On Vertices and Facets of Combinatorial 2-Level Polytopes

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Abstract. 2-level polytopes naturally appear in several areas of mathematics, including combinatorial optimization, polyhedral combinatorics, communication complexity, and statistics.
We investigate upper bounds on the product of the number of facets $f_{d-1}(P)$ and the number of vertices $f_0(P)$, where d is the dimension of a 2-level polytope P. This question was first posed in [3], where experimental results showed $f_0(P)f_{d-1}(P) \leq d2^{d+1}$ up to $d = 6$.
We show that this bound holds for all known (to the best of our knowledge) 2-level polytopes coming from combinatorial settings, including stable set polytopes of perfect graphs and all 2-level base polytopes of matroids. For the latter family, we also give a simple description of the facet-defining inequalities. These results are achieved by an investigation of related combinatorial objects, that could be of independent interest.

1 Introduction

Let $P \subseteq \mathbb{R}^d$ be a polytope. We say that P is 2-*level* if, for all facets F of P, all the vertices of P that are not vertices of F lie in the same translate of the affine hull of F. Equivalently, P is 2-level if and only if it has theta-rank 1 [9], or all its pulling triangulations are unimodular [16], or it has a slack matrix with entries that are only 0 or 1 [3]. Those last three definitions appear in papers from the semidefinite programming, statistics, and polyhedral combinatorics communities respectively, showing that 2-level polytopes naturally arise in many areas of mathematics.

Arguably, the most important reasons 2-level polytopes are interesting for researchers in polyhedral combinatorics and theoretical computer science are their connections with the theory of *linear extensions* and the prominent *log-rank conjecture* in communication complexity, since they generalize stable set polytopes of perfect graphs.

Because of all the reasons above, a complete understanding of 2-level polytopes would be desirable. Unfortunately, despite an increasing number of studies [3, 9–11], such an understanding has not been obtained yet: we do not have e.g. any decent bound on the number of d-dimensional 2-level polytopes or on their linear extension complexity, nor do we have a structural theory of their slack matrices, of the kind that has been developed for totally unimodular matrices (see e.g. [14]). On the positive side, many properties of 2-level polytopes have

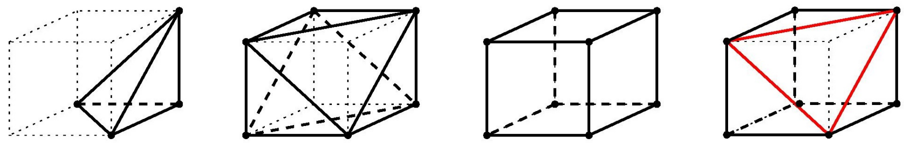

Fig. 1: The first three polytopes (the simplex, the cross-polytope and the cube) are clearly 2-level. The fourth one is not 2-level, due to the highlighted facet.

been shown. For instance, each d-dimensional 2-level polytope is affinely isomorphic to a 0/1 polytope [9], hence it has at most 2^d vertices. Interestingly, one can show that a d-dimensional 2-level polytope has at most 2^d facets [9]. This makes 2-level polytopes quite different from "random" 0/1 polytopes, that have $(d/\log d)^{\Theta(d)}$ facets [2]. In fact, 2-level polytopes seem to be a very restricted subclass of 0/1 polytopes, as experimental results from [3] have shown.

The goal of this paper is to shed some light on the relationship between the number of vertices and the number of facets of a 2-level polytope. Experimental evidence from [3] up to dimension 6 suggests the existence of a trade-off between those two numbers, in a very strong sense: a d-dimensional 2-level polytope can have at most 2^d vertices and facets, but their product seems to be upper bounded by a number much smaller than 2^{2d}. More formally, for a polytope P and $i \in \mathbb{Z}_+$, let $f_i(P)$ be the number of its i-dimensional faces. The following was posed as a question in [3], and we turn it here into a conjecture.

Conjecture 1 (Vertex/facet trade-off). *Let P be a d-dimensional 2-level polytope. Then $f_0(P)f_{d-1}(P) \leq d2^{d+1}$. Moreover, equality is achieved if and only if P is affinely isomorphic to the cross-polytope or the cube.*

It is immediate to check that the cube and the cross-polytope (its polar) indeed verify $f_0(P)f_{d-1}(P) = d2^{d+1}$. The conjecture above essentially states that those basic polytopes maximize $f_0(P)f_{d-1}(P)$ among all 2-level polytopes of a fixed dimension.

