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Real Variable(s) Functions — *The BBM formula revisited*, by Haïm Brezis and Hoai-Minh Nguyen, communicated on 10 June 2016.

To the memory of Ennio De Giorgi with emotion and admiration

Abstract. — In this paper, we revise the BBM formula due to J. Bourgain, H. Brezis, and P. Mironescu in [1].

Key words: Sobolev spaces, BV functions, non-local approximations, maximal functions

Mathematics Subject Classification: 46E35, 46E30, 26D15

1. Introduction

We first recall the BBM formula due to J. Bourgain, H. Brezis, and P. Mironescu [1], see also [3], (with a refinement by J. Davila [5]). Let $d \geq 1$ be an integer. Throughout this paper, (ρ_n) denotes a sequence of radial mollifiers in the sense that

$$\rho_n \in L^1_{loc}(0, +\infty), \quad \rho_n \geq 0, \tag{1.1}$$

$$\int_0^\infty \rho_n(r) r^{d-1}\, dr = 1 \quad \forall n, \tag{1.2}$$

and

$$\lim_{n \to +\infty} \int_\delta^\infty \rho_n(r) r^{d-1}\, dr = 0 \quad \forall \delta > 0. \tag{1.3}$$

Even though the next assumption is required only for a few results, it is convenient to assume that

$$\rho_n(r) = 0 \quad \text{for all } r > 1,\ n \in \mathbb{N}. \tag{1.4}$$

Set, for $p \geq 1$,

$$I_{n,p}(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|)\, dx\, dy \leq +\infty, \quad \forall u \in L^1_{loc}(\mathbb{R}^d). \tag{1.5}$$

For $u \in L^1_{loc}(\mathbb{R}^d)$, define, for $p > 1$,

$$I_p(u) = \begin{cases} \gamma_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p & \text{if } \nabla u \in L^p(\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.6}$$

and, for $p = 1$,

$$I_1(u) = \begin{cases} \gamma_{d,1} \int_{\mathbb{R}^d} |\nabla u| & \text{if } \nabla u \text{ is a finite measure,} \\ +\infty & \text{otherwise,} \end{cases} \tag{1.7}$$

where, for any $e \in \mathbb{S}^{d-1}$ and $p \geq 1$,

$$\gamma_{d,p} = \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma. \tag{1.8}$$

In the case $p = 1$, we have

$$\gamma_{d,1} = \int_{\mathbb{S}^{d-1}} |\sigma \cdot e| \, d\sigma = \begin{cases} \dfrac{2}{d-1} |\mathbb{S}^{d-2}| = 2|B^{d-1}| & \text{if } d \geq 3, \\ 4 & \text{if } d = 2, \\ 2 & \text{if } d = 1. \end{cases} \tag{1.9}$$

The BBM formula asserts that, for $p \geq 1$,

$$\lim_{n \to +\infty} I_{n,p}(u) = I_p(u) \quad \forall u \in L^1_{loc}(\mathbb{R}^d). \tag{1.10}$$

Applying (1.10) with $p = 1$, $u = \mathbb{1}_E$ (the characteristic function of a measurable set E), and $\rho_n(r) = C_d n^{(d+1)/2} r e^{-nr^2}$, we obtain

$$\lim_{n \to +\infty} n^{(d+1)/2} \int_{E^c} \int_E e^{-n|x-y|^2} \, dx \, dy = A_d \operatorname{Per}(E).$$

By comparison the De Giorgi formula [6, 7] for the perimeter involves a derivative and asserts that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} |\nabla W_n(x)| \, dx = B_d \operatorname{Per}(E),$$

where

$$W_n(x) = n^{d/2} \int_E e^{-n|x-y|^2} \, dy,$$

and A_d, B_d, and C_d are positive constants depending only on d.

Define, for $p \geq 1$, $n \in \mathbb{N}$, and $u \in L^1_{loc}(\mathbb{R}^d)$,

$$D_{n,p}(u)(x) := \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dy \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.11}$$

Note that, see [1],

$$\int_{\mathbb{R}^d} D_{n,p}(u)(x)\, dx \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla u|^p(x)\, dx \quad \text{for } n \in \mathbb{N},$$

and hence

$$D_{n,p}(x) < +\infty \quad \text{for a.e. } x \in \mathbb{R}^d \tag{1.12}$$

if $p > 1$ and $\nabla u \in L^p(\mathbb{R}^d)$ or $p = 1$ and ∇u is a finite measure. From the BBM formula, we have, for $p \geq 1$,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} D_{n,p}(u)(x) = I_p(u) \quad \text{for } u \in L^1_{loc}(\mathbb{R}^d). \tag{1.13}$$

On the other hand, an easy computation (see [1, formula (6)]) gives, for $p \geq 1$, $u \in C^1_{\mathrm{c}}(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$,

$$\lim_{n \to \infty} D_{n,p}(u)(x) = \gamma_{d,p} |\nabla u|^p(x).$$

In this paper, we investigate the mode convergence of $D_{n,p}(u)$ to $\gamma_{d,p}|\nabla u|^p$ as $n \to +\infty$ for non smooth u. Our main results are the following

THEOREM 1. *Let $d \geq 1$, $p \geq 1$, and $u \in W^{1,p}_{loc}(\mathbb{R}^d)$. Then*

