A refined estimate for the topological degree

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Abstract

We sharpen an estimate of [4] for the topological degree of continuous maps from a sphere $\mathbb{S}^d$ into itself in the case $d \geq 2$. This provides the answer for $d \geq 2$ to a question raised by Brezis. The problem is still open for $d = 1$.

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1 Introduction

Motivated by the theory of Ginzburg Landau equations (see, e.g., [1]), Bourgain, Brezis, and the author established in [4]:

Theorem 1. *Let $d \geq 1$. For every $0 < \delta < \sqrt{2}$, there exists a positive constant $C(\delta)$ such that, for all $g \in C(\mathbb{S}^d, \mathbb{S}^d)$,*

$$|\deg g| \leq C(\delta) \underset{|g(x)-g(y)|>\delta}{\int_{\mathbb{S}^d}\int_{\mathbb{S}^d}} \frac{1}{|x-y|^{2d}}\, dx\, dy. \tag{1.1}$$

Here and in what follows, for $x \in \mathbb{R}^{d+1}$, $|x|$ denotes its Euclidean norm in $\mathbb{R}^{d+1}$.

The constant $C(\delta)$ depends also on d but for simplicity of notation we omit d. Estimate (1.1) was initially suggested by Bourgain, Brezis, and Mironescu in [2]. It was proved in [3] in the case where $d = 1$ and δ is sufficiently small. In [9], the author improved (1.1) by establishing that (1.1) holds for $0 < \delta < \ell_d = \sqrt{2 + \frac{2}{d+1}}$ with a constant $C(\delta)$ independent of δ. It was also shown there that (1.1) does not hold for $\delta \geq \ell_d$.

This note is concerned with the behavior of $C(\delta)$ as $\delta \to 0$. Brezis [7] (see also [6, Open problem 3]) conjectured that (1.1) holds with

$$C(\delta) = C\delta^d, \tag{1.2}$$

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for some positive constant C depending only on d. This conjecture is somehow motivated by the fact that (1.1)-(1.2) holds "in the limit" as $\delta \to 0$. More precisely, it is known that (see [8, Theorem 2])

$$\lim_{\delta\to 0} \int_{\mathbb{S}^d}\int_{\mathbb{S}^d \atop |g(x)-g(y)|>\delta} \frac{\delta^d}{|x-y|^{2d}}\,dx\,dy = K_d \int_{\mathbb{S}^d} |\nabla g(x)|^d\,dx \text{ for } g \in C^1(\mathbb{S}^d)$$

for some positive constant K_d depending only on d and that

$$\deg g = \frac{1}{|\mathbb{S}^d|}\int_{\mathbb{S}^d} \operatorname{Jac}(g) \text{ for } g \in C^1(\mathbb{S}^d, \mathbb{S}^d),$$

by Kronecker's formula.

In this note, we confirm Brezis' conjecture for $d \ge 2$. The conjecture is still open for $d = 1$. Here is the result of the note.

Theorem 2. *Let $d \ge 2$. There exists a positive constant $C = C(d)$, depending only on d, such that, for all $g \in C(\mathbb{S}^d, \mathbb{S}^d)$,*

$$|deg\ g| \le C \int_{\mathbb{S}^d}\int_{\mathbb{S}^d \atop |g(x)-g(y)|>\delta} \frac{\delta^d}{|x-y|^{2d}}\,dx\,dy \quad \text{ for } 0 < \delta < 1. \tag{1.3}$$

2 Proof of Theorem 2

The proof of Theorem 2 is in the spirit of the approach in [4, 9]. One of the new ingredients of the proof is the following result [10, Theorem 1], which has its roots in [5]:

Lemma 1. *Let $d \ge 1$, $p \ge 1$, let B be an open ball in $\mathbb{R}^d$, and let f be a real bounded measurable function defined in B. We have, for all $\delta > 0$,*

$$\frac{1}{|B|^2}\int_B\int_B |f(x)-f(y)|^p\,dx\,dy \le C_{p,d}\left(|B|^{\frac{p}{d}-1}\int_B\int_{B \atop |f(x)-f(y)|>\delta} \frac{\delta^p}{|x-y|^{d+p}}\,dx\,dy + \delta^p\right), \tag{2.1}$$

for some positive constant $C_{p,d}$ depending only on p and d.

In Lemma 1, $|B|$ denotes the Lebesgue measure of B.

We are ready to present

Proof of Theorem 2. We follow the strategy in [4, 9]. We first assume in addition that $g \in C^1(\mathbb{S}^d, \mathbb{S}^d)$. Let B be the open unit ball in $\mathbb{R}^{d+1}$ and let $u : B \to B$ be the average extension of g, i.e.,

$$u(X) = \fint_{B(x,r)} g(s)\,ds \text{ for } X \in B, \tag{2.2}$$

where $x = X/|X|$, $r = 2(1-|X|)$, and $B(x,r) := \{y \in \mathbb{S}^d; |y-x| \le r\}$. In this proof, $\fint_D g(s)\,ds$ denotes the equantity $\frac{1}{|D|}\int_D g(s)\,ds$ for a measurable subset D of $\mathbb{S}^d$ with positive (d-dimensional

Hausdorff) measure. Fix $\alpha = 1/2$ and for every $x \in \mathbb{S}^d$, let $\rho(x)$ be the length of the largest radial interval coming from x on which $|u| > \alpha$ (possibly $\rho(x) = 1$). In particular, if $\rho(x) < 1$, then

$$\left| \fint_{B(x,2\rho(x))} g(s)\, ds \right| = 1/2. \tag{2.3}$$

By [4, (7)], we have

$$|\deg g| \le C \int_{\substack{\mathbb{S}^d \\ \rho(x)<1}} \frac{1}{\rho(x)^d}\, dx. \tag{2.4}$$

Here and in what follows, C denotes a positive constant which is independent of x, ξ, η, g, and δ, and can change from one place to another.

