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# Non-convex, non-local functionals converging to the total variation

## *Convergence de fonctionnelles non convexes et non locales vers la variation totale*

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A B S T R A C T

We present new results concerning the approximation of the total variation, $\int_\Omega |\nabla u|$, of a function $u$ by non-local, non-convex functionals of the form

$$\Lambda_\delta(u) = \int_\Omega \int_\Omega \frac{\delta\varphi\big(|u(x)-u(y)|/\delta\big)}{|x-y|^{d+1}}\,\mathrm{d}x\,\mathrm{d}y,$$

as $\delta \to 0$, where $\Omega$ is a domain in $\mathbb{R}^d$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying some appropriate conditions. The mode of convergence is extremely delicate, and numerous problems remain open. The original motivation of our work comes from Image Processing.



R É S U M É

Nous présentons des résultats nouveaux concernant l'approximation de la variation totale $\int_\Omega |\nabla u|$ d'une fonction $u$ par des fonctionnelles non convexes et non locales de la forme

$$\Lambda_\delta(u) = \int_\Omega \int_\Omega \frac{\delta\varphi\big(|u(x)-u(y)|/\delta\big)}{|x-y|^{d+1}}\,\mathrm{d}x\,\mathrm{d}y,$$

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quand $\delta \to 0$, où $\Omega$ est un domaine de $\mathbb{R}^d$ et $\varphi : [0, +\infty) \to [0, +\infty)$ est une fonction croissante vérifiant certaines hypothèses. Le mode de convergence est extrêmement délicat et de nombreux problèmes restent ouverts. La motivation provient du traitement d'images.


## 1. Introduction

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be non-decreasing, and continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$. Assume that $\varphi(0) = 0$ and that $\varphi(t) = \varphi(t_-)$ for all $t > 0$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain of $\mathbb{R}^d$. Given a measurable function $u$ on $\Omega$, and $\delta > 0$, we define the following non-local functionals:

$$\Lambda(u) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{d+1}} \, dx \, dy \le +\infty \quad \text{and} \quad \Lambda_\delta(u) := \delta \Lambda(u/\delta).$$

We make the following three basic assumptions on $\varphi$:

$$\varphi(t) \le at^2 \text{ in } [0, 1] \text{ for some positive constant } a, \tag{1}$$

$$\varphi(t) \le b \text{ in } \mathbb{R}_+ \text{ for some positive constant } b, \tag{2}$$

and

$$\gamma_d \int_0^\infty \varphi(t) t^{-2} \, dt = 1, \text{ where } \gamma_d := 2|B^{d-1}|; \tag{3}$$

here $B^{d-1}$ denotes the unit ball in $\mathbb{R}^{d-1}$ and $|B^{d-1}|$ denotes its $(d-1)$-Hausdorff measure (with $\gamma_d = 2$ when $d = 1$). Condition (3) is a normalization condition prescribed in order to have (7) below with constant 1 in front of $\int_\Omega |\nabla u|$. Denote

$$\mathbf{A} = \{\varphi; \ \varphi \text{ satisfies (1)–(3)}\}. \tag{4}$$

Note that $\Lambda$ is **never convex** when $\varphi \in \mathbf{A}$.

Here are three examples of functions $\varphi$ that we have in mind. They all satisfy (1) and (2). In order to achieve (3), we choose $\varphi = c_i \tilde{\varphi}_i$, where $\tilde{\varphi}_i$ is taken from the list below and $c_i$ is an appropriate constant:

$$\tilde{\varphi}_1(t) = \begin{cases} 0 & \text{if } t \le 1 \\ 1 & \text{if } t > 1, \end{cases} \qquad \tilde{\varphi}_2(t) = \begin{cases} t^2 & \text{if } t \le 1 \\ 1 & \text{if } t > 1, \end{cases} \qquad \text{and} \qquad \tilde{\varphi}_3(t) = 1 - e^{-t^2}.$$

Example 1 is extensively studied in [3,6,10–14] (see also [5,15]). Examples 2 and 3 are motivated by Image Processing (see [8,17]).

In this note, we are concerned with modes of convergence of $\Lambda_\delta$ to the total variation as $\delta \to 0$. The convergence to the total variation of a sequence of **convex** non-local functionals $J_\varepsilon$, defined by

$$J_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|) \, dx \, dy, \tag{5}$$

where $\rho_\varepsilon$ is a sequence of radial mollifiers, was originally analyzed by J. Bourgain, H. Brezis and P. Mironescu and thoroughly investigated in [1,2,4,9].

