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Theory for serial correlations of interevent intervals

T. Schwalger[a] and B. Lindner

Max-Planck-Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, 01187 Dresden, Germany

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Abstract. We consider stochastic systems with m internal states in which discrete events (e.g. hopping events between metastable states or firing events of neurons) occur at a state-dependent rate. Transitions between states are possible with certain fixed rates. Because the state immediately after an event depends in general on the history of the process, the intervals between two consecutive events ("residence times") are correlated among each other, i.e. the residence time sequence constitutes a nonrenewal process. We construct a general kinetic scheme that accounts for the number of events at a given time. The count statistics is used to derive a general expression for the correlation coefficient of residence times with a certain lag. We apply the theoretical result to a simple neuron model with discrete threshold states leading to negative interspike interval correlations.

1 Introduction

The calculation of escape-time and residence-time distributions is a prominent problem in the theory of nonlinear stochastic systems with many applications in physics, biology, and chemistry. The classical setup for this problem is to prepare an ensemble of realizations at time $t = 0$ in a metastable state and to calculate the density of times t at which realizations escape for the first time from this state. Depending on the kind of fluctuations (white or colored, Gaussian or non-Gaussian), on the presence of additional driving terms (e.g. periodic or stochastic driving), on the spatial dimensions of the problem, and on the exact preparation of the ensemble, one obtains different probability densities and mean values of the escape time in this metastable state. Phenomena like stochastic resonance [1–4] and coherence resonance [5–9] have been characterized by various features of the escape time density and the way it depends on parameters like the noise intensity or the time scale of an external driving. So, at a first glance, it may look like as if the residence time distribution is the main statistics of interest and captures the entire statistics of the escape process.

Generally, the picture of an ensemble prepared in the metastable state, however, is incomplete. Many processes generate a *sequence of events*, for instance, a series of spikes as generated by an excitable neuron or a series of transitions between different metastable states as, for instance, a Brownian particle in a bistable or periodic potential. The long-term statistics of such series of events can be captured by a phase description. It was one of the many important contributions of Lutz Schimansky-Geier and his co-workers [10–14] to work out the phase description for various nonlinear stochastic systems. The phase statistics was used by Lutz Schimansky-Geier to study, for instance, synchronization between a nonlinear stochastic system's output and an external driving in the context of stochastic resonance.

[a] e-mail: tilo@pks.mpg.de

An alternative to a phase description is provided by the sequence of interevent intervals $\{\ldots, I_{k-1}, I_k, I_{k+1}\ldots\}$ (here $I_k = t_k - t_{k-1}$ where t_k is the time instant of the kth event). For neural spiking, for instance, this sequence would correspond to the well-known interspike intervals; for Brownian motion in a bistable potential, the interevent intervals correspond to the residence times of the particle in one of the metastable states. As pointed out above, a lot of research has focussed on the first-order statistics of these interevent intervals (escape times). The correlations between them, however, have received only little attention. One reason for this could be that in simplest case of a one-dimensional overdamped dynamical system driven by white Gaussian noise, the intervals are independent and the time instances of the events form a so-called renewal process [15]. Nonrenewal behavior, i.e. the existence of correlations in the sequence of interevent intervals, is expected in any slightly more complicated setup, e.g. if the system is driven by colored instead of white noise, if intrinsic feedback or other slow internal variables are added. A common quantifier of correlations is the serial correlation coefficient, given by

$$\rho_n = \frac{\langle I_k I_{k+n}\rangle - \langle I_k\rangle^2}{\langle I_k^2\rangle - \langle I_k\rangle^2}. \tag{1}$$

In neurons, nonrenewal spiking has been theoretically predicted and observed in experiments [16–26]. Correlations in the interspike interval sequence can arise because of (i) slow external inputs and (ii) feedback of the generated spikes into the spike generator. The first case is analytically tractable only for certain simple neuron models and simple driving processes [20,27] and leads often to positive correlations [28]. The second case, however, realized by an intrinsic feedback, for instance, by spike-frequency adaptation [29] leads to negative correlations over only the first few lags and is much harder to deal with analytically [30]. These negative correlations have attracted particular attention because it has been shown that they may contribute to an enhanced neural information transmission [31,32]. Hence, a theory for the serial correlation in a neuron model with intrinsic feedback is highly desirable.

We have recently put forward a method to calculate an approximation of the serial correlation coefficient of the residence times in a bistable system that is driven by a dichotomous noise [33,34]. The main idea of this approximation is to map the transitions between metastable states to a Markovian hopping process on a two-dimensional lattice: the vertical coordinate denotes the state of the external driving and the horizontal coordinate denotes the count of events (progressing always to the right). In this framework, the interevent intervals correspond to the residence times in one vertical layer. The intervals can be correlated because the transition rates to the next layer differ for the two states within a vertical layer. For such a discrete system, the master equation can be solved and from the moments of sums of residence times over several layers, one can calculate the serial correlation coefficient. By means of this discrete theory, several nontrivial predictions for the dependence of ρ_n on system and driving parameters could be made which were confirmed by simulation results of the original continuous dynamics [33,34].

