RECOVERING A POTENTIAL FROM CAUCHY DATA VIA COMPLEX GEOMETRICAL OPTICS SOLUTIONS

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ABSTRACT. This paper is devoted to the problem of recovering a potential q in a domain in $\mathbb{R}^d$ for $d \geq 3$ from the Dirichlet to Neumann map. This problem is related to the inverse Calderón conductivity problem via the Liouville transformation. It is known from the work of Haberman and Tataru [11] and Nachman and Lavine [17] that uniqueness holds for the class of conductivities of one derivative and the class of $W^{2,d/2}$ conductivities respectively. The proof of Haberman and Tataru is based on the construction of complex geometrical optics (CGO) solutions initially suggested by Sylvester and Uhlmann [22], in functional spaces introduced by Bourgain [2]. The proof of the second result, in the work of Ferreira et al. [10], is based on the construction of CGO solutions via Carleman estimates. The main goal of the paper is to understand whether or not an approach which is based on the construction of CGO solutions in the spirit of Sylvester and Uhlmann and involves only standard Sobolev spaces can be used to obtain these results. In fact, we are able to obtain a new proof of uniqueness for the Calderón problem for 1) a slightly different class as the one in [11], and for 2) the class of $W^{2,d/2}$ conductivities. The proof of statement 1) is based on a new estimate for CGO solutions and some averaging estimates in the same spirit as in [11]. The proof of statement 2) is on the one hand based on a generalized Sobolev inequality due to Kenig et al. [14] and on another hand, only involves standard estimates for CGO solutions [22]. We are also able to prove the uniqueness of a potential for 3) the class of $W^{s,3/s}$ ($\supsetneq W^{2,3/2}$) conductivities with $3/2 < s < 2$ in three dimensions. As far as we know, statement 3) is new.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 3)$ with C^1 boundary and let $q \in L^{d/2}(\Omega)$, an assumption that will be weaken later. We consider the Dirichlet to Neumann map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by

$$\Lambda_q(f) = g,$$

where

$$g = \frac{\partial v}{\partial \eta}\Big|_{\partial\Omega},$$

and $v \in H^1(\Omega)$ is the unique solution to the system

$$\begin{cases} \Delta v - qv = 0 & \text{in } \Omega, \\ v = f & \text{on } \partial\Omega. \end{cases}$$

Here and in what follows η denotes a unit normal vector directed into the exterior of Ω. We assume here that 0 is not a Dirichlet eigenvalue for this problem; this implies Λ_q is well-defined (this assumption is not essential and is discussed later in Remark 1). In this paper, we are interested in the injectivity of Λ_q for $d \geq 3$. This problem has a connection to

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the inverse conductivity problem posed by Calderón in [6]. In [6] Calderón asked whether one can determine $\gamma \in L^\infty(\Omega)$ with $\mathrm{essinf}_\Omega \gamma > 0$ from its Dirichlet to Neumann map $DtN_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by

$$DtN_\gamma(f) = \gamma \frac{\partial u}{\partial \eta},$$

where $u \in H^1(\Omega)$ is the unique solution to the equation

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega \text{ and } u = f \text{ on } \partial\Omega.$$

In the same paper, Calderón proved the injectivity of the derivative of the map $\gamma \to DtN_\gamma$ at $\gamma =$ constant. Kohn and Vogelius [15, 16] showed that if $\partial\Omega$ is C^∞ then Λ_q determines q and all its derivatives on $\partial\Omega$ and then used this to prove uniqueness for the class of piecewise analytic coefficients. Sylvester and Uhlmann [22] proved that Λ_q uniquely determines q if $q \in C^\infty$; their method also gave the injectivity of Λ_q for $q \in L^\infty$ (see also [20]). In [22], they introduced the concept of *complex geometrical optics* (CGO) solutions which plays an important role in establishing the uniqueness for inverse problems for $d \geq 3$. In one direction, the L^∞ uniqueness result was improved by Chanillo, and Kenig and Jerrison in [7] and Lavine and Nachman in [17]. In [7], Chanillo established the injectivity of Λ_q for $q \in L^{d/2}$ with small norm and (in the same paper) Kenig and Jerrison obtained the injectivity of Λ_q for $q \in L^p$ for any $p > d/2$. In [17], the authors announced the injectivity of Λ_q holds for $q \in L^{d/2}$. Recently, this result has been extend by Ferreira et al. in [10] for compact Riemannian manifolds with boundary which are conformally embedded in a product of the Euclidean line and a simple manifold. Their technique is based on Carleman estimates. In another direction, the injectivity of Λ_q was established for $q \in B^{-s}_{\infty,2}$ [1] $(0 < s < 1/2)$, $q \in B^{-1/2}_{\infty,2}$, and for $q \in W^{-1/2,s}$ $(s > 2d)$ by Brown in [4], Päivärinta et al. in [19], and Brown and Torres in [3], respectively. Recently, Haberman and Tataru in [11] established the injectivity of DtN_γ (Calderón's problem) for $\gamma \in C^1(\Omega)$ or $\gamma \in W^{1,\infty}(\overline{\Omega})$ with a smallness assumption on the derivative. The corresponding uniqueness result for Λ_q would hold for $q \in W^{-1,\infty}$ with some kind of smallness assumption; however, obtaining this conclusion from their approach is not clear to us. The approach in [4, 19, 3] is via CGO solutions. The approach due to Haberman and Tataru is also via CGO solutions; the novelty in their approach stems from their use of weighted spaces and averaging arguments. Some refinements for piecewise smooth potentials q can be found in references therein (see also [12]). We note that the result of Lavine and Nachman is not a consequence of the one of Haberman and Tataru and vice versa since $L^{d/2}(\Omega) \not\in W^{-1,\infty}(\Omega)$ and $W^{-1,\infty}(\Omega) \not\in L^{d/2}(\Omega)$. In dimension 2, the injectivity of Λ_q was established by Astala and Päivärinta in [1]. Previous contributions in the $2d$ case can be found in [21, 5] and references therein.

The standard method to establish uniqueness for the Calderón problem is to prove the injectivity of Λ_q. This can be done by the Liouville transform and using the fact that one can recover the boundary data from the Dirichlet to Neumann map since

$$\Delta v - qv = 0 \text{ in } \Omega \tag{1.1}$$

if and only if

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega,$$

where $u = \gamma^{1/2} v$ and $q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$. It is known that (see e.g. [13, (5.0.4)]) if

$$\Lambda_{q_1} = \Lambda_{q_2},$$

[1] $B^s_{p,q}$ denotes the Besov spaces.

then

$$\int_\Omega (q_1 - q_2) v_1 v_2 = 0 \tag{1.2}$$

for any $v_i \in H^1(\Omega)$ $(i = 1, 2)$ a solution of the equation

$$\Delta v_i - q_i v_i = 0 \text{ in } \Omega.$$

The crucial idea of Sylvester and Uhlmann in [22] is to the find a (large) class of solutions of the equation

$$\Delta v - qv = 0 \text{ in } \mathbb{R}^d$$

of the form

$$v = (1 + w)e^{x\cdot\xi/2} \text{ in } \mathbb{R}^d,$$

where $\xi \in \mathbb{C}^d$ with $\xi \cdot \xi = 0$ and $|\xi|$ is large. Since $\xi \cdot \xi = 0$, it follows that

$$\Delta w + \xi \cdot \nabla w - qw = q \text{ in } \mathbb{R}^d; \tag{1.3}$$

here one extends q appropriately on $\mathbb{R}^d$ and denotes the extension also by q. Their key observation is

$$\lim_{|\xi|\to 0} \|w\|_{H^1(B_r)} = 0 \text{ for } r > 0, \tag{1.4}$$

which is a consequence of the following fundamental estimate established in [22]:

$$\|W\|_{H^1(B_r)} \le \frac{C_r}{|\xi|} \|f\|_{H^1} \quad \forall\, r > 0, \tag{1.5}$$

if f has compact support, where W is the solution to the equation

$$\Delta W + \xi \cdot \nabla W = f \text{ in } \mathbb{R}^d. \tag{1.6}$$

By appropriate choices of ξ_1 and ξ_2 for the associated v_1 and v_2 with $\xi_1 + \xi_2 = 2k$, a constant vector in $\mathbb{C}^d$, then using (1.2) and (1.4), they show that

$$\int_\Omega (q_1 - q_2) e^{k\cdot x} = 0 \text{ for all } k \in \mathbb{C}^d.$$

This in turn implies

$$q_1 = q_2.$$

In [3, 4], the authors improved this estimate for solutions to (1.6) in a Besov space where f has $-1/2$ derivatives. The proof in [19] is based on a different way of constructing CGO solutions.

We next discuss the approach due to Haberman and Tataru in [11]. The key point in [11] is to consider solutions to (1.6) in $X_\xi^{1/2}$ with $f \in X_\xi^{-1/2}$, where

$$\|f\|_{X_\xi^s} := \Big\| \big| |k|^2 + k \cdot \xi \big|^s \hat{f}(k) \Big\|_{L^2} \quad \text{ for } s \in \mathbb{R}.$$

These special function spaces have roots from the work of Bourgain in [2]. Their key estimates involves various quantities related to L^2-norm of a function by its norm in X_ξ^s with $s = -1/2$ or $1/2$. This is given in [11, Lemma 2.2]. Another ingredient in their proof is an averaging estimate for solutions to (1.6), [11, Lemma 3.1].

The work of Kenig and Jerison in [7] is in the spirit of [22] but uses a generalized Sobolev inequality, due to Kenig et al. in [14]. This Sobolev inequality for W, a solution to (1.6), is of the form

$$\|W\|_{L^p} \le C\|f\|_{L^{p'}}, \tag{1.7}$$

if $1 < p < +\infty$ and $1 < p' := pd/(d+2) < +\infty$. In [7] the requirement $p > d/2$ is used to showed that

$$\|W\|_{L^q} \le C|\xi|^{-\alpha}\|V\|_{L^q},$$

where $\alpha = 2 - d/p$ and $(q-2)/q = 1/p$, and W is the solution to equation (1.8) below. This estimate was used in their iteration process to obtain solutions to (1.3).

The construction of CGO solutions by Ferreira et al. in [10] is quite different and based on a limiting Carleman's estimate originating in the work of [9].

The goal of the paper is to introduce an approach, which is based on the construction of CGO solutions in the spirit of Sylvester and Uhlmann and involves only standard Sobolev spaces, to prove the following results:

i) Λ_q uniquely determines q if $q = \operatorname{div} g_1 + g_2$ where $\inf_{\phi \in [C(\bar\Omega)]^d} \|g_1 - \phi\|_{L^\infty}$ is small, $g_1 \in L^\infty(\Omega) \cap C^0(\overline{\Omega}_\delta)$ for some $\delta > 0$, $\hat g_1 \in L^p$ for some $p < 2$, and $g_2 \in L^d$. Here $\overline{\Omega}_\delta = \overline{\{\operatorname{dist}(x, \partial\Omega) < \delta\} \cap \Omega}$ (Theorem 1).

ii) Λ_q uniquely determines q for $q \in L^{d/2}$ (Theorem 2).

iii) Λ_q uniquely determines q if $q = \operatorname{div} g_1 + g_2$ where $g_1 \in W^{t, 3/(t+1)}(\Omega)$ for some $t > 1/2$ and $g_2 \in L^{3/2}(\Omega)$ in three dimensions (Theorem 3).

To this end, we extend results of Sylvester and Uhlmann in [22] on the stability of solutions to (1.6) for one negative order. The proof is different from the one in [22] and quite simple. The same approach also implies similar results as in [22]. The proof of *i*) is mainly based on a new observation on the stability of the following equation (see Lemma 2)

$$\Delta W + \xi \cdot \nabla W = qV \text{ in } \mathbb{R}^d, \tag{1.8}$$

which is the key for the iteration process to obtain a solution to (1.3), and an averaging argument for initial data (see Lemma 4) in the same spirit of [11]. The (new) proof of *ii*) in this paper is (only) based on a combination of the generalized Sobolev inequality and the standard approach used in [22] (see Proposition 4); however, the iteration process used to obtain solutions to (1.3) is quite tricky. The proof of *iii*) is based on an averaging argument on **both initial data and the kernel** (see Lemmas 5 and 6).

Statement i) is slightly different from what one can derive directly from the results of Haberman and Tataru. Statement ii) is Lavine and Nachman's result. Statment iii) **implies the results of Lavine and Nachmann in three dimensions and yields uniqueness for a larger class of conductivities**. As a consequence, we give a new proof for Haberman and Tataru's result under a mild additional assumption (Corollary 1), Lavine and Nachman's result, and prove the uniqueness of Calderón's problem for the class of $W^{s,3/s}$ (for some $s > 3/2$) conductivities in three dimensions (Corollary 3); this last result is new as far as we know.

