On Some Weighted Stokes Problems. Applications on Smagorinsky Models

Jacques Rappaz and Jonathan Rochat

Abstract In this paper we study existence and uniqueness of weak solutions for some non-linear weighted Stokes problems using convex analysis. The characterization of these considered equations is that the viscosity depends on the strain rate of the velocity field with a weight being a positive power of the distance to the boundary of the domain. These non-linear relations can be seen as a first approach of mixing-length eddy viscosity from turbulent modeling. A well known model is von Karman's on which the viscosity depends on the square of the distance to the boundary of the domain. Numerical experiments conclude the work and show properties from the theory.

Key words: Stokes Equations · Weighted Sobolev Spaces · Finite Element Method

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1 Introduction

Turbulent flows have an importance in many domains, including technology and industry. While measurements are sometimes difficult to make, the use of numerical simulations of such flows in the process industries can be very useful: it allows optimization of activities and has led to reductions in the cost of products and process development. The Navier-Stokes equations offer an accurate description of these flows, whose Reynolds number is large. The resolution of these equations is consequently a challenging task as the mesh required to obtain most of the structure of these flows should be very thin.

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To overcome these difficulties, many turbulent models appears such as Large Eddy Simulation (LES), mainly described in [19], that assume that the inertial scale of the flow have been captured by an adequate grid. Another simpler approach we consider here is the Reynold Averaged Navier-Stokes (RANS) model. This kind of models assumes that the period of the mean solution is several order of magnitude larger than the turbulence fluctuations. A type of simple model often used by engineering is a mixing-length model called "Smagorinsky Modelling" (see [19]). In practise, these models consist in changing the initial viscosity of the fluid by a turbulent viscosity depending of the velocity, transforming the initial linear elliptic term in the Navier-Stokes equations by a non-linear one.

If $\mathbf{u}$ and p are the velocity and the pressure of a stationary incompressible fluid of density ρ, submitted to a force $\mathbf{f}$, flowing in a cavity $\Omega \subset \mathbb{R}^n$, $n = 2,3$, with a Lipschitz boundary $\partial\Omega$, stationary Navier-Stokes equations on Ω take on the form

$$\begin{cases} -\text{ div } (2\mu\varepsilon(\mathbf{u})) + \nabla p = \mathbf{F}(\mathbf{u}) & \text{in } \Omega, \\ \qquad\qquad \text{div } \mathbf{u} \qquad = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

with $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, $\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ and $\mathbf{F}(\mathbf{u}) = \mathbf{f} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}$. In this paper we will treat Smagorinsky models in which the viscosity depends on $|\varepsilon(\mathbf{u})|$ and takes the form

$$\mu(|\varepsilon(\mathbf{u})|) = \mu_L + \kappa^\alpha \rho l^{2-\alpha} d_{\partial\Omega}^\alpha |\varepsilon(\mathbf{u})| \tag{2}$$

where $\mu_L > 0$ corresponds to a laminar viscosity, $\kappa = 0.41$ is the von Karman constant, $l > 0$ is a characteristic length of the domain, $\alpha \geq 0$ is a real number, $d_{\partial\Omega}(x)$ is the distance of a point $x \in \Omega$ to the boundary $\partial\Omega$ and $|\varepsilon(\mathbf{u})| = \left(\sum_{i,j}\varepsilon_{ij}(\mathbf{u})^2\right)^{\frac{1}{2}}$.

The cases with $\alpha = 0$ can be treated in usual Sobolev spaces and their analysis can be found in several papers [3, 21]. At the opposite, the cases with $\alpha > 0$ have to be treated in weighted Sobolev spaces and present several difficulties. In particular, we will give some comments on a very popular model for a fluid flow in between to close plates (Von Karman model) in which $\alpha = 2$.

We proceed in this paper to an analysis of Problem (1) with a viscosity given by (2) and $\alpha < 2$. To do it, we have to start by considering the simpler Stokes problem with a given $\mathbf{F}$ function. By using several known results concerning weighted Sobolev spaces [17, 5, 15, 18], we establish some theoretical results on the existence and uniqueness of a velocity field of equation (1) when $\mathbf{F}$ is given. We show how is important the role of the laminar viscosity μ_L when Von Karman model is used and its impact on numerical results when we use a finite element method to discretize Problem (1). The uniqueness of the pressure is sometimes an open question.

2 Main Existence Theorem

In this section we prove that the problem (1) for a given **F** function has a unique solution related to velocity in a space with free divergence. Let Ω be an bounded open subset of $\mathbb{R}^n$, $n = 2,3$, with a Lipschitz boundary $\partial\Omega$. We first introduce some adequate weighted functional spaces on Ω to define a weak problem from the equation (1) concerning the velocity. We then prove by convex analysis the existence of such a velocity field. The section finishes with some results concerning the pressure.

2.1 Suitable functional spaces

Let $d_{\partial\Omega}(x) = dist(x, \partial\Omega) = \min_{y\in\partial\Omega} |x-y|$ the distance between $x \in \Omega$ and $\partial\Omega$. For $1 \leq p < \infty$ and $\alpha > 0$, we denote the weighted Sobolev space of order one as

$$W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega}) = \left\{ v \in L^p(\Omega, d^{\alpha}_{\partial\Omega}) \mid \frac{\partial v}{\partial x_i} \in L^p(\Omega, d^{\alpha}_{\partial\Omega}), \quad \forall i = 1,\ldots,n \right\}$$

where $L^p(\Omega, d^{\alpha}_{\partial\Omega}) = \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega |v|^p d^{\alpha}_{\partial\Omega} dx < \infty \right\}$ provided with norm $\|v\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} := \left(\int_\Omega |v|^p d^{\alpha}_{\partial\Omega} dx \right)^{\frac{1}{p}}$. We thus endowed $W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})$ with the norm

$$\|v\|_{W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})} := \left(\int_\Omega |v|^p d^{\alpha}_{\partial\Omega} dx + \int_\Omega |\nabla v|^p d^{\alpha}_{\partial\Omega} dx \right)^{\frac{1}{p}}. \tag{3}$$

Lemma 2.1 *For all* $1 < p < \infty$ *and* $\alpha \geq 0$, $W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})$ *endowed with the norm* (3) *is a reflexive Banach space*

Proof. The properties of spaces $W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})$ are deduced from the ones of the spaces $L^p(\Omega, d^{\alpha}_{\partial\Omega})$ (see [17] or Theorem 1.3 in [11]). The reflexivity is due to the uniform convexity of these spaces (Theorem III.29 in [4]).

