\rangle_{\mathcal{H}} = \langle u_n(t,\cdot),\mathbb{A}u_n(t,\cdot)\rangle_{\mathcal{H}} + \langle f(t,\cdot),u_n(t,\cdot)\rangle_{\mathcal{H}}. \tag{3.33}$$

Integrating (3.33) from 0 to t and using (3.30), we obtain that, in $(0, T)$,

$$\frac{1}{2}\langle Mu_n(t,\cdot), u_n(t,\cdot)\rangle_{\mathcal{H}} + \int_0^t \langle (\Lambda \star u_n)(s,\cdot), u_n(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s = \frac{1}{2}\langle Mu_n(0,\cdot), u_n(0,\cdot)\rangle_{\mathcal{H}} + \int_0^t \langle f(s,\cdot), u_n(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s. \tag{3.34}$$

We derive from (3.29) that, in $(0, T)$,

$$\langle Mu_n(t,\cdot), u_n(t,\cdot)\rangle_{\mathcal{H}} \le \langle Mu_{0,n}, u_{0,n}\rangle_{\mathcal{H}} + 2\int_0^t \|f(s,\cdot)\|_{\mathcal{H}}\|u_n(s,\cdot)\|_{\mathcal{H}}\,\mathrm{d}s. \tag{3.35}$$

By Grönwall's inequality (see Lemma 3.1 below) and assumptions (2.6), one gets from (3.35)

$$\langle Mu_n(t,\cdot), u_n(t,\cdot)\rangle_{\mathcal{H}} \le \left(\langle Mu_{n,0}, u_{n,0}\rangle_{\mathcal{H}}^{1/2} + C\int_0^T \|f(s)\|_{\mathcal{H}}\,\mathrm{d}s\right)^2, \text{ in } (0, T), \tag{3.36}$$

where C is a positive constant depending only on the ellipticity of M. Since $\|u_{n,0}\|_{\mathcal{H}} \le \|u_0\|_{\mathcal{H}}$ by (3.27), the sequence $(u_n)_{n\in\mathbb{N}}$ is hence bounded in $L^\infty(0, T; \mathcal{H})$. Up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ weakly star converges to $u \in L^\infty(0, T; \mathcal{H})$. It is clear from (3.36) that (3.9) holds and, for $k \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Mu(t,\cdot), \phi_k\rangle_{\mathcal{H}} + \langle (\Lambda \star u)(t,\cdot), \phi_k\rangle_{\mathcal{H}} = \langle u(t,\cdot), \mathbb{A}\phi_k\rangle_{\mathcal{H}} + \langle f(t,\cdot), \phi_k\rangle_{\mathcal{H}}, \quad \text{in } (0, T). \tag{3.37}$$

Since (ϕ_k) is dense in $\mathcal{V}$, we derive that for $\phi \in \mathcal{V}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Mu(t,\cdot), \phi\rangle_{\mathcal{H}} + \langle (\Lambda \star u)(t,\cdot), \phi\rangle_{\mathcal{H}} = \langle u(t,\cdot), \mathbb{A}\phi\rangle_{\mathcal{H}} + \langle f(t,\cdot), \phi\rangle_{\mathcal{H}}, \quad \text{in } (0, T). \tag{3.38}$$

One can also check that the initial condition (3.11) holds.

We finally establish the uniqueness of u. It suffices to show that if $u \in L^\infty(0, T; \mathcal{H})$ is a weak solution to (3.6) on $[0, T]$ with $u_0 = 0$ and $f = 0$, then $u = 0$. Set

$$U(t, x) := \int_0^t u(s, x)\,\mathrm{d}s, \quad \text{for } t \in [0, T],\ x \in \mathbb{R}^3. \tag{3.39}$$

Integrating (3.10) from 0 to t and using the fact that $u(t = 0, \cdot) = 0$, we obtain that, for all $v \in \mathcal{V}$ and almost every $t \in [0, T]$,

$$\langle Mu(t,\cdot), v\rangle_{\mathcal{H}} + \int_0^t \langle (\Lambda \star u)(s,\cdot), v\rangle_{\mathcal{H}}\,\mathrm{d}s = \langle U(t,\cdot), \mathbb{A}v\rangle_{\mathcal{H}}. \tag{3.40}$$

Using the fact that

$$\partial_t U(t,\cdot) = u(t,\cdot), \quad \text{for a.e. } t \in (0, T), \tag{3.41}$$

we derive that, for all $v \in \mathcal{H}$,

$$\langle M\partial_t U(t,\cdot), v\rangle_{\mathcal{H}} + \int_0^t \langle (\Lambda \star u)(s,\cdot)\,\mathrm{d}s, v\rangle_{\mathcal{H}} = \langle U(t,\cdot), \mathbb{A}v\rangle_{\mathcal{H}}, \quad \text{in } (0, T). \tag{3.42}$$

We claim that

$$\int_0^t (\Lambda \star u)(s,\cdot)\,\mathrm{d}s = (\Lambda \star U)(t,\cdot), \quad \text{for almost every } t \in (0, T). \tag{3.43}$$

Indeed, by Fubini's theorem, one gets, for almost every $t \in (0, T)$,

$$\begin{aligned} \int_0^t (\Lambda \star u)(s,\cdot)\,\mathrm{d}s &= \int_0^t \left[\int_0^s \Lambda(\tau,\cdot) u(s-\tau,\cdot)\,\mathrm{d}\tau \right] \mathrm{d}s = \int_0^t \Lambda(\tau,\cdot) \left[\int_\tau^t u(s-\tau,\cdot)\,\mathrm{d}s \right] \mathrm{d}\tau \\ &= \int_0^t \Lambda(\tau,\cdot) \left[\int_0^{t-\tau} u(\tau',\cdot)\,\mathrm{d}\tau' \right] \mathrm{d}\tau = (\Lambda \star U)(t,\cdot). \end{aligned} \tag{3.44}$$

