

# SHAPE FROM BANDWIDTH: THE 2-D ORTHOGONAL PROJECTION CASE

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## ABSTRACT

Could bandwidth—one of the most classic concepts in signal processing—have a new purpose? In this paper, we investigate the feasibility of using bandwidth to infer shape from a single image. As a first analysis, we limit our attention to orthographic projection and assume a 2-D world.

We show that, under certain conditions, a single image of a surface, painted with a bandlimited texture, is enough to deduce the surface up to an equivalence class. This equivalence class is unavoidable, since it stems from surface transformations that are invisible to orthographic projections.

A proof of concept algorithm is presented and tested with both a simulation and a simple practical experiment.

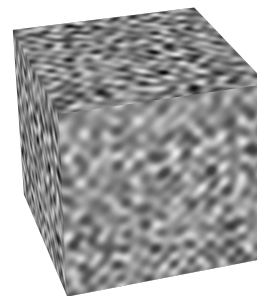
**Index Terms**— Shape estimation, bandwidth, warped bandlimited signals.

## 1. INTRODUCTION

A surface, with a bandlimited texture painted on it, appears sharper when viewed from a small grazing angle. In other words, we observe minimum local bandwidth when the surface is fronto-parallel to the viewing plane. Figure 1 demonstrates this effect by showing a rendering of a cube that has the same bandlimited texture painted on each side.

In this paper, we exploit this fact to develop an algorithm that approximates the shape of a painted surface, from a single image. As a first analysis, we limit our attention to the 2-D orthogonal projection case; i.e., our world is 2-D, therefore by surface we mean a curve in  $\mathbb{R}^2$ , and we observe an orthographic projection. With these assumptions, as depicted in Fig. 2, the camera observes a warped version of the texture painted on the surface. Therefore, if we paint the surface with a 1-D bandlimited texture, we observe a warped-bandlimited signal—a familiar beast to the signal processing savant (see for instance [1, 2, 3, 4]).

In the theory of warped-bandlimited signals, there is a conjecture [5, 6], which, if true, would lead to an important uniqueness result. In this paper, we use these ideas to develop a uniqueness result for the shape from bandwidth problem. More precisely, we show that, if the arc-length of the surface is equal to an entire function over a non-zero interval and if the surface has a stationary point in the field of view, there is a



**Fig. 1:** An image of bandlimited white noise “painted” onto a cube. Notice how the front face, which is closer to fronto-parallel to the image plane, has lower local bandwidth; i.e., it is less sharp. Similarly, the top face is less sharp than the side.

unique warping and bandlimited texture that could have generated the observation. Furthermore, there is a unique equivalence class of surfaces (to be defined) that corresponds to this warping.

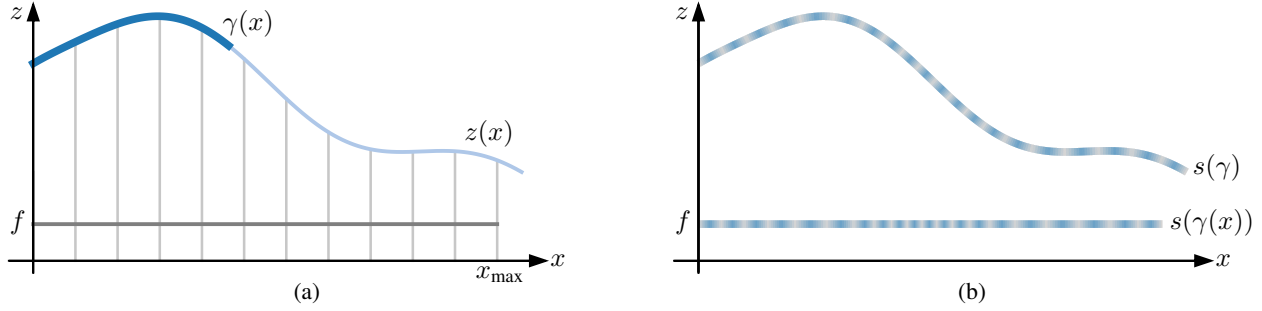
As well as this unicity result, we develop a very simple algorithm to approximate this equivalence class of surfaces, which we verify with a simple practical experiment. The main aim of this paper, however, is not to provide a practical algorithm; instead, we hope to show, for the first time, that bandwidth could be used to infer shape.

Of course, there are many existing cues used to sense shape and depth, based on both single and multiple images [7, 8, 9, 10]. Of this multitude of techniques, shape from shading and shape from texture are the most similar. However, the proposed technique is fundamentally different, since it relies on bandwidth, rather than lighting information or the orientation of texture elements, to infer surface normals.