Let us describe the ideas of the proof of each conclusion in more detail. Without loss of generality one may assume that $\operatorname{supp} q \subset B_1$. Here and in what follows $B_r(a)$ denotes the ball centered at a of radius r, and B_r denotes $B_r(0)$. Concerning i), our new key estimate for solutions to (1.8) is

$$\|\nabla W\|_{L^2(B_r)} + |\xi| \cdot \|W\|_{L^2(B_r)} \le C_r\big(\|g_1\|_{L^\infty} + \|g_2\|_{L^d}\big)\Big(\|\nabla V\|_{L^2(B_1)} + |\xi| \cdot \|V\|_{L^2(B_1)}\Big),$$

if $q = \operatorname{div} g_1 + g_2$ and $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset\subset B_1$, see Lemma 2. The proof of this inequality is based on an estimate for solutions to (1.6) in which $f \in H^{-1}$ in the spirit (1.5) and is presented in Lemma 1. The proof of Lemma 1 is quite elementary and different from the proof in [22]. After this, we employ some average estimates, as in [11]. We remark that we will need $g_1 \in C^0(\overline{\Omega}_\delta)$ to ensure the existence of a trace when turning the elliptic PDE (1.1) into the integral (1.2). Concerning ii), we first split q into $f+g$ where f is smooth and $\|g\|_{L^{d/2}}$ is small. Using the generalize Sobolev inequality (1.7) and the standard estimates for CGO solutions (1.5), we are able to reach

$$\lim_{|\xi|\to\infty} \|w\|_{H^1(B_r)} = 0$$

where w is the solution to (1.3). The iteration process to obtain the existence of w and the estimate of w mentioned above are rather tricky in this case. Concerning iii), our key ingredient are 1) the following estimate for solutions to (1.8)

$$\|W\|_{H^1(B_r)} \le E(q,\xi)\|V\|_{H^1}$$

for some $E(q,\xi)$ (see Lemma 5), and 2) the observation that, roughly speaking, if $q \in H^{-1/2}$ with compact support then $E(q,\xi) \to 0$ as $\xi \to \infty$ for a large set of ξ's (see Proposition 5). At this point we both average as in [11] and also average $E(q,\xi)$; the estimate for solutions of (1.8) depends on the direction of ξ and q.

We state these results explicitly. Concerning i), using the construction of CGO solutions in the spirit of Sylvester and Uhlmann in standard Sobolev spaces and some new observations (Lemma 2, see also 2.6), we can reach

Theorem 1. *Let $d \ge 3$, Ω be a smooth bounded subset of $\mathbb{R}^d$. Let $g_1, h_1 \in L^\infty(\Omega)\cap C^0(\overline{\Omega}_\delta)$ for any $\delta > 0$, $g_2, h_2 \in L^d(\Omega)$ be such that*

$$\|\mathcal{F}(1_\Omega g_1)\|_{L^p} + \|\mathcal{F}(1_\Omega h_1)\|_{L^p} < \infty^2 \textit{ for some } 1 < p < 2. \tag{1.9}$$

Set

$$q_1 = \operatorname{div} g_1 + g_2 \quad \textit{and} \quad q_2 = \operatorname{div} h_1 + h_2.$$

Assume that

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then there exists a positive constant c such that if

$$\inf_{\phi\in C(\bar\Omega)} \|g_1 - \phi\|_{L^\infty} + \inf_{\phi\in C(\bar\Omega)} \|h_1 - \phi\|_{L^\infty} \le c. \tag{1.10}$$

then

$$q_1 = q_2.$$

As a consequence, we obtain the following result which is slightly weaker from the one of Haberman and Tataru in [11].

Corollary 1. *Let $d \ge 3$, Ω be a smooth bounded subset of $\mathbb{R}^d$, $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega)\cap C^1(\Omega_\delta)$ for some $\delta > 0$ be such that*

$$1/\lambda \le \gamma_1(x), \gamma_2(x) \le \lambda \textit{ for a.e. } x \in \Omega,$$

for some $\lambda > 0$ and

$$\mathcal{F}\big(1_\Omega \nabla \ln \gamma_i\big) \in L^p \textit{ for some } 1 < p < 2. \tag{1.11}$$

Assume that

$$DtN_{\gamma_1} = DtN_{\gamma_2},$$

[2] Here 1_Ω denotes the characteristic function of Ω and $\mathcal{F}$ denotes the Fourier transform. This technical condition arises from our averaging estimates in Lemma 4 (see Remark 3).

then there exists a positive constant c such that if

$$\inf_{\phi \in [C(\bar{\Omega})]^d} \|\nabla \ln \gamma_1 - \varphi\|_{L^\infty} + \inf_{\phi \in [C(\bar{\Omega})]^d} \|\nabla \ln \gamma_2 - \varphi\|_{L^\infty} \le c, \tag{1.12}$$

then

$$\gamma_1 = \gamma_2.$$

Assumption (1.11) is a mild condition since it holds holds for $p = 2$ since $g_1, h_1 \in L^\infty(\Omega)$. Assumption (1.11) is not required in [11]. The requirement that $\gamma_1, \gamma_2 \in C^1(\Omega_\delta)$ does not appear in [11]. Statement (1.12) is stronger than their results; however, their method can derive (1.12) as well.

Concerning ii), we give a new proof of

Theorem 2. *Let $d \ge 3$, Ω be a smooth bounded subset of $\mathbb{R}^d$. Let $q_1, q_2 \in L^{d/2}(\Omega)$. Assume that*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

As a consequence of Theorem 2, one obtains

Corollary 2. *Let $d \ge 3$, Ω be a smooth bounded subset of $\mathbb{R}^d$, $\gamma_1, \gamma_2 \in W^{2,d/2}(\Omega)$ be such that*

$$1/\lambda \le \gamma_1(x), \gamma_2(x) \le \lambda \text{ for a.e. } x \in \Omega,$$

for some $\lambda > 0$. Assume that

$$DtN_{\gamma_1} = DtN_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

Concerning iii), we obtain the following new result

Theorem 3. *Let Ω be a smooth bounded subset of $\mathbb{R}^3$, $g_1, h_1 \in W^{t,3/(t+1)}(\Omega)$ for some $t > 1/2$, $g_2, h_2 \in L^{3/2}(\Omega)$. Set*

$$q_1 = \operatorname{div} g_1 + g_2 \text{ and } q_2 = \operatorname{div} h_1 + h_2.$$

Assume that

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Here is a consequence of Theorem 3.

Corollary 3. *Let Ω be a smooth bounded subset of $\mathbb{R}^3$, $\gamma_1, \gamma_2 \in W^{s,3/s}(\Omega)$ for some $s > 3/2$ be such that*

$$1/\lambda \le \gamma_1(x), \gamma_2(x) \le \lambda \text{ for a.e. } x \in \Omega,$$

for some $\lambda > 0$. Assume that

$$DtN_{\gamma_1} = DtN_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

Remark 1. *In Theorems 1, 2, and 3,* 0 *is assumed not a Dirichlet eigenvalue for the potential problems. Then the fact that* $\Lambda_{q_1} = \Lambda_{q_2}$ *implies* $q_1 = q_2$*. In fact this assumption can be weaken as follows. Assume that*

$$\frac{\partial v_1}{\partial \eta} = \frac{\partial v_2}{\partial \eta},$$

for any $v_1, v_2 \in H^1(\Omega)$ *such that*

$$\Delta v_i - q_i v_i = 0 \text{ in } \Omega \text{ for } i = 1, 2, \text{ and } v_1 = v_2 \text{ on } \partial\Omega.$$

Then $q_1 = q_2$ *under the same conditions on* q_i*,* $i = 1, 2$*. In fact, we prove Theorems 1, 2, and 3 under this weaker assumption.*

The paper is organized as follows. In Section 2, we establish new estimates for CGO solutions in the spirit of Sylvester and Uhlmann. In Section 3 we establish Theorem 1 and Corollary 1. This is established by generating CGO solutions via a direct iteration method and averaging methods. We then turn to the proof of Theorem 2 and Corollary 2 in Section 4. Section 5 handles the proof of Theorem 3 and Corollary 3. Finally, in Appendix A we provide a few results on averaging of the kernel $K_\xi(x)$ to (1.8) that are used crucially in our CGO arguments, and in Appendix B we establish that $\gamma_1 = \gamma_2$ on $\partial\Omega$ if $DtN_{\gamma_1} = DtN_{\gamma_2}$ and γ_1, γ_2 belong only to $W^{1,1}(\partial\Omega)$. We recall again that all above results are only obtained via the construction of CGO solutions in standard Sobolev spaces and averaging arguments.

2. New estimates for CGO solutions in the spirit of Sylvester and Uhlmann

In this section, we recall and extend the fundamental estimates due to Sylvester and Uhlmann in [22] concerning solutions of the equation

$$\Delta w + \xi \cdot \nabla w = f \tag{2.1}$$

where $\xi \in C^d$ and $\xi \cdot \xi = 0$.

Given $\xi \in \mathbb{C}^d$ with $|\xi| > 2$ and $\xi \cdot \xi = 0$, define

$$\widehat{K_\xi}(k) = \frac{1}{-|k|^2 + i\xi \cdot k} \quad \text{for } k \in \mathbb{R}^d.$$

Then for $f \in H^{-1}(\mathbb{R}^d)$ with compact support, $K_\xi * f$ is a solution to the equation

$$\Delta w + \xi \cdot \nabla w = f \text{ in } \mathbb{R}^d,$$

and

$$\widehat{K_\xi * f} = \widehat{K}_\xi \cdot \hat{f} \in L^1 + L^2.$$

We recall the following fundamental results due to Sylvester and Uhlmann in [22].

Proposition 1 (Sylvester-Uhlmann)**.** *Let* $-1 < \delta < 0$*,* $\xi \in \mathbb{C}^d$ *with* $|\xi| > 2$ *and* $\xi \cdot \xi = 0$*, and let* $f \in L^2_{loc}(\mathbb{R}^d)$*. Then*

$$\|K_\xi * f\|_{H^k_\delta} \le \frac{C}{|\xi|} \|f\|_{H^k_{1+\delta}} \quad \text{for } k \ge 0, \tag{2.2}$$

$$\|K_\xi * f\|_{H^{k+1}_\delta} \le C \|f\|_{H^k_{1+\delta}} \quad \text{for } k \ge 0. \tag{2.3}$$

for some positive constant C *independent of* ξ *and* f*.*

Here

$$\|v\|_{L^2_\delta} := \|(1+|\cdot|^2)^\delta v(\cdot)\|_{L^2}$$

and

$$\|v\|_{H^k_\delta} := \sum_{|\alpha|=0}^{k} \|(1+|\cdot|^2)^\delta D^\alpha v(\cdot)\|_{L^2}.$$

These estimates play an important role in their proof of the uniqueness of smooth potentials [22] and in the proofs of the improvements in [4, 19, 10].

We will extend the above results to negative derivatives and to the case with two derivative difference, which are crucial for the proof of Theorem 1. Our proof for negative derivatives and the two derivative difference is rather elementary. The same proof also gives the following estimates, for $f \in L^2(\mathbb{R}^d)$ with compact support,

$$\|K_\xi * f\|_{H^k(B_r)} \le \frac{C_r}{|\xi|}\|f\|_{H^k} \quad \text{for } k \ge 0, \tag{2.4}$$

and

$$\|K_\xi * f\|_{H^{k+1}(B_r)} \le C_r\|f\|_{H^k} \quad \text{for } k \ge 0 \tag{2.5}$$

Here C is a positive number independent of ξ and f. These estimates are slightly weaker than the original ones of Sylvester and Uhlmann in (2.2) and (2.3); however, they are sufficient for establishing the uniqueness of smooth potential in [22]. Here is the extension:

Lemma 1. *Let $R > 0$, $\xi \in \mathbb{C}^d$ with $|\xi| > 2$ and $\xi \cdot \xi = 0$, and let $f \in H^{-1}(\mathbb{R}^d)$ with* $\operatorname{supp} f \subset B_R$. *Then*

$$\|K_\xi * f\|_{L^2(B_r)} \le C_r\|f\|_{H^{-1}} \tag{2.6}$$

and

$$\|K_\xi * f\|_{H^{k+1}(B_r)} \le C_r|\xi| \cdot \|f\|_{H^{k-1}}, \quad \textit{for } k \ge 0, \tag{2.7}$$

for some C_r which depends on r and R but is independent of ξ and f.