The asymptotic analysis of $\Lambda_\delta$ is **much more delicate** than the one of $J_\varepsilon$, because two basic properties satisfied by $J_\varepsilon$ (which played an important role in [1]) are **not** fulfilled by $\Lambda_\delta$:

i) there is **no** constant $C$ such that

$$\Lambda_\delta(u) \le C \int_\Omega |\nabla u| \quad \forall u \in C^1(\bar{\Omega}), \ \forall \delta > 0, \tag{6}$$

ii) $\Lambda_\delta(u)$ is **not** a convex functional.

## 2. Statement of the main results

Concerning the pointwise limit of $\Lambda_\delta$ as $\delta \to 0$, i.e. the convergence of $\Lambda_\delta(u)$ for fixed $u$, we prove that, for every $\varphi \in \mathbf{A}$,

$$\Lambda_\delta(u) \text{ converges, as } \delta \to 0, \text{ to } TV(u) = \int_\Omega |\nabla u| \quad \forall u \in \bigcup_{p>1} W^{1,p}(\Omega). \tag{7}$$

If $u \in W^{1,1}(\Omega)$, we can only assert that, for every $\varphi \in \mathbf{A}$,

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u|.$$

Surprisingly, for every $d \geq 1$ and for every $\varphi \in \mathbf{A}$, one can construct a function $u \in W^{1,1}(\Omega)$ such that

$$\lim_{\delta \to 0} \Lambda_\delta(u) = +\infty.$$

This kind of "pathology" was originally discovered by A. Ponce and presented in [10] for $\varphi = c_1\tilde{\varphi}_1$ (for another example, see [7]). One may also construct (see [7]) functions $u \in W^{1,1}(\Omega)$ such that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \text{and} \quad \limsup_{\delta \to 0} \Lambda_\delta(u) = +\infty.$$

When dealing with functions $u \in BV(\Omega)$, the situation becomes even more intricate. It may happen, for some $\varphi \in \mathbf{A}$ and some $u \in BV(\Omega)$, that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) < \int_\Omega |\nabla u|.$$

All these facts suggest that the mode of convergence of $\Lambda_\delta$ to $TV$ as $\delta \to 0$ is delicate and that a theory of pointwise convergence is out of reach. It turns out that $\Gamma$-convergence (in the sense of E. De Giorgi) is the appropriate framework to analyze the asymptotic behavior of $\Lambda_\delta$ as $\delta \to 0$.

Our main result is the following.

**Theorem 1.** *For every $\varphi \in \mathbf{A}$, there exists a constant $K = K(\varphi) \in (0, 1]$, which is independent of $\Omega$, such that, as $\delta \to 0$,*

$$\Lambda_\delta \ \Gamma\text{-converges to } \Lambda_0 \text{ in } L^1(\Omega), \tag{8}$$

*where $\Lambda_0$ is defined on $L^1(\Omega)$ by*

$$\Lambda_0(u) = K \int_\Omega |\nabla u| \text{ for } u \in BV(\Omega), \text{ and } +\infty \text{ otherwise.}$$

The proof of Theorem 1 is extremely involved and it would be desirable to simplify it. When $\varphi = c_1\tilde{\varphi}_1$ and $\Omega = \mathbb{R}^d$, Theorem 1 is originally due to H.-M. Nguyen [11,13]. One of the key ingredients was the following earlier result, basically due to J. Bourgain and H.-M. Nguyen [3, Lemma 2].

**Lemma 1.** *Let $\Omega = (0, 1)$, $\varphi = c_1\tilde{\varphi}_1$. There exists a constant $k > 0$ such that*

$$\liminf_{\delta \to 0} \Lambda_\delta(u) \geq k|u(t_2) - u(t_1)|,$$

*for every $u \in L^1(\Omega)$, and for all Lebesgue points $t_1$, $t_2 \in (0, 1)$ of $u$.*

Furthermore, one can show that

$$\inf_{\varphi \in \mathbf{A}} K(\varphi) > 0.$$

One of the most intriguing remaining questions is

**Open Problem 1.** *Is it true that for every $\varphi \in \mathbf{A}$, $K(\varphi) < 1$ in Theorem 1?*

It has been proved in [11] (see also [7]) that $K(c_1\tilde{\varphi}_1) < 1$. However, the answer to Open Problem 1 is **not** known for $\varphi = c_2\tilde{\varphi}_2$ and $\varphi = c_3\tilde{\varphi}_3$, even when $d = 1$.

Motivated by questions arising in Image Processing (see, e.g., [7,8,16,17]), we consider the problem

$$\inf_{u \in L^q(\Omega)} E_\delta(u), \tag{9}$$

where

$$E_\delta(u) = \lambda \int_\Omega |u - f|^q + \Lambda_\delta(u), \tag{10}$$

$q \geq 1$, $f \in L^q(\Omega)$ is given, and $\lambda$ is a fixed positive constant. Our goal is twofold: investigate the existence of minimizers for $E_\delta$ (for fixed $\delta$) and analyze their behavior as $\delta \to 0$. The existence of a minimizer in (9) is not obvious since $\Lambda_\delta$ is **not convex** and one cannot invoke the standard tools of Functional Analysis. Our main result in this direction is the following.

