Quantization Design for Distributed Optimization

Ye Pu, Melanie N. Zeilinger and Colin N. Jones

Abstract

We consider the problem of solving a distributed optimization problem using a distributed computing platform, where the communication in the network is limited: each node can only communicate with its neighbours and the channel has a limited data-rate. A common technique to address the latter limitation is to apply quantization to the exchanged information. We propose two distributed optimization algorithms with an iteratively refining quantization design based on the inexact proximal gradient method and its accelerated variant. We show that if the parameters of the quantizers, i.e. the number of bits and the initial quantization intervals, satisfy certain conditions, then the quantization error is bounded by a linearly decreasing function and the convergence of the distributed algorithms is guaranteed. Furthermore, we prove that after imposing the quantization scheme, the distributed algorithms still exhibit a linear convergence rate, and show complexity upper-bounds on the number of iterations to achieve a given accuracy. Finally, we demonstrate the performance of the proposed algorithms and the theoretical findings for solving a distributed optimal control problem.

I. Introduction

Distributed optimization methods for networked systems that have many coupled sub-systems and must act based on local information, are critical in many engineering problems, e.g. resource allocation, distributed estimation and distributed control problems. The algorithms are required to solve a global optimization problem in a distributed fashion subject to communication constraints.

Inexact distributed optimization methods are attracting increasing attention, since these techniques have the potential to deal with errors, for instance caused by inexact solution of local problems as well as unreliable or limited communication, e.g., transmission failures and quantization errors. Previous work has aimed at addressing the questions of how such errors affect the algorithm and under what conditions the convergence of the distributed algorithms can be guaranteed. In [7], the authors propose an inexact decomposition algorithm for solving distributed optimization problems by employing smoothing techniques and an excessive gap condition. In our previous work [12], we have proposed an inexact splitting method, named the inexact fast alternating minimization algorithm, and applied it to distributed optimization problems, where local computation errors as well as errors resulting from limited communication are allowed, and convergence conditions on the errors are derived based on a complexity upper-bound. Some other related references for inexact optimization algorithms include [6], [10] and [14]. In [14], an inexact proximal-gradient method, as well as its accelerated version, are introduced. The proximal gradient method, also known as the iterative shrinkage-thresholding algorithm (ISTA) [1], has two main steps: the first one is to compute the gradient of the smooth objective and the second one is to solve the proximal minimization. The conceptual idea of the inexact proximal-gradient method is to allow errors in these two steps, i.e. an error in the calculation of the gradient and an error in the proximal minimization. The results in [14] show convergence properties of the inexact proximal-gradient method and provide conditions on the errors, under which convergence of the algorithm can be guaranteed.

We consider a distributed optimization problem, where each sub-problem has a local cost function that involves both local and neighbouring variables, and local constraints on local variables. The problem is solved in a distributed manner with only local communication, i.e. between neighbouring sub-systems. In addition, the communication bandwidth between neighbouring sub-systems is limited. In order to meet the limited communication data-rate, the information exchanged between the neighbouring sub-systems needs to be quantized. The quantization process results in inexact iterations throughout the distributed optimization algorithm, which effects its convergence. Related work includes [3], [9], [15] and [11], which study the effects of quantization on the performance of averaging or distributed optimization algorithms.

We propose two distributed optimization algorithms with progressive quantization design building on the work in [14] and [15]. The main idea behind the proposed methods is to apply the inexact gradient method to the distributed optimization problem and to employ the error conditions, which guarantee convergence to the global optimum, to design a progressive quantizer. Motivated by the linear convergence upper-bound of the optimization algorithm, the range of the quantizer is set to reduce linearly at a rate smaller than one and larger than the rate of the algorithm, in order to refine the information exchanged in the network with each iteration and achieve overall converge to the global optimum. The proposed quantization method is computationally cheap and consistent throughout the iterations as every node implements the same quantization procedure.

Y. Pu and C.N. Jones are with the Automatic Control Lab, École Polytechnique Fédérale de Lausanne, EPFL-STI-IGM-LA Station 9 CH-1015 Lausanne, Switzerland, e-mail: {y.pu,colin.jones}@epfl.ch.

M.N. Zeilinger is with the Empirical Inference Department, Max Planck Institute for Intelligent Systems, 72076 Tübingen, Germany, e-mail: melanie.zeilinger@tuebingen.mpg.de.

This work has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013)/ ERC Grant Agreement n. 307608. The research of M. N. Zeilinger has received funding from the EU FP7 under grant agreement no. PIOF-GA-2011-301436-"COGENT".

This work extends the initial ideas presented in [13] for designing a quantization scheme for unconstrained distributed optimization. In particular, the paper makes the following main extensions and contributions:

- Constrained optimization problems: We consider distributed optimization problems with convex local constraints. To handle the constraints, two projection steps are required. One is applied before the information exchange, and the other after. The reason to have a second projection is that after the information exchange, the quantized value received by each agent can be an infeasible solution subject to the local constraints. The second projection step therefore guarantees that at each iteration every agent has a feasible solution for the computation of the gradient. We present conditions on the number of bits and the initial quantization intervals, which guarantee convergence of the algorithms. We show that after imposing the quantization scheme including the two projections, the algorithms preserve the linear convergence rate, and furthermore derive complexity upper-bounds on the number of iterations to achieve a given accuracy. In addition, we provide a discussion about how the minimum number of bits and the corresponding minimum initial quantization intervals can be obtained.
- Accelerated algorithm: We propose an accelerated variant of the distributed optimization algorithm with quantization refinement based on the inexact accelerated proximal-gradient method. With the acceleration step, the algorithm preserves the linear convergence rate, but the constant of the rate will be improved.
- Distributed optimal control example: We demonstrate the performance of the proposed method and the theoretical results for solving an distributed optimal control example.

II. Preliminaries

A. Notation

Let $v \in \mathbb{R}^{n_v}$ be a vector. $\|v\|$ and $\|v\|_\infty$ denote the l_2 and infinity norms of v, respectively. Note that $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{n_v}\|v\|_\infty$. Let $\mathbb{C}$ be a subset of $\mathbb{R}^{n_v}$. The projection of any point $v \in \mathbb{R}^{n_v}$ onto the set $\mathbb{C}$ is denoted by $\text{Proj}_{\mathbb{C}}(v) := \text{argmin}_{\mu\in\mathbb{C}} \|\mu - v\|$. Let $f : \Theta \to \Omega$ be a strongly convex function; σ_f denotes the convexity modulus $f(v) \geq f(\mu) + \langle \partial f(\mu), v - \mu\rangle + \frac{\sigma_f}{2}\|v - \mu\|^2$ for any $v, \mu \in \Theta$, where $\partial f(\cdot)$ denotes the set of sub-gradients of the function f at a given point. $L(f)$ denotes a Lipschitz constant of the function f, i.e. $\|f(v) - f(\mu)\| \leq L(f)\|v - \mu\|$, $\forall v, \mu \in \Theta$. The proximity operator is defined as

$$\text{prox}_f(v) = \text{argmin}_w \quad f(w) + \frac{1}{2}\|w - v\|^2 \ . \tag{1}$$

We refer to [2] and [8] for details on the definitions and properties above. The proximity operator with an extra subscript ϵ, i.e. $\mu = \text{prox}_{f,\epsilon}(v)$, means that a maximum computation error ϵ is allowed in the proximal objective function:

$$f(\mu) + \frac{1}{2}\|\mu - v\|^2 \ \leq \ \epsilon + \text{min}_w \left\{ f(w) + \frac{1}{2}\|w - v\|^2 \right\} \tag{2}$$

B. Inexact Proximal-Gradient Method

In this section, we will introduce the inexact proximal-gradient method (inexact PGM) proposed in [14]. Inexact PGM is presented in Algorithm 1. It addresses optimization problems of the form given in Problem 2.1 and requires Assumption 2.2 for convergence with a linear rate.

Problem 2.1:

$$\min_{x\in\mathbb{R}^{n_x}} \quad \Phi(x) = \phi(x) + \psi(x) \ .$$

Assumption 2.2:

- ϕ is a strongly convex function with a convexity modulus σ_ϕ and Lipschitz continuous gradient with Lipschitz constant $L(\nabla\phi)$.
- ψ is a lower semi-continuous convex function, not necessarily smooth.

Algorithm 1 Inexact Proximal-Gradient Method

Require: Require $x^0 \in \mathbb{R}^{n_x}$ and $\tau < \frac{1}{L(\nabla\phi)}$
for $k = 0, 1, \cdots$ **do**
$x^{k+1} = \text{prox}_{\tau\psi,\epsilon^k}(x^k - \tau(\nabla\phi(x^k) + e^k))$
end for

Inexact PGM in Algorithm 1 allows two kinds of errors: $\{e^k\}$ represents the error in the gradient calculations of ϕ, and $\{\epsilon^k\}$ represents the error in the computation of the proximal minimization in (2) at every iteration k. The following proposition states the convergence property of inexact PGM.

Proposition 2.3 (Proposition 3 in [14]): Let $\{x^k\}$ be generated by inexact PGM defined in Algorithm 1. If Assumption 2.2 holds, then for any $k \geq 0$ we have:

$$\|x^{k+1} - x^\star\| \leq (1-\gamma)^{k+1} \cdot (\|x^0 - x^\star\| + \Gamma^k) \ , \tag{3}$$

where $\gamma = \frac{\sigma_\phi}{L(\nabla\phi)}$ and x^0 and $x^\star$ denote the initial sequence of Algorithm 1 and the optimal solution of Problem 2.1, respectively, and

$$\Gamma^k = \sum_{p=0}^{k} (1-\gamma)^{-p-1} \cdot \left(\frac{1}{L(\nabla\phi)} \|e^p\| + \sqrt{\frac{2}{L(\nabla\phi)}} \sqrt{\epsilon^p} \right) \ .$$

As discussed in [14], the upper-bound in Proposition 2.3 allows one to derive sufficient conditions on the error sequences $\{e^k\}$ and $\{\epsilon^k\}$ for convergence of the algorithm to the optimal solution x^*, where $\mu = 1-\gamma$:

- If the series $\{\|e^k\|\}$ and $\{\sqrt{\epsilon^k}\}$ decrease at a linear rate with the constant $\kappa < \mu$, then $\|x^k - x^\star\|$ converges at a linear rate with the constant μ.
- If the series $\{\|e^k\|\}$ and $\{\sqrt{\epsilon^k}\}$ decrease at a linear rate with the constant $\mu < \kappa < 1$, then $\|x^k - x^\star\|$ converges at the same rate with the constant κ.
- If the series $\{\|e^k\|\}$ and $\{\sqrt{\epsilon^k}\}$ decrease at a linear rate with the constant $\kappa = \mu$, then $\|x^k - x^\star\|$ converges at a rate of $O(k \cdot \mu^k)$.