In this paper, we extend the theory for the kinetic scheme from Refs. [33,34] from two horizontal lines of states to m lines of states and also allow for more general transitions within one vertical layer and for the transitions from one vertical layer to the next one. We present general formulas that should be applicable in a number of simple cases. As a specific application, we consider the problem of a spiking neuron with intrinsic feedback. In this example, the states within a layer represent an approximation of an internal variable (e.g. a dynamical threshold in an integrate-and-fire neuron) rather than an external driving (as the telegraph noise for the bistable system in Ref. [33,34]). We derive expressions for the serial correlation coefficient for two different models and verify them by stochastic simulations of the discrete scheme. Further applications and other extensions of the theory are discussed in the Conclusions.

2 General framework

We consider a system which generates events at times t_n and thus increase the event count $N(t)$ by one as indicated in the coarse-grained view of Fig. 1(a). Interevent intervals can be correlated because the event-generating dynamics is not Markovian – there is memory in the system or

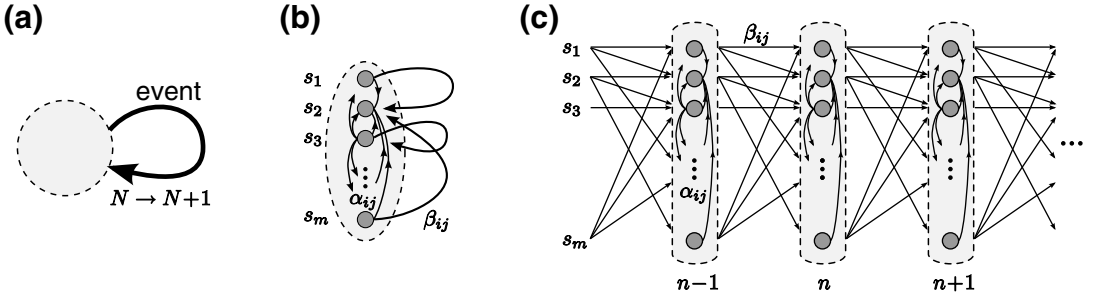

Fig. 1. (a) Non-Markovian system that generates discrete events. The temporal occurence of events is described by a count variable $N(t)$. (b) Markov approximation of the system by m discrete states s_i. There are internal transitions between states with constant rates α_{ij} and state transitions that are associated with an event (large arrows with constant rates β_{ij}). (c) Transitions in the two-dimensional state-space (s_i, n): matrix (α_{ij}) (internal transition) and matrix (β_{ij}) (events) are the same for every layer n.

in the external driving of the system. In many cases, we may approximate the non-Markovian dynamics by a Markovian dynamics on m discrete states s_i $(i = 1, \ldots, m)$ that we will call *internal* states irrespective of the origin of the memory (external driving or intrinsic feedback). These states are shown in Fig. 1(b). We can have two different kinds of transitions between these states: those that are associated with an event (they make up the thick arrow shown in Fig. 1(a)) and those that are purely internal. We can rearrange the scheme as shown in Fig. 1(c) and now keep track of the number of events by a second dimension. Purely internal transitions happen with rates α_{ij} from state s_j to s_i; transition rates that are associated with an event are given by β_{ij}. The stochastic evolution of the dynamics is given by the two-dimensional hopping process $(S(t), N(t))$ on the lattice shown in Fig. 1(c). Furthermore, at time $t = 0$ we require that $N(0) = 0$, i.e. $N(t)$ yields the number of events in the time window $(0, t]$.

2.1 Master equation

Both α_{ij} and β_{ij} completely define the Markovian dynamics of the discrete state variable $S(t)$, which yields the value of the internal state at time t (Fig. 1(b)). In order to calculate the interevent interval correlations, however, it is necessary to keep track of the count variable $N(t)$ as well. Therefore, let $p_i(n, t)$ be the probability that $S(t) = s_i$ and $N(t) = n$. Then, the vector $\mathbf{p}(n, t)$ made up by the elements $p_i(n, t)$ satisfies the master equation

$$\dot{\mathbf{p}}(n, t) = \mathbf{A}\mathbf{p}(n, t) + \mathbf{B}\mathbf{p}(n - 1, t), \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $m \times m$ matrices with elements

$$A_{ij} = \alpha_{ij} - \delta_{i,j} \sum_k (\alpha_{ki} + \beta_{ki}), \qquad B_{ij} = \beta_{ij}. \tag{3}$$

Next, consider the marginal probability $p_i(t) = \sum_{n=0}^{\infty} p_i(n, t)$ to find the system in the internal state s_i (irrespective of the number of events that have occured). The initial probabilites are $p_i(n, 0) = p_i(0)\delta_{n,0}$, because at time $t = 0$ no event has occured yet (that is how we defined $N(t)$ above, namely as the number of events in the time interval $(0, t]$). We will consider two different ensembles resulting from two different initial conditions $p_i(0)$.