## 2. SURFACE ESTIMATION

We assume the setup depicted in Fig. 2, as well as Lambertian reflection, and consider the process of estimating the surface, from the observed signal. The process can be broken down into two steps: given the observed signal, estimating the warping and, given the warping, estimating the surface<sup>1</sup>.

<sup>1</sup>We use the terms warping and arc-length interchangeable, since it is the arc-length,  $\gamma(x)$ , that warps the texture.



**Fig. 2:** Problem setup. A 1-D surface (i.e. curve) is viewed by an orthographic camera: (a) depicts a top-down view of the surface,  $z(x)$ , its arc-length,  $\gamma(x)$ , and the camera’s image plane,  $z = f$  for  $0 \leq x \leq x_{\max}$ ; (b) depicts the bandlimited texture,  $s(\gamma)$ , painted onto the surface, and the warped-bandlimited signal observed on the image plane,  $u(x) = s(\gamma(x))$ . In (b), one can observe how the local bandwidth of  $u = s \circ \gamma$  depends on the slope of the surface.

We start by showing that the latter problem of estimating the surface from the warping leads to an equivalence class of surface reconstructions; i.e., given a particular warping, there is no way to distinguish between a family of surfaces. Any reconstruction is thus up to this equivalence class.

Then, for the problem of estimating the warping from the observed signal, we develop a uniqueness result and a simple recovery algorithm.

### 2.1. Estimating the surface from the warping

To start with, suppose we had access to the warping  $\gamma(x)$ . Using the properties of arc-length, we could easily calculate the absolute value of the gradient of the surface:

$$\left| \frac{dz}{dx} \right| = \sqrt{\left( \frac{d\gamma}{dx} \right)^2 - 1}. \quad (1)$$

Reconstructing a 2-D surface from its normals is a key step in shape from shading, shape from texture and photometric stereo. Many successful techniques have been developed for this task: [11, 12], which typically enforce integrability. However, in our simplified setup, we have the 1-D version of this problem, where loss of integrability is not an issue. We can thus utilise the following simple estimate of the surface:

$$\hat{z}(x) = \int_0^x \left| \frac{dz}{d\bar{x}} \right| d\bar{x}.$$

From this reconstruction formula, we see that there are a number of ambiguities associated with reconstructing a surface from the warping. Let’s define the equivalence relation  $\sim$  such that  $z_1 \sim z_2$  if and only if

$$\int_0^x \left| \frac{dz_1}{d\bar{x}} \right| d\bar{x} = \int_0^x \left| \frac{dz_2}{d\bar{x}} \right| d\bar{x}.$$

This equivalence class stems from two types of transformation, which when applied to the surface, do not affect the observed image: translations in the  $z$ -direction and sign flips

of the gradient. Figures 3a-c show three surfaces, belonging to the same equivalence class<sup>2</sup>. The surface  $z_2$  is continuous and differentiable: it has been obtained, from  $z_1$ , by a global translation in the  $z$ -direction and a single sign flip of the gradient at its stationary point. The surface  $z_3$  is more complex: in addition to a global translation, there is a second translation, leading to the discontinuity, and a single sign flip of the gradient, which has created another non-differentiable point. However, the absolute value of its derivative, at all points where it exists, is the same as  $z_1$ ’s.

In what follows, we restrict the surface to be differentiable, so its derivative is continuous. In this case, we know that sign flips of the surface can only occur at its stationary points and there can only be a single global offset. If a warping leads to a  $|dz/dx|$ , which has  $N$  zeros, we can find  $2^{N+1}$  possible surfaces, corresponding to each possible sign-flip. One of these is the true surface, up to a global offset.