Remark 2.4: Compared to [14], we modify the index of the sequence in Algorithm 1 from x_k to x_{k+1} and the corresponding index in Proposition 2.3, such that in Section III the quantization errors have the same index as the quantized sequences.

C. Inexact Accelerated Proximal-Gradient Method

In this section, we introduce an accelerated variant of inexact PGM, named the inexact accelerated proximal-gradient method (inexact APGM) proposed in [14]. Compared to inexact PGM, it addresses the same problem class in Problem 2.1 and requires the same assumption in Assumption 2.2 for linear convergence, but involves one extra linear update in Algorithm 2, which improves the constant of the linear convergence rate from $(1-\gamma)$ to $\sqrt{1-\sqrt{\gamma}}$.

Algorithm 2 Inexact Accelerated Proximal-Gradient Method

Require: Initialize $x^0 = y^0 \in \mathbb{R}^{n_x}$ and $\tau < \frac{1}{L(\nabla\phi)}$
  for $k = 0, 1, \cdots$ **do**
    $x^{k+1} = \text{prox}_{\tau\psi,\epsilon^k}(y^k - \tau(\nabla\phi(y^k) + e^k))$
    $y^{k+1} = x^{k+1} + \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}(x^{k+1} - x^k)$
  end for

Proposition 4 of [14] presents a complexity upper-bound on the sequence $\{\Phi(x^{k+1}) - \Phi(x^\star)\}$, where the sequence $\{x^{k+1}\}$ is generated by inexact APGM. The following proposition extends this result and states a complexity upper-bound on $\|x^{k+1} - x^\star\|$.

Proposition 2.5: Let $\{x^k\}$ be generated by inexact APGM defined in Algorithm 2. If Assumption 2.2 holds, then for any $k \geq 0$ we have:

$$\|x^{k+1} - x^\star\| \leq (1-\sqrt{\gamma})^{\frac{k+1}{2}} \cdot \left(\frac{2\sqrt{\Phi(x^0) - \Phi(x^\star)}}{\sqrt{\sigma_\phi}} + \Theta^k \right) \ , \tag{4}$$

where $\gamma = \frac{\sigma_\phi}{L(\nabla\phi)}$, x^0 and $x^\star$ denote the initial sequence of Algorithm 1 and the optimal solution of Problem 2.1, respectively, and

$$\Theta^k = \frac{2}{\sigma_\phi} \cdot \sum_{p=0}^{k} (1-\sqrt{\gamma})^{\frac{-p-1}{2}} \cdot \left(\|e^p\| + (\sqrt{2L(\nabla\phi)} + \sqrt{\frac{\sigma_\phi}{2}}) \cdot \sqrt{\epsilon^p} \right) \ .$$

The proof of Proposition 2.5 will be given in the appendix in Section V-A. The upper-bound in Proposition 2.5 provides similar sufficient conditions on the error sequences $\{e^k\}$ and $\{\epsilon^k\}$ for the convergence of Algorithm 2, which are obtained by replacing $\mu = 1-\gamma$ in the sufficient conditions for Algorithm 1 in Section II-B with $\mu = \sqrt{1-\sqrt{\gamma}}$.

D. Uniform quantizer

Let x be a real number. A uniform quantizer with a quantization step-size Δ and the mid-value $\bar{x}$ can be expressed as

$$Q(x) = \bar{x} + \text{sgn}(x - \bar{x}) \cdot \Delta \cdot \left\lfloor \frac{\|x - \bar{x}\|}{\Delta} + \frac{1}{2} \right\rfloor \ , \tag{5}$$

where $\text{sgn}(\cdot)$ is the sign function. The parameter Δ is equal to $\Delta = \frac{l}{2^n}$, where l represents the size of the quantization interval and n is the number of bits sent by the quantizer. In this paper, we assume that n is a fixed number, which means that the quantization interval is set to be $[\bar{x} - \frac{l}{2}, \bar{x} + \frac{l}{2}]$. The quantization error is upper-bounded by

$$\|x - Q(x)\| \leq \frac{\Delta}{2} = \frac{l}{2^{n+1}} \ . \tag{6}$$

For the case that the input of the quantizer and the mid-value are not real numbers, but vectors with the same dimension n_x, the quantizer Q is composed of n_x independent scalar quantizers in (5) with the same quantization interval l and corresponding mid-value. In this paper, we design a uniform quantizer denoted as $Q^k(\cdot)$ with changing quantization interval l^k and mid-value $\bar{x}^k$ at every iteration k of the optimization algorithm.

III. Distributed optimization with limited communication

In this section, we propose two distributed optimization algorithms with progressive quantization design based on the inexact PGM algorithm and its accelerated variant. The main challenge is that the communication in the distributed optimization algorithms is limited and the information exchanged in the network needs to be quantized. We propose a progressive quantizer with changing parameters, which satisfies the communication limitations, while ensuring that the errors induced by quantization satisfy the conditions for convergence.

A. Distributed optimization problem

In this paper, we consider a distributed optimization problem on a network of M sub-systems (nodes). The sub-systems communicate according to a fixed undirected graph $G = (\mathcal{V}, \mathcal{E})$. The vertex set $\mathcal{V} = \{1, 2, \cdots, M\}$ represents the sub-systems and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ specifies pairs of sub-systems that can communicate. If $(i, j) \in \mathcal{E}$, we say that sub-systems i and j are neighbours, and we denote by $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$ the set of the neighbours of sub-system i. Note that $\mathcal{N}_i$ includes i. We denote d as the degree of the graph G. The optimization variable of sub-system i and the global variable are denoted by x_i and $x = [x_1^T, \cdots, x_M^T]^T$, respectively. For each sub-system i, the local variable has a convex local constraint $x_i \in \mathbb{C}_i \subseteq \mathbb{R}^{n_{m_i}}$. The constraint on the global variable x is denoted by $\mathbb{C} = \prod_{1 \leq i \leq M} \mathbb{C}_i$. The dimension of the local variable x_i is denoted by m_i and the maximum dimension of the local variables is denoted by $\bar{m}$, i.e. $\bar{m} := \max_{1 \leq i \leq M} m_i$. The concatenation of the variable of sub-system i and the variables of its neighbours is denoted by $x_{\mathcal{N}_i}$, and the corresponding constraint on $x_{\mathcal{N}_i}$ is denoted by $\mathbb{C}_{\mathcal{N}_i} = \prod_{j \in \mathcal{N}_i} \mathbb{C}_j$. With the selection matrices E_i and F_{ji}, they can be represented as $x_{\mathcal{N}_i} = E_i x$ and $x_i = F_{ji} x_{\mathcal{N}_j}$, $j \in \mathcal{N}_i$, which implies the relation between the local variable x_i and the global variable x, i.e. $x_i = F_{ji} E_j x$, $j \in \mathcal{N}_i$. Note that E_i and F_{ji} are selection matrices, and therefore $\|E_i\| = \|F_{ji}\| = 1$. We solve a distributed optimization problem of the formulation in Problem 3.1:

Problem 3.1:

$$\begin{aligned} \min_{x,\, x_{\mathcal{N}_i}} \quad & f(x) = \sum_{i=1}^{M} f_i(x_{\mathcal{N}_i}) \\ s.t. \quad & x_i \in \mathbb{C}_i \ , \ x_i = F_{ji} x_{\mathcal{N}_j} \ , \ j \in \mathcal{N}_i \ , \ x_{\mathcal{N}_i} = E_i x \ , \ i = 1, 2, \cdots, M \ . \end{aligned}$$

Assumption 3.2: We assume that the global cost function $f(\cdot)$ is strongly convex with a convexity modulus σ_f and Lipschitz continuous gradient with Lipschitz constant L, i.e. $\|\nabla f(x_1) - \nabla f(x_2)\| \leq L \|x_1 - x_2\|$ for any x_1 and x_2.

Assumption 3.3: The local constraint $\mathbb{C}_i$ is a convex set, for $i = 1, \cdots, M$.

Assumption 3.4: We assume that every local cost function $f_i(\cdot)$ has Lipschitz continuous gradient with Lipschitz constant L_i, and denote L_{max} as the maximum Lipschitz constant of the local functions, i.e. $L_{max} := \max_{1 \leq i \leq M} L_i$.

B. Qualitative description of the algorithm

In this section, we provide a qualitative description of the distributed optimization algorithm with quantization refinement to introduce the main idea of the approach. We apply the inexact PGM algorithm to the distributed optimization problem in Problem 3.1, where the two objectives in Problem 2.1 are chosen as $\phi = \sum_{i=1}^{M} f_i(x_{\mathcal{N}_i})$ and $\psi = \sum_{i=1}^{M} I_{\mathbb{C}_i}(x_i)$, where $I_{\mathbb{C}_i}$ denotes the indicator function on the set $\mathbb{C}_i$. The parameter γ is equal to

$$\gamma = \frac{\sigma_f}{L} \ . \tag{7}$$

The communication in the network is limited: each sub-system in the network can only communicate with its neighbours, and at each iteration, only a fixed number of bits can be transmitted. Only considering the first limitation, the distributed optimization algorithm resulting from applying the inexact PGM algorithm to Problem 3.1 is represented by the blue boxes in Fig. 1. At iteration k, sub-system i carries out four main steps:

1. Send the local variable to its neighbours;
2. Compute the local gradient;

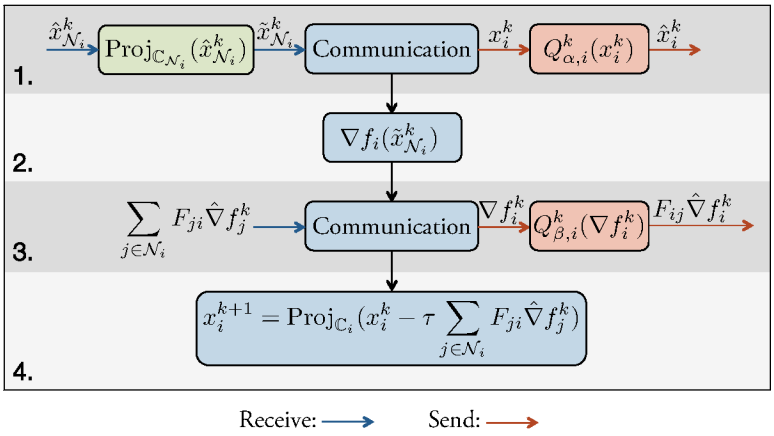

Figure 1: Distributed algorithm with quantization refinement for subsystem i at iteration k

3. Send the local gradient to its neighbours;
4. Update the local variable and compute the projection of the updated local variable on the local constraint.