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|^p}{|h|^p} \rho_n(|h|)\, dh = 0 \quad \textit{for a.e. } x \in \mathbb{R}^d. \tag{1.14}$$

Consequently,

$$\lim_{n \to +\infty} D_{n,p}(u)(x) = \gamma_{d,p} |\nabla u|^p(x) \quad \textit{for a.e. } x \in \mathbb{R}^d. \tag{1.15}$$

REMARK 1. When $\rho_n(r) = d\varepsilon_n^{-d} \mathbb{1}_{(0,\varepsilon_n)}$ for a sequence of $(\varepsilon_n) \to 0_+$, assertion (1.14) is part of the classical L^p-differentiability theory of Calderón-Zygmund; the same comment applies to assertion (1.18) below. Theorem 1 is due to D. Spector [11, Theorem 1.7] under the additional assumption that ρ_n is non-increasing for every n. His argument is much more complicated than ours (in addition he relies on the L^{p^*}-differentiability of $W^{1,p}$ functions, see e.g., [8, Theorem 2 on page 262]).

We now turn to the L^1-convergence of $D_{n,p}$.

PROPOSITION 1. *Let $d \geq 1$, $p \geq 1$, and $u \in L^1_{loc}(\mathbb{R}^d)$ with $\nabla u \in L^p(\mathbb{R}^d)$. Then*

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|^p}{|h|^p} \rho_n(|h|)\, dh\, dx = 0. \tag{1.16}$$

Consequently,

$$\lim_{n \to +\infty} D_{n,p}(u) = \gamma_{d,p} |\nabla u|^p \quad \text{in } L^1(\mathbb{R}^d). \tag{1.17}$$

REMARK 2. Assertion (1.17) was proved in [1].

Theorem 1 (resp. Proposition 1) is established in Section 2 (resp. Section 3) where we also present some variants, generalizations, and pathologies related to these results.

The case $p = 1$ and $u \in BV_{loc}(\mathbb{R}^d)$ is more delicate. In this case instead of Theorem 1, we have

THEOREM 2. *Let $d \geq 1$ and $u \in BV_{loc}(\mathbb{R}^d)$. Then*

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla^{ac} u(x) \cdot h|}{|h|} \rho_n(|h|)\, dh = 0 \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.18}$$

Consequently,

$$\lim_{n \to +\infty} D_{n,1}(u)(x) = \gamma_{d,1} |\nabla^{ac} u|(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.19}$$

Here and in what follows, for $u \in BV_{loc}(\mathbb{R}^d)$, we denote $\nabla^{ac} u$ and $\nabla^s u$ the absolutely continuous part and the singular part of ∇u.

REMARK 3. A version of Proposition 1 for $u \in BV(\mathbb{R}^d)$ has been established by A. Ponce and D. Spector [9, Proposition 2.1]. Here is their result: Let $d \geq 1$, and $u \in BV(\mathbb{R}^d)$. Then

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla^{ac} u(x) \cdot h|}{|h|} \rho_n(|h|)\, dh = \gamma_{d,1} |\nabla^s u| \text{ in the sense of measures.}$$

Theorem 2 is established in Section 4. In the last section, we present miscellaneous facts related to the above results.

2. CONVERGENCE ALMOST EVERYWHERE IN THE SOBOLEV CASE

We will use the following elementary lemma (see [4, Lemma 1]):

LEMMA 1. *Let $d \geq 1$, $r > 0$, $x \in \mathbb{R}^d$, and $f \in L^1_{loc}(\mathbb{R}^d)$. We have*

$$\int_{\mathbb{S}^{d-1}} \int_0^r |f(x + s\sigma)| \, ds \, d\sigma \leq C_d r M(f)(x), \tag{2.1}$$

for some positive constant C_d depending only on d.

Here $M(f)$ denotes the maximal function of f. We now give the

PROOF OF THEOREM 1. We first present the proof for $u \in W^{1,p}(\mathbb{R}^d)$. We claim that, for all $u \in W^{1,p}(\mathbb{R}^d)$,

$$D_{n,p}(u)(x) \leq CM(|\nabla u|^p)(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{2.2}$$

Here and in what follows, C denotes a positive constant depending only on d. We have, for a.e. $x \in \mathbb{R}^d$, $\sigma \in \mathbb{S}^{d-1}$, and $r > 0$,

$$u(x + r\sigma) - u(x) = \int_0^r \nabla u(x + s\sigma) \cdot \sigma \, ds.$$

Using polar coordinates, Hölder's inequality, and Fubini's theorem, we obtain, for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_n(|h|) \, dh \\ &\leq \int_0^\infty \rho_n(r) r^{d-1} \frac{1}{r} \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla u(x + s\sigma) \cdot \sigma|^p \, ds \, d\sigma \, dr \\ &= \int_0^\infty \rho_n(r) r^{d-1} \frac{1}{r} \int_{B(x,r)} |\nabla u(y)|^p |y|^{1-d} \, dy \, dr. \end{aligned}$$

Applying Lemma 1, we obtain (2.2).