We now implement ideas involving Lemma 1 applied with $p = 1$. We have, by (2.3),

$$\fint_{B(x,2\rho(x))} \fint_{B(x,2\rho(x))} |g(\xi) - g(\eta)|\, d\xi\, d\eta \ge \fint_{B(x,2\rho(x))} \left| g(\xi) - \fint_{B(x,2\rho(x))} g(\eta)\, d\eta \right| d\xi \ge C.$$

This yields, for some $1 \le j_0 \le d+1$,

$$\fint_{B(x,2\rho(x))} \fint_{B(x,2\rho(x))} |g_{j_0}(\xi) - g_{j_0}(\eta)|\, d\xi\, d\eta \ge C,$$

where g_j denotes the j-th component of g. It follows from (2.1) that, for some $\delta_0 > 0$ (δ_0 depends only on d) and for $0 < \delta < \delta_0$,

$$\rho(x)^{1-d} \int_{B(x,2\rho(x))} \int_{\substack{B(x,2\rho(x)) \\ |g_{j_0}(\xi) - g_{j_0}(\eta)|>\delta}} \frac{\delta}{|\xi-\eta|^{d+1}}\, d\xi\, d\eta \ge C,$$

which implies

$$\sum_{j=1}^{d+1} \rho(x)^{1-d} \int_{B(x,2\rho(x))} \int_{\substack{B(x,2\rho(x)) \\ |g_j(\xi) - g_j(\eta)|>\delta}} \frac{\delta}{|\xi-\eta|^{d+1}}\, d\xi\, d\eta \ge C. \tag{2.5}$$

Since

$$\rho(x)^{1-d} \int_{B(x,2\rho(x))} \int_{\substack{B(x,2\rho(x)) \\ |\xi-\eta|>C_1\rho(x)\delta}} \frac{\delta}{|\xi-\eta|^{d+1}}\, d\xi\, d\eta < \frac{C}{2(d+1)},$$

if $C_1 > 0$ is large enough (the largeness of C_1 depends only on C and d), it follows from (2.5) that

$$\sum_{j=1}^{d+1} \rho(x)^{1-d} \int_{B(x,2\rho(x))} \int_{\substack{B(x,2\rho(x)) \\ |g_j(\xi) - g_j(\eta)|>\delta \\ |\xi-\eta|\le C\rho(x)\delta}} \frac{\delta}{|\xi-\eta|^{d+1}}\, d\xi\, d\eta \ge C. \tag{2.6}$$

We derive from (2.4) and (2.6) that, for $0 < \delta < \delta_0$,

$$|\deg g| \le C \int_{\substack{\mathbb{S}^d \\ \rho(x)<1}} \frac{1}{\rho(x)^{2d-1}}\, dx \sum_{j=1}^{d+1} \int_{B(x,2\rho(x))} \int_{\substack{B(x,2\rho(x)) \\ |g_j(\xi) - g_j(\eta)|>\delta \\ |\xi-\eta|\le C\rho(x)\delta}} \frac{\delta}{|\xi-\eta|^{d+1}}\, d\xi\, d\eta.$$

This implies, by Fubini's theorem, that, for $0 < \delta < \delta_0$,

$$|\deg g| \le C \sum_{j=1}^{d+1} \int_{\mathbb{S}^d} \int_{\substack{\mathbb{S}^d \\ |g_j(\xi)-g_j(\eta)|>\delta}} \frac{\delta}{|\xi-\eta|^{d+1}}\, d\xi\, d\eta \int_{\substack{\rho(x)\ge C|\xi-\eta|/\delta \\ 2\rho(x)>|x-\xi|}} \frac{1}{\rho(x)^{2d-1}}\, dx. \tag{2.7}$$

We have

$$\int_{\substack{2\rho(x)>|x-\xi| \\ \rho(x)\ge C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}}\, dx \le \int_{\substack{2\rho(x)>|x-\xi| \\ |x-\xi|>C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}}\, dx + \int_{\substack{\rho(x)\ge C|\xi-\eta|/\delta \\ |x-\xi|\le C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}}\, dx$$

$$\le \int_{|x-\xi|>C|\xi-\eta|/\delta} \frac{C}{|x-\xi|^{2d-1}}\, dx + \int_{|x-\xi|\le C|\xi-\eta|/\delta} \frac{C\delta^{2d-1}}{|\xi-\eta|^{2d-1}}\, dx.$$

Finally, we use the assumption that $d \ge 2$. Since $d > 1$, it follows that

$$\int_{\substack{\rho(x)>|x-\xi| \\ \rho(x)\ge C|\xi-\eta|/\delta}} \frac{1}{\rho(x)^{2d-1}}\, dx \le \frac{C\delta^{d-1}}{|\xi-\eta|^{d-1}}. \tag{2.8}$$

Combining (2.7) and (2.8) yields, for $0 < \delta < \delta_0$,

$$|\deg g| \le C \sum_{j=1}^{d+1} \int_{\mathbb{S}^d} \int_{\substack{\mathbb{S}^d \\ |g_j(\xi)-g_j(\eta)|>\delta}} \frac{\delta^d}{|\xi-\eta|^{2d}}\, d\xi\, d\eta. \tag{2.9}$$

Assertion (1.3) is now a direct consequence of (2.9) for $\delta < \delta_0$ and (1.1) for $\delta_0 \le \delta < 1$.

The proof in the case $g \in C(\mathbb{S}^d, \mathbb{S}^d)$ can be derived from the case $g \in C^1(\mathbb{S}^d, \mathbb{S}^d)$ via a standard approximation argument. The details are omitted. □

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