From (3.6), we derive that

$$M(x)\partial_t U(t,x) + (\Lambda \star U)(t,x) = \mathbb{A}U(t,x), \qquad \text{for } t \in (0,T),\ x \in \mathbb{R}^3 \tag{3.45}$$

and hence $U \in L^1(0,T;\mathcal{V})$. Multiplying (3.45) by $U(t,\cdot)$, integrating with respect to x, and using Fubini's theorem as well as the fact that $\langle v, \mathbb{A}v\rangle_{\mathcal{H}} = 0$ for all $v \in \mathcal{V}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \langle MU(t,\cdot), U(t,\cdot)\rangle_{\mathcal{H}} + \langle (\Lambda \star U)(t,\cdot), U(t,\cdot)\rangle_{\mathcal{H}} = 0, \quad \text{for almost every } t \in [0,T]. \tag{3.46}$$

Integrating this equation from 0 to t gives

$$\frac{1}{2} \langle MU(t,\cdot), U(t,\cdot)\rangle_{\mathcal{H}} + \int_0^t \langle (\Lambda \star U)(s,\cdot), U(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s = 0, \quad \text{for almost every } t \in [0,T]. \tag{3.47}$$

We derive from (3.29) that $\|U(t)\|_{\mathcal{H}}^2 \le 0$ for almost every $t \in [0,T]$. It follows that

$$U(t,\cdot) = 0, \qquad \text{for almost every } t \in [0,T]. \tag{3.48}$$

This in turn implies that $u = 0$. The proof is complete. □

In the proof of Theorem 3.1, we use the following Grönwall inequality:

Lemma 3.1. *Let $T > 0$, $\tau \in (0,1)$, $\alpha, \beta \ge 0$ and let ξ and ϕ be two non-negative, measurable functions defined in $(0,T)$ such that*

$$\xi(t) \le \alpha + \beta \int_0^t \phi(s)\xi(s)^\tau\,\mathrm{d}s, \quad \textit{for almost every } t \in (0,T). \tag{3.49}$$

We have

$$\xi(t) \le \left(\alpha^{1-\tau} + (1-\tau)\beta \int_0^t \phi(s)\,\mathrm{d}s \right)^{1/(1-\tau)}, \quad \textit{for almost every } t \in (0,T). \tag{3.50}$$

Proof. The proof of this result is standard. Set

$$G(t) := \alpha + \beta \int_0^t \phi(s)\xi(s)^\tau\,\mathrm{d}s, \quad \text{for } t \in (0,T). \tag{3.51}$$

Then $G'(t) = \beta\phi(t)\xi(t)^\tau \le \beta\phi(t)G(t)^\tau$ for $t \in (0,T)$ and consequently

$$G(t)^{-\tau} G'(t) \le \beta\phi(t), \quad \text{for } t \in (0,T). \tag{3.52}$$

Integrating this with respect to t and using the fact $G(t) \ge \xi(t)$ for $t \in (0,T)$ yield the conclusion. □

Remark 3.4. In [31], the authors used Lorentz's model to study approximate cloaking via a change of variables for the acoustic waves in the time domain. Wave equations that are non-local in time also appeared in a very different context in [30], the one of generalized impedance boundary conditions for conducting obstacles. The proof of Theorem 3.1 is inspired from these works.

Remark 3.5. Assume that (2.6), (2.10), (2.13) and (2.15) hold. Let $u \in L^\infty(-\infty, +\infty; \mathcal{H})$ be a weak solution to

$$M(x)\partial_t u(t,x) + (\Lambda * u)(t,x) = \mathbb{A}u(t,x) + f(t,x), \quad \text{for } t \in R,\ x \in \mathbb{R}^3. \tag{3.53}$$

Note that the time convolution $*$ is considered here, and not the operator $\star$ defined by (3.1). Assume that $f(t,\cdot) = 0$ for $t < t_1$ and in addition that $u \in L^1(-\infty, t_1; \mathcal{V})$ and

$$\liminf_{t\to-\infty} \|u(t,)\|_{\mathcal{H}} = 0. \tag{3.54}$$

Then $u(t,\cdot) = 0$ for $t < t_1$. The definition of weak solutions to (3.53) is similar to the one given in Definition 3.1: u is required to satisfy the following equation, in the distributional sense,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Mu(t,\cdot), v\rangle_{\mathcal{H}} + \langle(\Lambda * u)(t,\cdot), v\rangle_{\mathcal{H}} = \langle u(t,\cdot), \mathbb{A}v\rangle_{\mathcal{H}} + \langle f(t,\cdot), v\rangle_{\mathcal{H}}, \quad \text{in } (-\infty,\infty), \tag{3.55}$$

for all $v \in \mathcal{V}$. Indeed, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Mu(t,\cdot), u(t,\cdot)\rangle_{\mathcal{H}} + \langle(\Lambda * u)(t,\cdot), u(t,\cdot)\rangle_{\mathcal{H}} = 0, \quad \text{in } (-\infty, t_1). \tag{3.56}$$

This implies, by (3.54),

$$\frac{1}{2}\langle Mu(t,\cdot), u(t,\cdot)\rangle_{\mathcal{H}} + \int_{-\infty}^{t} \langle(\Lambda * u)(t,\cdot), u(t,\cdot)\rangle_{\mathcal{H}} = 0, \quad \text{in } (-\infty, t_1). \tag{3.57}$$

Similar to (3.29), we obtain, for $t < t_1$,

$$\int_{-\infty}^{t} \langle(\Lambda * u)(s,\cdot), u(s,\cdot)\rangle_{\mathcal{H}}\, \mathrm{d}s \ge 0. \tag{3.58}$$

Therefore, $u(t,\cdot) = 0$ for $t < t_1$.