The proof of (1.14) now goes as follows. Set

$$\Omega(u) := \left\{ x \in \mathbb{R}^d; \limsup_{n \to +\infty} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|^p}{|h|^p} \rho_n(|h|) \, dh > 0 \right\}.$$

Note that if $u \in C^1_c(\mathbb{R}^d)$ then (1.14) holds for all $x \in \mathbb{R}^d$. This implies

$$|\Omega(v)| = 0 \quad \text{for all } v \in C^1_c(\mathbb{R}^d).$$

It follows that

$$\Omega(u) = \Omega(u - v) \quad \text{for all } v \in C^1_c(\mathbb{R}^d). \tag{2.3}$$

Recall that, see e.g., [12, Theorem 1 on page 5], for $f \in L^1(\mathbb{R}^d)$, we have

$$|\{x \in \mathbb{R}^d; M(f)(x) > \varepsilon\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^d} |f|. \tag{2.4}$$

Using (2.2) and (2.4), we obtain

$$\left|\left\{x \in \mathbb{R}^d \int_{\mathbb{R}^d} \frac{|(u-v)(x+h)-(u-v)(x)-\nabla(u-v)(x)\cdot h|^p}{|h|^p}\rho_n(|h|)\,dh > \varepsilon\right\}\right| \tag{2.5}$$
$$\leq \frac{C}{\varepsilon}\int_{\mathbb{R}^d} |\nabla(u-v)(x)|^p\,dx \quad \text{for all } \varepsilon > 0.$$

Combining (2.3) and (2.5) yields (1.14). Assertion (1.15) follows from (1.14) by the triangle inequality after noting that, for every $V \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \frac{|V\cdot h|^p}{|h|^p}\rho_n(|h|)\,dh = \int_0^\infty \int_{\mathbb{S}^{d-1}} |V\cdot\sigma|^p \rho_n(r) r^{d-1}\,d\sigma\,dr = \gamma_{d,p}|V|^p.$$

We now turn to the proof in the case $u \in W^{1,p}_{loc}(\mathbb{R}^d)$. Given $R > 1$, let $\varphi \in C^1_c(\mathbb{R}^d)$ be such that $\varphi = 1$ in $B(0, 2R)$. We have $\varphi u \in W^{1,p}(\mathbb{R}^d)$. Applying the above result to φu, we obtain

$$\lim_{n\to+\infty} D_{n,p}(\varphi u)(x) = \gamma_{d,p}|\nabla(\varphi u)|^p(x) \quad \text{for a.e. } x \in B(0,R).$$

Since $D_{n,p}(u)(x) = D_{n,p}(\varphi u)(x)$ for $x \in B_R$ by (1.4) and $\varphi(x)u(x) = u(x)$ in B_R, it follows that

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla(u)|^p(x) \quad \text{for a.e. } x \in B(0,R).$$

Since $R > 1$ is arbitrary, the conclusion follows. $\square$

Here is a natural question related to Theorem 1. Suppose for example that $u \in W^{1,1}(\mathbb{R}^d)$ and u has compact support. Is it true that for every $1 < p < +\infty$,

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla u|^p(x) \quad \text{for a.e. } x \in \mathbb{R}^d?$$

Surprisingly, the answer is delicate and some pathologies may occur as seen in our next result.

THEOREM 3. *Let $d \geq 1$ and $u \in W^{1,1}_{loc}(\mathbb{R}^d)$. We have*

1. *If $d = 1$, then, for $p > 1$,*

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{1,p}|u'|^p(x) \quad \textit{for a.e. } x \in \mathbb{R}. \tag{2.6}$$

2. *If $d \geq 2$, $p \leq d/(d-1)$, and ρ_n is non-increasing, then*

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla u|^p(x) \quad \textit{for a.e. } x \in \mathbb{R}^d. \tag{2.7}$$

3. *If* $d \geq 2$ *and* $p > 1$, *then*

$$\liminf_{n\to+\infty} D_{n,p}(u)(x) \geq \gamma_{d,p}|\nabla u|^p(x) \quad \textit{for a.e. } x \in \mathbb{R}^d. \tag{2.8}$$

Moreover, strict inequality in (2.8) *can occur:*

4. *If* $d \geq 2$, *there exist* $u \in W^{1,1}(\mathbb{R}^d)$ *with compact support, a set* $A \subset \mathbb{R}^d$ *of positive measure, and a sequence of non-increasing functions* (ρ_n) *such that, for every* $n \in \mathbb{N}$,

$$D_{n,p}(u)(x) = +\infty \quad \textit{for a.e. } x \in A, \quad \textit{for all } p > d/(d-1). \tag{2.9}$$

Note that there is no contradiction between (1.12) and (2.9); the u which we construct here does not satisfy the condition $\nabla u \in L^p(\mathbb{R}^d)$.

Remark 4. Statement (2.7) is due to D. Spector [11, Theorem 1.7]. In fact, he proves a more general result: if $u \in W^{1,q}(\mathbb{R}^d)$ $(d \geq 2)$ with $1 \leq q < d$, $p \leq q^* = qd/(d-q)$, and ρ_n is non-increasing then (2.7) holds.

Remark 5. We do not know whether (2.7) holds without the additional assumption that ρ_n is non-increasing.