To handle the second limitation, we design two uniform quantizers (the salmon-pink boxes) for the two communication steps for each sub-system $Q^k_{\alpha,i}$ and $Q^k_{\beta,i}$ using a varying quantization interval and mid-value to refine the exchanged information at each iteration. Motivated by the second sufficient condition on the error sequences $\{e^k\}$ and $\{\epsilon^k\}$ for the convergence of the inexact PGM algorithm discussed in Section II-B (if the sequences $\{\|e^k\|\}$ and $\{\sqrt{\epsilon^k}\}$ decrease at a linear rate with the constant $(1-\gamma) < \kappa < 1$, then $\|x^k - x^\star\|$ converges with the same rate), the quantization intervals are set to be a linearly decreasing function $l^k_{\alpha,i} = C_\alpha \kappa^k$ and $l^k_{\beta,i} = C_\beta \kappa^k$, with $(1-\gamma) < \kappa < 1$ and two constants C_α and C_β as the initial intervals. We know that if for every k, the values x^k_i and ∇f_i fall inside the quantization intervals, the quantization errors converge at the same linear rate with the constant κ. In Section III-C, we will show that by properly choosing the number of bits n and the initial intervals C_α and C_β, it can be guaranteed that x^k_i and ∇f_i fall inside the quantization intervals at every iteration and the quantization errors decrease linearly.

We add an extra re-projection step (green box) into the algorithm, because the quantized value $\hat{x}^k_{\mathcal{N}_i}$ can be an infeasible solution with respect to the constraints $\mathbb{C}_{\mathcal{N}_i}$. The re-projection step guarantees that at each iteration the gradient is computed based on a feasible solution. Using the convexity of the constraints, we can show that the error caused by the re-projected point $\tilde{x}^k_{\mathcal{N}_i} = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}^k_{\mathcal{N}_i})$ is upper-bounded by the quantization error. To summarize, all the errors induced by the limited communication in the distributed algorithm are upper bounded by a linearly decreasing function with the constant κ, which implies that the distributed algorithm with quantization converges to the global optimum and the linear convergence rate is preserved. These results will be shown in detail in Section III-C.

C. Distributed algorithm with quantization refinement

In this section, we propose a distributed algorithm with a progressive quantization design in Algorithm 3. For every sub-system i, there are two uniform quantizers $Q^k_{\alpha,i}$ and $Q^k_{\beta,i}$ using the formulation introduced in Section II-D with a fixed number of bits n, changing quantization intervals $l^k_{\alpha,i}$ and $l^k_{\beta,i}$ and changing mid-values $\bar{x}^k_{\alpha,i}$ and $\bar{\nabla} f^k_{\beta,i}$ for transmitting x^k_i, and ∇f^k_i at every iteration k. At iteration k, the quantization intervals are set to be $l^k_{\alpha,i} = C_\alpha \kappa^k$ and $l^k_{\beta,i} = C_\beta \kappa^k$, and the mid-values are set to be the previous quantized values $\bar{x}^k_{\alpha,i} = \hat{x}^{k-1}_i$ and $\bar{\nabla} f^k_{\beta,i} = \hat{\nabla} f^{k-1}_i$. The two parameters $C_\alpha = l^0_{\alpha,i}$ and $C_\beta = l^0_{\beta,i}$ denote the initial quantization intervals.

In this paper, $\hat{\cdot}$ is used to denote a quantized value, e.g. $\hat{x}^k_i = Q^k_{\alpha,i}(x^k_i)$ and $\tilde{\cdot}$ is used to denote a re-projected value, e.g. $\tilde{x}^k_{\mathcal{N}_i} = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}^k_{\mathcal{N}_i})$. The quantization errors are denoted by $\alpha^k_i = \hat{x}^k_i - x^k_i$ and $\beta^k_i = \hat{\nabla} f^k_i - \nabla f^k_i$.

Remark 3.5: We want to highlight Step 4 in Algorithm 3, because it is the key step that allows us to extend the algorithm in [13] for solving an unconstrained distributed optimization problem to constrained problems. The re-projection step ensures that the point used to compute the gradient at each iteration is a feasible solution subject to the constraints $\mathbb{C}_{\mathcal{N}_i}$, which is a necessary condition for the convergence of the algorithm.

In the following, we present four lemmas that link Algorithm 3 to the inexact PGM and prove that Algorithm 3 converges linearly to the global optimum despite the quantization errors. Lemma 3.6 states that due to the fact that the constraints are convex, the error between the re-projected point and the original point $\|\tilde{x}^k_{\mathcal{N}_i} - x^k_{\mathcal{N}_i}\| \leq \|\hat{x}^k_{\mathcal{N}_i} - x^k_{\mathcal{N}_i}\|$ is upper-bounded by the quantization error. Lemma 3.7 shows that the inexactness resulting from quantization in Algorithm 3 can be considered as the error in the gradient calculation $\{e^k\}$ and the error in the computation of the proximal minimization $\{\epsilon^k\}$ in Algorithm 1. Lemma 3.9 states that if at each iteration the values x^k_i and ∇f^k_i fall inside the quantization intervals, then the errors caused by quantization decrease linearly and the algorithm converges to the global optimum at the same rate. Lemma 3.13 gives conditions

Algorithm 3 Distributed algorithm with quantization refinement

Require: Initialize $\hat{x}_i^{-1} = x_i^0 = 0$, $\hat{\nabla} f_i^{-1} = \nabla f_i(\text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(0))$, $(1-\gamma) < \kappa < 1$ and $\tau < \frac{1}{L}$.

for $k = 0, 1, 2, \cdots$ **do**

For sub-system i, $i = 1, 2, \cdots, M$ do in parallel:

1: Update the parameters of quantizer $Q_{\alpha,i}^k$: $l_{\alpha,i}^k = C_\alpha \kappa^k$ and $\bar{x}_{\alpha,i}^k = \hat{x}_i^{k-1}$

2: Quantize the local variable: $\hat{x}_i^k = Q_{\alpha,i}^k(x_i^k) = x_i^k + \alpha_i^k$

3: Send $\hat{x}_i^k$ to all the neighbours of sub-system i

4: Compute the projection of $\hat{x}_{\mathcal{N}_i}^k$: $\tilde{x}_{\mathcal{N}_i}^k = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}_{\mathcal{N}_i}^k)$

5: Compute $\nabla f_i^k = \nabla f_i(\tilde{x}_{\mathcal{N}_i}^k)$

6: Update the parameters of quantizer $Q_{\beta,i}^k$: $l_{\beta,i}^k = C_\beta \kappa^k$ and $\bar{\nabla} f_{\beta,i}^k = \hat{\nabla} f_i^{k-1}$

7: Quantize the gradient: $\hat{\nabla} f_i^k = Q_{\beta,i}^k(\nabla f_i^k) = \nabla f_i^k + \beta_i^k$

8: Send $\hat{\nabla} f_i^k$ to all the neighbours of sub-system i

9: Update the local variable: $x_i^{k+1} = \text{Proj}_{\mathbb{C}_i}(x_i^k - \tau \sum_{j \in \mathcal{N}_i} F_{ji} \hat{\nabla} f_j^k)$

end for

on the number of bits and the initial quantization intervals, which guarantee that x_i^k and ∇f_i^k fall inside the quantization intervals for each iteration. Once we prove the three lemmas, we are ready to present the main result in Theorem 3.14.

Lemma 3.6: Let $\mathbb{C}$ be a convex subset of $\mathbb{R}^{n_v}$ and $\mu \in \mathbb{C}$. For any point $v \in \mathbb{R}^{n_v}$, the following holds:

$$\|\mu - \text{Proj}_{\mathbb{C}}(v)\| \leq \|\mu - v\| \ . \tag{8}$$

Proof: Since $\mu \in \mathbb{C}$, we have $\text{Proj}_{\mathbb{C}}(\mu) = \mu$. Lemma 3.6 follows directly from Proposition 2.2.1 in [2]. ■

Lemma 3.7: Algorithm 3 is equivalent to applying the inexact proximal-gradient method in Algorithm 1 to Problem 3.1 with $\phi = \sum_{i=1}^M f_i(x_{\mathcal{N}_i})$, $\psi = \sum_{i=1}^M I_{\mathbb{C}_i}(x_i)$,

$$e^k = \sum_{i=1}^M E_i^T \nabla f_i(\tilde{x}_{\mathcal{N}_i}^k) + \sum_{i=1}^M E_i^T \beta_i^k - \sum_{i=1}^M E_i^T \nabla f_i(x_{\mathcal{N}_i}^k) \ ,$$

and $\epsilon^k = \frac{1}{2}\|x^k - \tilde{x}^k\|^2$. Furthermore, $\|e^k\|$ and $\sqrt{\epsilon^k}$ are upper-bounded by

$$\|e^k\| \leq \sum_{i=1}^M L_i \cdot \sum_{j \in \mathcal{N}_i} \|\alpha_j^k\| + \sum_{i=1}^M \|\beta_i^k\| \ , \tag{9}$$

and

$$\sqrt{\epsilon^k} \leq \frac{\sqrt{2}}{2} \sum_{i=1}^M \|\alpha_i^k\| \ . \tag{10}$$

The proof of Lemma 3.7 will be provided in the appendix in Section V-B.

Remark 3.8: Lemma 3.7 shows that the errors $\|e^k\|$ and $\sqrt{\epsilon^k}$ are upper-bounded by functions of the quantization errors $\|\alpha_i^k\|$ and $\|\beta_i^k\|$. We want to emphasize that the quantization errors $\|\alpha_i^k\|$ and $\|\beta_i^k\|$ are not necessarily bounded by a linear function with the rate κ. They are bounded only if the values x_i^k and ∇f_i fall inside the quantization intervals that are decreasing at a linear rate. Otherwise, the quantization errors $\|\alpha_i^k\|$ and $\|\beta_i^k\|$ can be arbitrarily large.

From the discussion in Section II-B, we know that if $\|e^k\|$ and $\sqrt{\epsilon^k}$ decrease linearly at a rate larger than $(1-\gamma)$, then $\|x^k - x^\star\|$ converges linearly at the same rate as $\|e^k\|$. Lemma 3.9 provides the first step towards achieving this goal. It shows that if the values of x_i^k and ∇f_i^k always fall inside the quantization interval, then the computational error of the gradient $\|e^k\|$ and the computational error of the proximal operator $\sqrt{\epsilon^k}$ as well as $\|x^k - x^\star\|$ decrease linearly with the constant κ.