For arbitrary weight ω, the books [17, 11] give a well overview of these spaces that found applications in a large scale of problems such as $p-$Laplacian [7] or degenerated elliptic problem [8]. Generally, the chosen weight ω belongs to the Muckenhoupt class A_p (see [1, 6]). For weights which are a positive power of the distance to the boundary, they belong to the Muckenhoupt class if $0 \leq \alpha < p-1$ (see [7]). Publications on the space generated with such weights are less frequent but some papers and books treat many properties of these spaces, see for example [5, 8, 2, 18]. One of the important property is that the embedding

$$W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega}) \hookrightarrow L^p(\Omega, d^{\alpha}_{\partial\Omega}) \tag{4}$$

is continuous and compact, as it is shown in the theorem 3.8 from [18].

Another characterization is that we can define a continuous and bounded trace operator $Tr : W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega}) \to L^p(\partial\Omega)$ if $1 < p < \infty$ and $0 \leq \alpha < p-1$ (see theorem 9.15 in [17]). In that case, the space $W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})$ (the closure of $C^{\infty}_0(\Omega)$ in $W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})$) for the norm (3)) can be identified with the space of functions in $W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})$ vanishing on the boundary:

$$W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega}) = \{v \in W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega}) : Tr(v) = 0\}.$$

Moreover, as the problem (1) involve vector fields $\mathbf{u} : \Omega \to \mathbb{R}^n$, $\mathbf{u} = (u_1, \ldots, u_n)$, we denote the following norms for $\mathbf{u} \in [W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})]^n$ as:

$$\|\mathbf{u}\|_{W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})} := (\sum_{i=1}^{n} \|u_i\|^p_{W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})})^{\frac{1}{p}},$$

$$\|\nabla\mathbf{u}\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} := (\sum_{i,j=1}^{n} \|\frac{\partial u_i}{\partial x_j}\|^p_{L^p(\Omega, d^{\alpha}_{\partial\Omega})})^{\frac{1}{p}} \quad \text{and}$$

$$\|\varepsilon(\mathbf{u})\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} := (\sum_{i,j=1}^{n} \|\varepsilon_{ij}(\mathbf{u})\|^p_{L^p(\Omega, d^{\alpha}_{\partial\Omega})})^{\frac{1}{p}}.$$

These above definitions and characterizations allows us to prove an important result:

Proposition 2.2 (Korn Inequality) *Let* $0 \leq \alpha < p-1$. *There exists a generic constant* $C > 0$ *such that*

$$\|\nabla\boldsymbol{u}\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} \leq C\|\varepsilon(\boldsymbol{u})\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})}, \quad \forall\boldsymbol{u} \in [W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})]^n. \tag{5}$$

Proof. The structure of the proof follows mainly the procedure developed in [16]. First of all, Theorem 6 in [15] states for $-1 \leq \alpha < p-1$ the existence of a constant $C > 0$ such that

$$\|\nabla\mathbf{u}\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} \leq C\left\{\|\mathbf{u}\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} + \|\varepsilon(\mathbf{u})\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})}\right\}, \quad \forall\mathbf{u} \in [W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})]^n. \tag{6}$$

Consequently, it remains to prove that there exists a generic constant $C > 0$ such that

$$\|\mathbf{u}\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})}, \quad \forall\mathbf{u} \in [W^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})]^n.$$

By contradiction, we assume that there exists a sequence $(\mathbf{u}_l)_{l=1}^{\infty} \in [W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})]^n$ satisfying

$$\|\mathbf{u}_l\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} = 1 \quad \text{and} \lim_{l\to\infty} \|\varepsilon(\mathbf{u}_l)\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} = 0. \tag{7}$$

By using (6) and (7), the sequence $\{\mathbf{u}_l\}_l^{\infty}$ is bounded in $[W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})]^n$ and by compacity (4), it is not restrictive to assume there exists $\mathbf{u} \in [W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})]^n$ such that

$$\lim_{l\to\infty} \|\mathbf{u}_l - \mathbf{u}\|_{L^p(\Omega, d^{\alpha}_{\partial\Omega})} = 0 \quad \text{and} \quad \mathbf{u}_l \rightharpoonup \mathbf{u} \text{ weakly in } [W^{1,p}_0(\Omega, d^{\alpha}_{\partial\Omega})]^n. \tag{8}$$

Relations (7) and (8) imply $\varepsilon(\mathbf{u}) = 0$. Then from [16], the function $\mathbf{u}$ belongs to a class of polynomial of degree one. Since $\mathbf{u}$ is vanishing on the boundary, then $\mathbf{u} \equiv 0$. This contradicts the fact that $\|\mathbf{u}\|_{L^p(\Omega,d^{\alpha}_{\partial\Omega})} = 1$.

Remark 2.3 *In order to study Stokes problem* (1) *with viscosity* (2)*, we will see below that we need to work with weighted Sobolev spaces with p=3. In this particular case, inequality* (5) *takes the following form: for* $0 \leq \alpha < 2$*, there exists a generic constant* $C > 0$ *such that*

$$\|\nabla \boldsymbol{u}\|_{L^3(\Omega,d^{\alpha}_{\partial\Omega})} \leq C\|\varepsilon(\boldsymbol{u})\|_{L^3(\Omega,d^{\alpha}_{\partial\Omega})}, \quad \forall \boldsymbol{u} \in [W^{1,3}_0(\Omega,d^{\alpha}_{\partial\Omega})]^n.$$

In the following we consider $0 \leq \alpha < 2$. The problem (1) involves homogeneous Dirichlet conditions and takes into account two viscosity terms $\operatorname{div}(2\mu_0\varepsilon(\mathbf{u}))$ and $\operatorname{div}(2\kappa^{\alpha}\rho l^{2-\alpha}d^{\alpha}_{\partial\Omega}|\varepsilon(\mathbf{u})|\varepsilon(\mathbf{u}))$, see (2). Consequently, when we will consider a weak formulation of Problem (1) (see section 3.2) we have to work in the two following Banach spaces $H^1_0(\Omega)$ and $W^{1,3}_0(\Omega,d^{\alpha}_{\partial\Omega})$. Let us remark that there exists α_0 with $0 \leq \alpha_0 < 2$ such that $W^{1,3}_0(\Omega,d^{\alpha}_{\partial\Omega}) \subset H^1_0(\Omega)$ when $0 \leq \alpha < \alpha_0$ (see [17]) but it is not the case when α is close to 2. Thus, if we want to analyse von Karman model corresponding to $\alpha = 2$, we have to define the space

$$X_{\alpha} = H^1_0(\Omega) \cap W^{1,3}_0(\Omega,d^{\alpha}_{\partial\Omega})$$

endowed with the following norm $\|v\|_{X_{\alpha}} = \|v\|_{H^1(\Omega)} + \|v\|_{W^{1,3}(\Omega,d^{\alpha}_{\partial\Omega})}$.