Proof. As in the proof of Theorem 1, one may assume that $u \in W^{1,1}(\mathbb{R}^d)$. We first prove (2.6). Since, for a.e. $x \in \mathbb{R}$ and $r > 0$,

$$|u(x+r) - u(x)| \leq \int_x^{x+r} |u'(s)|\, ds,$$

we have

$$D_{n,p}(u)^{1/p}(x) \leq CM(u')(x).$$

Assertion (2.6) now follows as in the proof of Theorem 1 by noting that, for $u \in C^1_c(\mathbb{R})$,

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{1,p}|u'|^p(x) \quad \text{for } x \in \mathbb{R}^d.$$

We next turn to the proof of (2.8). Using polar coordinates, we have, for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} D_{n,p}(u)(x) &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \left| \int_0^1 \nabla u(x + tr\sigma) \cdot \sigma\, dt \right|^p \rho_n(r) r^{d-1}\, d\sigma\, dr \\ &\geq \int_{\mathbb{S}^{d-1}} \left| \int_0^\infty \int_0^1 \nabla u(x + tr\sigma) \cdot \sigma \rho_n(r) r^{d-1}\, dt\, dr \right|^p d\sigma. \end{aligned} \tag{2.10}$$

We claim that, for a.e. $\sigma \in \mathbb{S}^{d-1}$ and for a.e. $x \in \mathbb{R}^d$,

$$\lim_{n\to+\infty} \int_0^\infty \int_0^1 \nabla u(x + tr\,\sigma) \cdot \sigma \rho_n(r) r^{d-1}\, dt\, dr = \nabla u(x) \cdot \sigma. \tag{2.11}$$

Assuming this and applying Fatou's lemma, we derive from (2.10) and (2.11) that, for a.e. $x \in \mathbb{R}^d$,

$$\liminf_{n\to+\infty} D_{n,p}(u)(x) \geq \gamma_{p,d} |\nabla u|^p(x);$$

which is (2.8). To complete the proof of (2.8), it remains to prove (2.11). For $v \in W^{1,1}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}^{d-1}$, set

$$M(\nabla v, \sigma, x) = \sup_{r>0} \fint_0^r |\nabla v(x + s\sigma) \cdot \sigma|\, ds. \tag{2.12}$$

Given $v \in W^{1,1}(\mathbb{R}^d)$ and $\sigma \in \mathbb{S}^{d-1}$, we claim that for all $\varepsilon > 0$, there exists a positive constant C independent of v, ε, and σ such that

$$|\{x \in \mathbb{R}^d; M(\nabla v, \sigma, x) > \varepsilon\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^d} |\nabla v(y)|\, dy. \tag{2.13}$$

Using Fubini's theorem, we derive from (2.13) that

$$|\{(x,\sigma) \in \mathbb{R}^d \times \mathbb{S}^{d-1}; M(\nabla v, \sigma, x) > \varepsilon\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^d} |\nabla v(y)|\, dy. \tag{2.14}$$

Using (2.14), one can now obtain assertion (2.11) as in the proof of Theorem 1 by noting that for all $u \in C^1_c(\mathbb{R}^d)$,

$$\lim_{n\to+\infty} \int_0^\infty \int_0^1 \nabla u(x + tr\,\sigma) \cdot \sigma \rho_n(r) r^{d-1}\, dt\, dr = \nabla u(x) \cdot \sigma \quad \text{for all } x \in \mathbb{R}^d.$$

We next establish (2.13). For simplicity of notation, we assume that $\sigma = e_d := (0, \dots, 0, 1)$. We have, by Fubini's theorem,

$$|\{x \in \mathbb{R}^d; M(\nabla v, e_d, x) > \varepsilon\}| = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \mathbb{1}_{\{x\in\mathbb{R}^d;\, M(\nabla v, e_d, x) > \varepsilon\}}\, dx_d\, dx'. \tag{2.15}$$

It follows from the theory of maximal functions (see (2.4)) that

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \mathbb{1}_{\{x\in\mathbb{R}^d;\, M(\nabla v, e_d, x) > \varepsilon\}}\, dx_d\, dx' \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\partial_{x_d} v(x', x_d)|\, dx_d\, dx'. \tag{2.16}$$

Combining (2.15) and (2.16) yields

$$|\{x \in \mathbb{R}^d; M(\nabla v, e_d, x) > \varepsilon\}| \le \frac{C}{\varepsilon} \int_{\mathbb{R}^d} |\nabla v(x)|\, dx;$$

which is (2.13). The proof of (2.8) is complete.

We finally establish (2.9). Let (δ_n) be a positive sequence converging to 0 such that $\delta_n < 1/2$ for all n, and define

$$\rho_n(t) = \delta_n t^{\delta_n - 1} \mathbb{1}_{(0,1)}(t). \tag{2.17}$$

Set $u(x) = \varphi(x)|x|^{(1-d)} \ln^{-2}|x|$ for some $\varphi \in C^1_c(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $|x| < 2$. It is clear that $u \in W^{1,1}(\mathbb{R}^d)$ and for $x \in \mathbb{R}^d$ with $1/4 < |x| < 1/2$,

$$\int_{|y|<1/8} |u(x) - u(y)|^p\, dy = +\infty$$

since $p > d/(d-1)$ and $\rho_n(|y - x|) \ge \delta_n (1/8)^{\delta_n - 1}$ for $|y| < 1/8$ and $1/4 < |x| < 1/2$. It follows that, for $1/4 < |x| < 1/2$,

$$D_{n,p}(u)(x) = +\infty \quad \forall n.$$

The proof is complete. $\square$

3. Convergence in norm

We present two proofs of Proposition 1.