Lemma 3.9: For any parameter κ satisfying $(1-\gamma) < \kappa < 1$ and a $k \geq 0$, if for all $0 \leq p \leq k$ the values of x_i^p and ∇f_i^p generated by Algorithm 3 fall inside of the quantization intervals of $Q_{\alpha,i}^p$ and $Q_{\beta,i}^p$, i.e. $\|x_i^p - \bar{x}_{\alpha,i}^p\|_\infty \leq \frac{l_{\alpha,i}^p}{2}$ and $\|\nabla f_i^p - \bar{\nabla} f_{\beta,i}^p\|_\infty \leq \frac{l_{\beta,i}^p}{2}$, then the error sequences $\|e^p\|$ and $\sqrt{\epsilon^p}$ satisfy

$$\|e^p\| \leq C_1 \kappa^p \ , \quad \sqrt{\epsilon^p} \leq C_2 \kappa^p \ , \tag{11}$$

where $C_1 = \frac{M\sqrt{m}(L_{max} d C_\alpha + \sqrt{d} C_\beta)}{2^{n+1}}$ and $C_2 = \frac{\sqrt{2}}{2} \cdot \frac{M\sqrt{m} C_\alpha}{2^{n+1}}$, and $\|x^{p+1} - x^\star\|$ satisfies

$$\|x^{p+1} - x^\star\| \leq \kappa^{p+1} \left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L} C_2)\kappa}{L(\kappa + \gamma - 1)(1-\gamma)} \right] \ . \tag{12}$$

The proof of Lemma 3.9 will be provided in the appendix in Section V-C. From Lemma 3.9, we know that the last missing piece is to show that the values x_i^k and ∇f_i^k fall inside the quantization interval at every iteration k. The following assumption presents conditions on the number of bits n and the initial quantization intervals C_α and C_β, which guarantee that for each iteration x_i^k and ∇f_i^k in Algorithm 3 fall inside the changing quantization intervals and the quantization errors decrease linearly with the constant κ, which further implies that the Algorithm 3 converges to the global optimum linearly with the same rate κ.

Assumption 3.10: Consider the quantizers $Q_{\alpha,i}^k$ and $Q_{\beta,i}^k$ in Algorithm 3. We assume that the parameters of the quantizers, i.e. the number of bits n and the initial quantization intervals C_α and C_β satisfy

$$a_1 + a_2\frac{C_\alpha}{2^{n+1}} + a_3\frac{C_\beta}{2^{n+1}} \leq \frac{C_\alpha}{2} \tag{13}$$

$$b_1 + b_2\frac{C_\alpha}{2^{n+1}} + b_3\frac{C_\beta}{2^{n+1}} \leq \frac{C_\beta}{2} \ , \tag{14}$$

with

$$a_1 = \frac{(\kappa+1)\|x^0 - x^\star\|}{\kappa} \ , \quad a_2 = \frac{M\sqrt{\bar{m}}\kappa(\kappa+1)(dL_{\max}+\sqrt{L}) + M\sqrt{\bar{m}}L(\kappa+\gamma-1)(1-\gamma)}{L\kappa(\kappa+\gamma-1)(1-\gamma)} \ , \quad a_3 = \frac{M\sqrt{d\bar{m}}(\kappa+1)}{L(\kappa+\gamma-1)(1-\gamma)} \ ,$$

$$b_1 = \frac{L_{\max}(\kappa+1)\|x^0 - x^\star\|}{\kappa} \ , \quad b_2 = \frac{L_{\max}M\sqrt{\bar{m}}\kappa(\kappa+1)(dL_{max}+\sqrt{L}) + L_{max}d\sqrt{\bar{m}}L(\kappa+1)(\kappa+\gamma-1)(1-\gamma)}{L\kappa(\kappa+\gamma-1)(1-\gamma)} \ ,$$

$$b_3 = \frac{L_{\max}M\sqrt{d\bar{m}}\kappa(\kappa+1) + L\sqrt{d\bar{m}}(\kappa+\gamma-1)(1-\gamma)}{L\kappa(\kappa+\gamma-1)(1-\gamma)} \ .$$

Remark 3.11: The parameters of the quantizers n, C_α and C_β are all positive constants. Assumption 3.10 can always be satisfied by increasing n, C_α and C_β.

Remark 3.12: For a fixed n, inequalities (13) and (14) represent two polyhedral constraints on C_α and C_β. Therefore, the minimal C_α and C_β can be computed by solving a simple LP problem, i.e. minimizing $C_\alpha + C_\beta$ subject to $C_\alpha \geq 0$, $C_\beta \geq 0$, and inequalities (13) and (14). Since the minimal n is actually the minimal one guaranteeing that the LP problem has a feasible solution, the minimal n can be found by testing feasibility of the LP problem.

Lemma 3.13: If Assumption 3.10 is satisfied and $(1-\gamma) < \kappa < 1$, then for any $k \geq 0$ the values of x_i^k and ∇f_i^k in Algorithm 3 fall inside of the quantization intervals of $Q_{\alpha,i}^k$ and $Q_{\beta,i}^k$, i.e. $\|x_i^k - \bar{x}_{\alpha,i}^k\|_\infty \leq \frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$.

The proof of Lemma 3.13 will be provided in the appendix in Section V-D. After showing Lemma 3.7, Lemma 3.9 and Lemma 3.13, we are ready to present the main theorem.

Theorem 3.14: If Assumptions 3.2, 3.4 and 3.10 hold and $(1-\gamma) < \kappa < 1$, then for $k \geq 0$ the sequence $\{x^k\}$ generated by Algorithm 3 converges to the optimum linearly with the constant κ and satisfies

$$\|x^{k+1} - x^\star\| \leq \kappa^{k+1}\left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)(1-\gamma)}\right] \ . \tag{15}$$

with $C_1 = \frac{M\sqrt{\bar{m}}(L_{max}dC_\alpha + \sqrt{d}C_\beta)}{2^{n+1}}$ and $C_2 = \frac{\sqrt{2}}{2}\cdot\frac{M\sqrt{\bar{m}}C_\alpha}{2^{n+1}}$.

Proof: Since Assumption 3.2, 3.4 and 3.10 hold, Lemma 3.13 states that for each iteration the values x_i^k and ∇f_i^k in Algorithm 3 fall inside of the quantization intervals of $Q_{\alpha,i}^k$ and $Q_{\beta,i}^k$. Then from Lemma 3.9, we know that the error sequences $\|e^k\|$ and $\sqrt{\epsilon^k}$ satisfy $\|e^k\| \leq C_1\kappa^k$ and $\sqrt{\epsilon^k} \leq C_2\kappa^k$, and by Lemma 3.7 the sequence x^k generated by Algorithm 3 satisfies inequality (15). ∎

Recalling the complexity bound in Proposition 2.3, we know that for the case without errors the algorithm converges linearly with the constant $1-\gamma$. After imposing quantization on the algorithm, it still converges to the global optimum linearly but with a larger constant $\kappa > 1-\gamma$. We conclude that with the proposed quantization design, the linear convergence of the algorithm is preserved, but the constant of the convergence rate has to be enlarged in order to compensate for the deficiencies from limited communication.

D. Accelerated distributed algorithm with quantization refinement

In this section, we propose an accelerated variant of the distributed algorithm with quantization refinement in Algorithm 4 based on the inexact accelerated proximal gradient method in Algorithm 2. Compared to Algorithm 3, Algorithm 4 has an extra accelerating Step 5 $\tilde{y}_{\mathcal{N}_i}^k = \tilde{x}_{\mathcal{N}_i}^k + \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}(\tilde{x}_{\mathcal{N}_i}^k - \tilde{x}_{\mathcal{N}_i}^{k-1})$, and at each iteration the gradient ∇f_i^k is computed based on $\tilde{y}_{\mathcal{N}_i}^k$. The accelerating step improves the constant of the linear convergence rate of the algorithms from $1-\gamma$ to $\sqrt{1-\sqrt{\gamma}}$, and changes the condition on the quantization parameter κ to $\sqrt{1-\sqrt{\gamma}} < \kappa < 1$.

Algorithm 4 Accelerated distributed algorithm with quantization refinement

Require: Initialize $\hat{x}_i^{-1} = x_i^{-1} = x_i^0 = 0$, $\tilde{x}_{\mathcal{N}_i}^{-1} = 0$, $\hat{\nabla} f_i^{-1} = \nabla f_i(\text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(0))$, $\sqrt{1-\sqrt{\gamma}} < \kappa < 1$ and $\tau < \frac{1}{L}$.

for $k = 0, 1, 2, \cdots$ **do**

For sub-system i, $i = 1, 2, \cdots, M$ do in parallel:

1: Update the parameters of quantizer $Q_{\alpha,i}^k$: $l_{\alpha,i}^k = C_\alpha \kappa^k$ and $\bar{x}_{\alpha,i}^k = \hat{x}_i^{k-1}$

2: Quantize the local variable: $\hat{x}_i^k = Q_{\alpha,i}^k(x_i^k) = x_i^k + \alpha_i^k$

3: Send $\hat{x}_i^k$ to all the neighbours of sub-system i

4: Compute the projection of $\hat{x}_{\mathcal{N}_i}^k$: $\tilde{x}_{\mathcal{N}_i}^k = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}_{\mathcal{N}_i}^k)$

5: Accelerating update: $\tilde{y}_{\mathcal{N}_i}^k = \tilde{x}_{\mathcal{N}_i}^k + \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}(\tilde{x}_{\mathcal{N}_i}^k - \tilde{x}_{\mathcal{N}_i}^{k-1})$ and $y_i^k = x_i^k + \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}(x_i^k - x_i^{k-1})$

6: Compute $\nabla f_i^k = \nabla f_i(\tilde{y}_{\mathcal{N}_i}^k)$

7: Update the parameters of quantizer $Q_{\beta,i}^k$: $l_{\beta,i}^k = C_\beta \kappa^k$ and $\bar{\nabla} f_{\beta,i}^k = \hat{\nabla} f_i^{k-1}$

8: Quantize the gradient: $\hat{\nabla} f_i^k = Q_{\beta,i}^k(\nabla f_i^k) = \nabla f_i^k + \beta_i^k$

9: Send $\hat{\nabla} f_i^k$ to all the neighbours of sub-system i

10: Update the local variable: $x_i^{k+1} = \text{Proj}_{\mathbb{C}_i}(y_i^k - \tau \sum_{j \in \mathcal{N}_i} F_{ji} \hat{\nabla} f_j^k)$

end for

Lemma 3.15: Algorithm 4 is equivalent to applying the inexact accelerated proximal-gradient method in Algorithm 2 to Problem 3.1 with $\phi = \sum_{i=1}^M f_i(x_{\mathcal{N}_i})$, $\psi = \sum_{i=1}^M I_{\mathbb{C}_i}(x_i)$,

$$e^k = \sum_{i=1}^M E_i^T \nabla f_i(\tilde{y}_{\mathcal{N}_i}^k) + \sum_{i=1}^M E_i^T \beta_i^k - \sum_{i=1}^M E_i^T \nabla f_i(y_{\mathcal{N}_i}^k) \ ,$$

and $\epsilon^k = \frac{1}{2}\|x^k - \tilde{x}^k\|^2$. Furthermore, $\|e^k\|$ and $\sqrt{\epsilon^k}$ are upper-bounded by

$$\|e^k\| \leq \sum_{i=1}^M L_i \cdot \sum_{j \in \mathcal{N}_i} \left(\frac{2}{1+\sqrt{\gamma}}\|\alpha_j^k\| + \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}\|\alpha_j^{k-1}\|\right) + \sum_{i=1}^M \|\beta_i^k\| \ . \tag{16}$$

and

$$\sqrt{\epsilon^k} \leq \frac{\sqrt{2}}{2} \sum_{i=1}^M \|\alpha_i^k\| \ . \tag{17}$$