Lemma 2.4 *The normed space* $(X_{\alpha}, \|\cdot\|_{X_{\alpha}})$ *is a reflexive Banach space.*

Proof. The proof is a consequence of the compact embedding

$$H^1_0(\Omega) \hookrightarrow L^3(\Omega) \subset L^3(\Omega,d^{\alpha}_{\partial\Omega})$$

and (4). In particular, we prove that each bounded sequence in X_{α} has a weakly convergent subsequence in X_{α}.

Lemma 2.5 *The space* X_{α} *endowed with the semi-norm*

$$|\boldsymbol{v}|_{X_{\alpha}} := \|\nabla v\|_{L^2(\Omega)} + \|\nabla v\|_{L^3(\Omega,d^{\alpha}_{\partial\Omega})}.$$

is a reflexive Banach space.

Proof. The proof is a consequence of Lemma 2.4 and from the equivalence of the norm $\|\cdot\|_{X_{\alpha}}$ and the semi-norm $|\cdot|_{X_{\alpha}}$ since $H^1_0(\Omega) \hookrightarrow L^3(\Omega,d^{\alpha}_{\partial\Omega})$ and by Poincaré's inequality.

2.2 On the velocity of Stokes problem

In this section we consider the non-linear Stokes problem (1) with viscosity (2) in which $\mathbf{F} \in [L^{\frac{4}{3}}(\Omega)]^n$ does not depend on $\mathbf{u}$. The index α verifies $0 \leq \alpha < 2$.

As in problem (1) we are looking for a free divergence velocity field, we take now the space

$$\mathbf{X}_{\alpha,div} = \{\mathbf{v} \in X_\alpha^n,\ \text{div}\ \mathbf{v} = 0\}$$

endowed with the norm $|\mathbf{v}|_{\mathbf{X}_{\alpha,div}} = \|\nabla \mathbf{v}\|_{L^2(\Omega)} + \|\nabla \mathbf{v}\|_{L^3(\Omega, d_{\partial\Omega}^\alpha)}$. By multiplying the Stokes equation in problem (1) by a test velocity field $\mathbf{v} \in \mathbf{X}_{\alpha,div}$ and integrating by part, we obtain a weak formulation of the problem (1)-(2) for the velocity:
find $\mathbf{u} \in \mathbf{X}_{\alpha,div}$ such that

$$\int_\Omega (2\mu(|\varepsilon(\mathbf{u})|)\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}))dx = \int_\Omega (\mathbf{F} \cdot \mathbf{v})dx, \quad \forall \mathbf{v} \in \mathbf{X}_{\alpha,div}. \tag{9}$$

We use convex arguments to show existence and uniqueness of a solution to (9). Let us define the functional $J : X_{\alpha,div} \to \mathbb{R}$ by:

$$J(\mathbf{u}) = \int_\Omega [2A(x, |\varepsilon(\mathbf{u}(x))|) - \mathbf{F}(x) \cdot \mathbf{u}(x)]dx,$$

where $A : (x,s) \in \Omega \times \mathbb{R} \to A(x,s) \in \mathbb{R}$ is given by

$$A(x,s) = \frac{\mu_L}{2}s^2 + \frac{1}{3}\kappa^\alpha \rho l^{2-\alpha} d_{\partial\Omega}^\alpha(x) s^3.$$

Lemma 2.6 *The functional J is Gâteau-differentiable and its derivative at* $\boldsymbol{u}$ *in the direction* $\boldsymbol{v}$ *is*

$$DJ_{\boldsymbol{u}}(\boldsymbol{v}) = \int_\Omega (2\mu(|\varepsilon(\boldsymbol{u})|)\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}))dx - \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} dx.$$

Proof. It is easy to verify that for $\beta \geq 2$:

$$\lim_{t\to 0} \frac{|\varepsilon(\mathbf{u}+t\mathbf{v})|^\beta - |\varepsilon(\mathbf{u})|^\beta}{t} = \beta |\varepsilon(\mathbf{u})|^{\beta-2}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}).$$

Taking in account that

$$\frac{\partial}{\partial s}A(x,s) = \mu_L s + \kappa^\alpha \rho l^{2-\alpha} d_{\partial\Omega}^\alpha s^2,$$

we obtain

$$\begin{aligned}\lim_{t\to 0} \frac{J(\mathbf{u}+t\mathbf{v}) - J(\mathbf{u})}{t} &= \int_\Omega [2\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + 2\kappa^\alpha \rho l^{2-\alpha} d_{\partial\Omega}^\alpha |\varepsilon(\mathbf{u})| \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})]dx \\ &- \int_\Omega \mathbf{F} \cdot \mathbf{v} dx.\end{aligned}$$

In the following we are going to prove that the functional J is continuous, strictly convex and coercive. Existence and uniqueness of a velocity field of the problem (9) will then follow from results in [10].

Lemma 2.7 *Let $f,g \in L^r(\Omega, d^\alpha_{\partial\Omega})$ with $1 \le r < \infty$. Then*

$$\int_\Omega |d^\alpha_{\partial\Omega}(|f|^r - |g|^r)|dx \le r\||f|+|g|\|^{r-1}_{L^r(\Omega,d^\alpha_{\partial\Omega})}\|f-g\|_{L^r(\Omega,d^\alpha_{\partial\Omega})}.$$

Proof. The proof is similar from Lemma 4 in [9]. Generalization with weighted space is done using Holder inequality for weighted Lebesgue space: if p,q are such that $\frac{1}{p}+\frac{1}{q}=1$ and if $h \in L^p(\Omega, d^\alpha_{\partial\Omega})$, $l \in L^q(\Omega, d^\alpha_{\partial\Omega})$, then we have

$$\int_\Omega d^\alpha_{\partial\Omega} hl dx \le \|h\|_{L^p(\Omega,d^\alpha_{\partial\Omega})}\|l\|_{L^q(\Omega,d^\alpha_{\partial\Omega})}.$$