First proof of Proposition 1 via Theorem 1. By Theorem 1, we have

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla u(x)|^p \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{3.1}$$

On the other hand, by the BBM formula,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} D_{n,p}(u)(x)\, dx = \gamma_{d,p} \int_{\mathbb{R}^d} |\nabla u(x)|^p\, dx. \tag{3.2}$$

Recall that (see e.g., [2, page 113]) if $f_n(x) \to f(x)$ for a.e. $x \in \mathbb{R}^d$, and $\|f_n\|_{L^1(\mathbb{R}^d)} \to \|f\|_{L^1(\mathbb{R}^d)}$, then $f_n \to f$ in $L^1(\mathbb{R}^d)$. We deduce from (3.1) and (3.2) that

$$D_{n,p}(u) \to \gamma_{d,p}|\nabla u|^p \quad \text{in } L^1(\mathbb{R}^d) \text{ as } n \to +\infty. \qquad \square$$

Direct proof of Proposition 1. We have, see [1],

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x)\cdot h|^p}{|h|^p} \rho_n(|h|)\, dh\, dx \le C_{p,d} \int_{\mathbb{R}^d} |\nabla u(x)|^p$$

and, for $v \in C_c^1(\mathbb{R}^d)$,

$$\lim_{n \to +\infty} D_{n,p}(v)(x) = \gamma_{d,p} |\nabla v(x)|^p \quad \text{in } L^1(\mathbb{R}^d) \text{ as } n \to +\infty.$$

The conclusion now follows by a standard approximation argument. □

4. Convergence almost everywhere in the BV case

Let $d \geq 1$, μ be a Radon measure defined on $\mathbb{R}^d$, and $0 < R \leq +\infty$. Denote

$$M_R(\mu)(x) = \sup_{0 < s \leq R} \frac{|\mu|(B(x,s))}{|B(x,s)|} \quad \text{and} \quad M(\mu)(x) = M_\infty(\mu)(x).$$

We begin this section with

Lemma 2. *Let $d \geq 1$, μ be a positive Radon measure defined in $\mathbb{R}^d$, and let $(\chi_k)_{k \geq 1}$ be a sequence of mollifier such that $\operatorname{supp} \chi_k \subset B(0, 1/k)$ and $0 \leq \chi_k \leq Ck^d$ for some positive constant C depending only on d. Set $\mu_k = \mu * \chi_k$. We have, for $x \in \mathbb{R}^d$ and for $r > 0$,*

$$\frac{1}{r} \int_{B(x,r)} |y - x|^{1-d} \, d\mu(y) \leq C M_r(\mu)(x) \tag{4.1}$$

and, for every k,

$$\frac{1}{r} \int_{B(x,r)} |y - x|^{1-d} \, d\mu_k(y) \leq C M(\mu)(x), \tag{4.2}$$

for some positive constant C depending only on d.

Proof. Without loss of generality, one may assume that $x = 0$. We have

$$\begin{aligned}
\frac{1}{r} \int_{B(0,r)} |y|^{1-d} \, d\mu(y) &= \frac{1}{r} \sum_{m=0}^{\infty} \int_{B(0, 2^{-m}r) \setminus B(0, 2^{-(m+1)}r)} |y|^{1-d} \, d\mu(y) \\
&\leq \frac{C}{r} \sum_{m=0}^{\infty} 2^{-m(1-d)} r^{1-d} \int_{B(0, 2^{-m}r) \setminus B(0, 2^{-(m+1)}r)} d\mu(y) \\
&\leq \frac{C}{r} \sum_{m=0}^{\infty} 2^{-m} r M_r(\mu)(0) = C M_r(\mu)(0);
\end{aligned}$$

which is (4.1).

We next prove (4.2). As above, we obtain

$$\frac{1}{r} \int_{B(0,r)} |y|^{1-d} \, d\mu_k(y) \leq \frac{C}{r} \sum_{m=0}^{\infty} 2^{-m(1-d)} r^{1-d} \int_{B(0, 2^{-m}r) \setminus B(0, 2^{-(m+1)}r)} d\mu_k(y). \tag{4.3}$$

We claim that

$$\int_{B(0,2^{-m}r)\setminus B(0,2^{-(m+1)}r)} d\mu_k(y) \le C2^{-md}r^d M(\mu)(0). \tag{4.4}$$

Combining (4.3) and (4.4) yields (4.2)

It remains to prove (4.3). We have

$$\begin{aligned}\int_{B(0,2^{-m}r)\setminus B(0,2^{-(m+1)}r)} d\mu_k(y) &\le \int_{B(0,2^{-m}r)\setminus \overline{B(0,2^{-(m+2)}r)}} d\mu_k(y) \\ &= \sup_{\varphi\in C_c(B(0,2^{-m}r)\setminus\overline{B(0,2^{-(m+2)}r)});\,|\varphi|\le 1} \int_{\mathbb{R}^d}\varphi\, d\mu_k.\end{aligned} \tag{4.5}$$

We have

$$\int_{\mathbb{R}^d}\varphi\, d\mu_k = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(z)\chi_k(z-y)\,dz\,d\mu(y) \tag{4.6}$$