In both cases, we imagine that the process $S(t)$ has been initiated at an arbitrary value at $t = -\infty$, so that at the origin $t = 0$ the process is stationary. In what we call the *stationary* ensemble, we take $p_i(0)$ to be the stationary probability distribution of $S(t)$, which we denote by the vector $\mathbf{p}^{(s)}$.

We can also consider the values of $S(t)$ conditioned by the occurrence of an event: in particular, we consider the sequence $\{S_k\}$ of $S(t)$ right after the occurrence of an event, i.e.

$$S_k = \lim_{\varepsilon \to 0^+} S(t_k + \varepsilon) \tag{4}$$

where t_k are the instances of the kth event. For the second, the so-called *conditional* ensemble, the initial distribution is taken as the distribution $\hat{\mathbf{p}}$ of the stationary sequence $\{S_k\}$.

The first distribution $\mathbf{p}^{(s)}$ is simply the marginal distribution $\mathbf{p}(t) = \sum_n \mathbf{p}(n,t)$ in the steady state. An equation for this density is obtained by summing Eq. (2) over n and setting $\dot{\mathbf{p}}(t) = 0$ which yields

$$(\mathbf{A} + \mathbf{B})\mathbf{p}^{(s)} = 0 \tag{5}$$

or

$$\mathbf{C}\mathbf{p}^{(s)} = \mathbf{p}^{(s)}, \qquad \mathbf{C} = -\mathbf{A}^{-1}\mathbf{B}. \tag{6}$$

Eq. (6) states that $\mathbf{C}$ has an eigenvalue $\lambda_1 = 1$ with a corresponding eigenvector that is the stationary distribution $\mathbf{p}^{(s)}$.

Regarding the second distribution $\hat{\mathbf{p}}$, we note that the stationary distribution of states immediately after an event is proportional to the stationary currents from the state $n-1$ into the states (n, s_i). Hence, the stationary and the conditional distributions are related by

$$\hat{\mathbf{p}} \propto \mathbf{B}\mathbf{p}^{(s)}. \tag{7}$$

The factor of proportionality is obtained by normalization. In the following, symbols which refer to the conditional ensemble will be marked with a hat.

2.2 Residence time statistics

The count statistics of events is completely given by the marginal distribution

$$p(n,t) = \sum_{i=1}^{m} p_i(n,t). \tag{8}$$

Furthermore, we define the quantities

$$\tau_i^{(n)} = \int_0^\infty p_i(n,t)\,dt,\ \hat{\tau}_i^{(n)} = \int_0^\infty \hat{p}_i(n,t)\,dt,\ \tau^{(n)} = \int_0^\infty p(n,t)\,dt,\ \hat{\tau}^{(n)} = \int_0^\infty \hat{p}(n,t)\,dt. \tag{9}$$

As can be easily seen[1], $\tau_i^{(n)}$ and $\hat{\tau}_i^{(n)}$ yield the average time the system spends in the state (s_i, n) for the two different initial distributions. Likewise, $\tau^{(n)}$ and $\hat{\tau}^{(n)}$ constitute the respective average residence times in the state n. Because in the conditional ensemble the distribution of states is the same at the beginning of each interval, i.e. immediately after each event, $\hat{\tau}_i^{(n)}$ does not depend on n, i.e. $\hat{\tau}_i^{(n)} = \hat{\tau}_i$ and $\hat{\tau}^{(n)} = \hat{\tau}$.

The statistics of interevent intervals of a stationary point process is linked to the count statistics by [35]

$$\langle I_k \rangle = \hat{\tau}, \quad \langle I_k^2 \rangle = 2\hat{\tau}\tau^{(0)}, \quad \langle I_k I_{k+n} \rangle = \hat{\tau}\tau^{(n)}. \tag{10}$$

Hence, the serial correlation coefficient, Eq. (1), can be rewritten as

$$\rho_n = \frac{\tau^{(n)} - \hat{\tau}}{2\tau^{(0)} - \hat{\tau}}, \quad n \geq 1. \tag{11}$$

[1] From a realization of the process $(S(t), N(t))$ we obtain the total time the trajectory spends in state (s_i, n) by integrating $\int_0^\infty \delta_{N(t),n}\delta_{S(t),s_i}\,dt$ where $\delta_{x,y}$ is one if $x = y$ and zero otherwise. The average of the integral (the mean total residence time in (s_i, n)) reduces to the integral over the average $\langle \delta_{N(t),n}\delta_{S(t),s_i} \rangle$. The latter is by definition the probability distribution $p_i(n,t)$.

This equation illustrates that the serial correlation coefficient is proportional to the difference of mean residence times of the two ensembles. Clearly, the difference vanishes as $n \to \infty$, because in this limit the initial distributions are forgotten and both ensembles become equivalent.