3.2. Proof of Theorem 3.2

In the case where u is *regular enough*, the argument is quite standard using the two observations (3.29) and (3.30). As far as we know, the proof of finite-speed propagation for energy solutions is not presented in standard references on partial differential equations. To overcome the lack of the regularity of u, we implement the strategy used in the proof of the uniqueness part of Theorem 3.1. For simplicity of notations, we assume that $a = 0$ and we denote $c_{a,R}$ by c in this proof.

Set

$$U(t,x) := \int_0^t u(s,x)\, \mathrm{d}s, \quad \text{for } t \in [0,T),\ x \in \mathbb{R}^3. \tag{3.59}$$

Integrating (3.10) from 0 to t and using the fact that $u(t=0,\cdot) = u_0$, we obtain that, for all $v \in \mathcal{V}$ and for almost every $t \in (0,T)$,

$$\langle Mu(t,\cdot), v\rangle_{\mathcal{H}} - \langle Mu_0, v\rangle_{\mathcal{H}} + \int_0^t \langle(\Lambda \star u)(s,\cdot), v\rangle_{\mathcal{H}}\, \mathrm{d}s = \langle U(t,\cdot), \mathbb{A}v\rangle_{\mathcal{H}} + \langle F(t), v\rangle_{\mathcal{H}}, \tag{3.60}$$

where

$$F(t,\cdot) := \int_0^t f(s,\cdot)\, \mathrm{d}s, \quad \text{for } t \in [0,T). \tag{3.61}$$

As in (3.44), we have

$$\int_0^t (\Lambda \star u)(s,\cdot)\, \mathrm{d}s = (\Lambda \star U)(t,\cdot), \quad \text{for almost every } t \in [0,T). \tag{3.62}$$

Since

$$\partial_t U(t,\cdot) = u(t,\cdot), \quad \text{for almost every } t \in (0,T), \tag{3.63}$$

we derive, for all $v \in \mathcal{H}$, that in $(0,T)$

$$\langle M\partial_t U(t,\cdot), v\rangle_{\mathcal{H}} + \langle (\Lambda \star U)(s,\cdot)\,\mathrm{d}s, v\rangle_{\mathcal{H}} = \langle U(t,\cdot), \mathbb{A}v\rangle_{\mathcal{H}} + \langle F(t,\cdot), \phi_k\rangle_{\mathcal{H}} + \langle Mu_0, v\rangle_{\mathcal{H}}. \tag{3.64}$$

It follows that

$$M(x)\partial_t U(t,x) + (\Lambda \star U)(t,x) = \mathbb{A}U(t,x) + F(t,x) + Mu_0(x), \quad \text{for } t \in (0,T),\ x \in \mathbb{R}^3. \tag{3.65}$$

From (3.63), we obtain

$$U \in L^1(0,T;\mathcal{V}). \tag{3.66}$$

We claim

$$U(t,\cdot) = 0, \quad \text{in } B_{R-ct} \text{ and for } t \in (0, R/c). \tag{3.67}$$

Since $u_0 = 0$ in B_R, it is clear that the conclusion follows from claim (3.67) and the definition of U.

It remains to prove (3.67). Multiplying the equation of U (3.65) by $U(t,x)$, integrating with respect to x in B_{R-ct}, and using the facts that $u_0 = 0$ in B_{R-ct} and $F(t,\cdot) = 0$ in B_{R-ct} for almost every $t \in (0,R/c)$, we have, for almost every $t \in (0,R/c)$,

$$\int_{B_{R-ct}} M\partial_t U(t,x) \cdot U(t,x)\,\mathrm{d}x + \int_{B_{R-ct}} (\Lambda \star U)(t,x) \cdot U(t,x)\,\mathrm{d}x = \int_{B_{R-ct}} \mathbb{A}U(t,x) \cdot U(t,x)\,\mathrm{d}x. \tag{3.68}$$

The divergence theorem gives, for almost every $t \in (0,R/c)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R-ct}} MU(t,x) \cdot U(t,x)\,\mathrm{d}x = \int_{B_{R-ct}} M\partial_t U(t,x) \cdot U(t,x)\,\mathrm{d}x - \frac{c}{2}\int_{\partial B_{R-ct}} MU(t,x) \cdot U(t,x)\,\mathrm{d}x. \tag{3.69}$$

It follows from (3.68) that, for almost every $t \in (0,R/c)$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R-ct}} MU(t,x)\cdot U(t,x)\,\mathrm{d}x + \int_{B_{R-ct}} (\Lambda \star U)(t,x)\cdot u(t,x)\,\mathrm{d}x \\ = -\frac{c}{2}\int_{\partial B_{R-ct}} MU(t,x)\cdot U(t,x)\,\mathrm{d}x + \int_{B_{R-ct}} \mathbb{A}U(t,x)\cdot U(t,x)\,\mathrm{d}x.\end{aligned} \tag{3.70}$$