Conjecture 1 has an interesting interpretation as an upper bound on the "size" of slack matrices of 2-level polytopes, since $f_0(P)$ (resp. $f_{d-1}(P)$) is the number of columns (resp. rows) of the (smallest) slack matrix of P. Many fundamental results on linear extensions of polytopes (including the celebrated upper bound on the extension complexity of the stable set polytope of perfect graphs [17]) are based on properties of slack matrices. We believe therefore that answering Conjecture 1 would be an interesting step towards a better understanding of 2-level polytopes.

Our contribution and organization of the paper. We show that Conjecture 1 holds true for all known classes (to the best of our knowledge) of

2-level polytopes coming from combinatorial settings. In most cases, this is deduced from properties of associated combinatorial objects, that are also shown in the current paper and we believe could be of independent interest. Detailed results and the organization of the paper are as follows. We introduce some common definitions and techniques in Section 2: those are enough to show that Conjecture 1 holds for Hanner polytopes. In Section 3 we give a simple but surprisingly sharp upper bound on the product of the numbers of stable sets and cliques of a graph. This is used to show that the conjecture holds for stable set polytopes of perfect graphs, order polytopes, and Hansen polytopes. In Section 4, we give a non-redundant description of facets of the base polytope of the 2-sum of matroids in terms of the facets of the base polytopes of the original matroids. This is used to obtain a compact description (in the original space) of 2-level base polytopes of matroids and a proof of Conjecture 1 for this class. In Section 5, we prove the conjecture for the cycle polytopes of certain binary matroids, which generalizes all cut polytopes that are 2-level. In Section 6 we give examples showing that Conjecture 1 does not trivially hold for all "well-behaved" 0/1 polytopes. **NOTE:** Because of space constraints, most proofs and some definitions are deferred to the journal version of the paper.

Related work. We already mentioned the paper [3] that provides an algorithm based on the enumeration of closed sets to list all 2-level polytopes, as well as papers [9, 11, 16] where equivalent definitions and/or families of 2-level polytopes are given. Among other results, in [9] it is shown that the stable set polytope of a graph G is 2-level if and only if G is perfect. A characterization of all base polytopes of matroids that are 2-level is given in [11], building on the decomposition theorem for matroids that are not 3-connected (see e.g. [13]).

2 Basics

We let $\mathbb{R}_+$ be the set of non-negative real numbers. For a set S and an element e, we denote by $A+e$ and $A-e$ the sets $A \cup \{e\}$ and $A \setminus \{e\}$, respectively. For a point $x \in \mathbb{R}^I$, where I is an index set, and a subset $J \subseteq I$, let $x(J) = \sum_{i \in J} x_i$.

For basic definitions about polytopes and graphs, we refer the reader to [18] and [6], respectively. The d-dimensional *cube* is $[-1,1]^d$, and the d-dimensional *cross-polytope* is its corresponding polar. Taking the polar of a polytope is a dual operation, that produces a polytope of the same dimension, where the number of vertices and the number of facets are swapped. Thus, a polytope and its polar will simultaneously satisfy or not satisfy Conjecture 1. A 0/1 polytope is the convex hull of a subset of the vertices of $[0,1]^d$. The following facts will be used many times:

Lemma 2. *[9] Let P be a 2-level polytope of dimension d. Then*

1. $f_0(P), f_{d-1}(P) \leq 2^d$.
2. *Any face of P is again a 2-level polytope.*

One of the most common operation with polytopes is the *Cartesian product*. Given two polytopes $P_1 \subseteq \mathbb{R}^{d_1}$, $P_2 \subseteq \mathbb{R}^{d_2}$, their Cartesian product is $P_1 \times P_2 = \{(x, y) \in \mathbb{R}^{d_1+d_2} : x \in P_1, y \in P_2\}$. This operation will be useful to us as it preserves the bound of Conjecture 1.

Lemma 3. *If two 2-level polytopes P_1 and P_2 satisfy Conjecture 1, then so does their Cartesian product.*

2.1 Hanner Polytopes

We start off with an easy example. *Hanner* polytopes can be defined as the smallest family that contains the $[-1, 1]$ segment of dimension 1, and is closed under taking polars and Cartesian products. These polytopes are 2-level and centrally symmetric, and from the previous observations it is straightforward that they verify Conjecture 1.