In order to compute ρ_n one has to determine $\tau^{(n)}$. Therefore, Eq. (2) is integrated over t yielding the system of difference equations

$$\boldsymbol{\tau}^{(n)} = \mathbf{C}\boldsymbol{\tau}^{(n-1)}, \qquad n \geq 1 \tag{12}$$

with initial condition

$$\boldsymbol{\tau}^{(0)} = -\mathbf{A}^{-1}\mathbf{p}^{(s)}. \tag{13}$$

Here, $\boldsymbol{\tau}^{(n)}$ denotes the vector with elements $\tau_i^{(n)}$. Note, that in the conditional ensemble

$$\hat{\boldsymbol{\tau}}^{(0)} = -\mathbf{A}^{-1}\hat{\mathbf{p}}^{(s)} \propto -\mathbf{A}^{-1}\mathbf{B}\mathbf{p}^{(s)} = \mathbf{C}\mathbf{p}^{(s)} = \mathbf{p}^{(s)}, \tag{14}$$

is an eigenvector of $\mathbf{C}$ to eigenvalue $\lambda_1 = 1$ (Eq. (6)) and hence $\hat{\boldsymbol{\tau}}^{(n)} = \mathbf{C}^n\hat{\boldsymbol{\tau}}^{(0)} = \hat{\boldsymbol{\tau}}^{(0)}$ is independent of n as stated above.

The formal solution of Eq. (12)–(13) is $\boldsymbol{\tau}^{(n)} = \mathbf{C}^n\boldsymbol{\tau}^{(0)}$, which amounts to finding the n-th power of $\mathbf{C}$. This standard task can be achieved, for instance, by transforming $\mathbf{C}$ to the Jordan normal form, which requires the determination of (generalized) eigenvectors. A particularly simple case arises if $\mathbf{C}$ possesses m linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, i.e. if $\mathbf{C}$ is diagonalizable. In this case, the general solution of Eq. (12) reads

$$\boldsymbol{\tau}^{(n)} = c_1\lambda_1^n\mathbf{v}_1 + \cdots + c_m\lambda_m^n\mathbf{v}_m, \tag{15}$$

where $c_1, \ldots, c_m$ are constants determined by the initial condition Eq. (13) and $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $\mathbf{C}$. Since $\boldsymbol{\tau}^{(n)}$ tends to $\hat{\boldsymbol{\tau}}$ in the limit $n \to \infty$ the eigenvalues λ_i, $i = 2, \ldots, m$ must satisfy $|\lambda_i| < \lambda_1 = 1$. Thus, the serial correlation coefficient is proportional to

$$\rho_n \propto \mathbf{1}(c_2\lambda_2^n\mathbf{v}_2 + \cdots + c_m\lambda_m^n\mathbf{v}_m), \tag{16}$$

where the product with the row vector $\mathbf{1} = (1, 1, \ldots, 1)$ yields the summation over all states s_i. Eq. (16) shows that the correlations between residence times decay geometrically fast[2].

3 Interspike intervals of neurons with inhibitory feedback

We will now apply the above described method to a simple neuron model with inhibitory feedback of the spike train. There are a number of biophysical realizations for such a feedback, however, it has been argued in [29] that many of them can be cast into the same first-order dynamics of some slow "adaptation" variable. This adaptation variable can be regarded as an internal variable of the neuron that rapidly increases upon spiking and decays exponentially between successive spikes. An increase of the adaptation variable leads to an exponential suppression of the firing rate. Furthermore, a rapid succession of two spikes (forming an ISI shorter than on average) increases the adaptation variable, such that the next interspike interval is longer than on average. Hence, an inhibitory feedback of the spike train becomes manifest by negative interspike interval correlations.

In neurons, the adaptation variable could represent either an inhibitory current that increases upon spikes [19,29,36] or a dynamic threshold [31]. In both cases, the adaptation variable indicates the momentary distance of the resting potential from the excitation threshold. Thus, the "escape" rate depends on the adaptation state. Both features, a state-dependent

[2] This observation is also valid in the general case, where $\mathbf{C}$ is in general not diagonizable. The reason for this is that the sequence $\{\boldsymbol{\tau}^{(n)}\}$ of processes described by Eq. (2) is closely related to the sequence $\{\mathbf{p}^{(n)}\}$ of the distributions of states S_n immediately after the events. Because S_n is a realization of a Markov chain, the distribution $\mathbf{p}^{(n)}$ converges geometrically fast to the limit distribution $\hat{\mathbf{p}}$. Indeed, the relationship between $\boldsymbol{\tau}^{(n)}$ and $\mathbf{p}^{(n)}$ is simply a linear map given by $\boldsymbol{\tau}^{(n)} = -\mathbf{A}^{-1}\mathbf{p}^{(n)}$. This follows from the fact that $\boldsymbol{\tau}^{(n)} = \int_0^\infty \mathbf{p}(t)\,dt$, where $\mathbf{p}(t)$ is the solution of $\dot{\mathbf{p}}(t) = \mathbf{A}\mathbf{p}(t)$ with $\mathbf{p}(0) = \mathbf{p}^{(n)}$.

firing rate and the spike-triggered dynamics of the adaptation variable, can be realized in our discrete-state Markov model[3]. Here, we consider the simple case of only four discrete adaptation states $s_1, \ldots, s_4$, representing e.g. discrete threshold values. The internal transitions realize the exponential decay of the adaptation variable in the order $s_4 \xrightarrow{3\alpha} s_3 \xrightarrow{2\alpha} s_2 \xrightarrow{\alpha} s_1$ (Fig. 2(a,d)). After each spike ("event") the internal state is not the same state as prior to spiking.