Integrating this identity from 0 to t with $t \in (0,R/c)$ and using the fact that $U(0,\cdot) = 0$, we obtain, for almost every $t \in (0,R/c)$,

$$\begin{aligned}\frac{1}{2}\int_{B_{R-ct}} MU(t,x)\cdot U(t,x)\,\mathrm{d}x + \int_0^t\int_{B_{R-cs}} (\Lambda \star U)(s,x)\cdot U(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ = -\frac{c}{2}\int_0^t\int_{\partial B_{R-cs}} MU(s,x)\cdot U(s,x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{B_{R-cs}} \mathbb{A}U(s,x)\cdot U(s,x)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.71}$$

In a similar fashion to (3.29), we have

$$\int_0^t\int_{B_{R-cs}} (\Lambda \star U)(s,x)\cdot U(s,x)\,\mathrm{d}x\,\mathrm{d}s \ge 0, \quad \text{for almost every } t \in (0,R/c). \tag{3.72}$$

Combining (3.71) and (3.72) yields, for almost every $t \in (0,R/c)$,

$$\frac{1}{2}\int_{B_{R-ct}} MU(t,x)\cdot U(t,x)\,\mathrm{d}x \le -\frac{c}{2}\int_0^t\int_{\partial B_{R-cs}} MU(s,x)\cdot U(s,x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{B_{R-cs}} \mathbb{A}U(s,x)\cdot U(s,x)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.73}$$

We claim that, for $0 < s < R/c$, one has

$$-\frac{c}{2} \int\limits_{\partial B_{R-cs}} MU(s,x) \cdot U(s,x)\,\mathrm{d}x + \int\limits_{B_{R-cs}} \mathbb{A}U(s,x) \cdot U(s,x)\,\mathrm{d}x \le 0. \tag{3.74}$$

Indeed, for $U = (E, H)^T$, one has

$$\int\limits_{B_{R-cs}} \mathbb{A}U(s,x) \cdot U(s,x)\,\mathrm{d}x = \int\limits_{B(0,R-cs)} [\operatorname{curl} H(s,x) \cdot E(s,x) - \operatorname{curl} E(s,x) \cdot H(s,x)]\,\mathrm{d}x$$
$$= - \int\limits_{\partial B_{R-cs}} \big(H(s,x) \times e_r\big) \cdot E(s,x)\,\mathrm{d}x \le \int\limits_{\partial B_{R-cs}} |H||E|\,\mathrm{d}x$$

and

$$MU(s,x) \cdot U(s,x) = \varepsilon_{\mathrm{rel}} E \cdot E + \mu_{\mathrm{rel}} H \cdot H \ge 2|\varepsilon_{\mathrm{rel}}^{1/2} E||\mu_{\mathrm{rel}}^{1/2} H|. \tag{3.75}$$

Assertion (3.74) now follows from the definition (3.16) of $c = c_{a,R}$.

We derive from (3.73) and (3.74) that

$$\frac{1}{2} \int\limits_{B_{R-ct}} MU(t,x) \cdot U(t,x)\,\mathrm{d}x \le 0, \quad \text{for almost every } t \in (0, R/c), \tag{3.76}$$

and claim (3.67) follows from the ellipticity of M. □

3.3. *Proof of Theorem 3.3*

In this proof, we use the notations from the one of Theorem 3.1. For $n \in \mathbb{N}^*$, set

$$v_n(t,x) := \partial_t u_n(t,x), \quad \text{for } t \in [0,T),\ x \in \mathbb{R}^3.$$

We recall that u_n is the approximate solution constructed by the Galerkin approach in the proof of Theorem 3.1. It follows from (3.25) that

$$v_n(t,x) = \sum_{k=1}^{n} d'_{n,k}(t)\phi_k(x), \quad \text{for } t \in [0,T),\ x \in \mathbb{R}^3 \tag{3.77}$$

(note that $d_{n,k}$ is Lipschitz with respect to $t \in [0,T]$). Differentiating (3.26) with respect to t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle M v_n(t,\cdot), \phi_k \rangle_{\mathcal{H}} + \langle (\Lambda \star v_n)(t,\cdot), \phi_k \rangle_{\mathcal{H}} = \langle v_n(t,\cdot), \mathbb{A}\phi_k \rangle_{\mathcal{H}} + \langle g_n(t,\cdot), \phi_k \rangle_{\mathcal{H}}, \quad \text{in } (0,T), \tag{3.78}$$

where

$$g_n(t,x) := \partial_t f(t,x) - \Lambda(t,x) u_{0,n}(x), \quad \text{for } t \in (0,T),\ x \in \mathbb{R}^3. \tag{3.79}$$

We have

$$\langle M \partial_t u_n(0,\cdot), \phi_k \rangle_{\mathcal{H}} = \langle u_n(0,\cdot), \mathbb{A}\phi_k \rangle_{\mathcal{H}} + \langle f(0,\cdot), \phi_k \rangle_{\mathcal{H}}, \quad \text{for } k \in \{1, \cdots, n\}. \tag{3.80}$$

It follows from (3.80) that $M^{1/2}\partial_t u_n(0,\cdot)$ is the projection of $M^{-1/2}\big(\mathbb{A}u_0 + f(0,\cdot)\big)$ into the space spanned by $\{M^{1/2}\phi_1, \cdots, M^{1/2}\phi_n\}$ in $\mathcal{H}$. This implies

$$\|v_n(0,\cdot)\|_{\mathcal{H}} = \|\partial_t u_n(0,\cdot)\|_{\mathcal{H}} \le C\Big(\|u_0\|_{\mathcal{V}} + \|f(0,\cdot)\|_{\mathcal{H}}\Big). \tag{3.81}$$