Proof: The proof follows the same flow of the proof of Lemma 3.7. The only difference is that at each iteration the gradient ∇f_i^k is computed based on $\tilde{y}_{\mathcal{N}_i}^k$, which is a linear combination of $\tilde{x}_{\mathcal{N}_i}^k$ and $\tilde{x}_{\mathcal{N}_i}^{k-1}$. Hence, the upper-bound on the computational error of the gradient $\|e^k\|$ is a function of the linear combination of $\|\alpha_i^{k-1}\|$, $\|\alpha_i^k\|$ and $\|\beta_i^k\|$. ■

Lemma 3.16: For any parameter κ satisfying $\sqrt{1-\sqrt{\gamma}} < \kappa < 1$ and a $k \geq 0$, if for all $0 \leq p \leq k$ the values of x_i^p and ∇f_i^p generated by Algorithm 4 fall inside of the quantization intervals of $Q_{\alpha,i}^p$ and $Q_{\beta,i}^p$, i.e. $\|x_i^k - \bar{x}_{\alpha,i}^k\|_\infty \leq \frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$, then the sequences $\|e^p\|$ and $\sqrt{\epsilon^p}$ satisfy

$$\|e^p\| \leq C_3 \kappa^p \ , \quad \sqrt{\epsilon^p} \leq C_4 \kappa^p \ . \tag{18}$$

where $C_3 = \frac{M\sqrt{m}(3L_{max}dC_\alpha + \kappa\sqrt{d}C_\beta)}{\kappa \cdot 2^{n+1}}$ and $C_4 = \frac{\sqrt{2}}{2} \cdot \frac{M\sqrt{m}C_\alpha}{2^{n+1}}$, and $\|x^{p+1} - x^\star\|$ satisfies

$$\|x^{p+1} - x^\star\| \leq \kappa^{p+1} \left[\frac{2\sqrt{\Phi(x^0) - \Phi(x^\star)}}{\sqrt{\sigma_\phi}} + \frac{(2C_3 + 2\sqrt{2L}C_4 + \sqrt{2\sigma_\phi}C_4)\kappa}{\sigma_\phi(\kappa - \sqrt{1-\sqrt{\gamma}}) \cdot \sqrt{1-\sqrt{\gamma}}}\right] \ . \tag{19}$$

Proof: The proof follows the same flow of the proof of Lemma 3.9 by replacing the upper-bounds on $\|e^k\|$ and $\sqrt{\epsilon^k}$ in Lemma 3.7 and the upper-bound on $\|x^{p+1} - x^\star\|$ in Proposition 2.3 by the ones in Lemma 3.15 and Proposition 2.5. In addition, the proof requires the fact that $\sqrt{1-\sqrt{\gamma}} < \kappa < 1$ and $1 < 1 + \sqrt{\gamma} < 2$. ■

Assumption 3.17: We assume that the number of bits n and the initial quantization intervals C_α and C_β satisfy

$$a_4 + a_5 \frac{C_\alpha}{2^{n+1}} + a_6 \frac{C_\beta}{2^{n+1}} \leq \frac{C_\alpha}{2} \tag{20}$$

$$b_4 + b_5 \frac{C_\alpha}{2^{n+1}} + b_6 \frac{C_\beta}{2^{n+1}} \leq \frac{C_\beta}{2} \ , \tag{21}$$

with

$$a_4 = \frac{2(\kappa+1)\sqrt{\Phi(x^0)-\Phi(x^\star)}}{\kappa\sqrt{\sigma_\phi}} \ ,$$

$$a_5 = \frac{6M\sqrt{\bar{m}}(\kappa+1)dL_{\max} + M\sqrt{\bar{m}}\kappa(\kappa+1)(2\sqrt{L}+\sqrt{\sigma_\phi}) + \sigma_\phi M\sqrt{\bar{m}}(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}}{\sigma_\phi\kappa(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}} \ ,$$

$$a_6 = \frac{2M\sqrt{d\bar{m}}(\kappa+1)}{\sigma_\phi(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}} \ ,$$

$$b_4 = \frac{2L_{\max}(2\kappa^2+3\kappa+1)\sqrt{\Phi(x^0)-\Phi(x^\star)}}{\kappa^2\sqrt{\sigma_\phi}} \ ,$$

$$b_5 = \frac{L_{max}\sqrt{\bar{m}}(2\kappa^2+3\kappa+1)}{\kappa^2}\left[d + \frac{6MdL_{max} + M\kappa(2\sqrt{L}+\sqrt{\sigma_\phi})}{\sigma_\phi(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}}\right] \ ,$$

$$b_6 = \frac{2L_{\max}M\sqrt{d\bar{m}}(2\kappa^2+3\kappa+1) + \sigma_\phi\sqrt{d\bar{m}}(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}}{\sigma_\phi\kappa(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}} \ .$$

Lemma 3.18: If Assumption 3.17 is satisfied and $\sqrt{1-\sqrt{\gamma}} < \kappa < 1$, then for any $k \geq 0$ the values of x_i^k and ∇f_i^k in Algorithm 4 fall inside of the quantization intervals of $Q_{\alpha,i}^k$ and $Q_{\beta,i}^k$, i.e. $\|x_i^k - \bar{x}_{\alpha,i}^k\|_\infty \leq \frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$.

The proof of Lemma 3.18 will be provided in the appendix in Section V-E.

Theorem 3.19: If Assumptions 3.2, 3.4 and 3.17 hold and $\sqrt{1-\sqrt{\gamma}} < \kappa < 1$, then for $k \geq 0$ the sequence $\{x^k\}$ generated by Algorithm 4 converges to the optimum linearly with the constant κ and satisfies

$$\|x^{k+1} - x^\star\| \leq \kappa^{k+1}\left[\frac{2\sqrt{\Phi(x^0)-\Phi(x^\star)}}{\sqrt{\sigma_\phi}} + \frac{(2C_3 + 2\sqrt{2L}C_4 + \sqrt{2\sigma_\phi}C_4)\kappa}{\sigma_\phi(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}}\right] \ , \tag{22}$$

with $C_3 = \frac{M\sqrt{\bar{m}}(3L_{max}dC_\alpha + \kappa\sqrt{d}C_\beta)}{\kappa\cdot 2^{n+1}}$ and $C_4 = \frac{\sqrt{2}}{2}\cdot\frac{M\sqrt{\bar{m}}C_\alpha}{2^{n+1}}$.

Proof: The proof follows directly from the proof of Theorem 3.14 by replacing Lemma 3.7, Lemma 3.9 and Lemma 3.13 by Lemma 3.15, Lemma 3.16 and Lemma 3.18. ■

IV. Numerical Example

This section illustrates the theoretical findings of the paper and demonstrates the performance of Algorithm 3 and Algorithm 4 for solving a distributed quadratic programming (QP) problem originating from the problem of regulating constrained distributed linear systems by model predictive control (MPC) in the form of Problem 4.1. For more information about distributed MPC, see e.g. [5], [4] and [12].

Problem 4.1:

$$\begin{aligned} \min_{z,u} \quad & \sum_{i=1}^{M}\sum_{t=0}^{N-1} l_i(z_i(t),u_i(t)) + \sum_{i=1}^{M} l_i^f(z_i(N)) \\ s.t. \quad & z_i(t+1) = A_{ii}z_j(t) + \sum_{j\in\mathcal{N}_i} B_{ij}u_j(t) \\ & u_i(t) \in \mathbb{U}_i \ , \ z_i(0) = \bar{z}_i \ , \ i = 1,2,\cdots,M \ , \end{aligned}$$

M and N denote the number of subsystems and the horizon of the MPC problem, respectively. The state and input sequences along the horizon of subsystem i are denoted by $z_i = [z_i^T(0), z_i^T(1), \cdots, z_i^T(N)]^T$ and $u_i = [u_i^T(0), u_i^T(1), \cdots, u_i^T(N-1)]^T$. The discrete-time linear dynamics of subsystem i are given by $z_i(t+1) = A_{ii}z_j(t) + \sum_{j\in\mathcal{N}_i} B_{ij}u_j(t)$, where A_{ii} and B_{ij} are the dynamic matrices. The initial state is denoted by $\bar{z}_i$. The control inputs of subsystem i are subject to local convex constraints $u_i(t) \in \mathbb{U}_i$. $l_i(\cdot,\cdot)$ and $l_i^f(\cdot)$ are strictly convex cost functions. From Problem 4.1, we can see that subsystem i is coupled with its neighbours in the linear dynamics.

We randomly generate a distributed MPC problem in the form of Problem 4.1. We first randomly generate a connected network with $M = 40$ sub-systems. Each sub-system has 3 states and 2 inputs. The dynamical matrices A_{ii} and B_{ij} are randomly generated, i.e. generally dense, and the local systems are controllable and unstable. The input constraint $\mathbb{U}_i$ for sub-system i is set to $\mathbb{U}_i = \{u_i | -0.4\cdot\mathbf{1} \leq u_i(t) \leq 0.3\cdot\mathbf{1}\}$, where $\mathbf{1}$ denotes the all-ones vector with the same dimension as u_i. The horizon of the MPC problem is set to $N = 11$. The local cost functions are chosen as quadratic functions $l_i(z_i(t),u_i(t)) = z_i^T(t)Q_iz_i(t) + u_i^T(t)R_iu_i(t)$ and $l_i^f(z_i(N)) = z_i^T(N)P_iz_i(N)$, where Q_i, R_i and P_i are identity matrices. The initial states $\bar{z}_i$ are chosen, such that more than 50% of the optimization variables are at the constraints at optimality.

Parameters	Algorithm 3	Algorithm 4
Constant of rate	$1-\gamma = 0.8093$	$\sqrt{1-\sqrt{\gamma}} = 0.7505$
κ	0.9333	0.7991
n_{min}	13	19

Table I: The parameters in Algorithm 3 and Algorithm 4 for solving Problem 4.2.