Specifically, we investigate two models: In the first model, each spike increases the adaptation variable by two states (Fig. 2(a)). This allows for two possible firing paths, $s_1 \xrightarrow{\beta_1} s_3$ and $s_2 \xrightarrow{\beta_2} s_4$, with the associated firing rates $\beta_1 > \beta_2$. In state s_3 and s_4 firing is not possible. In the second model, the adaptation variable increases by one state upon firing. Spiking is now possible in the states s_1, s_2 and s_3 with rates $\beta_1 > \beta_2 > \beta_3$, respectively (Fig. 2(d)). We will demonstrate our method for both models.

3.1 Neuron model A

The kinetic scheme of Fig. 2(a) leads to the following matrices

$$\mathbf{A} = \begin{pmatrix} -\beta_1 & \alpha & 0 & 0 \\ 0 & -(\beta_2+\alpha) & 2\alpha & 0 \\ 0 & 0 & -2\alpha & 3\alpha \\ 0 & 0 & 0 & -3\alpha \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \frac{\alpha}{\beta_2+\alpha} & \frac{\beta_2\alpha}{\beta_1(\beta_2+\alpha)} & 0 & 0 \\ \frac{\beta_1}{\beta_2+\alpha} & \frac{\beta_2}{\beta_2+\alpha} & 0 & 0 \\ \frac{\beta_1}{2\alpha} & \frac{\beta_2}{2\alpha} & 0 & 0 \\ 0 & \frac{\beta_2}{3\alpha} & 0 & 0 \end{pmatrix}. \tag{17}$$

The eigenvalues of $\mathbf{C}$ are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \mathbf{p}^{(s)} = \frac{1}{6\alpha^2 + 9\beta_1\alpha + 5\beta_1\beta_2} \left(6\alpha^2, 6\beta_1\alpha, 3\beta_1(\beta_2+\alpha), 2\beta_1\beta_2\right)^T \tag{18}$$

and

$$\mathbf{v}_2 = (0,0,1,0)^T, \qquad \mathbf{v}_3 = (0,0,0,1)^T. \tag{19}$$

From Eq. (7) we find the limiting distribution of states immediately after firing

$$\hat{\mathbf{p}} = \left(0, 0, \frac{\alpha}{\beta_2+\alpha}, \frac{\beta_2}{\beta_2+\alpha}\right)^T, \tag{20}$$

which, of course, has probability only in the states s_3 and s_4. Eq. (14) yields the vector of mean residence times in the conditional ensemble $\hat{\boldsymbol{\tau}}$ and summing over all states yields the mean interspike interval (cf. Eq. (10))

$$\langle I_k \rangle = \hat{\tau} = \frac{6\alpha^2 + 9\beta_1\alpha + 5\beta_1\beta_2}{6\beta_1(\beta_2+\alpha)\alpha}. \tag{21}$$

For the initial condition of the difference equation Eq. (12) we find from Eq. (13) and (18)

$$\boldsymbol{\tau}^{(0)} = \frac{1}{\hat{\tau}} \left(\frac{6\alpha^2 + (9\beta_1 + 6\beta_2)\alpha + 5\beta_1\beta_2}{6\beta_1^2(\beta_2+\alpha)^2}, \frac{9\alpha + 5\beta_2}{6\alpha(\beta_2+\alpha)^2}, \frac{3\alpha+5\beta_2}{12\alpha^2(\beta_2+\alpha)}, \frac{\beta_2}{9\alpha^2(\beta_2+\alpha)} \right)^T. \tag{22}$$

[3] Note, that the considered Markov model is designed for analyzing a time-homogeneous setting. In particular, it cannot be applied to the phenomenon of adaptation, which refers to the response to a time-dependent stimulus. Nevertheless, we will stick to the notion "adaptation variable", because our model is intended to approximate the stationary properties of certain adaptive neuron models.