By (3.78), $d'_{n,k} \in W^{1,1}(0,T)$. As in (3.36), we derive from (3.78) that

$$\|v_n(t,\cdot)\|_{\mathcal{H}}^2 \le C\left(\|u_0\|_{\mathcal{V}} + \|f(0,\cdot)\|_{\mathcal{H}} + \int_0^t \|\partial_s f(s,\cdot)\|_{\mathcal{H}} + \|\Lambda(s,\cdot)\|_{L^\infty}\|u_n(s.\cdot)\|_{\mathcal{H}}\,\mathrm{d}s\right)^2 \quad \text{in } (0,T). \tag{3.82}$$

This in turn yields

$$\|v(t,\cdot)\|_{\mathcal{H}}^2 \le C\left(\|u_0\|_{\mathcal{V}} + \|f(0,\cdot)\|_{\mathcal{H}} + \int_0^t \|\partial_s f(s,\cdot)\|_{\mathcal{H}} + \|\Lambda(s,\cdot)\|_{L^\infty}\|u(s.\cdot)\|_{\mathcal{H}}\,\mathrm{d}s\right)^2 \quad \text{in } (0,T), \tag{3.83}$$

and the conclusion follows. □

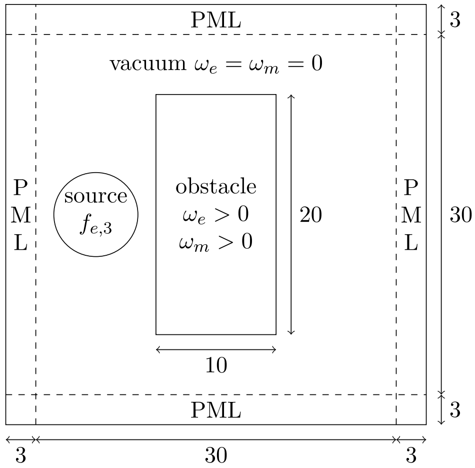

Fig. 1. Geometry of the problem (4.7).

4. Numerical results

We now perform some numerical simulations. In this section, we focus on the Drude's model without absorption described at the end of Section 2. More precisely, we consider $\varepsilon_{\text{rel}} = \mu_{\text{rel}} = 1$, $\widehat{\lambda_{em}} = \widehat{\lambda_{me}} = 0$,

$$\widehat{\lambda_{ee}}(\omega, x) = \frac{\mathbf{w}_e^2(x)}{-i\omega} \quad \text{and} \quad \widehat{\lambda_{mm}}(\omega, x) = \frac{\mathbf{w}_m^2(x)}{-i\omega}, \qquad \text{for } \omega \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{4.1}$$

or equivalently

$$\lambda_{ee}(t, x) = \mathbf{w}_e^2(x)\theta(t) \quad \text{and} \quad \lambda_{mm}(t, x) = \mathbf{w}_m^2(x)\theta(t), \qquad \text{for } t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{4.2}$$

where $\mathbf{w}_e$ and $\mathbf{w}_m$ are two functions defined later.

In this context, the problem (3.3) rewrites

$$\begin{cases} \partial_t E(t,x) + \mathbf{w}_e^2(x) \displaystyle\int_0^t E(s,x)\,ds = \operatorname{curl} H(t,x) + f_e(t,x), \\ \partial_t H(t,x) + \mathbf{w}_m^2(x) \displaystyle\int_0^t H(s,x)\,ds = -\operatorname{curl} E(t,x) + f_m(t,x), \\ E(t=0,\cdot) = E_0, \qquad H(t=0,\cdot) = H_0. \end{cases} \qquad \text{for } t > 0,\ x \in \mathbb{R}^3. \tag{4.3}$$

Define

$$J(t,x) := \int_0^t E(s,x)\,ds \quad \text{and} \quad K(t,x) := \int_0^t H(s,x)\,ds, \qquad \text{for } t \geq 0,\ x \in \mathbb{R}^3. \tag{4.4}$$

It is clear that $J(t=0,\cdot) = K(t=0,\cdot) = 0$ and that

$$\partial_t J(t,x) = E(t,x) \quad \text{and} \quad \partial_t K(t,x) = H(t,x), \qquad \text{for } t \geq 0,\ x \in \mathbb{R}^3. \tag{4.5}$$

From (4.3), one obtains the following local in time problem which is the advantage of the special structure of Drude's model:

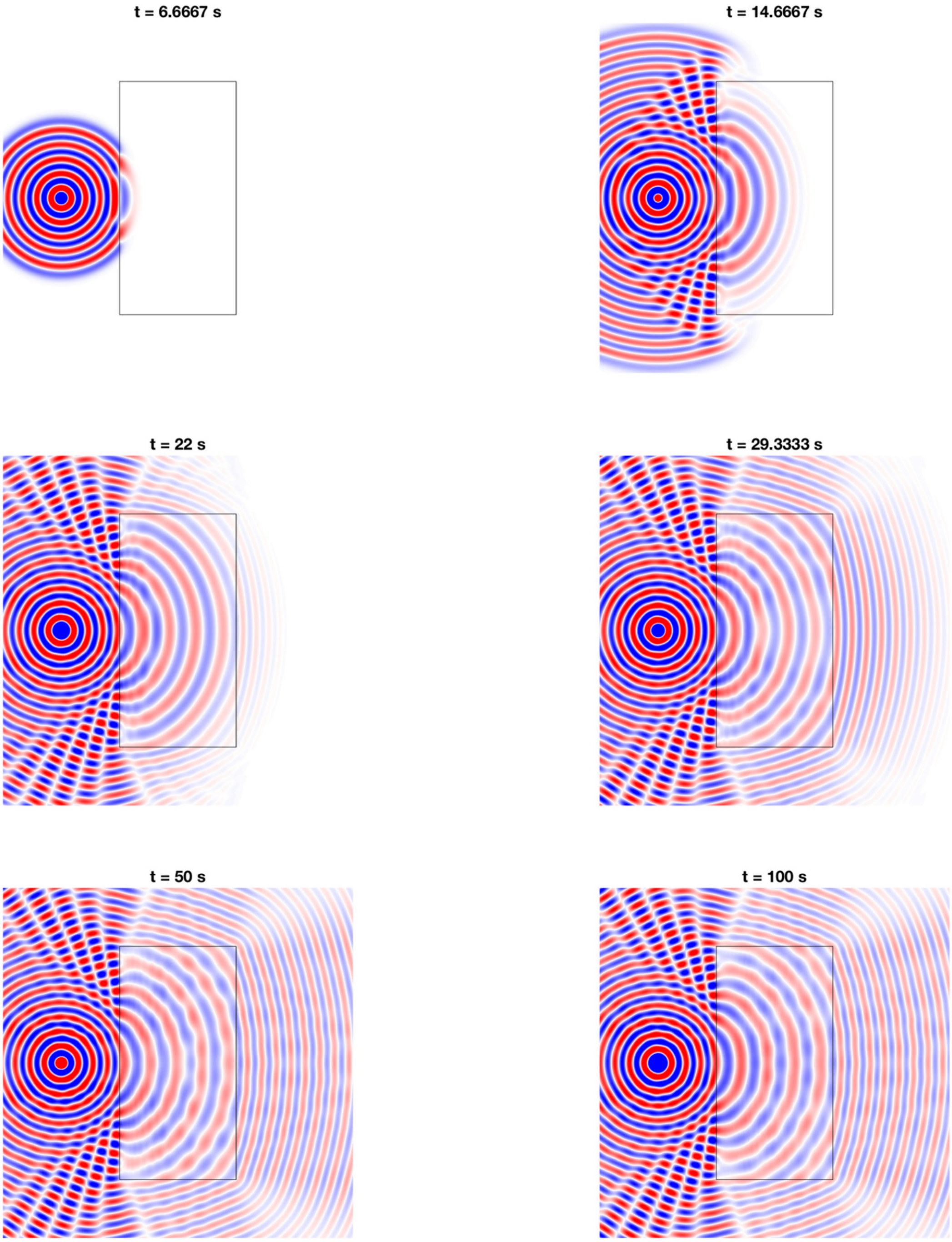

Fig. 2. Snapshots of E_3 at different times for the first experiment.

$$\begin{cases} \partial_t E(t,x) + \mathbf{w}_e^2(x) J(t,x) = \operatorname{curl} H(t,x) + f_e(t,x), \\ \partial_t H(t,x) + \mathbf{w}_m^2(x) K(t,x) = -\operatorname{curl} E(t,x) + f_m(t,x), \\ \partial_t J(t,x) = E(t,x), \\ \partial_t K(t,x) = H(t,x), \\ E(t=0,\cdot) = E_0, \qquad J(t=0,\cdot) = 0, \\ H(t=0,\cdot) = H_0, \qquad K(t=0,\cdot) = 0. \end{cases} \quad \text{for } t > 0,\ x \in \mathbb{R}^3. \tag{4.6}$$