Problem 4.2:

$$\min_{x\in\mathbb{R}^{n_x}} \; f(x) = \sum_{i=1}^{M} f_i(x_{\mathcal{N}_i}) = \sum_{i=1}^{M} x_{\mathcal{N}_i}^T H_i x_{\mathcal{N}_i} + h_i x_{\mathcal{N}_i}$$
$$s.t. \quad x_i \in \mathbb{C}_i \quad .$$

By eliminating all state variables distributed MPC problems of this class can be reformulated as a distributed QP of the form in Problem 4.2 with the local variables $x_i = u_i$ and the concatenations of the variables of subsystem i and its neighbours $x_{\mathcal{N}_i}$. Matrix H_i is dense and positive definite, and vector h_i is dense. The constraint $\mathbb{C}_i = \mathbb{U}_i^N$ is a polytopic set.

Table I shows the parameters chosen in Algorithm 3 and Algorithm 4, including the constants of the convergence rate of the algorithms, i.e. $\gamma = \frac{\sigma_f}{L}$ and $\sqrt{1-\sqrt{\gamma}}$, the decrease rates of the quantization intervals κ satisfying $1-\gamma \leq \kappa \leq 1$ for Algorithm 3 and $\sqrt{1-\sqrt{\gamma}} \leq \kappa \leq 1$ for Algorithm 4 and the minimum number of bits required for convergence n_{min}.

Fig. 2 shows the relationship between the number of bits n and the minimum initial quantization intervals C_α and C_β, which satisfy Assumption 3.10 for Problem 4.2. We see that the minimum number of bits required for convergence is equal to $n_{min} = 13$, and as the number of bits n increases, the required minimum C_α and C_β decrease.

Fig. 3 shows the performance of Algorithm 3 and Algorithm 4 for solving the distributed QP problem in Problem 4.2 originating from the distributed MPC problem. For Algorithm 3, n is set to 13 and 15, respectively, and the initial quantization intervals C_α and C_β are set to corresponding minimum values satisfying Assumption 3.10. For Algorithm 4, n is set to 19 and 23, and C_α and C_β to corresponding minimum values satisfying Assumption 3.17. In Fig. 3 we can observe that the proposed distributed algorithms with quantization converges to the global optimum linearly and the performance is improved when the number of bits n is increased. Due to the acceleration step, Algorithm 4 converges faster than Algorithm 3. However, Algorithm 4 requires a larger number of bits n to guarantee the convergence.

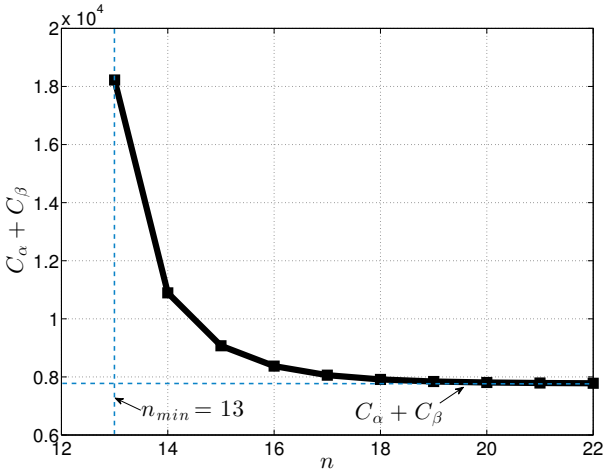

Figure 2: Relationship between the number of bits n and the minimum initial quantization intervals C_α and C_β satisfying Assumption 3.10 for Problem 4.2 originating from the distributed MPC problem.

V. Appendix

A. Proof of Proposition 2.5

Proof: By the strong convexity of the function ϕ, we know

$$\frac{\sigma_\phi}{2}\|x^{k+1} - x^\star\|^2 \leq \Phi(x^{k+1}) - \Phi(x^\star) \; .$$

From Proposition 4 in [14], it follows that

$$\|x^{k+1} - x^\star\|^2 \leq \frac{2}{\sigma_\phi}(1-\sqrt{\gamma})^{k+1}\left(\sqrt{2(\Phi(x^0)-\Phi(x^\star))} + \sqrt{\frac{2}{\sigma_\Phi}}\sum_{p=0}^{k}(\|e^p\| + \sqrt{2L(\nabla\phi)\epsilon^p})(1-\sqrt{\gamma})^{-\frac{p+1}{2}} + \sqrt{\sum_{p=0}^{k}\epsilon^p(1-\sqrt{\gamma})^{-p-1}}\right)^2 .$$

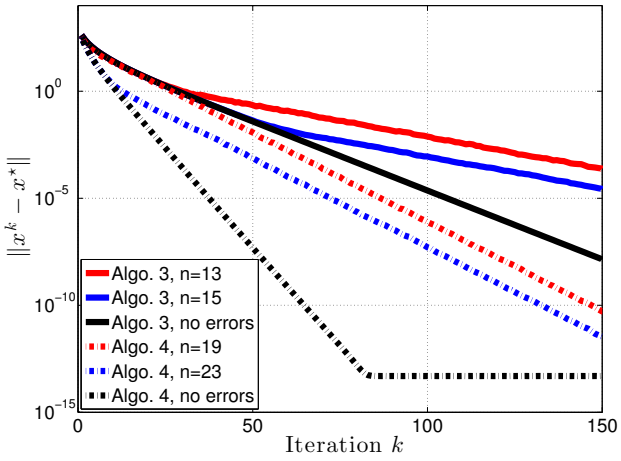

Figure 3: Comparison of the performance of Algorithm 3 and Algorithm 4 with different n and corresponding minimum C_α and C_β with the exact algorithms (no quantization errors) for Problem 4.2.

By the fact $\sqrt{v+\mu} \leq \sqrt{v} + \sqrt{\mu}$ for any $v, \mu \in \mathbb{R}_+$, we simplify the inequality above as

$$\|x^{k+1} - x^\star\|^2 \leq \frac{2}{\sigma_\phi}(1-\sqrt{\gamma})^{k+1}\left(\sqrt{2(\Phi(x^0)-\Phi(x^\star))} + \sqrt{\frac{2}{\sigma_\phi}}\sum_{p=0}^{k}(\|e^p\| + (\sqrt{2L(\nabla\phi)} + \sqrt{\frac{\sigma_\phi}{2}})\sqrt{\epsilon^p})(1-\sqrt{\gamma})^{-\frac{p+1}{2}}\right)^2 .$$

Taking the square-root of both sides of the inequality above, we get inequality (4). ■

B. Proof of Lemma 3.7

Proof: By the definition, the gradient computation error e^k in Algorithm 1 is equal to

$$\begin{aligned} e^k &= \hat{\nabla} f(\tilde{x}^k) - \nabla f(x^k) = \sum_{i=1}^{M} E_i^T \hat{\nabla} f_i(\tilde{x}^k_{\mathcal{N}_i}) - \sum_{i=1}^{M} E_i^T \nabla f_i(x^k_{\mathcal{N}_i}) \\ &= \sum_{i=1}^{M} E_i^T \nabla f_i(\tilde{x}^k_{\mathcal{N}_i}) + \sum_{i=1}^{M} E_i^T \beta_i^k - \sum_{i=1}^{M} E_i^T \nabla f_i(x^k_{\mathcal{N}_i}). \end{aligned}$$

Then,

$$\|e^k\| \leq \sum_{i=1}^{M} \|E_i^T\| \cdot L_i \cdot \|\tilde{x}^k_{\mathcal{N}_i} - x^k_{\mathcal{N}_i}\| + \sum_{i=1}^{M} \|E_i^T\| \|\beta_i^k\| \ .$$

Note that the matrix E_i is a selection matrix, then $\|E_i^T\| = 1$. Since $x^k_{\mathcal{N}_i} \in \mathbb{C}_{\mathcal{N}_i}$ and $\tilde{x}^k_{\mathcal{N}_i} = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}^k_{\mathcal{N}_i})$, Lemma 3.6 implies $\|\tilde{x}^k_{\mathcal{N}_i} - x^k_{\mathcal{N}_i}\| \leq \|\hat{x}^k_{\mathcal{N}_i} - x^k_{\mathcal{N}_i}\|$. Hence, we have

$$\|e^k\| \leq \sum_{i=1}^{M} L_i \cdot \|\hat{x}^k_{\mathcal{N}_i} - x^k_{\mathcal{N}_i}\| + \sum_{i=1}^{M} \|\beta_i^k\| \leq \sum_{i=1}^{M} L_i \cdot \sum_{j \in \mathcal{N}_i} \|\alpha_j^k\| + \sum_{i=1}^{M} \|\beta_i^k\| \ .$$

By definition in (2) and the fact that $x^k \in \mathbb{C}$ and $\tilde{x}^k = \text{Proj}_{\mathbb{C}}(\hat{x}^k)$, we know $\epsilon^k = \frac{1}{2}\|x^k - \tilde{x}^k\|^2$. Lemma 3.6 again implies $\|x^k - \tilde{x}^k\| \leq \|x^k - \hat{x}^k\|$. Hence, we have

$$\sqrt{\epsilon^k} = \frac{\sqrt{2}}{2}\|x^k - \tilde{x}^k\| \leq \frac{\sqrt{2}}{2}\|x^k - \hat{x}^k\| \leq \frac{\sqrt{2}}{2}\sum_{i=1}^{M} \|\alpha_i^k\| \ .$$

■

C. Proof of Lemma 3.9

Proof: From the property of the uniform quantizer, we know that if x_i^p and ∇f_i^p fall inside of the quantization intervals of $Q_{\alpha,i}^p$ and $Q_{\beta,i}^p$, then the quantization errors α_i^p and β_i^p are upper-bounded by

$$\|\alpha_i^p\| \leq \sqrt{m_i} \cdot \|\alpha_i^p\|_\infty \leq \sqrt{m_i} \cdot \frac{l_{\alpha,i}^p}{2^{n+1}} \leq \sqrt{\bar{m}} \cdot \frac{l_{\alpha,i}^p}{2^{n+1}} \ , \quad \|\beta_i^p\| \leq \sqrt{\sum_{j \in \mathcal{N}_i} m_i} \cdot \|\beta_i^p\|_\infty \leq \sqrt{\sum_{j \in \mathcal{N}_i} m_i} \cdot \frac{l_{\beta,i}^p}{2^{n+1}} \leq \sqrt{d\bar{m}} \cdot \frac{l_{\beta,i}^p}{2^{n+1}} \ ,$$

where $\bar{m} := \max_{1 \leq i \leq M} m_i$ and d denotes the degree of the graph of the distributed optimization problem. From Lemma 3.7, we have

$$\|e^p\| \leq \sum_{i=1}^M L_i \cdot \sum_{j \in \mathcal{N}_i} \frac{\sqrt{\bar{m}} \cdot l_{\alpha,j}^p}{2^{n+1}} + \sum_{i=1}^M \frac{\sqrt{d\bar{m}} \cdot l_{\beta,i}^p}{2^{n+1}} \ ,$$

and

$$\sqrt{\epsilon^k} \leq \frac{\sqrt{2}}{2} \sum_{i=1}^M \frac{\sqrt{\bar{m}} l_{\alpha,i}^p}{2^{n+1}} \ .$$