Theorem 4. *Hanner polytopes satisfy Conjecture 1.*

3 Graph Theoretical 2-Level Polytopes

We present a general result on the number of cliques and stable sets of a graph. Proofs of all theorems from the current section will be based on it.

Theorem 5 (Stable set/clique trade-off). *Let $G = (V, E)$ be a graph on n vertices, $\mathcal{C}$ its family of non-empty cliques, and $\mathcal{S}$ its family of non-empty stable sets. Then $|\mathcal{C}||\mathcal{S}| \leq n(2^n - 1)$. Moreover, equality is achieved if and only if G or its complement is a clique.*

Proof. Consider the function $f : \mathcal{C} \times \mathcal{S} \to 2^V$, where $f(C, S) = C \cup S$. For a set $W \subset V$, we bound the size of its pre-image $f^{-1}(W)$. This will imply a bound for $|C \times \mathcal{S}| = \sum_{W \subset V} |f^{-1}(W)|$. If W is a singleton, the only pair in its pre-image is (W, W). For $|W| \geq 2$, we claim that $|f^{-1}(W)| \leq 2|W|$.

There are at most $|W|$ intersecting pairs (C, S) in $f^{-1}(W)$. This is because the intersection must be a single element, $C \cap S = \{v\}$, and once it is fixed every element adjacent to v must be in C, and every other element must be in S.

There are also at most $|W|$ disjoint pairs in $f^{-1}(W)$, as we prove now. Fix one such disjoint pair (C, S), and notice that both C and S are non-empty proper subsets of W. All other disjoint pairs (C', S') are of the form $C' = C \setminus A \cup B$ and $S' = S \setminus B \cup A$, where $A \subseteq C$, $B \subseteq S$, and $|A|, |B| \leq 1$. Let X (resp. Y) denote the set formed by the vertices of C (resp. S) that are anticomplete to S (resp. complete to C). Clearly, either X or Y is empty. We settle the case $Y = \emptyset$, the other being similar. In this case $\emptyset \neq A \subseteq X$, so $X \neq \emptyset$. If $X = \{v\}$, then $A = \{v\}$ and we have $|S| + 1$ choices for B, with $B = \emptyset$ possible only if $|C| \geq 2$, because we cannot have $C' = \emptyset$. This gives at most $1 + |S| + |C| - 1 \leq |W|$ disjoint pairs (C', S') in $f^{-1}(W)$. Otherwise, $|X| \geq 2$ forces $B = \emptyset$, and the number of such pairs is at most $1 + |X| \leq 1 + |C| \leq |W|$.

We conclude that $|f^{-1}(W)| \leq 2|W|$, or one less if W is a singleton. Thus

$$|\mathcal{C} \times \mathcal{S}| \leq \sum_{k=0}^{n} 2k\binom{n}{k} - n = n2^n - n,$$

where the (known) fact $\sum_{k=0}^{n} 2k\binom{n}{k} = n2^n$ holds since

$$n2^n = \sum_{k=0}^{n}(k + (n-k))\binom{n}{k} = \sum_{k=0}^{n} k\binom{n}{k} + (n-k)\binom{n}{n-k} = 2\sum_{k=0}^{n} k\binom{n}{k}.$$

The bound is clearly tight for $G = K_n$ and $G = \overline{K_n}$. For any other graph, there is a subset W of 3 vertices that induces 1 or 2 edges. In both cases, $|f^{-1}(W)| = 5 < 2|W|$, hence the bound is loose. □

For a graph $G = (V, E)$, its stable set polytope STAB(G) is the convex hull of the characteristic vectors of all stable sets in G. It is known that STAB(G) is 2-level if and only if G is a *perfect graph* [9], or equivalently [5] if and only if

$$\text{STAB}(G) = \{x \in \mathbb{R}^V_+ : x(C) \leq 1 \text{ for all maximal cliques } C \text{ of } G\}.$$

Theorem 6. *Stable set polytopes of perfect graphs satisfy Conjecture 1.*

Given a $(d-1)$-dimensional polytope P, the *twisted prism* of P is the d-dimensional polytope defined as the convex hull of $\{(x, 1) : x \in P\}$ and $\{(-x, -1) : x \in P\}$. For a perfect graph G with $d-1$ vertices, its *Hansen* polytope Hans(G) is defined as the twisted prism of STAB(G). Hansen polytopes are 2-level and centrally symmetric.