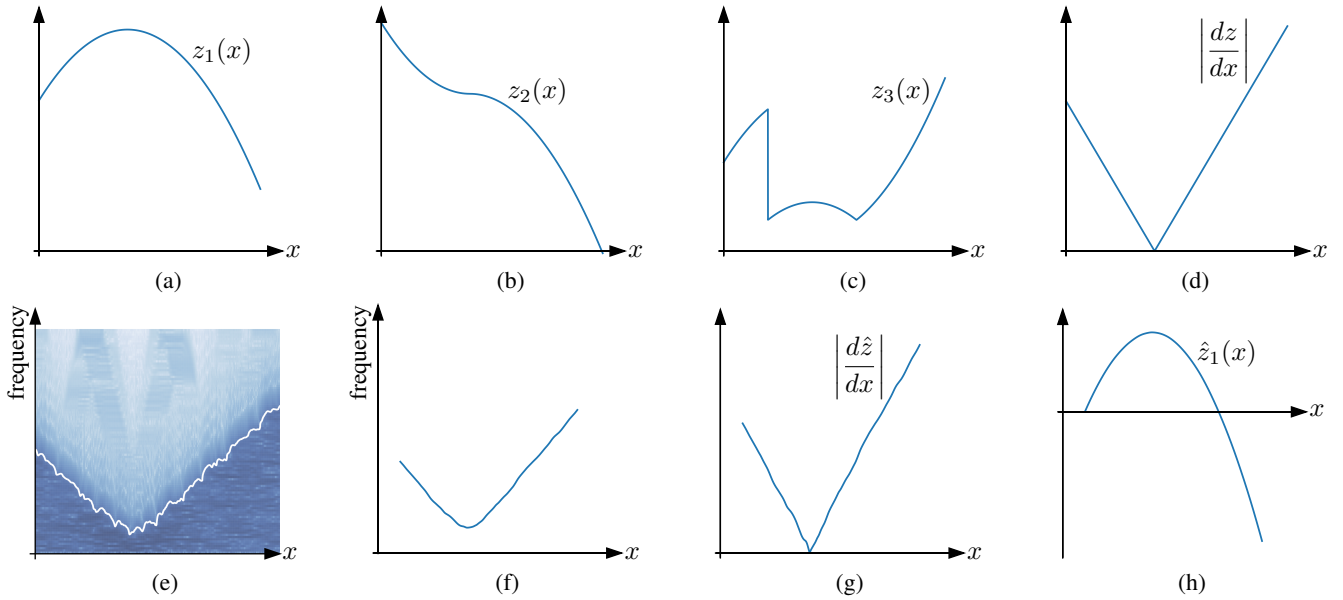
### 2.2. Estimating the warping

### 2.3. Uniqueness

The observed signal  $u$  belongs to the space of warped bandlimited signals, which can be defined as  $\mathcal{B} \circ \Gamma = \{s \circ \gamma : s \in \mathcal{B}, \gamma \in \Gamma\}$ ; here,  $\mathcal{B}$  is the space of bandlimited functions and  $\Gamma$  is the space of real-valued monotonic functions.

In [5], Clark conjectured that the only way a bandlimited signal could be warped into another bandlimited signal was if the warping was affine; i.e., if  $\gamma(x) = ax + b$ . If true, this conjecture would lead to an important uniqueness result: let  $\mathcal{B}_1$  be the space of unit bandwidth signals; then, given a signal  $u \in \mathcal{B} \circ \Gamma$ , Clark’s conjecture would imply that there could only be a unique pair,  $s \in \mathcal{B}_1$  and  $\gamma \in \Gamma$ , such that  $u = s \circ \gamma$ . Xia and Zhang proved a restricted version of the conjecture, in the case that the warping is entire [13]. Later, Azizi et al. used a peculiar counter-example, constructed by Y. Meyer, to show that the conjecture is, in fact, false [14]; however, in the

<sup>2</sup>For the time being, we neglect the effect of non-differentiable points.



**Fig. 3:** Surface estimation. (a) - (c) Three surfaces with the same absolute value of their derivative (at all points where it exists). (d) The absolute value of the derivative of these surfaces. (e) A spectrogram of the observed signal when  $z_1$  is painted with bandlimited white noise (log magnitude). 99.9% of the energy lies at those frequencies below the white line. (f) An estimate of the local bandwidth of the observed signal. This is obtained by convolving the white line in (e) with the spectrogram's window. (g) An approximation of the absolute value of the derivative of the surface. (h) The final surface estimation.

same paper, it is shown that no non-affine warping can warp all bandlimited signals into bandlimited signals.

Therefore, although the conjecture's uniqueness guarantee has been lost, we are left with two glimmers of light: from [13], we can obtain the uniqueness guarantee by further restricting the warping to be entire and, even if we do not further restrict the warping, in practice, we have never recovered the wrong  $s, \gamma$  pair<sup>3</sup>. Therefore, in all practical cases we believe that a warped bandlimited signal corresponds to a single warping, up to a scale factor. Furthermore, if we can correct for this unknown scale, it corresponds to a single element from the previously defined equivalence class of surfaces. As a direct extension of Xia and Zhang's result, we present the following lemma:

**Lemma 1** (Unique surface reconstruction). *Assume a non-constant texture  $s \in \mathcal{B}$  is painted on a surface  $z$ , with a stationary point in the field of view ( $0 \leq x \leq x_{\max}$ ,  $x_{\max} > 0$ ). Further, assume that  $\gamma$ , the arc-length of  $z$ , is equal to an entire function for a non-zero interval in the field of view. The signal  $u(x) = s(\gamma(x))$  could only have been generated by the original texture  $s$  and arc-length  $\gamma$ . Furthermore, this arc-length corresponds to a unique element from the previously defined equivalence class of surfaces.*

*Proof.* Let  $B \neq 0$  be the bandwidth of  $s$ . The local bandwidth of  $u$  is greater than or equal to  $B$ , with equality at stationary points of  $z$  (i.e. where the surface is fronto-parallel to the image plane). Since we observe a stationary point,  $u$  can be used to infer  $B$ .

<sup>3</sup>The fact that no non-affine warping can warp all bandlimited signals into bandlimited signals in some way formalises that non-uniqueness is rare. Clearly, it would be desirable to understand the measure of these occurrences to be more precise about the probability of recovery.

Now, since  $\gamma$  is entire, from [13, Theorem 1],  $u$  is bandlimited if and only if  $\gamma$  is affine. Now suppose there exists  $s_1, s_2$  with bandwidth  $B$  and  $\gamma_1, \gamma_2 \in \Gamma$ , such that  $u(x) = s_1(\gamma_1(x)) = s_2(\gamma_2(x))$ . The arc-lengths are monotonic so their inverses exist. Therefore,  $s_1(x) = s_2(\gamma_2(\gamma_1^{-1}(x))) = s_2(\gamma_3(x))$ , where  $\gamma_3(x) = \gamma_2(\gamma_1^{-1}(x))$ . Clearly,  $\gamma_3$  is monotonic and, since the derivative of  $\gamma_1$  is non-zero, it follows from the Lagrange inversion theorem that  $\gamma_1^{-1}$  is analytic; hence,  $\gamma_3$  equals an entire function on an interval. Therefore, from [13, Theorem 1],  $\gamma_3$  is affine. However, the only way an affine function can warp a bandlimited signal into another bandlimited signal of the same bandwidth is if  $\gamma_3(x) = x$ . Therefore,  $\gamma_1 = \gamma_2$  and  $s_1 = s_2$ .  $\square$

## 2.4. Recovery

We now propose a simple algorithm to demonstrate the feasibility of recovering the shape of a surface from its bandwidth.

To estimate the local bandwidth, let us define the following approximation of  $u$  around an arbitrary point  $x_0$ :

$$u(x; x_0) := s(\gamma(x_0) + \gamma'(x_0)(x - x_0)).$$

Furthermore, let us assume that this approximation is good within a window,  $w$ , of  $x_0$ :

$$w(x - x_0)u(x) \simeq u_w(x; x_0) := w(x - x_0)u(x; x_0).$$

Taking the Fourier transform yields

$$U_w(\omega; x_0) := \mathcal{F}[u_w(x; x_0)] = \mathcal{F}[w(x - x_0)] * \mathcal{F}[u(x; x_0)],$$

where  $\mathcal{F}[w(x - x_0)] = W(\omega)e^{-j\omega x_0}$  and

$$\mathcal{F}[u(x; x_0)] = \frac{1}{\gamma'(x_0)} S\left(\frac{\omega}{\gamma'(x_0)}\right) e^{j\omega(\gamma(x_0)/\gamma'(x_0) - x_0)}.$$

Therefore, if  $u(x)$  is bandlimited with bandwidth  $B$ ,  $u(x; x_0)$  has bandwidth  $\gamma'(x_0)B$ . Furthermore, if the window has a large enough cutoff frequency,  $u_w(x; x_0)$  has the same bandwidth. Throughout the paper, we have asserted that the local bandwidth of the signal is dependant on the slope of the surface. We now see that, in the 2-D orthographic case, it is proportional to the derivative of the arc-length, which, using (1), can easily be converted to the absolute value of the derivative of the surface.

Algorithm 1 summarises a simple process to estimate the surface using these ideas and Figs. 3e-3h show a simple reconstruction example. First, as depicted in Fig. 3e, a spectrogram of the observed signal is computed, using a single sample step between adjacent windows, and the bandwidth of each windowed region is approximated. This local bandwidth estimation is convolved with the spectrogram window, resulting in a local bandwidth estimate, like Fig. 3f. The reduced signal length is due to this convolution. We would like to divide the local bandwidth estimate by  $B$ , the original bandwidth of  $s$ , to obtain an approximation of the derivative of the warping. Although this is unknown, we can, like in the uniqueness result, make the assumption that  $z$  has a stationary point in the field of view. Then, the minimum observed local bandwidth is equal to the original bandwidth of  $s$  and, as depicted in Fig. 3g, we can correct to approximate the derivative of the arc-length. Finally, we can reconstruct the surface, up to the previously stated equivalence class.