Proof. We will prove (2.6); the proof of (2.7) as well (2.4) and (2.5) follow similarly. Set

$$\Gamma_\xi := \{k \in \mathbb{R}^d;\ -|k|^2 + i\xi \cdot k = 0\}.$$

It is clear that

$$|\hat{K}_\xi(k)| \le \frac{C}{|\xi| \operatorname{dist}(k, \Gamma_\xi)} \text{ if } |k| \le 2|\xi|, \text{ and } |\hat{K}_\xi(k)| \le \frac{C}{|k|^2} \text{ if } |k| \ge 2|\xi|, \tag{2.8}$$

In this proof, C denotes a positive constant independent of ξ and f. Define $K_{1,\xi}$ and $K_{2,\xi}$ as follows

$$\hat{K}_{1,\xi}(k) = \begin{cases} \hat{K}_\xi(k) & \text{if } \operatorname{dist}(k, \Gamma_\xi) \ge 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

and

$$\hat{K}_{2,\xi}(k) = \hat{K}_\xi(k) - \hat{K}_{1,\xi}(k), \tag{2.10}$$

and so

$$\|K_\xi * f\|_{L^2(B_r)} \le \|K_{1,\xi} * f\|_{L^2(B_r)} + \|K_{2,\xi} * f\|_{L^2(B_r)}. \tag{2.11}$$

Using Plancherel's theorem, we derive from (2.8) and (2.9) that

$$\|K_{1,\xi} * f\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{H^{-1}}. \tag{2.12}$$

Fix

$$\varphi \in C_0^\infty(\mathbb{R}^d) \text{ with } \varphi = 1 \text{ in } B_{R+r}. \tag{2.13}$$

Since $\operatorname{supp} f \subset B_R$, it follows that $f = \varphi f$; hence

$$\hat{f} = \hat{\varphi} * \hat{f}.$$

Define

$$\widetilde{f}(k) = \sup_{\eta \in B_4(k)} |\hat{f}(\eta)| \tag{2.14}$$

and

$$\widetilde{\varphi}(k) = \sup_{\eta \in B_4(k)} |\hat{\varphi}(\eta)|. \tag{2.15}$$

Since

$$|\hat{f}| * |\hat{\varphi}|(\eta) = \int_{\mathbb{R}^d} |\hat{f}(\zeta)||\hat{\varphi}(\eta - \zeta)|\, d\zeta,$$

it follows from the definition of $\widetilde{f}$ (2.14) and $\widetilde{\varphi}$ (2.15) that

$$\widetilde{f} \le |\hat{f}| * \widetilde{\varphi}. \tag{2.16}$$

From the choice of φ (2.13), we have

$$\begin{aligned} \|K_{2,\xi} * f\|^2_{L^2(B_r)} &\le \|\varphi(K_{2,\xi} * f)\|^2_{L^2(\mathbb{R}^d)} \\ &\le \int_{\mathbb{R}^d} \Big| \int_{\operatorname{dist}(\eta,\Gamma_\xi)\le 1} |\hat{\varphi}(k-\eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)|\, d\eta \Big|^2 dk. \end{aligned} \tag{2.17}$$

Using the fact that

$$\int_{|x|\le 1} \frac{1}{|x_1| + |x_2|}\, dx_1\, dx_2 < +\infty, \tag{2.18}$$

we obtain

$$\int_{\operatorname{dist}(\eta,\Gamma_\xi)\le 1} |\hat{\varphi}(k-\eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)|\, d\eta \le \frac{C}{|\xi|} \int_{\operatorname{dist}(\eta,\Gamma_\xi)\le 1} \widetilde{\varphi}(k-\eta)\widetilde{f}(\eta)\, d\eta. \tag{2.19}$$

In fact, for $|\xi| > 2$, there exists $0 < r \le 1$ (independent of ξ) such that for η with $\operatorname{dist}(\eta, \Gamma_\xi) \le r$, there exists an unique pair $(\eta_1, \eta_2) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $\eta_1 \in \Gamma_\xi$, $\eta_2 \perp T_{\Gamma_\xi}(\eta_1)$, the tangent plane of Γ_ξ at η_1, such that $|\eta_2| \le r$ and $\eta_1 + \eta_2 = \eta$. Then

$$\begin{aligned} &\int_{\operatorname{dist}(\eta,\Gamma_\xi)\le r} |\hat{\varphi}(k-\eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)|\, d\eta \\ &\quad \le C \int_{\eta_1\in\Gamma_\xi} \int_{|\eta_2|\le r; \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{\varphi}(k-\eta_1-\eta_2)| \cdot |\hat{K}_\xi(\eta_1+\eta_2)| \cdot |\hat{f}(\eta_1+\eta_2)|\, d\eta_2\, d\eta_1. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\eta_1\in\Gamma_\xi} \int_{|\eta_2|\le r; \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{\varphi}(k-\eta_1-\eta_2)| \cdot |\hat{K}_\xi(\eta_1+\eta_2)| \cdot |\hat{f}(\eta_1+\eta_2)|\, d\eta_2\, d\eta_1 \\ &\le \int_{\eta_1\in\Gamma_\xi} \sup_{|\eta_2|\le r} |\hat{\varphi}(k-\eta_1-\eta_2)| \sup_{|\eta_2|\le r} |\hat{f}(\eta_1+\eta_2)| \int_{|\eta_2|\le r; \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{K}_\xi(\eta_1+\eta_2)|\, d\eta_2\, d\eta_1. \end{aligned} \tag{2.20}$$

and, by (2.18),

$$\int_{|\eta_2|\le r; \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{K}_\xi(\eta_1+\eta_2)|\, d\eta_2 \le \frac{C}{|\xi|},$$

it follows that

$$\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le r} |\hat{\varphi}(k-\eta)|\cdot|\hat{K}_\xi(\eta)|\cdot|\hat{f}(\eta)|\, d\eta \le \frac{C}{|\xi|}\int_{\eta_1\in\Gamma_\xi} \sup_{|\eta_2|\le r}|\hat{\varphi}(k-\eta_1-\eta_2)| \sup_{|\eta_2|\le r}|\hat{f}(\eta_1+\eta_2)|\, d\eta_1. \tag{2.21}$$

On the other hand, by the definition of $\tilde{f}$ and $\tilde{\varphi}$,

$$\int_{\eta_1\in\Gamma_\xi} \sup_{|\eta_2|\le r}|\hat{\varphi}(k-\eta_1-\eta_2)| \sup_{|\eta_2|\le r}|\hat{f}(\eta_1+\eta_2)|\, d\eta_1 \le C\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}\widetilde{\varphi}(k-\eta)\widetilde{f}(\eta)\, d\eta. \tag{2.22}$$

A combination of (2.21) and (2.22) yields (2.19).

Applying Hölder's inequality, we derive from (2.17) and (2.19) that

$$\|K_{2,\xi}*f\|^2_{L^2(B_r)} \le \frac{C}{|\xi|^2}\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}|\widetilde{f}(\eta)|^2\, d\eta. \tag{2.23}$$

We now estimate the RHS of (2.23). Applying Hölder's inequality again, from (2.16) and the fact that $\tilde{\varphi}\in L^1$, we have

$$\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}|\widetilde{f}(\eta)|^2\, d\eta \le C\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}\int_{\mathbb{R}^d}\widetilde{\varphi}(\eta-k)|\hat{f}(k)|^2\, dk\, d\eta. \tag{2.24}$$

Using Fubini's theorem, we derive from (2.24) that

$$\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}|\widetilde{f}(\eta)|^2\, d\eta \le C\int_{\mathbb{R}^d}|\hat{f}(k)|^2\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}\widetilde{\varphi}(\eta-k)\, d\eta\, dk. \tag{2.25}$$

Since $\tilde{\varphi}\in\mathcal{S}$, the Schwartz class, it follows that

$$\int_{\mathbb{R}^d}|\hat{f}(k)|^2\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}\widetilde{\varphi}(\eta-k)\, d\eta\, dk \le C\Big(\int_{|k|\le 2|\xi|}|\hat{f}(k)|^2\, dk + \int_{|k|>2|\xi|}\frac{|\hat{f}(k)|^2}{|k|^2}\, dk\Big). \tag{2.26}$$

From (2.25) and (2.26), we obtain

$$\frac{1}{|\xi|^2}\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1}|\widetilde{f}(\eta)|^2\, d\eta \le C\|f\|^2_{H^{-1}}. \tag{2.27}$$

A combination of (2.23) and (2.27) yields,

$$\|K_{2,\xi}*f\|^2_{L^2(B_r)} \le C\|f\|^2_{H^{-1}}. \tag{2.28}$$

The conclusion follows from (2.11), (2.12), and (2.28). □

3. Proof of Theorem 1 and Corollary 1

In this section, we prove Theorem 1 and Corollary 1. The proof of Theorem 1 contains two main ingredients. The first one is a new useful inequality (Lemma 2) and its variant (Lemma 3) to solutions to (2.1) whose the proof is based on estimates presented in Section 2. The second one is an averaging estimate (Lemma 4) with respect to ξ for $K_\xi * q$ in the same spirit in [11] and is presented in Appendix A.

3.1. **Some useful lemmas.** The following lemma is new and interesting in itself. It plays an important role in our analysis. Its proof is quite elementary, and can be seen as the replacement of [11, Lemma 2.3].

Lemma 2. *Let* $d \ge 3$, $\xi \in \mathbb{C}^d$ ($|\xi| > 2$) *with* $\xi \cdot \xi = 0$, $g_1 \in [L^\infty(\mathbb{R}^d)]^d$, $g_2 \in L^d(\mathbb{R}^d)$ *and* $V \in H^1(\mathbb{R}^d)$ *be such that* $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset B_1$. *Set*

$$q = \operatorname{div} g_1 + g_2$$

and define

$$W = K_\xi * (qV).$$

We have

$$\begin{aligned} &\|\nabla W\|_{L^2(B_r)} + |\xi| \cdot \|W\|_{L^2(B_r)} \\ &\qquad \le C_r \big(\|g_1\|_{L^\infty} + \|g_2\|_{L^d}\big) \big(\|\nabla V\|_{L^2} + |\xi| \cdot \|V\|_{L^2}\big), \end{aligned} \tag{3.1}$$

for some positive constant C_r *independent of* ξ, g_1, g_2, *and* v.

Proof. We have

$$qV = \operatorname{div}(V g_1) - g_1 \cdot \nabla V + g_2 V \text{ in } \mathbb{R}^d. \tag{3.2}$$

Applying (2.5) with $k = 1$ and (2.7) with $k = 0$, we have

$$\|\nabla W\|_{L^2(B_r)} \le C_r \Big(|\xi| \cdot \| \operatorname{div}(V g_1)\|_{H^{-1}} + \|g_1 \cdot \nabla V\|_{L^2} + \|g_2 V\|_{L^2} \Big),$$

which implies

$$\|\nabla W\|_{L^2(B_r)} \le C_r \Big(|\xi| \cdot \|V g_1\|_{L^2} + \|g_1 \cdot \nabla V\|_{L^2} + \|g_2 V\|_{L^2} \Big).$$

It follows that

$$\|\nabla W\|_{L^2(B_r)} \le C_r \big(\|g_1\|_{L^\infty} + \|g_2\|_{L^d}\big) \big(|\xi| \cdot \|V\|_{L^2} + \|\nabla V\|_{L^2}\big). \tag{3.3}$$

Similarly, using (3.2) and applying (2.4) with $k = 0$, and (2.6), we obtain

$$|\xi| \cdot \|W\|_{L^2(B_r)} \le C_r \Big(|\xi| \cdot \| \operatorname{div}(g_1 V)\|_{H^{-1}} + \|g_1 \nabla V\|_{L^2} + \|g_2 V\|_{L^2} \Big),$$

which implies

$$|\xi| \cdot \|W\|_{L^2(B_r)} \le C_r \Big(|\xi| \cdot \|g_1 V\|_{L^2} + \|g_1 \nabla V\|_{L^2} + \|g_2 V\|_{L^2} \Big).$$

It follows that

$$|\xi| \cdot \|W\|_{L^2(B_r)} \le C_r \big(\|g_1\|_{L^\infty} + \|g_2\|_{L^d}\big) \big(|\xi| \cdot \|V\|_{L^2} + \|\nabla V\|_{L^2}\big). \tag{3.4}$$

A combination of (3.3) and (3.4) yields (3.1). □

When g_1 and g_2 are smooth, we can improve the conclusion in Lemma 2 as follows.

Lemma 3. *Let* $d \ge 3$, $\xi \in \mathbb{C}^d$ ($|\xi| > 2$) *with* $\xi \cdot \xi = 0$, $g_1 \in [C^2(\mathbb{R}^d)]^d$, $g_2 \in C^1(\mathbb{R}^d)$ *with* $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset B_1$, *and let* $V \in H^1(\mathbb{R}^d)$. *Set*

$$q = \operatorname{div} g_1 + g_2$$

and define

$$W = K_\xi * (qV).$$

We have, for $r > 0$,

$$\begin{aligned} &\|\nabla W\|_{L^2(B_r)} + |\xi| \cdot \|W\|_{L^2(B_r)} \\ &\qquad \le \frac{C_r}{|\xi|} \Big(\|g_1\|_{C^2} + \|g_2\|_{C^1}\Big) \Big(\|\nabla V\|_{L^2} + |\xi| \cdot \|V\|_{L^2}\Big). \end{aligned} \tag{3.5}$$

Here C_r is a positive constant depending only on r and d.

Proof. Applying (2.4) with $k=1$, we have

$$\begin{aligned}\|\nabla W\|_{L^2(B_r)} &\le \frac{C_r}{|\xi|}\Big(\|V \operatorname{div} g_1\|_{H^1} + \|V g_2\|_{H^1}\Big)\\ &\le \frac{C_r}{|\xi|}\|V\|_{H^1}\big(\|g_1\|_{C^2} + \|g_2\|_{C^1}\big).\end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned}\|W\|_{L^2(B_r)} &\le \frac{C_r}{|\xi|}\Big(\|V \operatorname{div} g_1\|_{L^2} + \|V g_2\|_{L^2}\Big)\\ &\le \frac{C_r}{|\xi|}\|V\|_{L^2}\big(\|g_1\|_{C^1} + \|g_2\|_{C^0}\big).\end{aligned} \tag{3.7}$$

A combination of (3.6) and (3.7) yields (3.5). □

3.2. **Construction of CGO solutions.** We begin this section with an estimate for the solution of the equation

$$\Delta w + \xi \cdot \nabla w - q w = q \text{ in } \mathbb{R}^d.$$

Proposition 2. *Let $\xi \in \mathbb{C}^d$ ($|\xi| > 2$) with $\xi\cdot\xi = 0$, $g_1 \in [L^\infty(\mathbb{R}^d)]^d$, $g_2 \in L^d(\mathbb{R}^d)$ with $\operatorname{supp} g_1, \operatorname{supp} g_2 \subset B_1$. Set $q = \operatorname{div} g_1 + g_2$. Then there exists a positive constant c such that if*

$$\inf_{\phi \in [C(\mathbb{R}^d)]^d, \operatorname{supp}\phi \subset B_1} \|g_1 - \phi\|_{L^\infty} \le c,$$

then there exists $w \in H^1_{loc}(\mathbb{R}^d)$ such that

$$w = K_\xi * (q + q w)$$

and

$$\begin{aligned}&\|\nabla (w - K_\xi * q)\|_{L^2(B_r)} + |\xi|\cdot\|w - K_\xi * q\|_{L^2(B_r)}\\ &\qquad\le C_r\Big(\|\nabla K_\xi * q\|_{L^2(B_2)} + |\xi|\cdot\|K_\xi * q\|_{L^2(B_2)}\Big) \quad \forall\, r > 0,\end{aligned} \tag{3.8}$$

for $|\xi|$ large enough [3].