Theorem 7. *Hansen polytopes satisfy Conjecture 1.*

Given a poset P on $[d]$, with order relation $<_P$, its order polytope $\mathcal{O}(P)$ is:

$$\mathcal{O}(P) = \{x \in [0,1]^d : \ x_i \leq x_j \ \forall \ i <_P j\}.$$

A subset $I \subseteq P$ is called an *upset* if $x \in I$ and $x <_P y$ imply $y \in I$. In [15] the following characterization of vertices of an order polytope is given.

Lemma 8. *The vertices of $\mathcal{O}(P)$ are the characteristic vectors of upsets of P. In particular, the number of vertices of $\mathcal{O}(P)$ is the number of upsets of P.*

From this result it is clear that $\mathcal{O}(P)$ is a 2-level polytope. Indeed, if all vertices of a polytope have 0/1 coordinates and all facet-defining inequalities can be written as $0 \leq c^T x \leq 1$ for integral vectors c, then the polytope is 2-level.

Given a poset P, we say that j *covers* i in P if $i <_P j$ and there is no k in P such that $i <_P k <_P j$. We say that i, j is a *covering pair* if j covers i or i covers j. P can be described by a graph called *Hasse Diagram* $G_P([d], E)$, with $ij \in E$ if and only i, j is a covering pair. This graphical representation and Theorem 5 are the main ingredients to prove the following.

Theorem 9. *Order polytopes satisfy Conjecture 1.*

4 2-Level Matroid Base Polytopes

We now give a non-redundant description of the base polytopes of the 2-sum $M_1 \oplus_2 M_2$ of matroids in terms of the facets of the base polytopes of M_1 and M_2. We then focus on 2-level matroids. We give an explicit description of the associated base polytopes, and prove that they verify Conjecture 1. For basic definitions and facts about matroids we refer to [13].

4.1 The Base Polytope of the 2-Sum of Matroids

We identify a matroid M by the couple $(E, \mathcal{B})$, where $E = E(M)$ is its ground set, and $\mathcal{B} = \mathcal{B}(M)$ is its base set. Given $M = (E, \mathcal{B})$ and a set $F \subseteq E$, the *restriction* $M|F$ is the matroid with ground set F and independent sets $\mathcal{I}(M|F) = \{I \in \mathcal{I}(M) : I \subseteq F\}$; and the *contraction* M/F is the matroid with ground set $M \setminus F$ and rank function $r_{M/F}(A) = r_M(A \cup F) - r_M(F)$. For an element $e \in E$, the *removal of* e is $M - e = M|(E - e)$. A set $F \subseteq E$ is a *flat* if it is maximal for its rank, i.e. $r(F) < r(F + x)$ for all $x \in E \setminus F$.

Consider matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$, with non-empty base sets. If $E_1 \cap E_2 = \emptyset$, we can define the *direct sum* $M_1 \oplus M_2$ as the matroid with ground set $E_1 \cup E_2$ and base set $\mathcal{B}_1 \times \mathcal{B}_2$. If, instead, $E_1 \cap E_2 = \{p\}$, where p is neither a loop nor a coloop in M_1 or M_2, we let the *2-sum* $M_1 \oplus_2 M_2$ be the matroid with ground set $E_1 \cup E_2 - p$, and base set $\{B_1 \cup B_2 - p \ : \ B_i \in \mathcal{B}_i \text{ for } i = 1, 2 \text{ and } p \in B_1 \triangle B_2\}$. A matroid is *connected* if it cannot be written as the direct sum of two matroids, each with fewer elements.

The *base polytope* $B(M) \subseteq \mathbb{R}^E$ of a matroid $M = (E, \mathcal{B})$ is given by the convex hull of the characteristic vectors of its bases. For a matroid M, the following is known to be a description of $B(M)$.

$$B(M) = \{x \in [0,1]^E : x(F) \leq r(F) \text{ for } F \subseteq E; \text{ and } x(E) = r(E) \}. \qquad (1)$$

When M is connected [7] give the following characterization of the facet-defining inequalities for (1). (We report the statement as it appears in [11].)

Theorem 10. *Let $M = (E, \mathcal{B})$ be a connected matroid. For every facet F of $B(M)$ there is a unique $S \subseteq E$, $S \neq \emptyset$, such that $F = B(M) \cap \{x \in \mathbb{R}^E : x(S) = r(S)\}$. Moreover, a non-empty subset S gives rise to a facet of $B(M)$ if and only if one of the these two conditions holds:*

1. *S is a flat such that $M|S$, M/S are connected;*
2. *$S = E - e$ for some $e \in E$ such that $M|S$, M/S are connected.*

The subsets S in 1. are called *flacets*, and they are in 1-to-1 correspondence with the facet-defining inequalities in (1) of the form $x(S) \leq r(S)$, including $x_e \leq 1$ for $e \in E$. For $S = E - e$ satisfying the conditions in 2., we refer to element e as defining a *non-negativity* facet. Indeed it can be easily seen that it defines the same facet as $x_e \geq 0$.