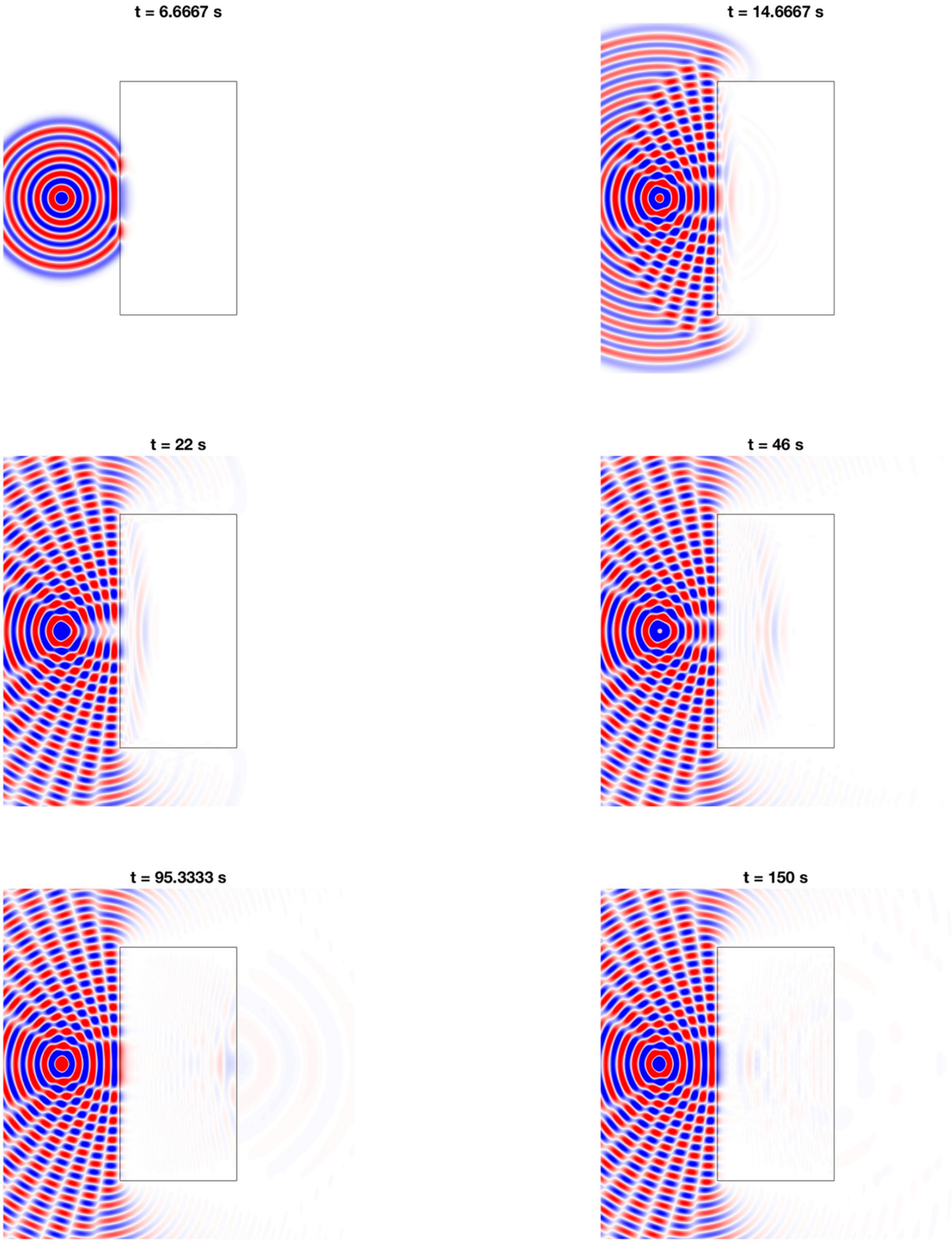

Fig. 3. Snapshots of E_3 at different times for the second experiment.

We are interested in simulations on (4.6) in the 2d setting for simplicity. We thus consider the case in which (E_0, H_0), (f_e, f_m), and $(\mathbf{w}_e, \mathbf{w}_m)$ do not depend on the third variable x_3 in space (here $x = (x', x_3) \in \mathbb{R}^3$ with $x' = (x_1, x_2) \in \mathbb{R}^2$). One can show that the four fields E, H, J and K are also independent of x_3 and that one has the two decoupled systems respectively called *transverse-electric* and *transverse-magnetic* modes. Here we focus on the transverse-electric modes, which are given as follows, for $t > 0$ and $x' = (x_1, x_2) \in \mathbb{R}^2$:

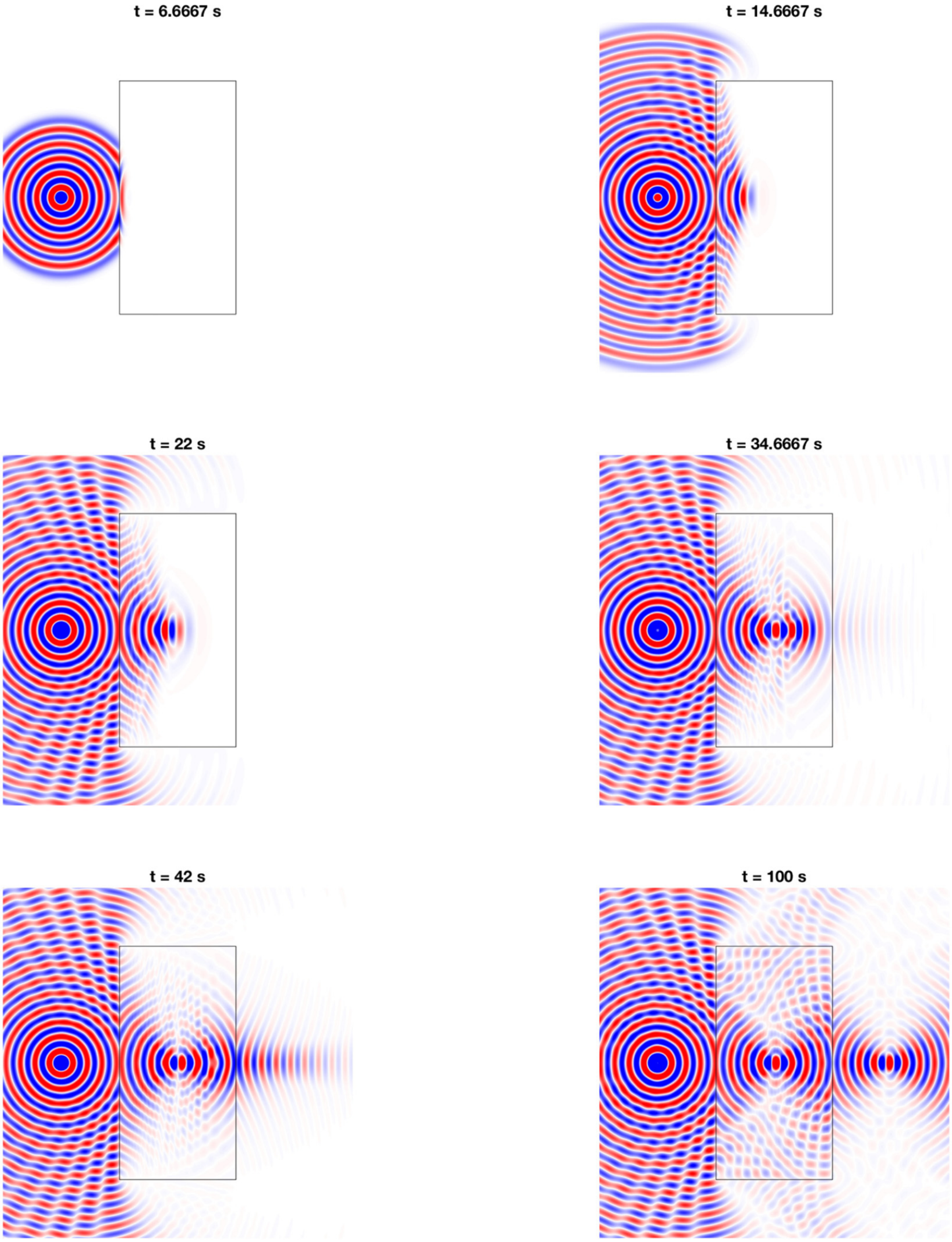

Fig. 4. Snapshots of E_3 at different times for the third experiment.

$$\begin{cases} \partial_t E_3(t,x') + \mathbf{w}_e^2(x_1,x_2) J_3(t,x') = \partial_{x_1} H_2(t,x') - \partial_{x_2} H_1(t,x') + f_{e,3}(t,x'), \\ \partial_t H_1(t,x') + \mathbf{w}_m^2(x') K_1(t,x') = -\partial_{x_2} E_3(t,x') + f_{m,1}(t,x'), \\ \partial_t H_2(t,x') + \mathbf{w}_m^2(x') K_2(t,x') = \partial_{x_1} E_3(t,x') + f_{m,2}(t,x'), \\ \partial_t J_3(t,x') = E_3(t,x'),\ \partial_t K_1(t,x') = H_1(t,x'),\ \partial_t K_2(t,x') = H_2(t,x'), \\ E_3(t=0,\cdot) = E_{0,3},\ H_1(t=0,\cdot) = H_{0,1},\ H_2(t=0,\cdot) = H_{0,2}, \\ J_3(t=0,\cdot) = 0,\ K_1(t=0,\cdot) = 0,\ K_2(t=0,\cdot) = 0. \end{cases} \tag{4.7}$$

The setting for the simulation is the following. The medium consists of a bounded rectangular obstacle filled with a Drude's material with positive constants ω_e and ω_m, which is surrounded by vacuum, i.e. $(\mathbf{w}_e, \mathbf{w}_m) = (\omega_e, \omega_m)$ inside the rectangle and $(0, 0)$ otherwise (see Fig. 1). We impose zero initial conditions for the electric and the magnetic fields:

$$E_3(t=0,\cdot) = H_1(t=0,\cdot) = H_2(t=0,\cdot) = 0, \tag{4.8}$$

and zero magnetic sources:

$$f_{m,1} = f_{m,2} = 0. \tag{4.9}$$

We choose

$$f_{e,3}(t, x_1, x_2) = \sin(\omega_* t) g(x_1, x_2), \quad \text{for } t > 0,\ (x_1, x_2) \in \mathbb{R}^2, \tag{4.10}$$

where g is a Gaussian given by

$$g(x_1, x_2) = \mathrm{e}^{-25(x_1+10)^2 - 25x_2^2}, \quad \text{for } (x_1, x_2) \in \mathbb{R}^2. \tag{4.11}$$

By selecting appropriately ω_e, ω_m and ω_*, the obstacle can have a negative permittivity, a negative permeability or even both.

Concerning the numerical methods, we use classical PMLs to artificially bound the computational domain and, for the numerical scheme, we use P^1–P^0 mixed finite elements (with mass lumping for efficiency) for the space discretization and centred finite difference approximations on staggered grids for the time discretization. The computations were done with FreeFem++ [13]. We refer to [43] for more details about these numerical methods.

We perform three numerical experiments.

- In the first one, we take $\omega_* = 5$, $\omega_e = 4$ and $\omega_m = 2$. With this choice, we have

  $$\widehat{\varepsilon}(\omega_*) \simeq 0.36 > 0 \quad \text{and} \quad \widehat{\mu}(\omega_*) \simeq 0.84 > 0.$$

  Here, the "effective" permittivity and permeability are both positive. Fig. 2 shows some snapshots of E_3 at different times. One can see that there is propagation inside the obstacle, but with different speeds (and consequently wavelengths). This is due to dispersion.
- In the second simulation, we take $\omega_* = 5$, $\omega_e = 6$ and $\omega_m = 2$. With this choice, we have

  $$\widehat{\varepsilon}(\omega_*) \simeq -0.44 < 0 \quad \text{and} \quad \widehat{\mu}(\omega_*) \simeq 0.84 > 0.$$

  Here, the "effective" permittivity and permeability are of opposite signs. Fig. 3 shows some snapshots of E_3 at different times. One can see that there is no propagation inside the obstacle: the field is exponentially decaying (after the transient wave has passed).
- In the third simulation, we take $\omega_* = 5$, $\omega_e = 5\sqrt{2}$ and $\omega_m = 5\sqrt{2}$. With this choice, we have

  $$\widehat{\varepsilon}(\omega_*) \simeq -1 < 0 \quad \text{and} \quad \widehat{\mu}(\omega_*) \simeq -1 < 0.$$

  Here, the "effective" permittivity and permeability are both negative. Fig. 4 shows some snapshots of E_3 at different times. There is propagation inside the obstacle. The field focuses inside the obstacle and re-focuses symmetrically to the source outside the obstacle on the right.

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