Proof. Let $g_{i,j}$, $1 \le i, j \le 2$, such that

$$g_{1,1} + g_{1,2} = g_1 \text{ and } g_{2,1} + g_{2,2} = g_2.$$

$$g_{1,2},\ g_{2,2} \text{ are smooth with compact support in } B_1,$$

$$\|g_{1,1}\|_{L^\infty} + \|g_{2,1}\|_{L^d} \le 2c,$$

Set

$$q = q_1 + q_2,$$

where

$$q_1 = \operatorname{div} g_{1,1} + g_{2,1}$$

and

$$q_2 = \operatorname{div} g_{1,2} + g_{2,2}.$$

Let $u_0 = 0$ and consider the following iteration process:

$$w_n = K_\xi * (q + q w_{n-1}) \quad \text{for } n \ge 1, \tag{3.9}$$

which implies

$$\Delta w_n + \xi\cdot\nabla w_n = q + q w_{n-1} \text{ in } \mathbb{R}^d, \text{ for } n \ge 1.$$

[3] The largeness of $|\xi|$ depends only on g_1 and g_2.

Define

$$w_{1,n} = K_\xi * (q_1 + q_1 w_{n-1}) \quad \text{and} \quad w_{2,n} = K_\xi * (q_2 + q_2 w_{n-1}).$$

Then

$$\Delta w_{1,n} + \xi \cdot \nabla w_{1,n} = q_1 + q_1 w_{n-1} \text{ in } \mathbb{R}^d,$$
$$\Delta w_{2,n} + \xi \cdot \nabla w_{2,n} = q_2 + q_2 w_{n-1} \text{ in } \mathbb{R}^d,$$

and

$$w_n = w_{1,n} + w_{2,n} \text{ in } \mathbb{R}^d. \tag{3.10}$$

Set

$$W_{n+1} = w_{n+1} - w_n, \quad W_{1,n+1} = w_{1,n+1} - w_{1,n}, \quad W_{2,n+1} = w_{2,n+1} - w_{2,n}.$$

It follows from Lemma 2 that

$$\|\nabla W_{1,n+1}\|_{L^2(B_r)} + |\xi| \cdot \|W_{1,n+1}\|_{L^2(B_r)} \le C_r\big(\|g_{1,1}\|_{L^\infty} + \|g_{2,1}\|_{L^d}\big)\Big(\|\nabla W_n\|_{L^2(B_2)} + |\xi| \cdot \|W_n\|_{L^2(B_2)}\Big), \tag{3.11}$$

and from Lemma 3 that

$$\|\nabla W_{2,n+1}\|_{L^2(B_r)} + |\xi| \cdot \|W_{2,n+1}\|_{L^2(B_r)} \le \frac{C_r}{|\xi|}\big(\|g_{1,2}\|_{C^2} + \|g_{2,2}\|_{C^1}\big)\Big(\|\nabla W_n\|_{L^2(B_2)} + |\xi| \cdot \|W_n\|_{L^2(B_2)}\Big). \tag{3.12}$$

A combination of (3.10), (3.11), and (3.12) yields

$$\|\nabla W_{n+1}\|_{L^2(B_r)} + |\xi| \cdot \|W_{n+1}\|_{L^2(B_r)} \le C_r\Big(\big(\|g_{1,1}\|_{L^\infty}+\|g_{2,1}\|_{L^d}\big)+\frac{1}{|\xi|}\big(\|g_{1,2}\|_{C^2}+\|g_{2,2}\|_{C^1}\big)\Big)\Big(\|\nabla W_n\|_{L^2(B_2)}+|\xi|\cdot\|W_n\|_{L^2(B_2)}\Big). \tag{3.13}$$

Choose c such that

$$cC_2 = 1/2.$$

Thus, if $|\xi|$ is large enough, then

$$C_2\Big(\big(\|g_{1,1}\|_{L^\infty} + \|g_{2,1}\|_{L^d}\big) + \frac{1}{|\xi|}\big(\|g_{1,2}\|_{C^2} + \|g_{2,2}\|_{C^1}\big)\Big) \le 3/4.$$

Hence, by a standard fixed point argument, it follows that

$$w_n \to w \text{ in } H^1(B_2).$$

This implies, by (3.13),

$$w_n \to w \text{ in } H^1(B_r) \quad \text{for all } r > 0,$$

and by (3.9)

$$w = K_\xi * (q + qw).$$

We derive from (3.13) that

$$\|\nabla(w - w_1)\|_{L^2(B_2)} + |\xi| \cdot \|w - w_1\|_{L^2(B_2)} \le C\big(\|\nabla w_1\|_{L^2(B_2)} + |\xi| \cdot \|w_1\|_{L^2(B_2)}\big).$$

Statement (3.8) now follows from (3.13). The proof is complete. □

To obtain some appropriate estimate for u_1 in Proposition 2, we use an averaging argument in the same spirit in [11]. More precisely, we have the following lemma whose proof is given in the appendix.

Lemma 4. *Let $d \ge 3$, $s > 2$, $k \in \mathbb{R}^d$ with $|k| \ge 2$, $1 \le p < 2$, and $R > 10$. We have*

$$\frac{1}{R}\int_{R/2}^{2R}\int_{\sigma_1 \in \mathbb{S}^{d-1}}\int_{\sigma_2 \in \mathbb{S}^{d-1}_{\sigma_1}} |\hat{K}_{s\sigma_2 - is\sigma_1}(k)|^p \, d\sigma_2 \, d\sigma_1 \, ds \le C \min\Big\{\frac{1}{R^p|k|^p}, \frac{1}{|k|^{2p}}\Big\}, \tag{3.14}$$

and

$$\frac{1}{R}\int_{R/2}^{2R}\int_{\sigma_1 \in \mathbb{S}^{d-1}}\int_{\sigma_2 \in \mathbb{S}^{d-1}_{\sigma_1}}\int_{\sigma_3 \in \mathbb{S}^{d-1}_{\sigma_1,\sigma_2}} \Big|\hat{K}_{\frac{s^2\sigma_2}{\sqrt{1+s^2}} + \frac{s\sigma_3}{\sqrt{1+s^2}} - is\sigma_1}(k)\Big|^p \, d\sigma_3 \, d\sigma_2 \, d\sigma_1 \, ds \le C \min\Big\{\frac{1}{R^p|k|^p}, \frac{1}{|k|^{2p}}\Big\}, \tag{3.15}$$

for some positive constant C depending only on d and p. Here

$$\mathbb{S}^{d-1}_{\sigma_1} := \big\{\sigma \in \mathbb{S}^{d-1};\ \sigma \cdot \sigma_1 = 0\big\} \tag{3.16}$$

and

$$\mathbb{S}^{d-1}_{\sigma_1,\sigma_2} := \big\{\sigma \in \mathbb{S}^{d-1};\ \sigma \cdot \sigma_1 = 0 \text{ and } \sigma \cdot \sigma_2 = 0\big\}. \tag{3.17}$$

Remark 2. *Let $\sigma_1 \in \mathbb{S}^{d-1}$, $\sigma_2 \in \mathbb{S}^{d-1}_{\sigma_1}$, and $\sigma_3 \in \mathbb{S}^{d-1}_{\sigma_1,\sigma_2}$. Set*

$$\xi_1 = s\sigma_2 - is\sigma_1 \quad \text{and} \quad \xi_2 = -\frac{s^2\sigma_2}{\sqrt{1+s^2}} + \frac{s\sigma_3}{\sqrt{1+s^2}} + is\sigma_1,$$

then

$$\xi_1 \cdot \xi_1 = \xi_2 \cdot \xi_2 = 0 \text{ and } \xi_1 + \xi_2 = \Big(s - \frac{s^2}{\sqrt{1+s^2}}\Big)\sigma_2 + \frac{s\sigma_3}{\sqrt{1+s^2}} \to \sigma_3$$

uniformly with respect to σ_1 and σ_2 as $s \to \infty$.

Remark 3. *Lemma 4 does not hold for $p = 2$. The requirements (1.9) and (1.11) in Theorem 1 and Corollary 1 are due to this point.*

Using Proposition 2 and Lemma 4, we can prove the following result:

Proposition 3. *Let $d \ge 3$, $g_1, h_1 \in [L^\infty(\mathbb{R}^d)]^d$, $g_2, h_2 \in L^d(\mathbb{R}^d)$ with supports in B_1 be such that $\hat{g}_1, \hat{h}_1 \in L^p(\mathbb{R}^d)$ for some $1 < p < 2$. Assume that*

$$\inf_{\phi \in [C(\bar{\Omega})]^d} \|g_1 - \phi\|_{L^\infty} + \inf_{\phi \in [C(\bar{\Omega})]^d} \|h_1 - \phi\|_{L^\infty} \le c,$$

where c is the constant in Proposition 2, or g_1, h_1 are continuous. Set

$$q_1 = \operatorname{div} g_1 + g_2 \text{ and } q_2 = \operatorname{div} h_1 + h_2.$$

Then for any $0 < \varepsilon < 1$, $n > 2$ large enough, and $\sigma \in \mathbb{S}^{d-1}$, there exist $\sigma_{1,\varepsilon}$, $\sigma_{2,\varepsilon}$, $\sigma_{3,\varepsilon} \in \mathbb{S}^{d-1}$, $s_\varepsilon \in (n, 4n)$, and $w_{1,\varepsilon}, w_{2,\varepsilon} \in H^1_{loc}(\mathbb{R}^d)$ such that

$$\sigma_{1,\varepsilon} \cdot \sigma_{2,\varepsilon} = \sigma_{1,\varepsilon} \cdot \sigma_{3,\varepsilon} = \sigma_{2,\varepsilon} \cdot \sigma_{3,\varepsilon} = 0, \tag{3.18}$$

$$|\sigma_{3,\varepsilon} - \sigma| \le \varepsilon, \tag{3.19}$$

$$w_{j,\varepsilon} = K_{\xi_{j,\varepsilon}} * (q_j + q_j w_{j,\varepsilon}) \quad \text{for } j = 1, 2,$$

and

$$\|\nabla w_{j,\varepsilon}\|_{L^2(B_r)} + s_\varepsilon \|w_{j,\varepsilon}\|_{L^2(B_r)} \le C_r/\varepsilon^{3d} \quad \text{for } j = 1, 2, \tag{3.20}$$

for some $C_r > 0$ independent of ε, s, and σ. Here

$$\xi_{1,\varepsilon} = s_\varepsilon\sigma_{2,\varepsilon} - is_\varepsilon\sigma_{1,\varepsilon} \quad \text{and} \quad \xi_{2,\varepsilon} = -\frac{s_\varepsilon^2\sigma_{2,\varepsilon}}{\sqrt{1+s_\varepsilon^2}} + \frac{s_\varepsilon\sigma_{3,\varepsilon}}{\sqrt{1+s_\varepsilon^2}} + is_\varepsilon\sigma_{1,\varepsilon}. \tag{3.21}$$

Proof. Applying Lemma 4, we have

$$\frac{1}{n}\int_n^{4n}\int_{\sigma_1\in\mathbb{S}^{d-1}}\int_{\sigma_2\in\mathbb{S}^{d-1}_{\sigma_1}}\Big(|\hat{K}_{s\sigma_2-is\sigma_1}(k)|^p + \int_{\sigma_3\in\mathbb{S}^{d-1}_{\sigma_1,\sigma_2}}\Big|\hat{K}_{-\frac{s^2\sigma_2}{\sqrt{1+s^2}}+\frac{s\sigma_3}{\sqrt{1+s^2}}+is\sigma_1}(k)\Big|^p\,d\sigma_3\Big)\,d\sigma_2\,d\sigma_1\,ds \le C\min\Big\{\frac{1}{n^p|k|^p},\frac{1}{|k|^{2p}}\Big\}.$$

This implies (3.18) and (3.19) hold for some $s_\varepsilon\in(n,4n)$ and $\sigma_{1,\varepsilon},\sigma_{2,\varepsilon},\sigma_{3,\varepsilon}\in\mathbb{S}^{d-1}$, and

$$\int_{\mathbb{R}^d}\Big(|\hat{K}_{\xi_{1,\varepsilon}}(k)|^p|\hat{q}_1(k)|^p+|\hat{K}_{\xi_{2,\varepsilon}}(k)|^p|\hat{q}_2(k)|^p\Big)(|k|^p+n^p)\,dk \le \frac{C}{\varepsilon^{3d}}\int_{\mathbb{R}^d}\frac{1}{|k|^p}\Big(|\hat{q}_1(k)|^p+|\hat{q}_2(k)|^p\Big)\,dk\le\frac{C}{\varepsilon^{3d}}, \tag{3.22}$$

where $\xi_{1,\varepsilon}$ and $\xi_{2,\varepsilon}$ are given by (3.21). It follows that

$$\|\nabla(K_{\xi_{j,\varepsilon}}*q_j)\|_{L^2(B_r)}+n\|K_{\xi_{j,\varepsilon}}*q_j\|_{L^2(B_r)}\le C_r/\varepsilon^{3d/p},$$

for all $r>0$ and for $j=1,2$. By Proposition 2, for n large enough, there exist $w_{j,\varepsilon}\in H^1_{loc}(\mathbb{R}^d)$ $(j=1,2)$ such that

$$w_{j,\varepsilon}=K_{\xi_{j,\varepsilon}}*(q_j+q_jw_{j,\varepsilon})\text{ in }\mathbb{R}^d.$$

and

$$\|\nabla w_{j,\varepsilon}\|_{L^2(B_r)}+s_\varepsilon\|w_{j,\varepsilon}\|_{L^2(B_r)}\le C_r/\varepsilon^{3d}.$$