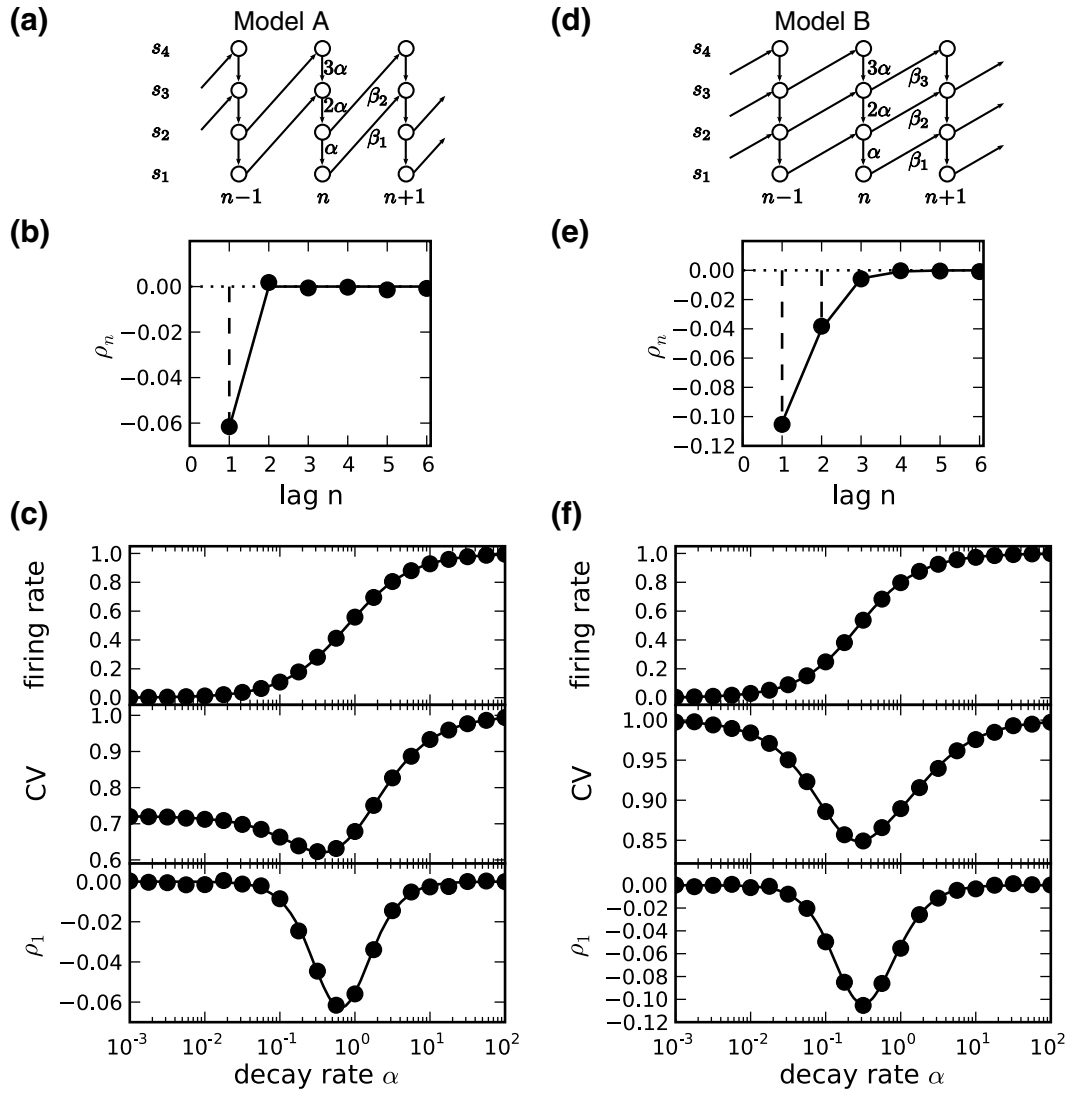

Fig. 2. Interspike interval statistics of neuron models with four adaptation states. (a) Kinetic scheme of neuron model A: The adaptation variable decays with rate α in between spiking events. Spiking is possible in the lower two states with rates $\beta_i = e^{-\gamma(i-1)}$, $i = 1, 2$ and $\gamma = 0.3$. Upon spiking the adaptation variable jumps by two states. (b) Serial correlation coefficient as a function of the lag for $\alpha = 0.56$: simulation (circles) and theory (solid line). (c) Firing rate, coefficient of variation and SCC at lag 1 as a function of the decay rate α of the adaptation variable: simulation (circles) and theory (solid line). (d)–(f) Neuron model B: Like model A but adaptation variable jumps by one state upon spiking. Spiking is possible in the lower three states with rates $\beta_i = e^{-\gamma(i-1)}$, $i = 1, 2, 3$. In (e) we chose $\alpha = 0.32$.

The matrix power $\mathbf{C}^n$ is given through a similarity transformation as

$$\mathbf{C}^n = \mathbf{P}\mathbf{J}^n\mathbf{P}^{-1} \quad \text{where} \quad \mathbf{P} = (\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_4, \mathbf{v}_4)\,. \tag{23}$$

The generalized eigenvector $\mathbf{v}_4$ is the solution of $\mathbf{C}\mathbf{v}_4 - \lambda_3\mathbf{I} = \mathbf{v}_3$ and reads

$$\mathbf{v}_4 = \left(-\frac{3\alpha}{\beta_1}, \frac{3\alpha}{\beta_2}, 0, 0\right)^T. \tag{24}$$