Since the quantization intervals are set to $l_{\alpha,i}^p = C_\alpha \kappa^p$ and $l_{\beta,i}^p = C_\beta \kappa^p$, it implies that

$$\|e^p\| \leq \frac{ML_{max} d\sqrt{\bar{m}} \cdot C_\alpha \kappa^p}{2^{n+1}} + \frac{M\sqrt{d\bar{m}} \cdot C_\beta \kappa^p}{2^{n+1}} = C_1 \kappa^p \ ,$$

and

$$\sqrt{\epsilon^k} \leq \frac{\sqrt{2}}{2} \cdot \frac{M\sqrt{\bar{m}} C_\alpha \kappa^p}{2^{n+1}} = C_2 \kappa^p \ ,$$

with $C_1 = \frac{M\sqrt{\bar{m}}(L_{max} d C_\alpha + \sqrt{d} C_\beta)}{2^{n+1}}$ and $C_2 = \frac{\sqrt{2}}{2} \cdot \frac{M\sqrt{\bar{m}} C_\alpha}{2^{n+1}}$, where $L_{max} := \max_{1 \leq i \leq M} L_i$. Since $(1-\gamma) < \kappa < 1$, Lemma 3.7 and Proposition 2.3 imply that for $0 \leq p \leq k$

$$\begin{aligned} \|x^{p+1} - x^\star\| &\leq (1-\gamma)^{p+1} \|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L} C_2)}{L} \sum_{q=0}^p \kappa^q (1-\gamma)^{p+1-q-1} \\ &\leq \kappa^{p+1} \left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L} C_2)}{L(1-\gamma)} \sum_{q=0}^p (\frac{1-\gamma}{\kappa})^{p+1-q} \right] . \end{aligned}$$

Since $0 < (1-\gamma) < \kappa < 1$, by using the property of geometric series, we get that the expression above is equal to

$$= \kappa^{p+1} \left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L} C_2)}{L(1-\gamma)} \cdot \frac{1 - (\frac{1-\gamma}{\kappa})^{p+1}}{1 - \frac{1-\gamma}{\kappa}} \right] \leq \kappa^{p+1} \left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L} C_2)\kappa}{L(\kappa + \gamma - 1)(1-\gamma)} \right] \ .$$

Hence, inequality (12) is proven. ■

D. Proof of Lemma 3.13

Proof: We will prove Lemma 3.13 by induction.

- Base case: When $k = 0$, since C_α and C_β are positive numbers and $\hat{x}_i^{-1}$ and x_i^0 are initialized to zero, it holds that $\|x_i^0 - \bar{x}_{\alpha,i}^0\|_\infty = \|x_i^0 - \hat{x}_i^{-1}\|_\infty = 0 \leq \frac{l_{\alpha,i}^0}{2} = \frac{C_\alpha}{2}$ and $\|\nabla f_i^0 - \bar{\nabla} f_{\beta,i}^0\|_\infty = \|\nabla f_i^0 - \hat{\nabla} f_i^{-1}\|_\infty = \|\nabla f_i(\tilde{x}_{\mathcal{N}_i}^0) - \nabla f_i(\text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(0))\| = 0 \leq \frac{l_{\beta,i}^0}{2} = \frac{C_\beta}{2}$.
- Induction step: Let $g \geq 0$ be given and suppose that $\|x_i^k - \bar{x}_{\alpha,i}^k\|_\infty \leq \frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$ for $0 \leq k \leq g$. We will prove that

$$\|x_i^{g+1} - \bar{x}_{\alpha,i}^{g+1}\|_\infty \leq \frac{l_{\alpha,i}^{g+1}}{2} \tag{23}$$

and

$$\|\nabla f_i^{g+1} - \bar{\nabla} f_{\beta,i}^{g+1}\|_\infty \leq \frac{l_{\beta,i}^{g+1}}{2} \tag{24}$$

for $i = 1, \cdots, M$. We first show (23). From Algorithm 3, we know

$$\begin{aligned}
\|x_i^{g+1} - \bar{x}_{\alpha,i}^{g+1}\|_\infty &= \|x_i^{g+1} - \hat{x}_i^g\|_\infty \\
&\leq \|x^{g+1} - \hat{x}^g\|_\infty \\
&= \|x^{g+1} - x^g - \sum_{i=1}^{M} E_i^T F_{ii}^T \alpha_i^g\|_\infty \\
&\leq \|x^{g+1} - x^g\|_\infty + \|\sum_{i=1}^{M} E_i^T F_{ii}^T \alpha_i^g\|_\infty \\
&\leq \|x^{g+1} - x^\star\|_\infty + \|x^g - x^\star\|_\infty + \|\sum_{i=1}^{M} E_i^T F_{ii}^T \alpha_i^g\|_\infty \ .
\end{aligned}$$

Since E_i and F_{ii} are selection matrices, then $\|E_i\| = \|F_{ii}\| = 1$. The term above is upper-bounded by

$$\leq \|x^{g+1} - x^\star\|_2 + \|x^g - x^\star\|_2 + \sum_{i=1}^{M} \|\alpha_i^g\|_2 \ .$$

By the assumption of the induction, we know $\|x_i^k - \bar{x}_{\alpha,i}^k\|_\infty \leq \frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$ for $0 \leq k \leq g$. Then, using Lemma 3.9, we obtain that the term above is upper-bounded by

$$\leq \kappa^{g+1}\left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)(1-\gamma)}\right] + \kappa^g\left[\|x^0 - x^\star\| + \frac{(C_1 + \sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)(1-\gamma)}\right] + \frac{M\sqrt{\bar{m}}C_\alpha\kappa^g}{2^{n+1}} \ .$$

By substituting $C_1 = \frac{M\sqrt{\bar{m}}(L_{max}dC_\alpha + \sqrt{d}C_\beta)}{2^{n+1}}$ and $C_2 = \frac{\sqrt{2}}{2} \cdot \frac{M\sqrt{\bar{m}}C_\alpha}{2^{n+1}}$ and using the parameters defined in Assumption 3.10, it follows that the expression above is equal to

$$= \kappa^{g+1}\left[a_1 + a_2\frac{C_\alpha}{2^{n+1}} + a_3 \cdot \frac{C_\beta}{2^{n+1}}\right] \ .$$

By inequality (13) in Assumption 3.10, the term above is bounded by $\frac{C_\alpha}{2}\kappa^{g+1}$. Thus, inequality (23) holds. In the following, we prove that inequality (24) is true.

$$\begin{aligned}
\|\nabla f_i^{g+1} - \bar{\nabla} f_{\beta,i}^{g+1}\|_\infty &= \|\nabla f_i^{g+1} - \hat{\nabla} f_i^g\|_\infty \\
&= \|\nabla f_i(\tilde{x}_{\mathcal{N}_i}^{g+1}) - \nabla f_i(\tilde{x}_{\mathcal{N}_i}^g) + \beta_i^g\|_\infty \\
&\leq \|\nabla f_i(\tilde{x}_{\mathcal{N}_i}^{g+1}) - \nabla f_i(\tilde{x}_{\mathcal{N}_i}^g)\|_\infty + \|\beta_i^g\|_\infty \\
&\leq \|\nabla f_i(\tilde{x}_{\mathcal{N}_i}^{g+1}) - \nabla f_i(\tilde{x}_{\mathcal{N}_i}^g)\|_2 + \|\beta_i^g\|_2 \\
&\leq L_i\|\tilde{x}_{\mathcal{N}_i}^{g+1} - \tilde{x}_{\mathcal{N}_i}^g\| + \|\beta_i^g\| \\
&\leq L_i\|x_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^g\| + L_i\|\tilde{x}_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^{g+1}\| + L_i\|\tilde{x}_{\mathcal{N}_i}^g - x_{\mathcal{N}_i}^g\| + \|\beta_i^g\|
\end{aligned}$$

Since $x_{\mathcal{N}_i}^{g+1}, x_{\mathcal{N}_i}^g \in \mathbb{C}_{\mathcal{N}_i}$, $\tilde{x}_{\mathcal{N}_i}^{g+1} = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}_{\mathcal{N}_i}^{g+1})$ and $\tilde{x}_{\mathcal{N}_i}^g = \text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(\hat{x}_{\mathcal{N}_i}^g)$, Lemma 3.6 implies $\|\tilde{x}_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^{g+1}\| \leq \|\hat{x}_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^{g+1}\|$ and $\|\tilde{x}_{\mathcal{N}_i}^g - x_{\mathcal{N}_i}^g\| \leq \|\hat{x}_{\mathcal{N}_i}^g - x_{\mathcal{N}_i}^g\|$. Hence, the term above is upper-bounded by

$$\begin{aligned}
&\leq L_i\|x_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^g\| + L_i\|\hat{x}_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^{g+1}\| + L_i\|\hat{x}_{\mathcal{N}_i}^g - x_{\mathcal{N}_i}^g\| + \|\beta_i^g\| \\
&\leq L_i\|x_{\mathcal{N}_i}^{g+1} - x_{\mathcal{N}_i}^g\| + L_i\sum_{j\in\mathcal{N}_i}(\|\alpha_j^{g+1}\| + \|\alpha_j^g\|) + \|\beta_i^g\| \\
&\leq L_i\|x^{g+1} - x^g\| + L_i\sum_{j\in\mathcal{N}_i}(\|\alpha_j^{g+1}\| + \|\alpha_j^g\|) + \|\beta_i^g\| \\
&\leq L_{\max}(\|x^{g+1} - x^\star\| + \|x^g - x^\star\|) + L_{\max}\sum_{j\in\mathcal{N}_i}(\|\alpha_j^{g+1}\| + \|\alpha_j^g\|) + \|\beta_i^g\| \ .
\end{aligned}$$

Again by the assumption of the induction, we know $\|x_i^k - \bar{x}_{\alpha,i}^k\|_\infty \leq \frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$ for $0 \leq k \leq g$.