**Theorem 2.** *Assume that $\varphi \in \mathbf{A}$ and $\varphi(t) > 0$ for all $t > 0$. Let $q \geq 1$ and $f \in L^q(\Omega)$. For each $\delta > 0$, there exists a minimizer $u_\delta$ of (9). Moreover, $u_\delta \to u_0$ in $L^q(\Omega)$ as $\delta \to 0$, where $u_0$ is the unique minimizer of the functional $E_0$ defined on $L^q(\Omega) \cap BV(\Omega)$ by*

$$E_0(u) := \lambda \int_\Omega |u - f|^q + K \int_\Omega |\nabla u|,$$

*and $0 < K \leq 1$ is the constant coming from Theorem 1.*

Note that the minimizers $u_\delta$ of (9) need not be unique, but the convergence assertion in Theorem 2 holds for any choice of minimizers. The proof of the existence of a minimizer for (9) relies on the following compactness lemma for **fixed** $\delta$, e.g., with $\delta = 1$.

**Lemma 2.** *Let $\varphi \in \mathbf{A}$ be such that $\varphi(t) > 0$ for all $t > 0$, and let $(u_n)$ be a bounded sequence in $L^1(\Omega)$ such that*

$$\sup_n \Lambda(u_n) < +\infty. \tag{11}$$

*There exists a subsequence $(u_{n_k})$ of $(u_n)$ and $u \in L^1(\Omega)$ such that $(u_{n_k})$ converges to $u$ in $L^1(\Omega)$.*

The proof of the convergence as $\delta \to 0$ in Theorem 2 relies heavily on the $\Gamma$-convergence of $\Lambda_\delta$ (Theorem 1), and also on the following compactness lemma (with roots in H.-M. Nguyen [14]).

**Lemma 3.** *Let $\varphi \in \mathbf{A}$, $(\delta_n) \to 0$, and let $(u_n)$ be a bounded sequence in $L^1(\Omega)$ such that*

$$\sup_n \Lambda_{\delta_n}(u_n) < +\infty. \tag{12}$$

*There exists a subsequence $(u_{n_k})$ of $(u_n)$ and $u \in L^1(\Omega)$ such that $(u_{n_k})$ converges to $u$ in $L^1(\Omega)$.*

The proofs of the results announced in this note are given in [7].

## References


[1] J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations: A Volume in Honour of A. Bensoussan's 60th Birthday, IOS Press, 2001, pp. 439–455.
[2] J. Bourgain, H. Brezis, P. Mironescu, Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, J. Anal. Math. 87 (2002) 77–101.
[3] J. Bourgain, H.-M. Nguyen, A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris, Ser. I 343 (2006) 75–80.
[4] H. Brezis, How to recognize constant functions. Connections with Sobolev spaces, Usp. Mat. Nauk 57 (2002) 59–74, A volume in honor of M. Vishik. English translation in Russ. Math. Surv. 57 (2002) 693–708.
[5] H. Brezis, New approximations of the total variation and filters in Imaging, Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. 26 (2015) 223–240.
[6] H. Brezis, H.-M. Nguyen, On a new class of functions related to VMO, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 157–160.
[7] H. Brezis, H.-M. Nguyen, Non-local functionals related to the total variation and applications in Image Processing, arXiv:1608.08204, 2016, submitted for publication.
[8] A. Buades, B. Coll, J.M. Morel, Image denoising methods. A new nonlocal principle, SIAM Rev. 52 (2010) 113–147.
[9] J. Davila, On an open question about functions of bounded variation, Calc. Var. Partial Differ. Equ. 15 (2002) 519–527.
[10] H.-M. Nguyen, Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689–720.
[11] H.-M. Nguyen, Γ-convergence and Sobolev norms, C. R. Acad. Sci. Paris, Ser. I 345 (2007) 679–684.
[12] H.-M. Nguyen, Further characterizations of Sobolev spaces, J. Eur. Math. Soc. 10 (2008) 191–229.
[13] H.-M. Nguyen, Γ-convergence, Sobolev norms, and BV functions, Duke Math. J. 157 (2011) 495–533.
[14] H.-M. Nguyen, Some inequalities related to Sobolev norms, Calc. Var. Partial Differ. Equ. 41 (2011) 483–509.
[15] H.-M. Nguyen, Estimates for the topological degree and related topics, J. Fixed Point Theory 15 (2014) 185–215.
[16] L.I. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D 60 (1992) 259–268.
[17] L.P. Yaroslavsky, M. Eden, Fundamentals of Digital Optics, Springer, 1996.