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Haldane quantum Hall effect for light in a dynamically modulated array of resonators: supplementary material

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This document provides supplementary information to "Haldane Quantum Hall effect for light in a dynamically modulated array of resonators," http://dx.doi.org/10.1364/optica.3.000200. The Floquet theory used to obtain the quasi-energy bands of our time-periodic system is presented. Some further details on the numerical simulations are also given.

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Floquet theory

We start from the most general, linear, time-independent Hamiltonian for photons on a lattice ($\hbar = 1$):

$$H_S = H_0 + H_J = \sum_i \omega_i a_i^\dagger a_i - \sum_{i \neq j} J_{ij} a_i^\dagger a_j, \tag{S1}$$

with $a_i^\dagger$, the photon creation operator, and no particular requirements for the couplings J_{ij} (these will come later depending on the chosen lattice geometry). We add an on-site, time-dependent, periodic modulation of the resonant frequency with a position-dependent intensity and phase

$$H_t = \sum_i A_i \cos(\Omega t + \phi_i) a_i^\dagger a_i. \tag{S2}$$

For times much larger than the period $T = 2\pi/\Omega$, it becomes meaningful to apply the Floquet theory of quasi-energies [1, 2]. In particular, the solutions to the time-dependent Schrödinger equation can be written as $|\psi_n(t)\rangle = \exp(-i\varepsilon_n t)|u_n(t)\rangle$, with $u_n(t)$ a T-periodic function which is a solution to

$$[H_S + H_t - i\partial_t]|u_n(t)\rangle = \varepsilon_n |u_n(t)\rangle \tag{S3}$$

The spectrum ε_n has a Brillouin-zone like structure with Ω the width of the first zone, i.e. for any solution $|u_n(t)\rangle$ of quasi-energy ε_n, and any integer m, $\exp(im\Omega t)|u_n(t)\rangle$ is also a solution, with eigenvalue $\varepsilon_n + m\Omega$. The states $|u_n(t)\rangle$ form a Hilbert space of T-periodic functions. The inner product in this space can be defined starting from the standard bra-ket inner product $\langle \bullet | \bullet \rangle$ for time-independent states, and reads

$$\langle\langle \bullet | \bullet \rangle\rangle T \int_0^T \mathrm{d}t \langle \bullet | \bullet \rangle, \tag{S4}$$

The states can be expanded on the Floquet basis given by

$$\begin{aligned} |\{n_i\}, m\rangle &= U_t(t)|\{n_i\}\rangle \exp(im\Omega t) \\ &= |\{n_i\}\rangle \exp\left(-\frac{i}{\Omega}\sum_i A_i \sin(\Omega t + \phi_i) n_i + im\Omega t\right), \end{aligned} \tag{S5}$$

where n_i denotes the occupation number of site i, and

$$U_t(t) = \exp\left(-i\int_{t_0}^t H_t(t')\mathrm{d}t'\right) \tag{S6}$$

is the time-evolution operator corresponding to the time-dependent Hamiltonian H_t, and we assume an adiabatic switching of the modulation in order to disregard the phase offset due to the starting time t_0. Since the Hamiltonian is particle-number-preserving, we need only consider the subspace of a single excitation in the system, $\sum_i n_i = 1 \; \forall \; \{n_i\}$. Equation (S3) is then an eigenvalue problem with matrix elements in this basis given by

$$\begin{aligned} &\langle\langle \{n_j'\}, m' | H_S + H_t - i\partial_t | \{n_i\}, m \rangle\rangle = \\ &\delta_{m,m'} \left[\langle \{n_j'\} | H_0 + m\Omega | \{n_i\} \rangle\right] + \\ &\int_0^T e^{i(m-m')\Omega t} \exp\left(\frac{iA_j}{\Omega}\sin(\Omega t + \phi_j) - \frac{iA_i}{\Omega}\sin(\Omega t + \phi_i)\right) \\ &\times \langle \{n_j\} | H_J | \{n_i\} \rangle. \end{aligned} \tag{S7}$$