Lemma 2.8 *The functional J is continuous for the norm $|\cdot|_{X_{\alpha,div}}$.*

Proof. Taking $\mathbf{v} \in \mathbf{X}_{\alpha,div}$ in a neighbourhood of a fixed $\mathbf{u} \in \mathbf{X}_{\alpha,div}$ and using lemma 2.7 with respectively $r=2, \alpha=0$ and $r=3, \alpha>0$, we have the existence of a constant $C>0$ (depending of $\mathbf{u}$) such that

$$\begin{aligned}&\int_\Omega 2|A(|\varepsilon(\mathbf{u})|)-A(|\varepsilon(\mathbf{v})|)|dx\\ =&\int_\Omega 2\left|\frac{\mu_L}{2}(|\varepsilon(\mathbf{u})|^2-|\varepsilon(\mathbf{v})|^2)+\frac{\kappa^\alpha\rho l^{2-\alpha}d^\alpha_{\partial\Omega}}{3}(|\varepsilon(\mathbf{u})|^3-|\varepsilon(\mathbf{v})|^3)\right|dx \le C|\mathbf{u}-\mathbf{v}|_{\mathbf{X}_\alpha}.\end{aligned}$$

Since $\mathbf{F} \in [L^{\frac{4}{3}}(\Omega)]^n$ and by Poincaré inequality, we have the existence of a constant $C_f>0$ such that

$$\begin{aligned}|J(\mathbf{u})-J(\mathbf{v})| &= \left|\int_\Omega 2|A(|\varepsilon(\mathbf{u})|)-A(|\varepsilon(\mathbf{v})|)|dx - \int_\Omega \mathbf{F}\cdot(\mathbf{u}-\mathbf{v})dx\right|\\ &\le C|\mathbf{u}-\mathbf{v}|_{X_\alpha} + \|\mathbf{F}\|_{L^{\frac{4}{3}}} C_p\|\nabla(\mathbf{u}-\mathbf{v})\|_{L^2} \le C_f|\mathbf{u}-\mathbf{v}|_{\mathbf{X}_\alpha}.\end{aligned}$$

where $C_f = (C + \|\mathbf{F}\|_{L^{\frac{4}{3}}} C_p)$. If $\mathbf{u} \in \mathbf{X}_{\alpha,div}$, then we have

$$\lim_{\mathbf{v}\in\mathbf{X}_{\alpha,div},\mathbf{v}\to\mathbf{u}} J(\mathbf{v}) = J(\mathbf{u}),$$

which finishes the proof.

Lemma 2.9 *The functional J is strictly convex on $\mathbf{X}_{\alpha,div}$.*

Proof. For $x \in \Omega$, the function $A(x,s)$ is strictly convex on $\mathbb{R}^+$ in s variable since

$$\frac{\partial^2}{\partial s^2}A(x,s) \ge \mu_L > 0 \text{ when } s>0.$$

For $0<\eta<1$ and $\xi \ne \nu \in \mathbb{R}^{n\times n}$, we have using triangle inequality

$$|\eta\xi+(1-\eta)\nu| \le \eta|\xi|+(1-\eta)|\nu|.$$

Since $A(x,s)$ is strictly convex and monotone in s variable,

$$A(x,|\eta\xi+(1-\eta)\nu|)\leq A(x,\eta|\xi|+(1-\eta)|\nu|)<\eta A(x,|\xi|)+(1-\eta)A(x,|\nu|)$$

which proves that $A(x,|\cdot|)$ is strictly convex. Let $\mathbf{u},\mathbf{v}\in\mathbf{X}_\alpha$ such that $\mathbf{u}\neq\mathbf{v}$ and $0<\eta<1$. Thus $\varepsilon(\mathbf{u})\neq\varepsilon(\mathbf{v})$ since $\varepsilon(\mathbf{u})-\varepsilon(\mathbf{v})=\varepsilon(\mathbf{u}-\mathbf{v})\neq 0$, see [16]. Moroever

$$\int_\Omega A(x,\eta|\varepsilon(\mathbf{u})|+(1-\eta)|\varepsilon(\mathbf{v})|)dx<\eta\int_\Omega A(x,|\varepsilon(\mathbf{u})|)dx+(1-\eta)\int_\Omega A(x,|\varepsilon(\mathbf{v})|)dx.$$

It follows that J is strictly convex on $\mathbf{X}_{\alpha,div}$.

Lemma 2.10 *For $0\leq\alpha<2$, the functional J is coercive on $\boldsymbol{X}_{\alpha,div}$ in the following sense:*

$$\lim_{\boldsymbol{u}\in\boldsymbol{X}_{\alpha,div};|\boldsymbol{u}|_{\boldsymbol{X}_{\alpha,div}}\to\infty}\frac{J(\boldsymbol{u})}{|\boldsymbol{u}|_{\boldsymbol{X}_{\alpha,div}}}\to\infty.$$

Proof. We have by definition of the function A and by the remark 2.3 the existence of $C_1,C_2>0$ such that

$$\begin{aligned}\int_\Omega A(x,|\varepsilon(\mathbf{u})|)dx&=\int_\Omega\left(\frac{\mu_L}{2}|\varepsilon(\mathbf{u})|^2+\frac{\rho\kappa^\alpha l^{2-\alpha}d^\alpha_{\partial\Omega}(x)}{3}|\varepsilon(\mathbf{u})|^3\right)dx\\&\geq C_1\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}+C_2\|\nabla\mathbf{u}\|^3_{L^3(\Omega,d^\alpha_{\partial\Omega})}.\end{aligned}$$

Since $\mathbf{F}\in[L^{\frac{4}{3}}(\Omega)]^n$, there exits $C_3>0$ such that

$$\int_\Omega|\mathbf{F}\cdot\mathbf{u}|dx\leq\|\mathbf{F}\|_{L^{\frac{4}{3}}}\|\mathbf{u}\|_{L^4}\leq C_3\|\mathbf{F}\|_{L^{\frac{4}{3}}}\|\mathbf{u}\|_{H^1}\leq C_3\|\mathbf{F}\|_{L^{\frac{4}{3}}}\|\nabla\mathbf{u}\|_{L^2}.$$