The matrix $\mathbf{J} = \mathbf{P}^{-1}\mathbf{C}\mathbf{P}$ is the Jordan normal form of $\mathbf{C}$, the n-th power of which is given by

$$\mathbf{J}^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{n,1} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Now we have all the ingredients for calculating the serial correlation coefficient using Eqs. (10), (22) and (23). The covariance $\mathrm{cov}(I_k, I_{k+n}) = \langle I_k I_{k+n}\rangle - \langle I_k\rangle^2$ is

$$\mathrm{cov}(I_k, I_{k+n}) = -\frac{\beta_2}{3\beta_1(\beta_2+\alpha)^2}\delta_{n,1} \tag{26}$$

and the variance is

$$\mathrm{var}(I_k) = \frac{36\alpha^4 + 72\beta_2\alpha^3 + 45\beta_1^2\alpha^2 + 26\beta_1^2\beta_2\alpha + 13\beta_1^2\beta_2^2}{36\beta_1^2\alpha^2(\beta_2+\alpha)^2}. \tag{27}$$

The ratio of the last two expressions yields

$$\rho_n = -\delta_{n,1}\frac{12\beta_1\beta_2\alpha^2}{36\alpha^4 + 72\beta_2\alpha^3 + 45\beta_1^2\alpha^2 + 26\beta_1^2\beta_2\alpha + 13\beta_1^2\beta_2^2}. \tag{28}$$

From the theoretical result it becomes clear that in the adaptation model A, adjacent interspike intervals are anti-correlated, whereas the correlations at higher lags vanish (Fig. 2b). Furthermore, the correlation coefficient depends only on the ratio of the rates such that we can set one of them to one without loss of generality; in the following $\beta_1 = 1$. In accordance with the inhibitory character of the feedback, we set $\beta_2 = \beta_1 e^{-\gamma}$ with $\gamma > 0$ (the firing rate is reduced for higher internal states), optimize with respect to γ and α, and find that $\rho_1 \geq -0.068$. Hence, the adaptation model A reproduces the expected negative interspike interval correlations, however, the absolute correlation strength is rather small[4].

Why are only adjacent intervals correlated, i.e. why do we have $\rho_n = 0$ for $n > 1$? One way to see this is that memory about interspike intervals is carried by the specific path the random walker takes on the lattice (s, n). Because of the topology, all realizations have to pass through the special state (s_2, n) which erases any memory about the residence time in $n-1$ for future residence times, as for instance that in $n+1$.

Fig. 2(c) compares the analytical expressions for the firing rate (the inverse mean interval), the coefficient of variation $\mathrm{CV} = \sqrt{\mathrm{var}(I_k)}/\langle I_k\rangle$ of the single interval, and the correlation coefficient ρ_1 as a function of α to numerical simulations of the kinetic scheme for some (suboptimal) choice of β_1 and β_2. While the firing rate just increases the faster the system decays through the cascade of internal states, the CV and ρ_1 both attain a minimum around $\alpha \approx \beta_1/2$; both minima are largely due to a nonmonotonic dependence of the variance on α (mean and covariance increase monotonically with α). It is plausible that for both $\alpha \to 0$ and $\alpha \to \infty$, the spike train becomes a renewal process. For small rate α, transitions to the next layer always go through β_2; at large rate, the system goes quickly to s_1 and the next spike occurs via the β_1 path. Only for intermediate rates, both paths between layers are used and thus correlations between the residence times in one layer (i.e. between interspike intervals) become possible.

[4] A closer inspection reveals that the global optimum is at negative γ yielding a slightly increased minimal correlation of $\rho_1 \approx -0.074$. Because $\gamma < 0$ would not correspond to the inhibitory feedback, we do not further consider this parameter choice.

3.2 Neuron model B

This model is illustrated in Fig. 2(d) and is described by the following matrices:

$$\mathbf{A} = \begin{pmatrix} -\beta_1 & \alpha & 0 & 0 \\ 0 & -(\beta_2+\alpha) & 2\alpha & 0 \\ 0 & 0 & -(\beta_3+2\alpha) & 3\alpha \\ 0 & 0 & 0 & -3\alpha \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \end{pmatrix}. \tag{29}$$

The calculation for this case is analogous to the previous one; we omit the rather tedious expressions of intermediated computational steps and just state the main results: In contrast to the previous model, the matrix $\mathbf{C} = -\mathbf{A}^{-1}\mathbf{B}$ has one non-vanishing eigenvalue λ_2 smaller than $\lambda_1 = 1$:

$$\lambda_2 = \frac{\beta_3\alpha}{(\alpha+\beta_2)(2\alpha+\beta_3)}. \tag{30}$$

The mean interspike interval reads

$$\langle I_k \rangle = \hat{\tau} = \frac{6\alpha^3 + 6\alpha^2\beta_1 + 3\alpha\beta_1\beta_2 + \beta_1\beta_2\beta_3}{6\alpha^3\beta_1 + 6\alpha^2\beta_1\beta_2 + 3\alpha\beta_1\beta_2\beta_3}. \tag{31}$$