The proof is complete. □

3.3. **Proof of Theorem 1.** Without loss of generality one may assume that $\Omega\subset B_{1/2}$. Let $g_{i,j}$, $1\le i,j\le 2$, such that

$$g_{1,1}+g_{1,2}=g_1\text{ and }h_{1,1}+h_{1,2}=h_1.$$

$$g_{1,1},\ h_{1,1}\text{ are smooth with compact support in }\Omega,$$

$$\|g_{1,2}\|_{L^\infty}+\|h_{1,2}\|_{L^d}\le 2c,$$

Extend $g_{1,1}$ and $h_{1,1}$ smoothly in $\mathbb{R}^d\setminus\Omega$ with compact support in B_1 and denote these extension by $G_{1,1}$ and $H_{1,1}$. Extend $g_{1,2}$, $h_{1,2}$, g_2, h_2 by 0 outside Ω and denote these extensions by $G_{1,2}$, $H_{1,2}$, G_2, H_2. Define

$$G_1=G_{1,1}+G_{1,2}\quad\text{and}\quad H_1=H_{1,1}+H_{1,2}.$$

Extend q_1 and q_2 in $\mathbb{R}^d$ by $\operatorname{div}G_1+G_2$ and $\operatorname{div}H_1+H_2$ and still denote these extensions by q_1 and q_2. Then q_1 and q_2 satisfy the assumptions of Proposition 3 since $F(1_\Omega)\in L^r(\mathbb{R}^d)$ for $r>2d/(d+1)$ (see [18, Theorem 1]). We claim that there exist $\sigma\in\mathbb{S}^{d-1}$ and $u_{1,n}$, $u_{2,n}\in H^1_{loc}(\mathbb{R}^d)$ such that

$$|\sigma-\sigma_0|\le\varepsilon, \tag{3.23}$$

$$\Delta w_{j,n}+\xi_{j,n}\cdot\nabla w_{j,n}-q_jw_{j,n}=q_j\text{ in }\mathbb{R}^d\quad\text{for }j=1,2, \tag{3.24}$$

$$w_{j,n}\to 0\text{ weakly in }H^1(B_2)\quad\text{for }j=1,2, \tag{3.25}$$

for some $\xi_{1,n},\xi_{2,n}\in\mathbb{C}^d$ with $\xi_{j,n}\cdot\xi_{j,n}=0$, $|\xi_{1,n}+\xi_{2,n}-\sigma|\to 0$ and $|\xi_{j,n}|\to\infty$ as $n\to\infty$.

Indeed, by Proposition 3, there exist $w_{j,n}\in H^1_{loc}(\mathbb{R}^d)$ $(j=1,\,2)$ such that

$$w_{j,n}=K_{\xi_{j,n}}*(q_j+q_jw_{j,n}).$$

Moreover,

$$\|\nabla w_{j,n}\|_{L^2(B_r)} + n\|w_{j,n}\|_{L^2(B_r)} \le C_r/\varepsilon^{3d/p},$$

for some $C_r > 0$ which depends only on d, g_i, h_i ($i = 1, 2$), and r. Here

$$\xi_{1,n} = s_n\sigma_{2,n} - is_n\sigma_{1,n}$$

and

$$\xi_{2,n} = s_n\Big(- \frac{s_\varepsilon \sigma_{2,n}}{\sqrt{1+s_n^2}} + \frac{\sigma_{3,n}}{\sqrt{1+s_n^2}}\Big) + is_n\sigma_{1,n}.$$

for some $s_n \in (n, 4n)$, and $\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n}$ such that

$$\sigma_{1,n}\cdot\sigma_{2,n} = \sigma_{1,n}\cdot\sigma_{3,n} = \sigma_{2,n}\cdot\sigma_{3,n} = 0,$$

$$|\sigma_{3,n} - \sigma_0| \le \varepsilon.$$

Without loss of generality one might assume that $\sigma_{3,n} \to \sigma$ for some $\sigma \in \mathbb{S}^{d-1}$. Then

$$\xi_{1,n} + \xi_{2,n} = \Big(s_n - \frac{s_n^2}{\sqrt{1+s_n^2}}\Big)\sigma_{2,n} + \frac{s_n\sigma_{3,n}}{\sqrt{1+s_n^2}} \to \sigma,$$

and the claim is proved.

We now apply the complex geometric optics approach introduced by Sylvester and Uhlmann in [22]. Define, for $j = 1, 2$,

$$v_{j,n} = (1 + w_{j,n})e^{\xi_{j,n}\cdot x/2}.$$

Since $w_{j,n}$ satisfies (3.24), it follows that

$$\Delta v_{j,n} - q_j v_{j,n} = 0 \text{ in } \mathbb{R}^d \quad \text{for } j = 1, 2.$$

We derive from (1.2) that

$$\int_{B_2} (q_1 - q_2)(1 + w_{1,n})(1 + w_{2,n})e^{\sigma_n\cdot x/2} = 0, \tag{3.26}$$

where

$$\sigma_n = \xi_{1,n} + \xi_{2,n} \to \sigma \text{ as } n \to \infty. \tag{3.27}$$

A combination of (3.24), (3.26), and (3.27) yields

$$\int_{B_2} (q_1 - q_2)e^{\sigma\cdot x/2} = 0.$$

Since $\sigma_0 \in \mathbb{S}^{d-1}$ and $\varepsilon > 0$ are arbitrary, it follows that

$$\int_{B_2} (q_1 - q_2)e^{\sigma_0\cdot x/2} = 0 \text{ for all } \sigma_0 \in \mathbb{S}^{d-1}.$$

This implies

$$q_1 = q_2,$$

and the proof is complete. □

3.4. **Proof of Corollary 1.** Let $u_i \in H^1(\Omega)$ $(i = 1, 2)$ be a solution to the equation

$$\mathrm{div}(\gamma_i \nabla u_i) = 0 \text{ in } \Omega.$$

Define

$$v_i = \gamma_i^{1/2} u_i \text{ in } \Omega.$$

Then $v_i \in H^1(\Omega)$ is a solution to the equation

$$\Delta v_i - q_i v_i = 0 \text{ in } \Omega,$$

where

$$q_i = \frac{\Delta \gamma_i^{1/2}}{\gamma_i^{1/2}} = \Delta t_j - |\nabla t_j|^2 \text{ in } \Omega.$$

Here t_i $(i = 1, 2)$ is given by

$$t_i = \ln \gamma_i^{1/2} \text{ in } \Omega.$$

Since $DtN_{\gamma_1} = DtN_{\gamma_2}$, it follows that

$$\int_\Omega (q_1 - q_2) v_1 v_2 = 0,$$

for all solutions v_i $(i = 1, 2)$ to the equation

$$\Delta v_i - q_i v_i = 0 \text{ in } \Omega.$$

Set

$$g_i = \nabla t_i \text{ and } h_i = -t_i^2,$$

then g_i and h_i satisfy the assumptions of Theorem 1. Applying Theorem 1, we have

$$q_1 = q_2 \text{ in } \Omega. \tag{3.28}$$

This implies,

$$\Delta(t_1 - t_2) = |\nabla t_1|^2 - |\nabla t_2|^2 \in L^2(\Omega).$$

Hence

$$\partial_\eta t_1 = \partial_\eta t_2 \text{ on } \partial\Omega, \tag{3.29}$$

We also have, by Proposition B1,

$$t_1 = t_2 \text{ on } \partial\Omega, \tag{3.30}$$

We derive from (3.28) and the definition of q_i that

$$\Delta(t_1 - t_2) - \nabla t \cdot \nabla(t_1 - t_2) = 0 \text{ in } \Omega,$$

where $t = t_1 + t_2 \in W^{1,\infty}(\Omega)$. This implies $t_1 = t_2$ by (3.29), (3.30), and the unique continuation principle. Therefore, the conclusion follows. □

4. Proof of Theorem 2 and Corollary 2

4.1. **Construction of CGO solutions.** We begin this section with an estimate for the solution to the equation

$$\Delta w + \xi \cdot \nabla w - qw = q \text{ in } \mathbb{R}^d,$$

for $q \in L^{d/2}(\mathbb{R}^d)$. This estimate will play an important role in the (new) proof of Theorem 2.

Proposition 4. *Let $d \geq 3$, and let $\xi \in \mathbb{C}^d$ with $\xi \cdot \xi = 0$, $q \in L^{d/2}(\mathbb{R}^d)$ with $\operatorname{supp} q \subset B_1$. For $|\xi|$ large enough, there exists $w \in H^1_{loc}(\mathbb{R}^d)$ such that*

$$w = K_\xi * (q + qw).$$

Moreover,

$$\lim_{|\xi| \to \infty} \|w\|_{L^{\frac{2d}{d-2}}(B_r)} = 0, \tag{4.1}$$

Proof. Let f and h be such that

$$q = f + h,$$

where

$$f \text{ is smooth with support in } B_{1/2}, \text{ and } \|h\|_{L^{d/2}} \text{ is small.}$$

Let $u_0 = 0$ and consider the following iteration process:

$$w_n = K_\xi * (q + qw_{n-1}) \text{ for } n \geq 1.$$

Define

$$w_{1,n} = K_\xi * (f + fw_{n-1})$$

and

$$w_{2,n} = K_\xi * (h + hw_{n-1}).$$

Then

$$w_n = w_{1,n} + w_{2,n}, \tag{4.2}$$

$$w_{1,n+1} - w_{1,n} = K_\xi * \big[f(w_n - w_{n-1})\big], \tag{4.3}$$

and

$$w_{2,n+1} - w_{2,n} = K_\xi * \big[h(w_n - w_{n-1})\big]. \tag{4.4}$$

Applying the generalized Sobolev's inequality [14, Theorem 2.1] (see also (1.7)) and using (4.4), we have

$$\begin{aligned}\|w_{2,n+1} - w_{2,n}\|_{L^{\frac{2d}{d-2}}} \leq & C\|\Delta(w_{2,n+1} - w_{2,n}) + \xi \cdot \nabla(w_{2,n+1} - w_{2,n})\|_{L^{\frac{2d}{d+2}}} \\ \leq & C\|h(w_n - w_{n-1})\|_{L^{\frac{2d}{d+2}}};\end{aligned}$$

which yields

$$\|w_{2,n+1} - w_{2,n}\|_{L^{\frac{2d}{d-2}}} \leq C\|h\|_{L^{d/2}}\|w_n - w_{n-1}\|_{L^{\frac{2d}{d-2}}(B_1)}. \tag{4.5}$$

We also have

$$\|w_{1,n+1} - w_{1,n}\|_{L^{\frac{2d}{d-2}}(B_r)} \leq C_r\|w_{1,n+1} - w_{1,n}\|_{H^1(B_r)},$$

and by (2.6),

$$\|w_{1,n+1} - w_{1,n}\|_{H^1(B_r)} \leq C_r\|f(w_n - w_{n-1})\|_{L^2(B_1)}.$$

This implies

$$\|w_{1,n+1} - w_{1,n}\|_{L^{\frac{2d}{d-2}}(B_r)} \leq C_r\|f\|_{L^\infty}\|w_n - w_{n-1}\|_{L^2(B_1)}. \tag{4.6}$$

A combination of (4.5) and (4.6) yields

$$\|w_{n+1} - w_n\|_{L^{\frac{2d}{d-2}}(B_r)} \leq C_r\Big(\|h\|_{L^{d/2}}\|w_n - w_{n-1}\|_{L^{\frac{2d}{d-2}}(B_1)} + \|f\|_{L^\infty}\|w_n - w_{n-1}\|_{L^2(B_1)}\Big). \tag{4.7}$$

On the other hand, by (2.4), it follows from (4.3) that

$$\begin{aligned}\|w_{1,n+1}-w_{1,n}\|_{L^2(B_r)} \leq &\frac{C_r}{|\xi|}\|f(w_n-w_{n-1})\|_{L^2(B_1)}\\ \leq &\frac{C_r}{|\xi|}\|f\|_{L^\infty}\|w_n-w_{n-1}\|_{L^2(B_1)}.\end{aligned} \tag{4.8}$$

A combination of (4.5) and (4.8) implies

$$\|w_{n+1}-w_n\|_{L^2(B_r)} \leq C_r\Big(\|h\|_{L^{d/2}}+\frac{\|f\|_{L^\infty}}{|\xi|}\Big)\|w_n-w_{n-1}\|_{L^{\frac{2d}{d-2}}(B_1)}. \tag{4.9}$$