Consequently, we have

$$J(\mathbf{u}):=\int_\Omega 2A(|\varepsilon(\mathbf{u})|)dx-\int_\Omega\mathbf{F}\cdot\mathbf{u}dx\geq\tilde{C}_1\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}+\tilde{C}_2\|\nabla\mathbf{u}\|^3_{L^3(\Omega,d^\alpha_{\partial\Omega})}-D\|\nabla\mathbf{u}\|_{L^2}$$

where $\tilde{C}_1,\tilde{C}_2,D$ are constants independant of $\mathbf{u}$. Finally we easily obtain

$$\lim_{\mathbf{u}\in\mathbf{X}_{\alpha,div};|\mathbf{u}|_{\mathbf{X}_{\alpha,div}}\to\infty}\frac{J(\mathbf{u})}{|\mathbf{u}|_{\mathbf{X}_{\alpha,div}}}\to\infty.$$

Proposition 2.11 *There exists a unique $\boldsymbol{u}\in\boldsymbol{X}_{\alpha,div}$ such that*

$$J(\boldsymbol{u})=\inf\{J(\boldsymbol{v}):\boldsymbol{v}\in\boldsymbol{X}_{\alpha,div}\}.$$

Moroever, $\boldsymbol{u}$ is the unique solution of the problem (9).

Proof. Corollary III.8 in [4] shows that the functional J is weakly lowest semi-continuous. The proof then follows from [10] using the reflexivity of $\mathbf{X}_{\alpha,div}$ and lemma 2.8-2.9-2.10. In particular, uniqueness comes from the strict convexity of J.

2.3 On the pressure of Stokes problem

In the previous section we focus on the existence of a divergence free velocity field $\mathbf{u}$. As the problem (1) involves the pressure, we study now existence of a solution of the mixed problem: find $(\mathbf{u}, p) \in \mathbf{X}_\alpha \times Y_\alpha$ such that

$$\begin{cases} \int_\Omega 2(\mu_L + \kappa^\alpha \rho l^{2-\alpha} d^\alpha_{\partial\Omega} |\varepsilon(\mathbf{u})|)\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}))dx - \int_\Omega p\,\mathrm{div}(\mathbf{v}) = \int_\Omega (\mathbf{F}\cdot\mathbf{v})dx, \\ \hfill \forall \mathbf{v} \in \mathbf{X}_\alpha, \\ \hfill \int_\Omega q\,\mathrm{div}(\mathbf{u}) = 0 \quad \forall q \in Y_\alpha, \end{cases} \tag{10}$$

with $0 \leq \alpha < 2$ and where Y_α is a space that should be defined. In particular, we investigate the existence of a pressure field $p \in Y_\alpha$ with Y_α an adequate functional space related to the velocity space $\mathbf{X}_\alpha$ that gives a sense of

$$\int_\Omega p\,\mathrm{div}\mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{X}_\alpha.$$

We start with some useful results:

Proposition 2.12 *The dual of the space $L^p(\Omega, d^\alpha_{\partial\Omega})$ can be identified with $L^q(\Omega, d^{-\alpha q/p}_{\partial\Omega})$, for $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Take a function $g \in L^p(\Omega, d^\alpha_{\partial\Omega})$ and define $\tilde{g}(x) = g(x) d^{\frac{\alpha}{p}}_{\partial\Omega}(x)$. Then $\tilde{g}$ is in $L^p(\Omega)$. We consider

$$B : L^p(\Omega, d^\alpha_{\partial\Omega}) \to L^p(\Omega) \text{ given by: } B(g) = \tilde{g}.$$

The operator B is linear and invertible, with $B^{-1}(\tilde{g}) = \tilde{g} d^{-\frac{\alpha}{p}}$. Suppose that K is in $L^p(\Omega, d^\alpha_{\partial\Omega})'$ (the dual space of $L^p(\Omega, d^\alpha_{\partial\Omega})$). We consider $\tilde{K} : L^p(\Omega) \to \mathbb{R}$ given by $\tilde{K}(\tilde{g}) = K(g)$ for all $g \in L^p(\Omega, d^\alpha_{\partial\Omega})$. We easily see that $\tilde{K}$ is a linear and continuous functional and thus there exists a unique $\tilde{u} \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\int_\Omega \tilde{u} v dx = \tilde{K}(v), \quad \forall v \in L^p(\Omega).$$

If we define $u = \tilde{u} d^{\frac{\alpha}{p}}_{\partial\Omega}$, it means that $u \in L^q(\Omega, d^{-\alpha q/p}_{\partial\Omega})$ and we have

$$K(g) = \tilde{K}(\tilde{g}) = \int_\Omega \tilde{u}\tilde{g} dx = \int_\Omega ug dx, \quad \forall g \in L^p(\Omega, d^\alpha_{\partial\Omega}).$$

We have shown that for each K in $L^p(\Omega, d^\alpha_{\partial\Omega})'$, there exists $u \in L^q(\Omega, d^{-\alpha q/p}_{\partial\Omega})$ unique such that

$$\int_\Omega ug dx = K(g), \quad \forall g \in L^p(\Omega, d^\alpha_{\partial\Omega}).$$

Consequently $\mathscr{K} : L^p(\Omega, d^{\alpha}_{\partial\Omega})' \to L^q(\Omega, d_{\partial\Omega}^{-\frac{\alpha q}{p}})$ given by $\mathscr{K}(K) = u$ is an isomorphism and $L^p(\Omega, d^{\alpha}_{\partial\Omega})'$ can be identified with $L^q(\Omega, d_{\partial\Omega}^{-\alpha q/p})$ within the "L^2 scalar product".