If $2^{-m}r < 1/k$, we have, for $\varphi\in C_c(B(0,2^{-m}r)\setminus\overline{B(0,2^{-(m+2)}r)})$ with $|\varphi|\le 1$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(z)\chi_k(z-y)\,dz\,d\mu(y) \\ &\quad\le \int_{|y|<2/k}\sup_y\int_{\mathbb{R}^d}|\varphi(z)|\chi_k(z-y)\,dz\,d\mu(y) \\ &\quad\le C(2^{-m}r)^d k^d\int_{|y|<2/k} d\mu(y) \le C2^{-md}r^d M(\mu)(0).\end{aligned} \tag{4.7}$$

Here we use the fact that $\operatorname{supp}\chi_k\subset B(0,1/k)$ and $0\le\chi_k\le Ck^d$. Similarly, if $1/k<2^{-m}r$, we have, for $\varphi\in C_c(B(0,2^{-m}r)\setminus\overline{B(0,2^{-(m+2)}r)})$ with $|\varphi|\le 1$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(z)\chi_k(z-y)\,dz\,d\mu(y)\,dy \\ &\quad\le \int_{|y|<2^{-m+2}r}\sup_y\int_{\mathbb{R}^d}|\varphi(z)|\chi_k(z-y)\,dz\,d\mu(y) \\ &\quad\le \int_{|y|<2^{-m+2}r} d\mu(y) \le C2^{-md}r^d M(\mu)(0).\end{aligned} \tag{4.8}$$

Combining (4.5), (4.6), (4.7), and (4.8), we obtain (4.4). The proof is complete. □

We recall that (see, e.g., [8])

$$\lim_{r\to 0}\frac{|\nabla^s u|(B(x,r))}{|B(x,r)|} = 0 \quad \text{for a.e. } x\in\mathbb{R}^d. \tag{4.9}$$

As a consequence of (4.9), one obtains

$$M(|\nabla^s u|)(x) < +\infty \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{4.10}$$

We now present the

Proof of Theorem 2. As in the proof of Theorem 1, one may assume that $u \in BV(\mathbb{R}^d)$. Let $(\chi_k)_{k\geq 1}$ be a sequence of smooth mollifiers such that $\operatorname{supp} \chi_k \subset B(0, 1/k)$ and $0 \leq \chi_k \leq Ck^d$. Here and in what follows, C denotes a positive constant depending only on d. Set, for $k \in \mathbb{N}_+$,

$$u_k = u * \chi_k, \quad V_k^s = \nabla^s u * \chi_k, \quad \text{and} \quad V_k^{ac} = \nabla^{ac} u * \chi_k.$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - V_k^{ac}(x) \cdot h|}{|h|} \rho_n(|h|)\, dh \\ &\quad = \int_0^\infty r^{d-1} \rho_n(r) \int_{\mathbb{S}^{d-1}} \frac{|u_k(x+r\sigma) - u_k(x) - rV_k^{ac}(x) \cdot \sigma|}{r}\, d\sigma\, dr. \end{aligned} \tag{4.11}$$

Since

$$u_k(x+r\sigma) - u_k(x) - rV_k^{ac}(x) \cdot \sigma = \int_0^r \nabla u_k(x+s\sigma) \cdot \sigma\, ds - rV_k^{ac}(x) \cdot \sigma$$

and

$$\nabla u_k(x) = V_k^s(x) + V_k^{ac}(x),$$

it follows from (4.11) that

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - V_k^{ac}(x) \cdot h|}{|h|} \rho_n(|h|)\, dh \\ &\quad \leq \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r}\, dr \int_{\mathbb{S}^{d-1}} \int_0^r |V_k^s(x+s\sigma)|\, ds\, d\sigma \\ &\qquad + \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r}\, dr \int_{\mathbb{S}^{d-1}} \int_0^r |V_k^{ac}(x+s\sigma) - V_k^{ac}(x)|\, ds\, d\sigma. \end{aligned} \tag{4.12}$$

We claim that, for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} &\lim_{k\to+\infty} \int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - V_k^{ac}(x) \cdot h|}{|h|} \rho_n(|h|)\, dh \\ &\quad = \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla^{ac} u(x) \cdot h|}{|h|} \rho_n(|h|)\, dh, \end{aligned} \tag{4.13}$$

$$\lim_{k\to+\infty}\int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |V_k^s(x+s\sigma)|\,ds\,d\sigma \tag{4.14}$$
$$= \int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{B(x,r)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy,$$

and

$$\lim_{k\to+\infty}\int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |V_k^{ac}(x+s\sigma)-V_k^{ac}(x)|\,ds\,d\sigma \tag{4.15}$$
$$= \int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|\,ds\,d\sigma.$$

Assuming these claims, we continue the proof. Combining (4.12), (4.13), (4.14), and (4.15) yields, for a.e. $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\frac{|u(x+h)-u(x)-\nabla^{ac}u(x)\cdot h|}{|h|}\rho_n(|h|)\,dh \tag{4.16}$$
$$\le \int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{B(x,r)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy$$
$$+ \int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|\,ds\,d\sigma.$$

Hence it suffices to prove that, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{n\to+\infty}\int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{B(x,r)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy = 0 \tag{4.17}$$

and

$$\lim_{n\to+\infty}\int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|\,ds\,d\sigma = 0. \tag{4.18}$$

Note that assertion (4.18) holds for every $x \in \mathbb{R}^d$ if $u \in C^1_{\mathrm{c}}(\mathbb{R}^d)$ and, by Lemma 2,

$$\int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|\,ds\,d\sigma \le CM(|\nabla^{ac}u|)(x).$$

As in the proof of Theorem 1, we have, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{n\to+\infty}\int_0^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|\,ds\,d\sigma = 0;$$

which is (4.18).