These matrix elements are then equal to

$$\langle\langle n_i', m' | H_S + H_t - i\partial_t | n_i, m \rangle\rangle = \delta_{m,m'} \left(m\Omega + \omega_i\right), \tag{S8}$$

when the particle stays on the same site, and to

$$\langle\langle n_j', m'|H_S + H_t - i\partial_t|n_{i\neq j}, m\rangle\rangle = \int_0^T e^{i(m-m')\Omega t} J_{ij} \times \exp\left(\frac{i}{\Omega}\left(A_j \sin(\Omega t + \phi_j) - A_i \sin(\Omega t + \phi_i)\right)\right), \tag{S9}$$

when the particle hops from site i to site j. In eqs. (S8) and (S9), we label by n_i the only non-zero occupation number of $\{n_i\}$. We can further use the Jacobi-Anger expansion to simplify eq. (S9) to

$$\langle\langle n_j', m'|H_S + H_t - i\partial_t|n_{i\neq j}, m\rangle\rangle = -\mathcal{J}_{m'-m}\left(\rho_{ij}\right) e^{i(m'-m)\phi_{ij}} J_{ij}, \tag{S10}$$

with $\mathcal{J}_n(x)$ the n-th Bessel function of the first kind, and the definition

$$\rho_{ij} e^{i\phi_{ij}} = (A_j e^{i\phi_j} - A_i e^{i\phi_i})/\Omega. \tag{S11}$$

Perturbation theory

Some additional intuition can be found in writing the Floquet perturbation theory [3, 4] for the effective time-independent Hamiltonian H_{eff} that describes the time evolution for timescales greater than T, in the sense that the time-evolution operator is

$$U(t+T,t) = \exp\left(-i\int_t^{t+T}(H_S + H_t)\mathrm{d}t\right) = \exp\left(-iH_{\text{eff}}T\right) \tag{S12}$$

For a Fourier-expanded time-periodic Hamiltonian,

$$H(t) = \sum_m H_m e^{im\Omega t}, \tag{S13}$$

we can write a perturbation expansion for H_{eff} in orders of $1/\Omega$. Up to first order, this reads

$$H_{\text{eff}} = H_{0\Omega} + H_{1\Omega} + \mathcal{O}\left(\frac{1}{\Omega^2}\right) = H_{m=0} + \frac{1}{\Omega}\sum_{m=1}^{\infty}\frac{1}{m}[H_m, H_{-m}] + \mathcal{O}\left(\frac{1}{\Omega^2}\right). \tag{S14}$$

In our dynamically-modulated lattice, after the unitary transformation

$$H' = U_t^\dagger[H_S + H_t - i\partial_t]U_t = \sum_i \omega_i a_i^\dagger a_i - \sum_m\sum_{ij}\mathcal{J}_m\left(\rho_{ij}\right) e^{im(\Omega t+\phi_{ij})} J_{ij} a_i^\dagger a_j,$$

with the definitions of $\mathcal{J}, \rho$, and ϕ as above, the Fourier components H_m can be easily read out. The zero-th order of the perturbative expansion of eq. (S14) is

$$H_{0\Omega} = H_0 + H_J' = \sum_i \omega_i a_i^\dagger a_i - \sum_{ij} J_{ij}' a_i^\dagger a_j, \qquad J_{ij}' = J_{ij}\mathcal{J}_0\left(\rho_{ij}\right), \tag{S15}$$

i.e. similar to the starting H_S of the static lattice, but with rescaled (but still real) couplings J_{ij}'. The first-order term is

$$\begin{aligned} H_{1\Omega} &= \sum_{m=1}^{\infty}\frac{(-1)^m}{\Omega m}\sum_{ijpq}\mathcal{J}_m(\rho_{ij})\mathcal{J}_m(\rho_{pq})J_{ij}J_{pq}e^{im(\phi_{ij}-\phi_{pq})} \times [a_i^\dagger a_q\delta_{jp} - a_p^\dagger a_j\delta_{iq}] \\ &= \sum_{ij} 2i\sum_{m=1}^{\infty}\frac{(-1)^m}{\Omega m}\sum_p \mathcal{J}_m(\rho_{ip})\mathcal{J}_m(\rho_{pj})J_{ip}J_{pj} \times \sin(m(\phi_{ip}-\phi_{pj}))a_i^\dagger a_j, \end{aligned} \tag{S16}$$