Definition 1. For all $\alpha \geq 0$ and $1 < p, q < \infty$, we denote

$$L^q_0(\Omega, d_{\partial\Omega}^{-\alpha q/p}) := \left\{ q \in L^q(\Omega, d_{\partial\Omega}^{-\alpha q/p}) | \int_\Omega q = 0 \right\}.$$

Lemma 2.13 *The spaces $[W_0^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})]^n$ and $L^p_0(\Omega, d^{\alpha}_{\partial\Omega})' := L^q_0(\Omega, d_{\partial\Omega}^{-\alpha q/p})$ satisfy the inf-sup condition: there exists $\tilde{C} > 0$ such that*

$$\inf_{q \in L^q_0(\Omega, d_{\partial\Omega}^{-\alpha q/p})} \sup_{\boldsymbol{v} \in [W_0^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})]^n} \frac{\int_\Omega q div(\boldsymbol{v}) dx}{\|q\|_{L^q_{d^{-\alpha q/p}}} \|\boldsymbol{v}\|_{W^{1,p}_{d^{\alpha}}}} > \tilde{C}.$$

Proof. From theorem 3.1 in [12], given $f \in L^p_0(\Omega, d^{\alpha}_{\partial\Omega})$, there exists a vector field $\mathbf{v} : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \mathbf{v} \in [W_0^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})]^n, \\ \text{div } \mathbf{v} = f, \\ \|\nabla \mathbf{v}\|_{W_0^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})} \leq c \|f\|_{L^p_0(\Omega, d^{\alpha}_{\partial\Omega})}. \end{cases} .$$

In other words, this show that the operator div: $[W_0^{1,p}(\Omega, d^{\alpha}_{\partial\Omega})]^n \to L^p_0(\Omega, d^{\alpha}_{\partial\Omega})$ is surjective. Lemma A.42 in [13] and proposition 2.12 conclude the proof.

We consider now the unique velocity field

$$\mathbf{u} \in \mathbf{X}_{\alpha,div} := \{\mathbf{v} \in [H^1_0(\Omega) \cap W_0^{1,3}(\Omega, d^{\alpha}_{\partial\Omega})]^n | \text{div}\mathbf{u} = 0\}$$

that solves

$$\int_\Omega (2\mu_L + \kappa^\alpha \rho l^{2-\alpha} d^{\alpha}_{\partial\Omega} |\varepsilon(\mathbf{u})|) \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})) dx = \int_\Omega (\mathbf{F} \cdot \mathbf{v}) dx, \quad \forall \mathbf{v} \in \mathbf{X}_{\alpha,div},$$

(see proposition 2.11). Since the inf-sup conditions is satisfied for the couple of spaces $[H^1_0(\Omega)]^n, L^2_0(\Omega)$, there exists a unique function $p_1 \in L^2_0(\Omega)$ such that

$$\int_\Omega p_1 \text{div}(\mathbf{v}) = \int_\Omega 2\mu_L \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})) dx - \int_\Omega (\mathbf{F} \cdot \mathbf{v}) dx, \quad \forall \mathbf{v} \in [H^1_0(\Omega)]^n.$$

On the other hand, and using Lemma 2.13 with $p = 3$ and $q = \frac{3}{2}$, we can also obtain a unique function $p_2 \in L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$ such that

$$\int_\Omega p_2 \text{div}(\mathbf{v}) = \int_\Omega 2\kappa^\alpha \rho l^{2-\alpha} d^{\alpha}_{\partial\Omega} |\varepsilon(\mathbf{u})|) \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})) dx, \quad \forall \mathbf{v} \in [W_0^{1,3}(\Omega, d^{\alpha}_{\partial\Omega})]^n.$$

Given now $\mathbf{X}_\alpha = [H_0^1(\Omega) \cap W_0^{1,3}(\Omega, d_{\partial\Omega}^{\alpha})]^n$ and $Y_\alpha := L_0^2(\Omega) \oplus L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$, we immediately deduce the following result for problem (10):

Theorem 2.14 *There exists* $(\boldsymbol{u}, p = p_1 + p_2) \in \boldsymbol{X}_\alpha \times Y_\alpha$ *such that relations* (10) *are satisfied.*

Remark 2.15 *In theorem 2.14, the pressure* $p \in Y_\alpha$ *is not necessary unique. In fact the second equation in* (10) *can be written as*

$$\int_\Omega q div(\boldsymbol{u}) = 0 \quad \forall q \in L_0^2(\Omega),$$
$$\int_\Omega q div(\boldsymbol{u}) = 0 \quad \forall q \in L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}}).$$

These two relations imply $div(\boldsymbol{u}) = 0$ *a.e. in* Ω *and are redundant. Thus as we are looking for p under the form* $p_1 + p_2 \in Y_\alpha$*, the decomposition could not be unique since in general* $L_0^2(\Omega)$ *is not included in* $L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$*.*

Nevertheless, uniqueness of the pressure is sometimes available. We start with a remark:

Remark 2.16 *Consider* $\Lambda = [0,1]$ *and the weight* $dist(x, \{0\}) = x$*. Thus if we take* $g \in L^3(\Lambda, d_{\{0\}}^{\alpha})$ *then we have:*

$$\int_0^1 |g|^2 dx = \int_0^1 (|g|^2 x^{\frac{2\alpha}{3}}) x^{-\frac{2\alpha}{3}} dx \leq \left(\int_0^1 |g|^3 x^\alpha dx \right)^{\frac{2}{3}} \left(\int_0^1 x^{-2\alpha} dx \right)^{\frac{1}{3}}.$$

The second integral $\left(\int_0^1 x^{-2\alpha} dx \right)^{\frac{1}{3}}$ *is bounded if* $0 \leq \alpha < \frac{1}{2}$*. When* $\alpha \geq \frac{1}{2}$*, this integral diverge. We then have* $\|g\|_{L^2(\Lambda)} \leq C \|g\|_{L^3_{d_{\{0\}}^{\alpha}}}$ *if* $0 \leq \alpha < \frac{1}{2}$*.*

The previous remark shows that if $0 \leq \alpha \leq \frac{1}{2}$, we have then $L^3(\Lambda, d_{\{0\}}^{\alpha}) \subset L^2(\Lambda)$. More generally, and using the proposition 6.5 in [17], we can show that there exists a number $\alpha_0 \leq \frac{1}{2}$ such that

$$W^{1,p}(\Omega, d_{\partial\Omega}^{\alpha}) \subset W^{1,p}(\Omega)$$

with continuous injection for $0 \leq \alpha < \alpha_0$. It means in particular that $L^2(\Omega) \subset L^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$ and thus $Y_\alpha := L_0^2(\Omega) \oplus L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$ becomes $Y_\alpha := L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$. Consequently, we can obtain the following result when $\mathbf{F}$ belongs to the dual space of $[W_0^{1,3}(\Omega, d_{\partial\Omega}^{\alpha})]^n$:

Theorem 2.17 *There exists* $0 < \alpha_0 \leq \frac{1}{2}$ *such that for all* $0 \leq \alpha < \alpha_0$ *and for all* $\boldsymbol{F} \in [L^{\frac{4}{3}}(\Omega)]^n$*, the problem* (10) *posesses a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{X}_\alpha \times Y_\alpha$*, with* $\boldsymbol{X}_\alpha = [W_0^{1,3}(\Omega, d_{\partial\Omega}^{\alpha})]^n$ *and* $Y_\alpha := L_0^{\frac{3}{2}}(\Omega, d_{\partial\Omega}^{-\frac{\alpha}{2}})$*.*