From (4.7) and (4.9), we obtain

$$\begin{aligned}\|w_{n+1}-w_n\|_{L^{\frac{2d}{d-2}}(B_r)} \leq C_r\Big(&\|h_1\|_{L^{d/2}}\|w_n-w_{n-1}\|_{L^{\frac{2d}{d-2}}(B_1)}\\ &+\|f_1\|_{L^\infty}\Big[\|h_2\|_{L^{d/2}}+\frac{\|f_2\|_{L^\infty}}{|\xi|}\Big]\|w_{n-1}-w_{n-2}\|_{L^{\frac{2d}{d-2}}(B_1)}\Big).\end{aligned} \tag{4.10}$$

Here f_1, f_2 and h_1, h_2 are such that

$$q = f_1 + h_1 = f_2 + h_2,$$

where

$$f_1, f_2 \text{ are smooth with support in } B_1, \text{ and } \|h_1\|_{L^{d/2}}, \|h_2\|_{L^{d/2}} \text{ are small.}$$

Appropriate choice of f_1, f_2 and h_1, h_2 implies that w_n converges to w in $H^1_{loc}(\mathbb{R}^d)$ and

$$w = K_\xi * (q + qw).$$

We next prove (4.1). By the same arguments used to obtain (4.7) and (4.9), we have

$$\|w-K_\xi * q\|_{L^{\frac{2d}{d-2}}(B_r)} \leq C_r\Big(\|h_1\|_{L^{d/2}}\|w\|_{L^{\frac{2d}{d-2}}(B_1)}+\|f_1\|_{L^\infty}\|w\|_{L^2(B_1)}\Big) \tag{4.11}$$

and

$$\|w-K_\xi * q\|_{L^2(B_r)} \leq C_r\Big(\|h_2\|_{L^{d/2}}\|w\|_{L^{\frac{2d}{d-2}}(B_1)}+\frac{\|f_2\|_{L^\infty}}{|\xi|}\|w\|_{L^2(B_1)}\Big). \tag{4.12}$$

We claim that

$$\lim_{|\xi|\to 0}\|K_\xi * q\|_{L^{\frac{2d}{d-2}}(B_1)} = 0. \tag{4.13}$$

Admitting (4.13), we will prove (4.1). In fact, from (4.12), and (4.13), we have $\|w\|_{L^2(B_r)} \to 0$ as $|\xi| \to \infty$. This implies, by (4.13), $\|w\|_{L^{\frac{2d}{d-2}}(B_r)} \to 0$ as $|\xi| \to \infty$.

It remains to prove (4.13). Let $q_1, q_2 \in L^{d/2}$ with the support in B_1 be such that

$$q = q_1 + q_2, \quad \|q_1\|_{L^{d/2}} \text{ is small, } \quad \text{and} \quad q_2 \text{ is smooth.}$$

We have

$$\|K_\xi * q\|_{L^{\frac{2d}{d-2}}(B_1)} \leq \|K_\xi * q_1\|_{L^{\frac{2d}{d-2}}(B_1)} + \|K_\xi * q_2\|_{L^{\frac{2d}{d-2}}(B_1)},$$

and, by the generalized Sobolev inequality,

$$\|K_\xi * q_1\|_{L^{\frac{2d}{d-2}}} \leq C\|q_1\|_{L^{\frac{d}{2}}},$$

and by (2.4),

$$\|K_\xi * q_2\|_{L^{\frac{2d}{d-2}}(B_1)} \leq \frac{C}{|\xi|}\|q_2\|_{C^2}.$$

By an appropriate choice of q_1 and q_2, it follows that

$$\lim_{|\xi|\to 0} \|K_\xi * q\|_{L^{\frac{2d}{d-2}}(B_1)} = 0;$$

claim (4.13) is proved. The proof is complete. □

4.2. **Proof of Theorem 2.** The proof is standard after Proposition 4. For the convenience of the reader, we present the proof. Without loss of generality one may assume that $\Omega \subset B_{1/2}$. Extend q_1 and q_2 by 0 in $\mathbb{R}^d \setminus \Omega$ and still denote these extensions by q_1 and q_2. Let $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{S}^{d-1}$ be such that

$$\sigma_1 \cdot \sigma_2 = \sigma_1 \cdot \sigma_3 = \sigma_2 \cdot \sigma_3 = 0.$$

Set

$$\xi_{1,n} = n\sigma_2 - in\sigma_1 \text{ and } \xi_{2,n} = n\Big(- \frac{n\sigma_2}{\sqrt{1+n^2}} + \frac{\sigma_3}{\sqrt{1+n^2}}\Big) + in\sigma_1.$$

For n large enough, by Proposition 4, there exist $w_{j,n} \in H^1_{loc}$ $(j = 1, 2)$ such that

$$w_{j,n} = K_{\xi_{j,n}} * (q_j + q_j u_{j,n}).$$

Moreover,

$$\lim_{n\to\infty} \|w_{j,n}\|_{L^{\frac{2d}{d-2}}(B_r)} = 0 \quad \text{for } j = 1, 2. \tag{4.14}$$

Define, for $j = 1, 2$,

$$v_{j,n} = (1 + w_{j,n}) e^{\xi_{j,n}\cdot x/2},$$

Then

$$\Delta v_{j,n} + q_j v_{j,n} = 0 \text{ in } \mathbb{R}^d \quad \text{for } j = 1, 2.$$

We derive from (1.2) that

$$\int_{B_2} (q_1 - q_2)(1 + w_{1,n})(1 + w_{2,n}) e^{\sigma_s \cdot x/2} = 0, \tag{4.15}$$

where

$$\sigma_s = \xi_{1,n} + \xi_{2,n} \to \sigma_3 \text{ as } s \to \infty. \tag{4.16}$$

A combination of (4.14), (4.15), and (4.16) yields

$$\int_{B_2} (q_1 - q_2) e^{\sigma \cdot x/2} = 0.$$

Since $\sigma_3 \in \mathbb{S}^{d-1}$ is arbitrary, it follows that

$$q_1 = q_2,$$

and the proof is complete. □

4.3. **Proof of Corollary 2.** The proof is similar to the one of Corollary 1. The details are left to the reader. □

5. Uniqueness of Calderon's problem for conductivities of class $W^{s,3/s}$ for $s > 3/2$ in $3d$

5.1. **Construction of CGO solutions.** We begin this section with

Lemma 5. *Let $\xi \in \mathbb{C}^3$ with $|\xi| > 2$ and $\xi \cdot \xi = 0$, $v \in H^1_{loc}(\mathbb{R}^3)$ and $q \in H^{-1/2}(\mathbb{R}^3)$ with $\operatorname{supp} q \subset B_1$. Define*

$$W = K_\xi * (qV).$$

We have

$$\|W\|^2_{H^1(B_r)} \le C_r \|V\|^2_{H^1(B_2)} \cdot E(q,\xi),$$

where

$$E(q,\xi) = \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \int_{4|\xi| \ge \operatorname{dist}(k,\Gamma_\xi) \ge |k|/|\xi|} \frac{|k|^2 |\hat{K}_\xi(k)|^2}{|k-\eta|^2}\, dk\, d\eta$$
$$+ \int_{\mathbb{R}^3} |\tilde{q}(\gamma)|^2 \int_{\operatorname{dist}(\eta,\Gamma_\xi) \le |\eta|/|\xi|} \frac{1}{|\eta-\gamma|^2}\, d\eta\, d\gamma + \|q\|^2_{H^{-1/2}},$$

where

$$\widetilde{q}(k) := \sup_{\eta \in B_4(k)} |\hat{q}(\eta)|.$$

Proof. Without loss of generality, one may assume that $\operatorname{supp} V \subset B_{3/2}$ and $r > 1$. Set

$$f = qV, \tag{5.1}$$

then

$$W = K_\xi * f.$$

We first prove

$$\|W\|_{L^2(B_r)} \le C_r \|V\|_{H^1} \|q\|_{H^{-1/2}}. \tag{5.2}$$

Applying (2.6), we have

$$\|W\|_{L^2(B_r)} \le C_r \|f\|_{H^{-1}}. \tag{5.3}$$

On the other hand,

$$\|f\|^2_{H^{-1}} \le \int_{\mathbb{R}^3} \frac{1}{|k|^2+1} \Big| \int_{\mathbb{R}^3} |\hat{q}(k-\eta)||\hat{V}(\eta)|\, d\eta \Big|^2 dk. \tag{5.4}$$

Since

$$\int_{\mathbb{R}^3} |\hat{q}(k-\eta)||\hat{V}(\eta)|\, d\eta = \int_{|\eta| \le |k|/2} |\hat{q}(k-\eta)||\hat{V}(\eta)|\, d\eta + \int_{|\eta| > |k|/2} |\hat{q}(k-\eta)||\hat{V}(\eta)|\, d\eta,$$

it follows that, by Hölder's inequality,

$$\frac{1}{2} \int_{\mathbb{R}^3} \frac{1}{|k|^2+1} \Big| \int_{\mathbb{R}^3} |\hat{q}(k-\eta)||\hat{V}(\eta)|\, d\eta \Big|^2 dk \tag{5.5}$$
$$\le \int_{\mathbb{R}^3} \int_{|\eta| \le |k|/2} \frac{|\hat{q}(k-\eta)|^2}{(k^2+1)(|\eta|^2+1)}\, d\eta \int_{|\eta| \le |k|/2} |\hat{V}(\eta)|^2 (|\eta|^2+1)\, d\eta\, dk$$
$$+ \int_{\mathbb{R}^3} \int_{|\eta| \ge |k|/2} \frac{|\hat{q}(k-\eta)|^2}{(|k-\eta|^2+1)^{1/2}}\, d\eta \int_{|\eta| \ge |k|/2} \frac{|\hat{V}(\eta)|^2 (|k-\eta|^2+1)^{1/2}}{|k|^2+1}\, d\eta\, dk.$$

We have, since $|k-\eta| \le |k|/2$ implies $2|\eta| \ge |k| \ge 2|\eta|/3$,

$$\int_{\mathbb{R}^3} \int_{|\eta| \le |k|/2} \frac{|\hat{q}(k-\eta)|^2}{(k^2+1)(|\eta|^2+1)} \, d\eta \, dk = \int_{\mathbb{R}^3} \int_{|k-\eta| \le |k|/2} \frac{|\hat{q}(\eta)|^2}{(k^2+1)(|k-\eta|^2+1)} \, dk \, d\eta \tag{5.6}$$

$$\le \int_{\mathbb{R}^3} \frac{|\hat{q}(\eta)|^2}{(1+|\eta|^2)^{1/2}} \int_{2|\eta| \ge |k| \ge 2|\eta|/3} \frac{1}{(|k-\eta|^2+1)(1+|\eta|^2)^{1/2}} \le C\|q\|^2_{H^{-1/2}}$$

and

$$\int_{\mathbb{R}^3} \int_{|\eta| \ge |k|/2} \frac{|\hat{V}(\eta)|^2(|k-\eta|^2+1)^{1/2}}{|k|^2+1} \, d\eta \, dk \le C\|V\|^2_{H^1}. \tag{5.7}$$

Using (5.4), (5.5), (5.6), and (5.7), we derive from (5.4) that

$$\|f\|_{H^{-1}} \le C\|V\|_{H^1}\|q\|_{H^{-1/2}}. \tag{5.8}$$

A combination of (5.3) and (5.8) yields (5.2).

It remains to establish the key estimate

$$\|\nabla W\|_{L^2(B_r)} \le C_r\|V\|_{H^1} \cdot E(q, \xi). \tag{5.9}$$

Set

$$\Gamma_\xi := \{k \in \mathbb{R}^3; \ -|k|^2 + i\xi \cdot k = 0\}.$$

Define $K_{1,\xi}$, $K_{2,\xi}$, and $K_{3,\xi}$ as follows

$$\hat{K}_{1,\xi}(k) = \begin{cases} \hat{K}_\xi(k) & \text{if } 4|\xi| \ge \operatorname{dist}(k, \Gamma_\xi) > |k|/|\xi|, \\ 0 & \text{otherwise}, \end{cases}$$

$$\hat{K}_{2,\xi}(k) = \begin{cases} \hat{K}_\xi(k) & \text{if } \operatorname{dist}(k, \Gamma_\xi) \le |k|/|\xi|, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\hat{K}_{3,\xi}(k) = \begin{cases} \hat{K}_\xi(k) & \text{if } \operatorname{dist}(k, \Gamma_\xi) > 4|\xi|, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\|\nabla(K_\xi * f)\|_{L^2(B_r)} \le \|\nabla(K_{1,\xi} * f)\|_{L^2(B_r)} + \|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)} + \|\nabla(K_{3,\xi} * f)\|_{L^2(B_r)}. \tag{5.10}$$

Since

$$|\hat{K}_\xi(k)| \le \frac{1}{|k|^2} \text{ for } \operatorname{dist}(k, \Gamma_\xi) \ge 4|\xi|,$$

it follows that

$$\|\nabla(K_{3,\xi} * f)\|_{L^2(B_r)} \le \|\nabla(K_{3,\xi} * f)\|_{L^2(\mathbb{R}^3)} \le C\|f\|_{H^{-1}}. \tag{5.11}$$

A combination of (5.8) and (5.11) yields

$$\|\nabla(K_{3,\xi} * f)\|_{L^2(B_r)} \le C\|V\|_{H^1}\|q\|_{H^{-1/2}}. \tag{5.12}$$

We next estimate the first two terms in the RHS of (5.10). We start with $\|\nabla(K_{1,\xi} * f)\|_{L^2(B_r)}$. Since

$$\|\nabla(K_{1,\xi} * f)\|^2_{L^2(B_r)} \le \|\nabla(K_{1,\xi} * f)\|^2_{L^2(\mathbb{R}^3)},$$

it follows from Plancherel's theorem that

$$\|\nabla(K_{1,\xi} * f)\|^2_{L^2(B_r)} \le C \int_{4|\xi| \ge |\hat{K}_\xi(k)| \ge k/|\xi|} |\hat{f}(k)|^2|k|^2|\hat{K}_\xi(k)|^2 \, dk. \tag{5.13}$$