We next establish (4.17). By Lemma 2, we have

$$\frac{1}{r}\int_{B(x,r)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy \le CM_r(|\nabla^s u|)(x).$$

It follows from (4.9) that

$$\lim_{n\to+\infty}\int_0^\infty r^{d-1}\rho_n(r)\frac{1}{r}\,dr\int_{B(x,r)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy = 0 \quad \text{for a.e. } x\in\mathbb{R}^d;$$

which is (4.17).

It remains to prove claims (4.13), (4.14), and (4.15). We begin with claim (4.13). We have

$$\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-V_k^{ac}(x)\cdot h|}{|h|}\rho_n(|h|)\,dh$$
$$=\int_0^\infty \rho_n(r)r^{d-1}\frac{1}{r}\,dr\int_{\mathbb{S}^{d-1}}|u_k(x+r\sigma)-u_k(x)-rV_k^{ac}(x)\cdot\sigma|\,d\sigma.$$

Using Lemma 2, we derive from (4.12) that

$$\frac{1}{r}\int_{\mathbb{S}^{d-1}}|u_k(x+r\sigma)-u_k(x)-rV_k^{ac}(x)\cdot\sigma|\,d\sigma \le CM(|\nabla u|)(x).$$

Since for a.e. $x\in\mathbb{R}^d$,

$$\lim_{k\to+\infty}\frac{1}{r}\int_{\mathbb{S}^{d-1}}|u_k(x+r\sigma)-u_k(x)-rV_k^{ac}(x)\cdot\sigma|\,d\sigma$$
$$=\frac{1}{r}\int_{\mathbb{S}^{d-1}}|u(x+r\sigma)-u(x)-r\nabla^{ac}u(x)\cdot\sigma|\,d\sigma \quad \text{for a.e. } r>0,$$

it follows from the dominated convergence theorem that, for a.e. $x\in\mathbb{R}^d$,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-V_k^{ac}(x)\cdot h|}{|h|}\rho_n(|h|)\,dh$$
$$=\int_{\mathbb{R}^d}\frac{|u(x+h)-u(x)-\nabla^{ac}u(x)\cdot h|}{|h|}\rho_n(|h|)\,dh;$$

which is (4.13).

The proof of (4.15) follows similarly. We finally establish (4.14). Fix $\tau>0$ (arbitrary). We have

$$\int_0^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{\mathbb{S}^{d-1}}\int_0^r |V_k^s(x+s\sigma)|\,ds\,d\sigma \tag{4.19}$$
$$= \int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)\setminus B(x,\tau)} |V_k^s(y)|\,|y-x|^{1-d}\,dy$$
$$+ \int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,\tau)} |V_k^s(y)|\,|y-x|^{1-d}\,dy$$
$$+ \int_0^\tau r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)} |V_k^s(y)|\,|y-x|^{1-d}\,dy.$$

We have, for a.e. $r > 0$,

$$\lim_{k\to+\infty}\frac{1}{r}\int_{B(x,r)\setminus B(x,\tau)} |V_k^s(y)|\,|y-x|^{1-d}\,dy = \frac{1}{r}\int_{B(x,r)\setminus B(x,\tau)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy$$

and, by Lemma 2,

$$\frac{1}{r}\int_{B(x,r)\setminus B(x,\tau)} |V_k^s(y)|\,|y-x|^{1-d}\,dy \le CM(|\nabla u|)(x).$$

It follows from the dominated convergence theorem that

$$\lim_{k\to+\infty}\int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)\setminus B(x,\tau)} |V_k^s(y)|\,|y-x|^{1-d}\,dy \tag{4.20}$$
$$= \int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)\setminus B(x,\tau)} |\nabla^s u(y)|\,|y-x|^{1-d}\,dy.$$

On the other hand, by Lemma 2,

$$\int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,\tau)} |V_k^s u(y)|\,|y-x|^{1-d}\,dy \tag{4.21}$$
$$\le CM(|\nabla u|)(x)\int_\tau^\infty r^{d-1}\rho_n(r)\tau/r\,dr$$

and

$$\int_0^\tau r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)} |V_k^s(y)|\,|y-x|^{1-d}\,dy \tag{4.22}$$
$$\le CM(|\nabla u|)(x)\int_0^\tau r^{d-1}\rho_n(r)\,dr.$$

Since

$$\lim_{\tau\to 0}\Big(\int_\tau^\infty r^{d-1}\rho_n(r)\tau/r\,dr + \int_0^\tau r^{d-1}\rho_n(r)\,dr\Big) = 0,$$

we obtain (4.14) from (4.19), (4.20), (4.21), and (4.22). The proof is complete. □

5. Miscellaneous results

5.1. On a characterization of $W^{1,1}(\mathbb{R}^d)$

The following result deals with a "converse" of Proposition 1. It is due to D. Spector in [10, Theorem 1.3] and [11, Theorem 1.4] in the case $\rho_n(r) = d\varepsilon_n^{-d}\mathbb{1}_{(0,\varepsilon_n)}$ for a sequence of $(\varepsilon_n)\to 0_+$ and to A. Ponce and D. Spector [9, Remark 5] for a general sequence (ρ_n). The proof we present here is more direct.