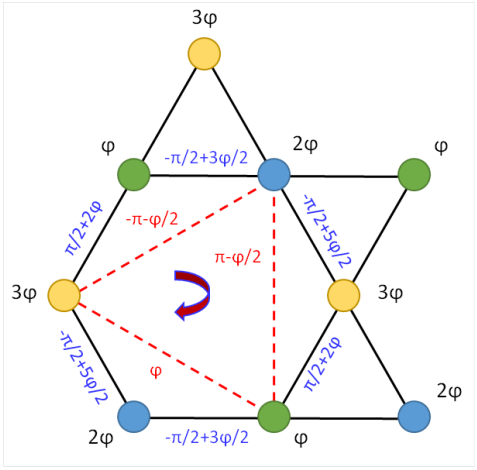

Fig. S1. Definition of various phases under a modulation of constant amplitude, and phases (marked in black): φ on site A, 2φ on site B, and 3φ on site C (see also to Fig. 2 in the main text). In blue, the phase ϕ_{ij} as defined in eq. (S11) is given for a clockwise hopping direction (indicated by the arrow in the center of the hexagon). With red, the phase $\phi_{ip} - \phi_{pj}$ entering eq. (S16) is given, for the same hopping direction.

The (purely imaginary) term after the first sum sign can obviously be interpreted as a new coupling amplitude, J_{ij}'', which is added to J_{ij}', thus introducing a complex phase. The obvious interpretation of these terms is hopping from one site to another through one intermediate site. In the same way, terms of higher order in $1/\Omega$ represent hopping through an increasing number of intermediate sites.

Kagomé lattice

The Kagomé lattice (illustrated in Fig. S1 here and Fig. 2 of the main text) has three lattice sites in the unit cell, which we label A, B, and C. The values of the couplings of eq. (11) in the main text computed through (eq. (S16)) are

$$\begin{aligned} t_{AC,0} &= \mathcal{J}_0(2\rho_\varphi), \quad t_{AB,0} = t_{BC,0} = \mathcal{J}_0(\rho_\varphi) \\ t_{AB,1} &= t'_{AB,1} = 2i\frac{J}{\Omega}\sum_m \frac{(-1)^m}{m}\mathcal{J}_m(\rho_{2\varphi})\mathcal{J}_m(\rho_\varphi)\sin(m(\varphi/2-\pi)) \\ t_{AC,1} &= t'_{AC,1} = 2i\frac{J}{\Omega}\sum_m \frac{(-1)^m}{m}\mathcal{J}_m(\rho_\varphi)\mathcal{J}_m(\rho_\varphi)\sin(m\varphi) \\ t_{BC,1} &= t'_{BC,1} = 2i\frac{J}{\Omega}\sum_m \frac{(-1)^m}{m}\mathcal{J}_m(\rho_{2\varphi})\mathcal{J}_m(\rho_\varphi)\sin(m(\pi+\varphi/2)), \end{aligned} \tag{S17}$$

where $\rho_\varphi = 2(A_0/\Omega)|\sin(\varphi/2)|$ and $\rho_{2\varphi} = 2(A_0/\Omega)|\sin(\varphi)|$ are the amplitudes computed through eq. (S16) for a phase difference between sites i and j of φ and 2φ, respectively. In Fig. S1, we illustrate the computation of the phases that enter the sine functions of eq. (S17). Starting from the modulation phases (marked in black), one first computes the value of ϕ_{ij} (marked in blue) as defined in eq. (S11) for all first neighbors, and then the values of $\phi_{ip} - \phi_{pj}$ (marked in red) that enter eq. (S16).