From (5.1), we have

$$\hat{f}(k) = \int_{\mathbb{R}^3} \hat{q}(\eta)\hat{V}(k-\eta)\, d\eta.$$

Applying Hölder's inequality, we obtain

$$|\hat{f}(k)|^2 \le \int_{\mathbb{R}^3} \frac{|\hat{q}(\eta)|^2}{|k-\eta|^2}\, d\eta \int_{\mathbb{R}^3} |(k-\eta)\hat{V}(k-\eta)|^2\, d\eta. \tag{5.14}$$

A combination of (5.13) and (5.14) yields

$$\begin{aligned}\int_{4|\xi|\ge|\hat{K}_\xi(k)|\ge k/|\xi|} & |\hat{f}(k)|^2|k|^2|\hat{K}_\xi(k)|^2\, dk \\ & \le C\|\nabla V\|_{L^2}^2 \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \int_{4|\xi|\ge|\hat{K}_\xi(k)|\ge|k|/|\xi|} \frac{|k|^2|\hat{K}_\xi(k)|^2}{|k-\eta|^2}\, dk\, d\eta.\end{aligned} \tag{5.15}$$

We next estimate $\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}$. Fix

$$\varphi \in C_0^\infty(\mathbb{R}^3) \text{ with } \varphi = 1 \text{ in } B_{2r}.$$

Define

$$\widetilde{f}(k) = \sup_{\eta\in B_4(k)} |\hat{f}(\eta)|, \tag{5.16}$$

and

$$\widetilde{\varphi}(k) = \sup_{\eta\in B_4(k)} |\hat{\varphi}(\eta)|. \tag{5.17}$$

Since

$$|\hat{f}| * |\hat{\varphi}|(\eta) = \int_{\mathbb{R}^d} |\hat{f}(\zeta)||\hat{\varphi}(\eta-\zeta)|\, d\zeta,$$

and $f = f\varphi$, it follows from the definition of $\widetilde{f}$ (5.16) and $\widetilde{\varphi}$ (5.17) that

$$\widetilde{f} \le |\hat{f}| * \widetilde{\varphi}.$$

Since

$$\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \le \|\nabla(\varphi\cdot[K_{2,\xi} * f])\|_{L^2(\mathbb{R}^3)}^2,$$

it follows that

$$\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \le \int_{\mathbb{R}^d} |k|^2 \left| \int_{\mathrm{dist}(\eta,\Gamma_\xi)\le|\eta|/|\xi|} |\hat{\varphi}(k-\eta)|\cdot|\hat{K}_\xi(\eta)|\cdot|\hat{f}(\eta)|\, d\eta \right|^2 dk. \tag{5.18}$$

Using the fact that $\hat{K}_\xi(\eta) \le C/\big(|\xi|\,\mathrm{dist}(\eta,\Gamma_\xi)\big)$ for $|\eta|\le 2|\xi|$ and

$$\int_{|x|\le 1} \frac{1}{|x_1|+|x_2|}\, dx < +\infty,$$

as in (2.19), we obtain

$$\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le 1} |\hat{\varphi}(k-\eta)|\cdot|\hat{K}_\xi(\eta)|\cdot|\hat{f}(\eta)|\, d\eta \le \frac{C}{|\xi|}\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le|\eta|/|\xi|} \widetilde{\varphi}(k-\eta)\cdot\widetilde{f}(\eta)\, d\eta. \tag{5.19}$$

Applying Hölder's inequality, we derive from (5.18) and (5.19) that

$$\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \le \frac{C}{|\xi|^2}\int_{\mathbb{R}^3}\int_{\mathrm{dist}(\eta,\Gamma_\xi)\le|\eta|/|\xi|} |k|^2\widetilde{\varphi}(k-\eta)|\widetilde{f}(\eta)|^2\, d\eta\, dk. \tag{5.20}$$

Since $|k|^2 \le C(|k-\eta|^2 + |\eta|^2)$ and $\widetilde{\varphi}$ decays fast at infinity, it follows from (5.17) and (5.20) that

$$\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \le C \int_{\mathrm{dist}(\eta,\Gamma_\xi) \le |\eta|/|\xi|} |\widetilde{f}(\eta)|^2 \, d\eta. \tag{5.21}$$

Since

$$\tilde{f}(\eta) \le \tilde{q} * |\hat{V}|(\eta),$$

it follows that

$$\int_{\mathrm{dist}(\eta,\gamma_\xi) \le |\eta|/|\xi|} |\tilde{f}(\eta)|^2 \, d\eta \le \int_{\mathrm{dist}(\eta,\gamma_\xi) \le |\eta|/s} \left| \int_{\mathbb{R}^3} \frac{|\tilde{q}(\gamma)|}{|\eta-\gamma|} |\eta-\gamma| |\hat{V}(\eta-\gamma)| \, d\gamma \right|^2 \, d\eta.$$

Using Hölder's inequality, we obtain

$$\int_{\mathrm{dist}(\eta,\Gamma_\xi) \le |\eta|/|\xi|} |\widetilde{f}(\eta)|^2 \, d\eta \le C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} |\tilde{q}(\gamma)|^2 \int_{\mathrm{dist}(\eta,\Gamma_\xi) \le |\eta|/|\xi|} \frac{1}{|\eta-\gamma|^2} \, d\eta \, d\gamma. \tag{5.22}$$

A combination of (5.21) and (5.22) yields

$$\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \le C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} |\tilde{q}(\gamma)|^2 \int_{\mathrm{dist}(\eta,\Gamma_\xi) \le |\eta|/|\xi|} \frac{1}{|\eta-\gamma|^2} \, d\eta \, d\gamma. \tag{5.23}$$

We derive from (5.12), (5.15), and (5.23) that (5.9) holds. The proof is complete. □

To use Lemma 5, we need to choose ξ such that $E(q,\xi)$ remains bounded. This can be done using the following average estimate for $E(q,\xi)$ whose proof is in the spirit of the one of Lemma 4 and is presented in the appendix.

Lemma 6. *Let $d=3$ and $R > 10$. We have*

$$\frac{1}{R} \int_{R/2}^{2R} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}_{\sigma_1}} E(q, s\sigma_2 - is\sigma_1) \, d\sigma_2 \, d\sigma_1 \, ds \le C \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \min\Big\{ \frac{\ln R}{R}, \frac{R \ln R}{|\eta|^2} \Big\} \, d\eta \tag{5.24}$$

and

$$\frac{1}{R} \int_{R/2}^{2R} \int_{\sigma_1 \in \mathbb{S}^2} \int_{\sigma_2 \in \mathbb{S}^2_{\sigma_1}} \int_{\sigma_3 \in \mathbb{S}^2_{\sigma_1,\sigma_2}} E\Big(q, \frac{s^2\sigma_2}{\sqrt{1+s^2}} + \frac{s\sigma_3}{\sqrt{1+s^2}} - is\sigma_1\Big) \, d\sigma_3 \, d\sigma_2 \, d\sigma_1 \, ds \\ \le C \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \min\Big\{ \frac{\ln R}{R}, \frac{R \ln R}{|\eta|^2} \Big\} \, d\eta. \tag{5.25}$$

We recall that, by (3.16) and (3.17),

$$\mathbb{S}^2_{\sigma_1} := \big\{ \sigma \in \mathbb{S}^2; \; \sigma \cdot \sigma_1 = 0 \big\}$$

and

$$\mathbb{S}^2_{\sigma_1,\sigma_2} := \big\{ \sigma \in \mathbb{S}^2; \; \sigma \cdot \sigma_1 = 0 \text{ and } \sigma \cdot \sigma_2 = 0 \big\}.$$

We will show that the RHS of (5.24) will behave like $\|q\|_{H^{-1/2}}$ for appropriate choice of s. For this end, we need the following lemma.

Lemma 7. *Let (a_n) be a non-negative sequence. Define*

$$b_n = \sum_{l=1}^{n} 2^{l-n} a_l.$$

Assume that $S = \sum_1^\infty a_n < +\infty$, then

$$\liminf_{n \to \infty} n b_n = 0.$$

Proof. The conclusion is a consequence of the following facts:

$$\sum_{n=1}^{\infty} b_n \le c \sum_{1}^{\infty} a_n < +\infty$$

for some positive constant c, and

$$\liminf_{n\to\infty} n b_n = 0,$$

if

$$\sum_{n=1}^{\infty} b_n < +\infty.$$

□

Applying Lemmas 5, 6, and 7, we can obtain the following result which is a variant of Propositions 2 and 3 in this setting.

Proposition 5. *Let $q_1, q_2 \in H^{-1/2}(\mathbb{R}^3)$ with support in B_1, and $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{S}^2$ be such that*

$$\sigma_1 \cdot \sigma_2 = \sigma_1 \cdot \sigma_3 = \sigma_2 \cdot \sigma_3 = 0.$$

For any $\varepsilon > 0$, there exist a sequence $s_n \to \infty$, $\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n} \in \mathbb{S}^2$ and $u_{i,n} \in H^1_{loc}(\mathbb{R}^3)$ such that

$$\sigma_{1,n} \cdot \sigma_{2,n} = \sigma_{1,n} \cdot \sigma_{3,n} = \sigma_{2,n} \cdot \sigma_{3,n} = 0,$$
$$|\sigma_{j,n} - \sigma_j| \le \varepsilon \text{ for } j = 1, 2, 3,$$

and

$$w_{j,n} = K_{\xi_{j,n}} * (q_j + q_j w_{j,n}) \text{ for } j = 1, 2.$$

Here

$$\xi_{1,n} = s_n \sigma_{2,n} - i s \sigma_{1,n} \quad \text{and} \quad \xi_{2,n} = -\frac{s_n^2 \sigma_{2,n}}{\sqrt{1+s_n^2}} + \frac{s_n \sigma_{3,n}}{\sqrt{1+s_n^2}} + i s_n \sigma_{1,n}.$$

Moreover,

$$\|w_{j,n}\|_{H^1(B_r)} \le C\varepsilon \|K_{\xi_{j,n}} * q_j\|_{H^1(B_1)} \quad \text{for } j = 1, 2,$$

and for large n.

Proof. For $\varepsilon > 0$, let $q_{j,1} \in C^\infty(\mathbb{R}^3)$ and $q_{j,2} \in C^\infty(\mathbb{R}^3)$ with supports in B_1 be such that, for $j = 1, 2$,

$$q_{j,1} + q_{j,2} = q_j,$$

and

$$\|q_{j,2}\|_{H^{-1/2}} \le \varepsilon^4. \tag{5.26}$$

Define

$$a_{j,n} = \int_{2^n \le |k| \le 2^{n+1}} \frac{|\hat{q}_{j,2}(k)|^2}{|k|}\, dk.$$

It is clear that

$$\sum_{n=1}^{\infty} a_{j,n} \le \|q_{j,2}\|^2_{H^{-1/2}}.$$

Set

$$b_{j,n} := \sum_{l=1}^{n} 2^{l-n} a_{j,l} \sim \int_{2^1 \le |k| \le 2^{n+1}} \frac{|\hat{q}_{j,2}(k)|^2}{2^{n+1}}\, dk.$$

By Lemma 7, there exists $n_k \to \infty$ such that

$$n_k b_{1,n_k} + n_k b_{2,n_k} \le c\big(\|q_{1,2}\|^2_{H^{-1/2}} + \|q_{2,2}\|^2_{H^{-1/2}}\big).$$

Applying Lemma 6, there exist $\sigma_{1,k}, \sigma_{2,k}, \sigma_{3,k} \in \mathbb{S}^2$ such that

$$\sigma_{1,k} \cdot \sigma_{2,k} = \sigma_{1,k} \cdot \sigma_{3,k} = \sigma_{2,k} \cdot \sigma_{3,k} = 0,$$

$$|\sigma_{j,k} - \sigma_j| \le \varepsilon \text{ for } j = 1, 2, 3,$$

and, by (5.26),

$$E(q_{1,2}, \xi_{1,k}) + E(q_{2,2}, \xi_{2,k}) \le C\varepsilon \text{ for large } k. \tag{5.27}$$

Here

$$\xi_{1,k} = s_k\sigma_{2,k} - is_k\sigma_{1,k} \quad \text{ and } \quad \xi_{2,k} = -\frac{s_k^2\sigma_{2,k}}{\sqrt{1+s_k^2}} + \frac{s_k\sigma_{3,k}}{\sqrt{1+s_k^2}} + is_k\sigma_{1,k}.$$

Let $w_{j,k,0} = 0$ $(j = 1, 2)$ and consider the following iteration process:

$$w_{j,k,n} = K_{\xi_{j,k}} * (q_j + q_j w_{j,k,n-1}) \text{ for } n \ge 1.$$

Then, for $n \ge 1$ and $j = 1, 2$,

$$\begin{aligned} w_{j,k,n+1} - w_{j,k,n} =& K_{\xi_{j,k}} * (q_j[w_{j,k,n} - w_{j,k,n-1}]) \\ =& K_{\xi_{j,k}} * (q_{j,1}[w_{j,k,n} - w_{j,k,n-1}]) + K_{\xi_{j,k}} * (q_{j,2}[w_{j,k,n} - w_{j,k,n-1}]). \end{aligned}$$