Then, Lemma 3.9 implies that the term above is upper-bounded by

$$\begin{aligned}
\leq & L_{\max}\kappa^{g+1}\left(\|x^0-x^\star\|+\frac{(C_1+\sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)}\right)+L_{\max}\kappa^{g}\left(\|x^0-x^\star\|+\frac{(C_1+\sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)}\right)\\
&+\frac{L_{\max}\sqrt{\bar{m}}\sum_{j\in\mathcal{N}_i}(l_{\alpha,j}^{g+1}+l_{\alpha,j}^{g})}{2^{n+1}}+\frac{\sqrt{d\bar{m}}l_{\beta,i}^{g}}{2^{n+1}}\\
\leq & L_{\max}\kappa^{g+1}\left(\|x^0-x^\star\|+\frac{(C_1+\sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)}\right)+L_{\max}\kappa^{g}\left(\|x^0-x^\star\|+\frac{(C_1+\sqrt{2L}C_2)\kappa}{L(\kappa+\gamma-1)}\right)\\
&+\frac{L_{\max}\sqrt{d\bar{m}}C_\alpha(\kappa^{g+1}+\kappa^{g})}{2^{n+1}}+\frac{\sqrt{d\bar{m}}C_\beta\kappa^{g}}{2^{n+1}}\ .
\end{aligned}$$

By substituting $C_1=\frac{M\sqrt{\bar{m}}(L_{max}dC_\alpha+\sqrt{d}C_\beta)}{2^{n+1}}$ and $C_2=\frac{\sqrt{2}}{2}\cdot\frac{M\sqrt{\bar{m}}C_\alpha}{2^{n+1}}$ and using the parameters defined in Assumption 3.10, it follows that the expression above is equal to

$$=\kappa^{g+1}\cdot\left[b_1+b_2\cdot\frac{C_\alpha}{2^{n+1}}+b_3\cdot\frac{C_\beta}{2^{n+1}}\right]\ .$$

By inequality (14) in Assumption 3.10, the term above is bounded by $\frac{C_\beta}{2}\kappa^{g+1}=\frac{l_{\beta,i}^{g+1}}{2}$. Thus, inequality (24) holds. We conclude that by the principle of induction, the values of x_i^k and ∇f_i^k in Algorithm 3 fall inside of the quantization intervals of $Q_{\alpha,i}^k$ and $Q_{\beta,i}^k$, i.e. $\|x_i^k-\bar{x}_{\alpha,i}^k\|_\infty\leq\frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k-\bar{\nabla}f_{\beta,i}^k\|_\infty\leq\frac{l_{\beta,i}^k}{2}$ for all $k\geq 0$. ■

E. Proof of Lemma 3.18

Proof: The proof is similar to the proof of Lemma 3.13. The difference is that at each iteration the gradient ∇f_i^k is computed based on $\tilde{y}_{\mathcal{N}_i}^k$, which is a linear combination of $\tilde{x}_{\mathcal{N}_i}^k$ and $\tilde{x}_{\mathcal{N}_i}^{k-1}$. We therefore only show a brief proof for the second step, i.e. the inequality $\|\nabla f_i^k-\bar{\nabla}f_{\beta,i}^k\|_\infty\leq\frac{l_{\beta,i}^k}{2}$ for any $k\geq 0$ by induction.

- Base case: When $k=0$, since C_β is positive a number, $\tilde{x}_{\mathcal{N}_i}^{-1}$ and x_i^0 are initialized to zero and $\hat{\nabla}f_i^{-1}=\nabla f_i(\text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(0))$, it holds that $\|\nabla f_i^0-\bar{\nabla}f_{\beta,i}^0\|_\infty=\|\nabla f_i^0-\hat{\nabla}f_i^{-1}\|_\infty=\|\nabla f_i(\tilde{y}_{\mathcal{N}_i}^0)-\nabla f_i(\text{Proj}_{\mathbb{C}_{\mathcal{N}_i}}(0))\|=0\leq\frac{l_{\beta,i}^0}{2}=\frac{C_\beta}{2}$.
- Induction step: Let $g\geq 0$ be given and suppose that $\|x_i^k-\bar{x}_{\alpha,i}^k\|_\infty\leq\frac{l_{\alpha,i}^k}{2}$ and $\|\nabla f_i^k-\bar{\nabla}f_{\beta,i}^k\|_\infty\leq\frac{l_{\beta,i}^k}{2}$ for $0\leq k\leq g$. We will prove

$$\|\nabla f_i^{g+1}-\bar{\nabla}f_{\beta,i}^{g+1}\|_\infty\leq\frac{l_{\beta,i}^{g+1}}{2}\ . \tag{25}$$

From the algorithm, we know

$$\begin{aligned}
\|\nabla f_i^{g+1}-\bar{\nabla}f_{\beta,i}^{g+1}\|_\infty&=\|\nabla f_i^{g+1}-\hat{\nabla}f_i^{g}\|_\infty\\
&=\|\nabla f_i(\tilde{y}_{\mathcal{N}_i}^{g+1})-\nabla f_i(\tilde{y}_{\mathcal{N}_i}^{g})+\beta_i^g\|_\infty\\
&\leq L_i\|y_{\mathcal{N}_i}^{g+1}-y_{\mathcal{N}_i}^{g}\|+L_i\|\hat{y}_{\mathcal{N}_i}^{g+1}-y_{\mathcal{N}_i}^{g+1}\|+L_i\|\hat{y}_{\mathcal{N}_i}^{g}-y_{\mathcal{N}_i}^{g}\|+\|\beta_i^g\|\ .
\end{aligned}$$

By substituting $\hat{y}_{\mathcal{N}_i}^g=\frac{2}{1+\sqrt{\gamma}}\hat{x}_{\mathcal{N}_i}^g-\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}\hat{x}_{\mathcal{N}_i}^{g-1}$, $y_{\mathcal{N}_i}^g=\frac{2}{1+\sqrt{\gamma}}x_{\mathcal{N}_i}^g-\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}x_{\mathcal{N}_i}^{g-1}$ and $L_{max}:=\max_{1\leq i\leq M}L_i$, and using the fact that $\frac{2}{1+\sqrt{\gamma}}\leq 2$ and $\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}\leq 1$, the expression above is upper-bounded by

$$\leq L_{\max}(2\|x^{g+1}-x^\star\|+3\|x^g-x^\star\|+\|x^{g-1}-x^\star\|)+L_{\max}\sum_{j\in\mathcal{N}_i}(2\|\alpha_j^{g+1}\|+3\|\alpha_j^g\|+\|\alpha_j^{g-1}\|)+\|\beta_i^g\|\ .$$

By the assumption of the induction and Lemma 3.16, we obtain that the above is upper-bounded by

$$\begin{aligned}
\leq & L_{\max}(2\kappa^{g+1}+3\kappa^g+\kappa^{g-1})\left[\frac{2\sqrt{\Phi(x^0)-\Phi(x^\star)}}{\sqrt{\sigma_\phi}}+\frac{(2C_3+2\sqrt{2L}C_4+\sqrt{2\sigma_\phi}C_4)\kappa}{\sigma_\phi(\kappa-\sqrt{1-\sqrt{\gamma}})\cdot\sqrt{1-\sqrt{\gamma}}}\right]\\
&+\frac{L_{\max}\sqrt{\bar{m}}d(2l_{\alpha,j}^{g+1}+3l_{\alpha,j}^{g}+l_{\alpha,j}^{g-1})}{2^{n+1}}+\frac{\sqrt{d\bar{m}}l_{\beta,i}^{g}}{2^{n+1}}\ .
\end{aligned}$$

By substituting $C_3=\frac{M\sqrt{\bar{m}}(3L_{max}dC_\alpha+\kappa\sqrt{d}C_\beta)}{\kappa\cdot 2^{n+1}}$ and $C_4=\frac{\sqrt{2}}{2}\cdot\frac{M\sqrt{\bar{m}}C_\alpha}{2^{n+1}}$ and using the parameters defined in Assumption 3.17, the expression becomes

$$=\kappa^{g+1}\cdot\left[b_4+b_5\cdot\frac{C_\alpha}{2^{n+1}}+b_6\cdot\frac{C_\beta}{2^{n+1}}\right]\ .$$

By inequality (21) in Assumption 3.17, the term above is bounded by $\frac{C_\beta}{2}\kappa^{g+1} = \frac{l_{\beta,i}^{g+1}}{2}$. Thus, the inequality $\|\nabla f_i^{g+1} - \bar{\nabla} f_{\beta,i}^{g+1}\|_\infty \leq \frac{l_{\beta,i}^{g+1}}{2}$ holds. The proof of the induction step is complete.

By the principle of induction, we conclude that the inequality$\|\nabla f_i^k - \bar{\nabla} f_{\beta,i}^k\|_\infty \leq \frac{l_{\beta,i}^k}{2}$ holds for any $k \geq 0$. ■

REFERENCES

[1] A. Beck and M. Teboulle. A fast iterative shrinkage thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, pages 183–202, 2009.

[2] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar. *Convex analysis and optimization*. Athena Scientific Belmont, 2003.

[3] R. Carli, F. Fagnani, P. Frasca, T. Taylor, and R. Zampieri. Average consensus on networks with transmission noise or quantization. In *European Control Conference*, pages 1852–1857, 2007.

[4] C. Conte, T. Summers, M.N. Zeilinger, M. Morari, and C.N. Jones. Computational aspects of distributed optimization in model predictive control. In *51th IEEE Conference on Decision and Control*, pages 6819–6824, 2012.

[5] C. Conte, N. R. Voellmy, M. N. Zeilinger, M. Morari, and C. N. Jones. Distributed synthesis and control of constrained linear systems. In *American Control Conference*, pages 6017–6022, 2012.

[6] O. Devolder, F. Glineur, and Y. Nesterov. First-order methods of smooth convex optimization with inexact oracle. *Mathematical Programming*, pages 1–39, 2013.

[7] Q. T. Dinh, I. Necoara, and M. Diehl. Fast inexact decomposition algorithms for large-scale separable convex optimization. *arXiv preprint arXiv:1212,4275*, 2012.

[8] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990.

[9] A. Kashyap, T. Basar, and R. Srikant. Quantized consensus. *Automatica*, 43:1192–1203, 2007.

[10] V. Nedelcu, I. Necoara, and I. Dumitrache. Complexity of an inexact augmented lagrangian method: Application to constrained MPC. In *19th World Congress of the International Federation of Automatic Control*, pages 2927–2932, 2014.

[11] A. Nedic, A. Olshevsky, A. Ozdaglar, and J.N. Tsitsiklis. Distributed subgradient methods and quantization effects. In *47th IEEE Conference on Decision and Control*, pages 4177–4184, 2008.

[12] Y. Pu, M.N. Zeilinger, and C. N. Jones. Inexact fast alternating minimization algorithm for distributed model predictive control. In *53th IEEE Conference on Decision and Control*, pages 5915–5921, 2014.

[13] Y. Pu, M.N. Zeilinger, and C. N. Jones. Quantization design for unconstrained distributed optimization. In *American Control Conference*, 2015.

[14] M. Schmidt, N. L. Roux, and F. Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In *25th Annual Conference on Neural Information Processing Systems*, pages 6819–6824, 2011.

[15] D. Thanou, E. Kokiopoulou, Y. Pu, and P. Frossard. Distributed average consensus with quantization refinement. *IEEE Transactions on Signal Processing*, 61:194–205, 2013.