3 Some comments on the Von Karman model

The turbulent viscosity of the popular von Karman model ($\alpha = 2$) for a fluid flow between to close plates [19] is given by:

$$\mu = \mu_L + \kappa^2 d_{\partial\Omega}^2 |\varepsilon(\mathbf{u})|.$$

In fact, the weight $d_{\partial\Omega}^{\alpha}$ does not belong to the Muckenhoupt classe A_3 when $\alpha = 2$ [1, 6]. This has two major consequences:

1. The space $W^{1,3}(\Omega, d_{\partial\Omega}^2)$ has in fact no trace on the boundary (an example is given in one dimension by $g(x) = \ln(|\ln(x)|)$, $x \in (0, \frac{1}{2})$, with $dist(x,0) = d_{\partial\Omega}$). Recall that in [17] a trace operator Tr: $W^{1,p}(\Omega, d_{\partial\Omega}^{\alpha}) \to L^p(\partial\Omega)$ is defined if $1 < p < \infty$ and $0 \leq \alpha < p-1$. In that case, the space $W_0^{1,p}(\Omega, d_{\partial\Omega}^{\alpha})$ (the closure of $C_0^{\infty}(\Omega)$ for the norm (3)) can be identified with the space of functions in $W^{1,p}(\Omega, d_{\partial\Omega}^{\alpha})$ whose $Tr(u)$ is vanishing on the boundary. For $\alpha \geq p-1$, the trace operator cannot be defined and for $\alpha > p-1$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p}(\Omega, d_{\partial\Omega}^{\alpha})$ is the space itself.
   Nevertheless, the space $W^{1,3}(\Omega, d_{\partial\Omega}^2)$ does not correspond to any of these cases and its characterization is more complicated (see section 8 in [17]).
2. The second Korn inequality in [15] is valid only for $\mathbf{u} \in [W^{1,3}(\Omega, d_{\partial\Omega}^{\alpha})]^n$ with $-1 \leq \alpha < 2$. This is an open question when $\alpha = 2$ and thus we cannot prove the first Korn inequality. Counterexample is expected in that case.

The direct consequence of these remarks is that when $\mu_L = 0$, the von Karman model is ill-posed. In this case the boundary condition $u = 0$ on $\partial\Omega$ has no meaning.
A main consequence is when a numerical method is used for obtaining an approximation of von Karman model with μ_L small (with respect to the numerical viscosity), the obtained results depend strongly on the mesh of the method as shown in the following section.

4 Numerical experiments

In this section, we provide some numerical experiments of the problem (1) with viscosity (2) using different values of α and μ_L. The following benchmark example in three dimensional case is considered: let $\Omega \subset \mathbb{R}^3$ be the rectangular parallelepiped with characteristic length $l = 0.1$ given by

$$\Omega = [0;1] \times [0;1] \times [0;0.1].$$

For $N \in \mathbb{N}$ we discretize Ω by splitting each side of that rectangular parallelepiped with N nodes. It gives N^3 hexahedron, all of which are subdivided into five tetrahedron. We obtain then a triangulation $\mathscr{T}_h$ of Ω composed of $5N^3$ tetrahedron K with $h = \frac{1}{N}$ being the reference mesh size.

Let $\mathbb{P}^1(K)$ be the space of polynomial of degree one on K. We define the following finite dimensional spaces:

$$\begin{aligned}\chi_h &= \{\mathbf{v}\in C^0(\Omega_R)^3 | \mathbf{v}_{|K} \in (\mathbb{P}^1(K)\oplus B_K)^3 \text{ and } \mathbf{v}_{|\partial\Omega_R} = \mathbf{0}\},\\ \Upsilon_h &= \{q\in C^0(\Omega_R) | q_{|K} \in \mathbb{P}^1(K) \text{ and } \int_\Omega q dx = 0\}.\end{aligned}$$

Here B_K denote the Bubble function on K. A renormalized version of the Problem (1)-(2) is discretized with this Galerkin approximation to obtain approximate solutions $(\mathbf{u}_h, p_h) \in \chi_h \times \Upsilon_h$. In particular we set $\nu = \frac{\mu}{\rho}$, with renormalized p and $\mathbf{f}$ divided by ρ $(p := p/\rho, \mathbf{f} := \mathbf{f}/\rho)$. In that case the turbulent kinematic viscosity is given by

$$\nu = \nu_L + \kappa^\alpha l^{2-\alpha} d^\alpha_{\partial\Omega} |\varepsilon(\mathbf{u})|.$$

The renormalized relation (1)-(2) is a non-linear problem which is solved by a Newton method based on the work [14]. The method is iterated to reach a velocity field $\mathbf{u}_h$ with a precision of $Tol_{New} = 1e^{-8}$.

Each iteration of that Newton method leads to solve a linear system given by the Galerkin matrix of the Stokes problem. This system is solved with GMRES algorithm [20] with ILU(2) preconditioner and a tolerance of $Tol_{GMRES} = 1e^{-8}$.

In all the following computations, we consider the following force field which generates a velocity field composed of two axial symmetric vortex:

$$\mathbf{F}(x,y,z) = \begin{pmatrix} 0.3*(y-0.5)^2 \\ 0.3*(-x+0.5)^2 \\ 0 \end{pmatrix}.$$

On figure 1, we display for different values of α and ν_L the maximum of the Euclidian norm of the velocity field u_{max}, the numerical kinematic viscosity ν_L (the numerical value of $l^{2-\alpha}\kappa^\alpha d^\alpha_{\partial\Omega}|\varepsilon(u_{max})|$) and the resulting Reynolds number $Re_T = \frac{u_{max} l}{\nu_T}$. The main observations are the following:

- For $\alpha = 0$ and different values of ν_L, the maximum value of the velocity converges as the mesh decreases and does not depend of ν_L.
- When $\alpha \in \{1,2\}$ the convergence is more difficult to obtain, especially in the case $\alpha = 2$. We observe that when N is increasing, the maximum value of the velocity increases too. Consequently, when the laminar viscosity ν_L is small with respect to the numerical viscosity ν_T, the obtained results depend strongly on the mesh. The same behavior is observed for the stationary Navier-Stokes equations corresponding to (1)- (2).