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#### Algorithm 1 Shape from bandwidth

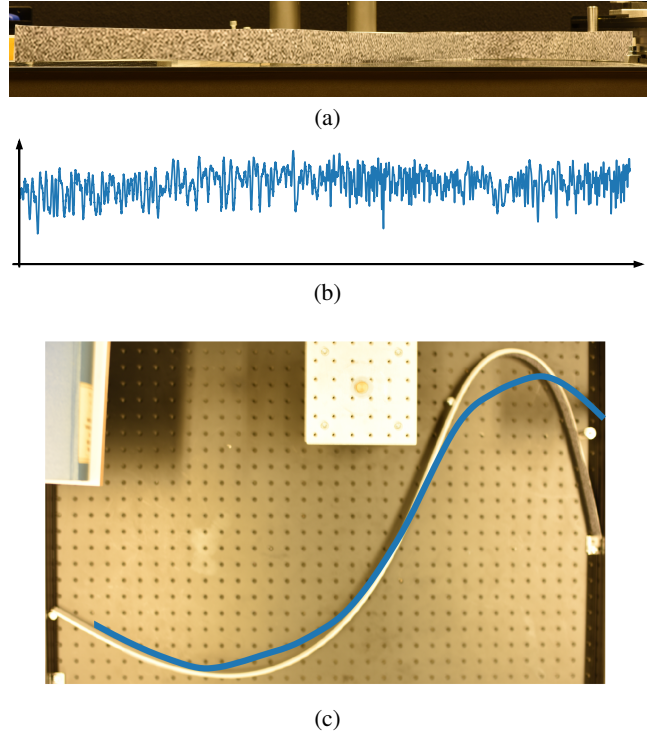
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- 1: Compute the spectrogram of the observed signal  $u$ .
  - 2: For each spatial value, compute the local bandwidth (taken to be the minimum frequency that contains almost all the energy; e.g., 99% of the energy). Convolve this estimate with the spectrogram window.
  - 3: Approximate  $d\gamma/dx$ , by dividing the local bandwidth estimate by the minimum observed bandwidth.
  - 4: Find all local minimum of  $d\gamma/dx$  that are approximately equal to one. These points are candidates for sign flips.
  - 5: Approximate the absolute value of derivative of the surface as  $|dz/dx| = \sqrt{(d\gamma/dx)^2 - 1}$ .
  - 6: Reconstruct all surfaces in the equivalence class by integrating  $|dz/dx|$ , for each possible sign flip.
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### 3. EXPERIMENTS

As seen in Figs. 3e-3h, Algorithm 1 works for simple synthetic simulations. The main inaccuracies come from the difficulty of estimating the true bandwidth of the original texture. This occurs because, in practice, we only have a finite number of samples around fronto-parallel regions; however, this could be improved by spatially adapting the window length.

To further test the algorithm, we performed a very simple practical experiment. We printed and attached bandlimited white noise to a curved surface and captured a single image



**Fig. 4:** Practical experiment. (a) An image of bandlimited white noise “painted” onto a surface. (b) A scanline of the above image. (c) A top-down view of the experimental setup, with the estimated surface superimposed on top.

using a Nikon D810 DSLR camera. Figure 4a shows the acquired image and Fig. 4b depicts one scanline of this image, which was input to the algorithm. Figure 4c depicts a top-down view of the setup with the estimated surface superimposed. Since the projection was perspective, an unknown scale ambiguity is introduced, which we have manually corrected. However, despite this model mismatch and the simplicity of Algorithm 1, the result is surprising accurate.

### 4. CONCLUSION AND FUTURE WORK

In this paper, we have, for the first time, utilised bandwidth to retrieve the shape of a painted surface from a single image.

As a first analysis, we only considered the 2D orthographic case and the extension to 3D and perspective projection are obvious topics of future work. Moreover, there are many unanswered theoretical questions. In particular, we have assumed that we have access to continuous-time signals and neglected any sampling effects. To be more precise, we could use the fact that, since the windowed signals,  $u_w(x; x_0)$ , have bandwidth  $\gamma'(x_0)B$ , they are completely described by  $2\gamma'(x_0)B$  samples. Understanding these types of sampling effects, will help us adaptively choose the window size, which is one of many modifications that needs to be made to develop a more practical reconstruction algorithm.

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