Throughout the rest of the section, we assume that $M_1(E_1, \mathcal{B}_1)$, $M_2(E_2, \mathcal{B}_2)$ are connected matroids, with, $E_1 \cap E_2 = \{p\}$, and we define $M = M_1 \oplus_2 M_2$.

It is well known that under these assumptions M is also connected. By the arguments above, characterizing $B(M)$ essentially boils down to characterizing flacets of $M_1 \oplus_2 M_2$.

Theorem 11. *Let F be a flacet of M. One of the following holds:*

1. $F = E_i \cup F' - p$, *where F' is a flacet of M_j containing p, and $i \neq j \in \{1, 2\}$.*
2. F *is a flacet of M_i not containing p for some $i \in \{1, 2\}$.*
3. $F = E_i - p$ *for some $i \in \{1, 2\}$.*

Conversely, let F_1 be a flacet of M_1, $F_1 \neq \{p\}$. Then

1. *If $p \in F_1$, $F = E_2 \cup F_1 - p$ is a flacet of M.*
2. *If $p \notin F_1$, F_1 is a flacet of M.*
3. *If M_2/p and $M_1 - p$ are connected, then $E_1 - p$ is a flacet of M.*

We remark that a statement similar to the first half of Theorem 11 for an analogous definition of 2-sum and flacets appeared in [4]. However, we were not able to convince ourselves that the proof from [4] is complete, and some of its statements appear to be wrong.

Corollary 12. *The following is a non-redundant description of $B(M)$:*

$$
\begin{array}{lll}
B(M) = \{x \in \mathbb{R}^E : & & \\
\quad x_e & \geq 0 & e \in E_i - p : M_i - e \text{ connected}, i = 1, 2 \\
\quad x(E_i \cup F - p) & \leq r(E_i \cup F - p) & F \text{ flacet of } M_j \ : \{p\} \subsetneq F, i \neq j \in \{1, 2\} \\
\quad x(F) & \leq r(F) & F \text{ flacet of } M_i \ : p \notin F, i \in \{1, 2\} \\
\quad x(E_i - p) & \leq r(E_i - p) & \text{if } M_i - p,\ M_j/p \text{ connected}, i \neq j \in \{1, 2\} \\
\quad x(E) & = r(E)\}. &
\end{array}
\tag{2}
$$

Corollary 13. *Let us write $f(M) = f_{d-1}(B(M))$, and similarly for M_1, M_2. Then $f(M_1) + f(M_2) - 2 \leq f(M) \leq f(M_1) + f(M_2) + 2$.*

4.2 Linear Description of 2-Level Matroid Base Polytopes

A matroid $M(E, \mathcal{B})$ is *uniform* if $\mathcal{B} = \binom{E}{k}$, where k is the rank of M. We denote the uniform matroid with n elements and rank k by $U_{n,k}$. Notice that, if M_1 and M_2 are uniform matroids with $|E(M_1) \cap E(M_2)| = 1$, then $M_1 \oplus_2 M_2$ is unique up to isomorphism, for any possible common element. Let $\mathcal{M}$ be the class of matroids whose base polytope is 2-level. $\mathcal{M}$ has been characterized in [11]:

Theorem 14. *The base polytope of a matroid M is 2-level if and only if M can be obtained from uniform matroids through a sequence of direct sums and 2-sums.*

The following lemma implies that we can, when looking at matroids in $\mathcal{M}$, decouple the operations of 2-sum and direct sum.

Lemma 15. *Let M be a matroid obtained by applying a sequence of direct sums and 2-sums from the matroids $M_1, \dots, M_k$. Then $M = M'_1 \times M'_2 \times ... \times M'_t$, where each of the M'_i is obtained by repeated 2-sums from some of the matroids $M_1, \dots, M_k$.*

Since the base polytope of the direct sum of matroids is the Cartesian product of the base polytopes, to obtain a linear description of $B(M)$ for $M \in \mathcal{M}$, we can focus on base polytopes of connected matroids obtained from the 2-sums of uniform matroids. A sequence of 2-sums can be represented via a tree (see Figure 2): the following is a version of [13, Proposition 8.3.5] tailored to our needs.