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		$\alpha = 0$											
		STOKES						NAVIER-STOKES					
		$\nu_L = 1e^{-5}$			$\nu_L = 1e^{-7}$			$\nu_L = 1e^{-5}$			$\nu_L = 1e^{-7}$		
		ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T
N	20	4.05e-4	4.84e-2	12	4.08e-4	4.86e-2	12	4.01e-4	4.83e-2	12	4.06e-4	4.85e-2	12
	40	4.22e-4	5.21e-2	12	4.29e-4	5.28e-2	12	4.21e-4	5.20e-2	12	4.27e-4	5.26e-2	12
	80	4.32e-4	5.32e-2	12	4.36e-4	5.38e-2	12	4.30e-4	5.30e-2	12	4.34e-4	5.33e-2	12

		$\alpha = 1$											
		STOKES						NAVIER-STOKES					
		$\nu_L = 1e^{-5}$			$\nu_L = 1e^{-7}$			$\nu_L = 1e^{-5}$			$\nu_L = 1e^{-7}$		
		ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T
N	20	2.54e-4	1.59e-1	62	2.57e-4	1.61e-1	63	2.46e-4	1.54e-1	63	2.48e-4	1.56e-1	63
	40	2.55e-4	1.84e-1	72	2.58e-4	1.88e-1	73	2.47e-4	1.83e-1	73	2.50e-4	1.86e-1	74
	80	2.55e-4	1.99e-1	78	2.58e-4	2.04e-1	79	2.49e-4	1.95e-1	78	2.51e-4	2.01e-1	80

		$\alpha = 2$											
		STOKES						NAVIER-STOKES					
		$\nu_L = 1e^{-5}$			$\nu_L = 1e^{-7}$			$\nu_L = 1e^{-5}$			$\nu_L = 1e^{-7}$		
		ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T	ν_T	u_{max}	Re_T
N	20	1.79e-4	2.01e-1	112	1.85e-4	2.07e-1	112	1.61e-4	1.95e-1	121	1.68e-4	2.01e-1	121
	40	1.88e-4	2.68e-1	142	1.95e-4	2.81e-1	144	1.64e-4	2.45e-1	149	1.66e-4	2.55e-1	152
	80	1.97e-4	3.20e-1	162	2.10e-4	3.45e-2	173	1.62e-4	2.79e-1	172	1.69e-4	2.98e-1	177

Fig. 1 Numerical resolution of the normalized non-linear stationary Stokes problem (1)-(2) and the stationary Navier-Stokes problem for different values of α and ν_L. The domain is a rectangular parallelepiped $\Omega = [0,1] \times [0,1] \times [0,0.1]$ with N nodes on each side for a total of $5N^3$ tetrahedra. The force is given by $f = (0.3*(y-0.5)^2, 0.3*(x-0.5)^2, 0)$ and we set $l = 0.1$.

References

1. AIMAR, H. CARENA, M. DURN, R and TOSCHI M. *Powers of distance to lower dimensional sets as Muckenoupt weights*. 2010 Mathematics Subject Classification.
2. AMBRIOSO, L, PINAMONTI, A, SPEIGHT, G. *Weighted Sobolev Spaces on Metric Measure Spaces*. Preprint, June 2014.
3. BARANGER, J, NAJIB, K. *Analyse numérique des écoulements quasi-Newtoniens dont la viscosité obéit à la loi puissance ou la loi de carreau* . Numer. Math. 58, 35-49 (1990).
4. BREZIS, H *Analyse fonctionnelle, théorie et applications*. Masson, Paris, 1983.
5. BROWN, R.C *Some Embeddings of Weighted Sobolev Spaces on Finite Measure and Quasibounded Domains*. J. of Inequal. and Appl., 1998, Vol. 2, pp: 325-356.
6. CAVALHEIRO, A.C *Existence of solutions for Dirichlet problem of some degenerate quasilinear elliptic equations*. Complex Variables and Elliptic Equations, Vol. 53, no 2, February 2008.
7. CAVALHEIRO, A.C *Existence Results for Dirichlet Problems with Degenerated p-Laplacian*. Opuscula Math. 33, no 3, 2013.

8. CAVALHEIRO, A.C *Weighted Sobolev Spaces and Degenerate Elliptic Equations*. Bol. Soc. Paran, Mat, 2008.
9. COLINGE, J. and RAPPAZ, J. *A strongly nonlinear problem arising in glaciology*. M2AN Math. Model Numer. Anal., 33(2):395-406, 1999.
10. DACOROGNA, B. *Introduction to the calculus of variations*. Imperial College Press, 2004.
11. DRABEK, P. KUFNER, A. NICOLOSI, F. *Quasilinear Elliptic Equations with Degenerations and Singularities*. Imperial College Press, 2004.
12. DURAN, R.G. and GARCIA F.L. *Solutions of the divergence operator and Korn inequality on domains with an external cusp*. Mathematical Subject Classification, 1991.
13. ERN, A, GUERMOND J-L.. *Theory and Practice of Finite Elements*. Springer. New-York, 2004.
14. JOUVET, G. *Modélisation, analyse mathématique et simulation numérique de la dynamique des glaciers*. PHD thesis, Ecole Polytechnique Fédérale de Lausanne, 2011. Thèse No 4677.
15. KALAMAJSKA, A. *Coercive inequalities on weighted sobolev spaces*. Colloquium Mathematicum, Vol. LXVI, 1993, FASC. 2.
16. KONDRAT'EV, V.A and O.A Olenik *Boundary value problems for systems of elasticity theory in unbounded domains and Korn inequalities*. Uspekhi Mat, Nauk 43(5) (1988), 55-98.
17. KUFNER, F. *Weighted Sobolev Spaces*. Elsevier, Paris, 1983.
18. OPIC, B and GURKA, P. *Continuous and Compact embeddings of weighted Sobolev spaces*. Czechoslovak Mathematical Journal, Vol.39 (1989), No. 1 78-94.
19. POPE, S. *Turbulent Flows*. Cambridge University Press, 2000.
20. QUARTERONI, A. and Valli, A. *Numerical Approximation of Partial Differential Equations*. Springer-Verlag, Berlin, Heidelberg, New-York, 1994.
21. SANDRI, D. *Sur l'approximation numérique des écoulements quasi-newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau*. Modélisation mathématique et analyse numérique, tome 27, number 2 (1993), p. 131-155.