Numerical simulations

We note that this perturbation theory discussion is only used for a better intuitive understanding of the effect, but the topo-

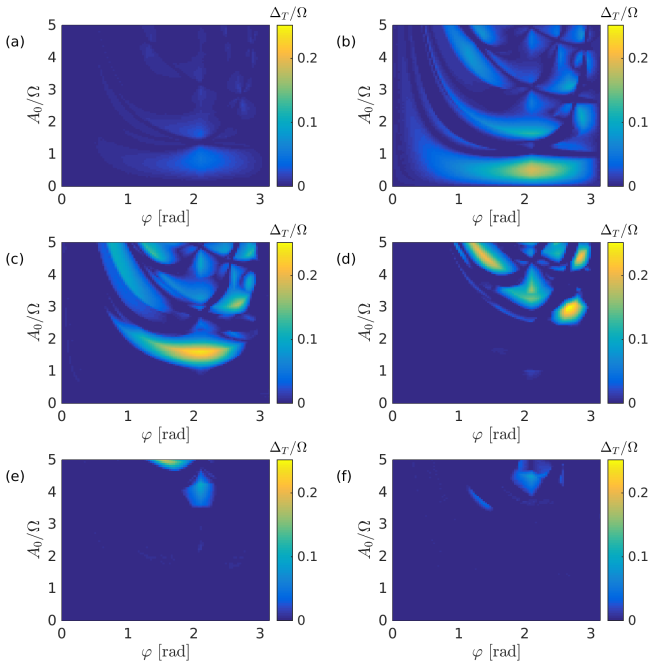

Fig. S2. The (largest) width of the opened band gap due to the dynamic modulation of frequency Ω vs. the amplitude A_0 and the phase angle φ for the Kagomé lattice with first-neighbor coupling (a): $J = 0.1\Omega$; (b): $J = 0.3\Omega$; (c): $J = 0.5\Omega$; (d): $J = 0.7\Omega$; (e): $J = 0.9\Omega$; (f): $J = 1.1\Omega$. The colorbar scale is the same in all panels.

logically non-trivial bands are present even for values of J that are comparable to Ω. This is why in our work we mostly use the full diagonalization on the Floquet basis, i.e. we diagonalize the matrix of eq. (S7). Numerically, we truncate the orders of $m\Omega$ by picking an $m_{\max}$ value and taking orders up to that, i.e. $|m|, |m'| \leq m_{\max}$. Convergence with respect to this parameter was always checked, and is reached very fast ($m_{\max} \approx 2$) for low values of J when the different orders of the Floquet bands are well separated in frequency space. As J becomes comparable to Ω, higher orders are needed, but convergence was always reached at $m_{\max} = 10$, at most. The diagonalization of the effective static Hamiltonian (from perturbation theory) was only used for panels (b) of Fig. 2 and panels (a) and (c) of Fig. 3 of the main text. As is seen and can be expected, this matches the exact Floquet diagonalization very well for low J/Ω, but becomes inadequate for higher values. Of course, that could in principle be fixed by higher orders in the perturbation expansion, but that would bring no further insight and is thus not really needed.

In Fig. S2 we show the width of the opened band gap Δ_T versus A_0 and φ, for J from 0.1Ω to 1.1Ω, computed by diagonalization on the Floquet basis. Panels (a) and (c) are shown in Fig. 3 of the main text (panels (b) and (d), respectively), but with a different scale of the colorbar. As J increases, the amplitude A_0 needed to open a gap grows as well. This can at least qualitatively be understood through the following considerations. As J becomes comparable to Ω, Floquet bands of different orders m start crossing, thus closing any potential band gaps. As A_0 increases, however, the re-scaled first-neighbor coupling J'_{ij} of eq. (S15) generally decreases, since ρ_{ij} is proportional to A_0, and the Bessel function $\mathcal{J}_0(x)$ has its maximum value at $x = 0$. Thus, the problem with mixing of bands of different orders can be avoided through a sufficiently large A_0.

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