Theorem 16. *Let M be obtained by a sequence of 2-sums operations from matroids $M_1, \dots, M_t$. Then there is a t-vertex tree $T = T(M)$ with edges labelled $e_1, \dots, e_{t-1}$ and vertices labelled $M_1, \dots, M_t$, such that*

1. $E(M_1) \cup E(M_2) \cup \dots \cup E(M_t) = E(M) \cup \{e_1, \dots, e_{t-1}\}$;
2. *if the edge e_i joins the vertices M_{j_1} and M_{j_2}, then* $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$;
3. *if no edge joins the vertices M_{j_1} and M_{j_2}, then* $E(M_{j_1}) \cap E(M_{j_2}) = \emptyset$.

Moreover, M is the matroid that labels the single vertex of the tree $T/e_1, \dots, e_{t-1}$ at the conclusion of the following process: contract the edges $e_1, \dots, e_{t-1}$ of T one by one in order; when e_i is contracted, its ends are identified and the vertex formed by this identification is labeled by the 2-sum of the matroids that previously labeled the ends of e_i.

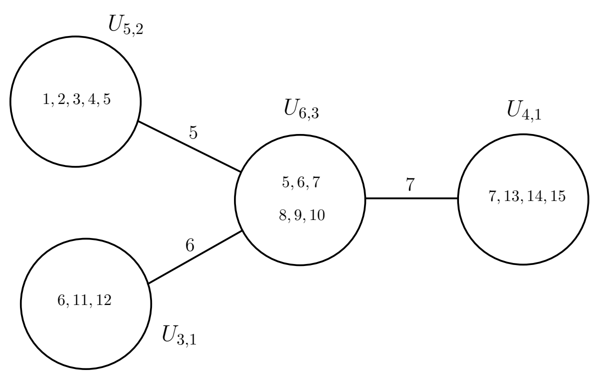

Fig. 2: An example of the tree structure of a matroid M that is a 2-sum of uniform matroids. Note that the elements $5, 6, 7$ will not be present in the ground set of M. From the picture it is easy to see that M is a matroid with 12 elements and rank 4. One basis of M is e.g. $\{1, 8, 9, 14\}$.

Observation 17. *If $M \in \mathcal{M}$ is connected and non-uniform, we can assume without loss of generality that every node in its tree structure given by Theorem*

16 is a uniform matroid with at least 3 elements. Each of those uniform matroid has no flacets besides its singletons.

For a connected matroid $M(E,\mathcal{B}) \in \mathcal{M}$, Theorem 16 reveals a tree structure $T(M)$, where every node represents a uniform matroid, and every edge represents a 2-sum operation. We now give a simple description of the associated base polytope. Let a be an edge of $T(M)$. The removal of a breaks T into 2 connected components C_a^1 and C_a^2. Let E_a^1 (resp. E_a^2) be the set of elements from E that belong to uniform matroids from C_a^1 (resp. C_a^2). All inequalities needed to describe $B(M)$ are the "trivial" inequalities $0 \leq x \leq 1$, plus $x(F) \leq r(F)$, where $F = E_a^1$ or E_a^2 for some edge a of $T(M)$. Thus the number of inequalities is linear in the number of elements.

Theorem 18. *Let $M = (E,\mathcal{B}) \in \mathcal{M}$ be a connected matroid obtained as 2-sums of uniform matroids $U_1 = U_{n_1,k_1}, \dots, U_t = U_{n_t,k_t}$. Let $T(N,A)$ be the tree structure of M according to Theorem 16. For each $a \in A$, let C_a^1, C_a^2, E_a^1, E_a^2 be defined as above. Then*

$$
\begin{aligned}
B(M) = \{x \in \mathbb{R}^E : \quad & x \geq 0 \\
& x \leq 1 \\
x(F) &\leq r(F) \quad \textit{for } F = E_a^i \textit{ for some } i \in \{1,2\} \textit{ and } a \in A, \\
x(E) &= r(E)\}.
\end{aligned}
$$

Moreover, if $F = E_a^i$ for $i \in \{1,2\}$ and some $a \in A$, then $r(F) = 1 - |C_a^i| + \sum_{j: U_j \in \mathcal{C}_a^i} k_j$.