Proposition 2. *Let $d\ge 1$ and $u\in L^1(\mathbb{R}^d)$. Then $u\in W^{1,1}(\mathbb{R}^d)$ if and only if there exists $U\in [L^1(\mathbb{R}^d)]^d$ such that*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x+h)-u(x)-U(x)\cdot h|}{|h|}\rho_n(|h|)\,dh\,dx = 0. \tag{5.1}$$

Proof. We already know that (5.1) holds for $u\in W^{1,1}(\mathbb{R}^d)$ with $\nabla u = U$ by Proposition 1. It remains to prove that if (5.1) holds, then $u\in W^{1,1}(\mathbb{R}^d)$. Let (χ_k) be a sequence of standard mollifiers. Define

$$u_k = u*\chi_k \quad\text{and}\quad U_k = U*\chi_k.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}&\frac{|u_k(x+h)-u_k(x)-U_k(x)\cdot h|}{|h|}\rho_n(|h|)\,dh\,dx\\ &=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}u(x+h-y)\chi_k(y)\,dy-\int_{\mathbb{R}^d}u(x-y)\chi_k(y)\,dy\\ &\qquad -\int_{\mathbb{R}^d}U(x-y)\cdot h\chi_k(y)\,dy\Big|\,|h|^{-1}\rho_n(|h|)\,dh\,dx.\end{aligned}$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}&\frac{|u_k(x+h)-u_k(x)-U_k(x)\cdot h|}{|h|}\rho_n(|h|)\,dh\,dx\\ &\le\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x+h-y)-u(x-y)-U(x-y)\cdot h|}{|h|}\chi_k(y)\,dy\rho_n(|h|)\,dh\,dx.\end{aligned}$$

A change of variables gives

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-U_k(x)\cdot h|}{|h|}\rho_n(|h|)\,dh\,dx$$
$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x+h)-u(x)-U(x)\cdot h|}{|h|}\rho_n(|h|)\,dh\,dx.$$

We derive from (5.1) that, for $k>0$,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-U_k(x)\cdot h|}{|h|}\rho_n(|h|)\,dh\,dx=0.$$

Since u_k is smooth, we obtain

$$U_k=\nabla u_k.$$

As $k\to+\infty$, $u_k\to u$ and $U_k\to U$ in $L^1(\mathbb{R}^d)$, so that $u\in W^{1,1}(\mathbb{R}^d)$ and $\nabla u=U$. □

5.2. The bounded domain case

Most of the above results hold when $\mathbb{R}^d$ is replaced by a smooth bounded domain Ω of $\mathbb{R}^d$. Define, for $p\geq 1$, $n\in\mathbb{N}$, and $u\in L^1_{loc}(\Omega)$,

$$D^{\Omega}_{n,p}(u)(x):=\int_{\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dy\quad\text{for a.e. } x\in\Omega. \tag{5.2}$$

Here is a typical result:

THEOREM 4. *Let $d\geq 1$, $p\geq 1$ and $u\in W^{1,p}(\Omega)$. Then*

$$\lim_{n\to+\infty}D^{\Omega}_{n,p}(u)(x)=\gamma_{d,p}|\nabla u|^p(x)\quad\textit{for a.e. } x\in\Omega. \tag{5.3}$$

PROOF. Let $\tilde u$ be an extension of u to $\mathbb{R}^d$ such that $\tilde u\in W^{1,p}(\mathbb{R}^d)$. Let $\omega\subset\subset\Omega$. We have, for $x\in\omega$,

$$D^{\Omega}_{n,p}(u)(x)=D_{n,p}(\tilde u)(x)-\int_{\mathbb{R}^d\setminus\Omega}\frac{|\tilde u(x)-\tilde u(y)|}{|x-y|}\rho_n(|x-y|)\,dy. \tag{5.4}$$

Applying Theorem 1 to $\tilde u$, we have for a.e. $x\in\omega$,

$$\lim_{n\to+\infty}D_{n,p}(\tilde u)(x)=\gamma_{d,p}|\nabla\tilde u|^p(x)=\gamma_{d,p}|\nabla u|^p(x). \tag{5.5}$$

Since ω is arbitrary, it suffices to prove that for a.e. $x \in \omega$,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d \setminus \Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x - y|} \rho_n(|x - y|)\, dy = 0. \tag{5.6}$$

Let $\varphi \in C^1(\mathbb{R}^d)$ be such that $\varphi = 1$ in $\mathbb{R}^d \setminus \Omega$ and $\varphi = 0$ in ω. Applying Theorem 1 to $\varphi\tilde{u}$, we obtain, for a.e. $x \in \omega$,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^d \setminus \Omega} \frac{|\tilde{u}(y)|}{|x - y|} \rho_n(|x - y|)\, dy = 0. \tag{5.7}$$

On the other hand, for a.e. $x \in \omega$,

$$\begin{aligned} &\lim_{n \to +\infty} \int_{\mathbb{R}^d \setminus \Omega} \frac{|\tilde{u}(x)|}{|x - y|} \rho_n(|x - y|)\, dy \\ &\quad = |u(x)| \lim_{n \to +\infty} \int_{\mathbb{R}^d \setminus \Omega} \frac{1}{|x - y|} \rho_n(|x - y|)\, dy = 0 \end{aligned} \tag{5.8}$$

Assertion (5.6) now follows from (5.7) and (5.8). □

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