Furthermore, the serial correlation coefficent is the ratio $\rho_n = \mathrm{cov}(I_k, I_{k+n})/\mathrm{var}(I_k)$ given by

$$\begin{aligned} \mathrm{var}(I_k) = \frac{\lambda_2}{9\alpha^3\beta_1^2\beta_3(2\alpha(\alpha+\beta_2)+\beta_2\beta_3)^2} \big[& 72\alpha^8 + 36\alpha^7(6\beta_2+\beta_3) \\ & + 36\alpha^6\left(2\beta_1^2 + \beta_2(4\beta_2+5\beta_3)\right) + 36\alpha^5\left(\beta_1\beta_2\beta_3 + \beta_1^2(3\beta_2+\beta_3) + \beta_2\beta_3(4\beta_2+\beta_3)\right) \\ & + 2\alpha^4\beta_2\left(12\beta_1\beta_3^2 + 18\beta_2\beta_3^2 + \beta 1^2(27\beta_2+46\beta_3)\right) + \alpha^3\beta_1^2\beta_2\left(18\beta_2^2 + 37\beta_2\beta_3 + 28\beta_3^2\right) \\ & + \alpha^2\beta_1^2\beta_2^2\beta_3(17\beta_2+10\beta_3) + \alpha\beta_1^2\beta_2^2\beta_3^2(6\beta_2+\beta_3) + \beta_1^2\beta_2^3\beta_3^3 \big], \end{aligned} \tag{32}$$

and for $n \geq 1$

$$\mathrm{cov}(I_k, I_{k+n}) = \frac{2\lambda_2}{9\alpha^2\beta_1^2\beta_3^2(2\alpha(\alpha+\beta_2)+\beta_2\beta_3)^2}\left(\nu_1\delta_{n1} - \nu_2\lambda_2^n\right), \tag{33}$$

where

$$\nu_1 = 3\alpha\beta_1\beta_2\beta_3(\alpha+\beta_2)(2\alpha+\beta_3)(2\alpha(\alpha+\beta_2)+\beta_2\beta_3) \tag{34}$$

$$\begin{aligned} \nu_2 = \beta_2 & \left(12\alpha^2(\alpha+\beta_2) + 3\alpha(4\alpha+\beta_1+4\beta_2)\beta_3 + (3\alpha+2\beta_1+3\beta_2)\beta_3^2\right) \\ & \times \left(\alpha^3(5\beta_1-3\beta_3) + \alpha^2\beta_1(7\beta_2-2\beta_3) + \beta_1\beta_2^2\beta_3 + 2\alpha\beta_1\beta_2(\beta_2+\beta_3)\right). \end{aligned} \tag{35}$$

In this case, as becomes also evident in Fig. 2e, the correlation coefficient is nonzero for any finite lag because not all paths go through the same internal state as in model A. Parametrizing the interlayer transition rates as before $\beta_i = e^{-\gamma(i-1)}$, we find that model B can show a modest correlation at lag one of about $\rho_1 \approx -0.12$ that is, however, twice as strong as the minimal correlation of model A.

The dependence of firing rate, CV, and correlation coefficient ρ_1 (Fig. 2(f)) is qualitatively similar to the one observed for model A (Fig. 2(c)). CV and ρ_1 attain a minimum at a rate smaller than $\beta_1 = 1$. Remarkably, the minimum of the CV is still close to a Poisson process ($\mathrm{CV}_{min} \approx 0.85$ compared to $\mathrm{CV}_{min} \approx 0.6$ for model A), whereas (as discussed above) the minimum in the correlation coefficient is deeper than for model A. Model B seems to be a more faithful representation of an excitable neuron with adaptation. We anticipate that generalizing model B with the addition of more internal states, one can obtain stronger negative correlations than observed here.

4 Conclusions

In this paper we have developed a Markovian framework for interevent interval correlations, generated by non-Markovian complex systems. We derived general algebraic expressions for the serial correlation coefficient and the first two moments of the interevent intervals. Within this approximation, the correlation coefficient decays exponentially fast with increasing lag; it is given by a superposition of different exponential functions. The developed framework is sufficiently general to describe approximately the effect of external drivings, of slow internal variables, and of intrinsic feedback of event generation on the interevent interval statistics of various systems.

As a specific example, we have applied our theory to two models of an excitable neuron with intrinsic feedback of spikes that can mimic an integrate-and-fire neuron with dynamical threshold or with an adaptation current. For these systems, we found a negative correlation of adjacent intervals as it is also observed in experiments on certain neurons [17,21]. However, in these models, we constraint the system to four internal states and achieved only rather modest correlations of $\rho_1 \approx -0.1$. We expect that with the addition of more internal states, stronger negative correlations are feasible. Another interesting task for future research is to compare the results of our scheme with simulations of the leaky integrate-and-fire model with dynamical threshold or with an adaptation current. Finally, the *inverse* problem might be interesting too: for a given experimental interevent statistics (including the interval correlations for different lags) one may ask for the minimal discrete model yielding a sufficient agreement with the data.

This paper is dedicated to our scientific teacher and friend, Lutz Schimansky-Geier, on the occasion of his 60th birthday.

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