Proof. Let us call a subset $C \subseteq N$ a *valid component* for T if $C = C_a^i$ for some $i \in \{1,2\}$ and $a \in A$, and denote the set of all valid components of T by $\mathcal{F}$. Each connected subtree of $T(N,A)$ represents a connected matroid obtained as 2-sums of uniform matroids. Thus, we can prove the theorem by induction on t. The statement on the rank is immediate. For $t = 1$, $\mathcal{F}$ is empty and thanks to Observation 17, the remaining inequalities are enough to describe $B(M)$. Now let $t > 1$. Thanks to Theorem 10, to prove the thesis it is enough to show that, if F is a flacet of M with $|F| \geq 2$, then $F \in \mathcal{F}$. First notice that we can write, without loss of generality, $M = M' \oplus_2 U_t$, where U_t corresponds to a leaf v_t of T and M' is obtained as 2-sums of $U_1, \dots, U_{t-1}$, hence it satisfies the inductive hypothesis. Note that the tree corresponding to M' is then $T - v_t$. Let us denote by v_l the only neighbor of v_t in T. Let $E' + p$, $E(U_t) = E_t + p$ be the ground sets of M', U_t respectively, where $E' = \bigcup_{i=1}^{t-1} E_i$, and $E_i = E \cap E(U_i)$ for $i = 1, \dots, t$. Clearly $p \in E(U_l)$. Now, since F is a flacet of M, we can apply Theorem 11 to get three possible cases. If F has non-empty intersection with both $E(M')$ and E_t, then we are in case 1 and either $F = E(U_t) \cup F' - p$ or $F = E' \cup F_t - p$, where F', F_t are flacets of M', U_t respectively, containing p. However, the latter case is not possible because of Observation 17, so the only possibility is that $F = E_t \cup F'$. By induction, F' belongs to $\mathcal{F}'$ defined for M' as in the statement of the theorem. Moreover, since F' contains p, its corresponding component C

in $T - v_t$ contains v_l and then $C + v_t$ is a valid component for T. Moreover $|F' \cap E_i| \in \{0, |E_i|\}$ for any $i = 1, \dots, t-1$, which implies $F \in \mathcal{F}$. Suppose now we are in case 2, i.e., F is strictly contained in one of E', E_t. Then F is a flacet of one of M', U_t, the latter not being possible again due to Observation 17. So F is a flacet of M' and it does not contain p, hence by induction hypothesis its corresponding component C does not contain v_l. But then C is a valid component of T and again $F \in \mathcal{F}$. Finally, if we are in case 3 then $F = E_t$ or $F = E$, and in both cases $F \in \mathcal{F}$. □

Theorem 19. *2-Level matroid base polytopes satisfy Conjecture 1.*

As the forest matroid of a graph G is in $\mathcal{M}$ if and only if G is series-parallel [11], we deduce the following.

Corollary 20. *Conjecture 1 is true for the spanning tree polytope of series-parallel graphs.*

5 Cut Polytope and Matroid Cycle Polytope

A cycle of a matroid M is a disjoint union of circuits. The *cycle polytope* $C(M)$ is given by the convex hull of the characteristic vectors of its cycles, and it is a generalization of the cut polytope $CUT(G)$ for a graph G [1]. In this section we prove Conjecture 1 for the cycle polytope $C(M)$ of the binary matroids M that have no minor isomorphic to F_7^*, R_{10}, $M_{K_5}^*$ and are 2-level. When those minors are forbidden, a complete linear description of the associated polytope is known (see [1]). This class includes all cut polytopes that are 2-level, and has been characterized in [8]:

Theorem 21. *Let M be a binary matroid with no minor isomorphic to F_7^*, R_{10}, $M_{K_5}^*$. Then $C(M)$ is 2-level if and only if M has no chordless cocircuit of length at least 5.*

Corollary 22. *The polytope $CUT(G)$ is 2-level if and only if G has no minor isomorphic to K_5 and no induced cycle of length at least 5.*

Recall that the cycle space of graph G is the set of its Eulerian subgraphs (subgraphs where all vertices have even degree), and it is known (see for instance [12]) to have a vector space structure over the field $\mathbb{Z}_2$. This statement and one of its proofs generalizes to the cycle space (the set of all cycles) of binary matroids.

Lemma 23. *Let M be a binary matroid with d elements and rank r. Then the cycles of M form a vector space $\mathcal{C}$ over $\mathbb{Z}_2$ with the operation of symmetric difference as sum. Moreover, $\mathcal{C}$ has dimension $d - r$.*

Corollary 24. *Let M be a binary matroid with d elements and rank r. Then M has exactly 2^{d-r} cycles.*

The only missing ingredient is a description of the facets of the cycle polytope for the class of our interest.

Theorem 25. *[1] Let M be a binary matroid, and let $\overline{\mathcal{C}}$ be its family of chordless cocircuits. Then M has no minor isomorphic to F_7^*, R_{10}, $M_{K_5}^*$ if and only if*

$$C(M) = \{x \in [0,1]^E : x(F) - x(C \setminus F) \leq |F| - 1 \text{ for } C \in \overline{\mathcal{C}}, F \subseteq C, |F| \text{ odd}\}.$$

Theorem 26. *Let M be a binary matroid with no minor isomorphic to F_7^*, R_{10}, $M_{K_5}^*$ and such that $C(M)$ is 2-level. Then $C(M)$ satisfies Conjecture 1.*

Proof. As remarked in [1] and [8], the following equations are valid for $C(M)$: a) $x_e = 0$, for e coloop of M; and b) $x_e - x_f = 0$, for $\{e, f\}$ cocircuit of M.

The first equation is due to the fact that a coloop cannot be contained in a cycle, and the second to the fact that circuits and cocircuits have even intersection in binary matroids. A consequence of this is that we can delete all coloops and contract e for any cocircuit $\{e, f\}$ without changing the cycle polytope: for simplicity we will just assume that M has no coloops and no cocircuit of length 2. In this case $C(M)$ has full dimension $d = |E|$. Let r be the rank of M. Corollary 24 implies that $C(M)$ has 2^{d-r} vertices. Let now T be the number of cotriangles (i.e., cocircuits of length 3) in M, and S the number of cocircuits of length 4 in M. Thanks to Theorem 25 and to the fact that M has no chordless cocircuit of length at least 5, we have that $C(M)$ has at most $2d + 4T + 8S$ facets. Hence the bound we need to show is:

$$2^{d-r}(2d + 4T + 8S) \leq d2^{d+1}, \text{ which is equivalent to } 2T + 4S \leq d(2^r - 1).$$

Since the cocircuits of M are circuits in the binary matroid M^*, whose rank is $d - r$, we can apply Corollary 24 to get $T + S \leq 2^r - 1$, where the -1 comes from the fact that we do not count the empty set. Hence, if $d \geq 4$,

$$2T + 4S \leq 4(T + S) \leq d(2^r - 1).$$

The bound is loose for $d \geq 5$. The cases with $d \leq 4$ can be easily verified, the only tight examples being affinely isomorphic to cubes and cross-polytopes. □

Corollary 27. *2-level cut polytopes satisfy Conjecture 1.*

6 Conclusions

In this paper, we showed that Conjecture 1 holds true for many important classes of 2-level polytopes. Whether the results and ideas from this paper can be extended to all 2-level polytopes remains open. Another natural question is whether 2-levelness is the "right" assumption for proving $f_{d-1}(P)f_0(P) \leq d2^{d+1}$, and whether this bound is also valid for more general classes of 0/1 polytopes. We provide here two examples showing that spanning tree and forest polytopes – two classes of "well-behaved" 0/1 polytopes – do not verify Conjecture 1.

Example 28 (Forest polytope of $K_{2,d}$). Conjecture 1 implies an upper bound of $d2^{2(d+1)} = O(4+\varepsilon)^d$ for $f_0(P)f_{d-1}(P)$, with P being the (full-dimensional) forest polytope of $K_{2,d}$ and any $\varepsilon > 0$. Each subgraph of $K_{2,d}$ that takes, for each node v of degree 2, at most one edge incident to v, is a forest. Those graphs are 3^d. Moreover, each induced subgraph of $K_{2,d}$ that takes the nodes of degree d plus at least 2 other nodes is 2-connected, hence it induces a (distinct) facet of P. Those are $2^d - (d+1)$. In total $f_0(P)f_{d-1}(P) = \Omega(6^d)$.

Example 29 (Spanning tree polytope of the skeleton of the 4-dimensional cube). Let G be the skeleton of the 4-dimensional cube, and P the associated spanning tree polytope. Numerical experiments show that $f_0(P)f_{d-1}(P) \geq 1.603 \cdot 10^{11}$, while the upper bound from Conjecture 1 is $\approx 1.331 \cdot 10^{11}$.

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