Applying (2.4) for the first part and applying Lemma 5 for the second part, we have

$$\|w_{j,k,n+1} - w_{j,k,n}\|^2_{H^1(B_r)} \le C_r\Big(E(q_{j,2}, \xi_{j,k})^{1/2} + \frac{\|q_{j,1}\|_{C^2}}{|\xi_{j,k}|}\Big)\|w_{j,k,n} - w_{j,k,n-1}\|^2_{H^1(B_1)}.$$

This implies

$$\sum_{m=1}^{n} \|w_{j,k,m+1} - w_{j,k,m}\|_{H^1(B_r)} \le c(r, f, h) \sum_{m=0}^{n-1} \|w_{j,k,m} - w_{j,k,m-1}\|_{H^1(B_1)}, \tag{5.28}$$

where $c(r, f, h) = C_r\Big(E(q_{i,2}, \xi_{i,k})^{1/2} + \frac{\|q_{j,1}\|_{C^2}}{|\xi_{i,k}|}\Big)$. From (5.27), for large k, we derive that $c(2, f, h) < 1/2$. For such a large k, we have

$$\sum_{m=1}^{n} \|w_{j,k,m+1} - w_{j,k,m}\|_{H^1(B_2)} \le 2c(r, f, h)\|w_{j,k,1} - w_{j,k,0}\|_{H^1(B_1)} = 2c(2, f, h)\|w_{j,k,1}\|_{H^1(B_1)}.$$

It follows that there exist $w_j \in H^1_{loc}(\mathbb{R}^3)$ such that

$$w_{j,k} = K_{\xi_{j,k}} * (q_{j,k} + q_j w_{j,k}),$$

and, by (5.27) and (5.28),

$$\|w_{j,k}\|_{H^1(B_r)} \le C\varepsilon\|w_{j,k,1}\|_{H^1(B_1)},$$

for large k large enough. The proof is complete. □

5.2. **Proof of Theorem 3.** Theorem 3 is a consequence of Proposition 5. The proof is standard and the details are left to the reader. We note that the condition $t > 1/2$ ensures the existence of the trace of g_1 on the boundary. □

5.3. **Proof of Corollary 3.** The proof is similar to the one of Corollary 1. The details are left to the reader. □

Appendix A. Some averaging estimates

A.1. **Proof of Lemma 4.** It is clear that

$$|\hat{K}_{s\sigma_2 - is\sigma_1}(k)|^p \leq \frac{C_p}{|k|^{2p}} \text{ for } |k| > 2s.$$

Hence to obtain (3.14), it suffices to prove that

$$\int_{\sigma_1 \in \mathbb{S}^{d-1}} \int_{\sigma_2 \in \mathbb{S}_{\sigma_1}} |\hat{K}_{s\sigma_2 - is\sigma_1}(k)|^p \, d\sigma_2 \, d\sigma_1 \leq \frac{C_p}{|k|^p s^p} \qquad \text{for } |k| \leq 2s.$$

Without loss of generality one may assume that $k = te_1\big(e_1 = (1, 0, \cdots, 0)\big)$. Set

$$\xi = s(\sigma_2 - i\sigma_1).$$

We have

$$\frac{1}{\big| - |k|^2 + ik \cdot \xi\big|^p} = \frac{1}{\big| - t^2 + iste_1 \cdot \sigma_1 + ste_1 \cdot \sigma_2\big|^p} \sim \frac{1}{|t^2 - st\sigma_1 \cdot e_1|^p + (st)^p |\sigma_2 \cdot e_1|^p}.$$

Let θ_1 be the angle between σ_1 and e_1 and let θ_2 be the angle between σ_2 and v where $v = e_1 - (e_1 \cdot \sigma_1)\sigma_1 = e_1 - \cos\theta_1 \sigma_1$. Note that $v \in \operatorname{span}\{\sigma_1, e_1\}$, v is orthogonal to σ_1, and $|v| = |\sin(\theta_1)|$. Using the spherical area element, we have

$$\begin{aligned} C_p \int_{\sigma_1 \in \mathbb{S}^{d-1}} \int_{\sigma_2 \in \mathbb{S}^{d-1}_{\sigma_1}} & \frac{1}{|t^2 - st\sigma_1 \cdot e_1|^p + (st)^p |\sigma_2 \cdot e_1|^p} \, d\sigma_2 \, d\sigma_1 \\ & \leq \int_0^{\pi/2} \int_0^{\pi} \frac{\theta_1^{d-2} \theta_2^{d-3}}{|t^2 - st\cos\theta_1|^p + (st)^p |\sin\theta_1 \cos\theta_2|^p} \, d\theta_2 \, d\theta_1 \\ & \quad + \int_{\pi/2}^{\pi} \int_0^{\pi} \frac{(\pi - \theta_1)^{d-2} \theta_2^{d-3}}{|t^2 - st\cos\theta_1|^p + (st)^p |\sin\theta_1 \cos\theta_2|^p} \, d\theta_2 \, d\theta_1. \end{aligned}$$

Here we use $|\sigma_2 \cdot e_1| = |\sigma_2 \cdot v| = |\sin\theta_1 \cos\theta_2|$. It follows that

$$\begin{aligned} \int_{\sigma_1 \in \mathbb{S}^{d-1}} \int_{\sigma_2 \in \mathbb{S}^{d-1}_{\sigma_1}} & \frac{1}{\big| - |k|^2 + ik \cdot \xi\big|^p} \, d\sigma_2 \, d\sigma_1 \\ & \leq \frac{C_p}{(st)^p} \int_0^{\pi/2} \int_0^{\pi} \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{t}{s} - \cos\theta_1|^p + |\sin\theta_1 \cos\theta_2|^p} \, d\theta_2 \, d\theta_1. \end{aligned}$$

Fix $0 < \delta < 2 - p$, and consider the case $t \leq s$. Then

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\pi} & \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{t}{s} - \cos\theta_1|^p + |\sin\theta_1 \cos\theta_2|^p} \, d\theta_2 \, d\theta_1 \\ & \leq C_p \int_0^{\pi/2} \int_0^{\pi} \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{t}{s} - \cos\theta_1|^{1-\delta} |\sin\theta_1 \cos\theta_2|^{p-1+\delta}} \, d\theta_2 \, d\theta_1. \end{aligned}$$

This implies

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\pi} & \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{t}{s} - \cos\theta_1|^p + |\sin\theta_1 \cos\theta_2|^p} \, d\theta_2 \, d\theta_1 \\ & \leq \int_0^{\pi/2} \frac{\theta_1^{d-2}}{|\frac{t}{s} - \cos\theta_1|^{1-\delta} |\sin\theta_1|^{p-1+\delta}} \, d\theta_1 \int_0^{\pi} \frac{1}{|\cos\theta_2|^{p-1+\delta}} d\theta_2 \end{aligned}$$

A computation yields
(A1)
$$\int_0^{\pi/2}\int_0^{\pi}\frac{\theta_1^{d-2}\theta_2^{d-3}}{|t^2-st\cos\theta_1|^p+(st)^p|\sin\theta_1\cos\theta_2|^p}\,d\theta_2\,d\theta_1\le C_p\int_0^{\pi/2}\frac{\theta_1^{d-1-p-\delta}}{|\frac{t}{s}-\cos\theta_1|^{1-\delta}}\,d\theta_1.$$

On the other hand, let θ_0,α_0 be such that $\cos\theta_0=\frac{t}{s}$ and $|\cos\theta_0-\cos(\alpha+\theta_0)|\le\frac{1}{2}$ for all $|\alpha|\le\alpha_0$. We have, since $d-1-p-\delta\ge 2-p-\delta>0$,

$$\text{(A2)}\quad C_p\int_0^{\pi/2}\frac{\theta_1^{d-1-p-\delta}}{|\frac{t}{s}-\cos\theta_1|^{1-\delta}}\,d\theta_1 \le\int_{|\theta-\theta_0|\le\alpha_0}\frac{1}{|\cos\theta_0-\cos\theta|^{1-\delta}}\,d\theta+\int_{[0,\pi/2]\setminus\{|\theta-\theta_0|\le\alpha_0\}}\frac{1}{|\cos\theta_0-\cos\theta|^{1-\delta}}\,d\theta.$$

We have
$$\int_{|\theta-\theta_0|\le\alpha_0}\frac{1}{|\cos\theta_0-\cos\theta|^{1-\delta}}\,d\theta\le\int_{|\theta-\theta_0|\le\alpha_0}\frac{C_p}{|\sin[(\theta_0+\theta)/2]|^{1-\delta}|\theta-\theta_0|^{1-\delta}}\,d\theta$$
$$\text{(A3)}\qquad\le\int_{|\theta-\theta_0|\le\alpha_0}\frac{C_p}{|\sin\theta_0|^{1-\delta}|\theta-\theta_0|^{1-\delta}}\,d\theta\le\frac{C_p}{(1-(\frac{t}{s}))^{\frac{1-\delta}{2}}}$$

and
$$\text{(A4)}\qquad\int_{[0,\pi/2]\setminus\{|\theta-\theta_0|\le\alpha_0\}}\frac{1}{|\frac{t}{s}-\cos\theta|^{1-\delta}}\,d\theta\le C_p.$$

A combination of (A1), (A2), (A3), and (A4) yields
$$\text{(A5)}\quad\int_0^{\pi/2}\int_0^{\pi}\frac{\theta_1^{d-2}\theta_2^{d-3}}{|t^2-st\cos\theta_1|^p+(st)^p|\sin\theta_1\cos\theta_2|^p}\,d\theta_2\,d\theta_1\le\frac{C_p}{(st)^p(1-(\frac{t}{s}))^{\frac{1-\delta}{2}}}+\frac{C_p}{(st)^p}.$$

For $s<t\le 2s$, we have
$$\int_0^{\pi/2}\frac{\theta_1^{d-1-p-\delta}}{|\frac{t}{s}-\cos\theta_1|^{1-\delta}}\,d\theta_1\le C.$$

Hence we also obtain (A5) in this case. Averaging (A5) in s yields bound (3.14).

We now establish (3.15). Define $v_1=e_1-(e_1\cdot\sigma_1)\sigma_1-(e_1\cdot\sigma_2)\sigma_2=v-(v\cdot\sigma_2)\sigma_2$ and let θ_3 be the angle between σ_3 and v_1. We have, since $\sigma_3\cdot e_1=\sigma_3\cdot v_1=|v_1|\cos\theta_3$,

$$\int_{\sigma_1\in\mathbb{S}^{d-1}}\int_{\sigma_2\in\mathbb{S}^{d-1}_{\sigma_1}}\int_{\sigma_3\in\mathbb{S}^{d-1}_{\sigma_1,\sigma_2}}\Big|\hat{K}_{\frac{s^2\sigma_2}{\sqrt{1+s^2}}+\frac{s\sigma_3}{\sqrt{1+s^2}}-is\sigma_1}(k)\Big|^p\,d\sigma_3\,d\sigma_2\,d\sigma_1$$
$$\le C_p\int_0^{3\pi/4}\int_0^{\pi}\int_0^{\pi}\frac{\theta_1^{d-2}\theta_2^{d-3}\theta_3^{d-4}}{|t^2-st\cos\theta_1|^p+(st)^p|\sin\theta_1\cos\theta_2-|v_1|\cos\theta_3/s|^p}\,d\theta_3\,d\theta_2\,d\theta_1.$$

Here $\int_0^\pi f(\theta_3)\theta_3^{d-4}\,d\theta_3:=f(\pi)+f(0)$ if $d=3$. We will only consider the case $d\ge 4$, the case $d=3$ follows similarly. We have
$$\int_0^{\pi}\frac{\theta_3^{d-4}\,d\theta_3}{|t^2-st\cos\theta_1|^p+(st)^p|\sin\theta_1\cos\theta_2-|v_1|\cos\theta_3/s|^p}\le\frac{C_p}{|t^2-st\cos\theta_1|^{p-1}t|v_1|}.$$

Since
$$|v_1|^2=|v|^2-|v\cdot\sigma_2|^2=\sin^2\theta_1\sin^2\theta_2,$$

it follows that

$$\int_0^\pi \frac{\theta_3^{d-4}\, d\theta_3}{|t^2 - st\cos\theta_1|^p + (st)^p |\sin\theta_1\cos\theta_2 - |v_1|\cos\theta_3/s|^p} \le \frac{C_p}{|t^2 - st\cos\theta_1|^{p-1} t |\sin\theta_2||\sin\theta_1|}.$$

This implies

$$\int_0^{3\pi/4}\int_0^\pi\int_0^\pi \frac{\theta_1^{d-2}\theta_2^{d-3}\theta_3^{d-4}}{|t^2 - st\cos\theta_1|^p + (st)^p |\sin\theta_1\cos\theta_2 - |v_1|\cos\theta_3/s|^p}\, d\theta_3\, d\theta_2\, d\theta_1$$

$$\le C_p \int_0^{3\pi/4}\int_0^\pi \frac{\theta_1^{d-2}\theta_2^{d-3}}{|t^2 - st\cos\theta_1|^{p-1} t |\sin\theta_1 \sin\theta_2|}\, d\theta_2\, d\theta_1.$$

We have, since $d \ge 4$,

$$\int_0^{3\pi/4}\int_0^\pi \frac{\theta_1^{d-2}\theta_2^{d-3}}{|t^2 - st\cos\theta_1|^{p-1} t |\sin\theta_1 \sin\theta_2|}\, d\theta_2\, d\theta_1 \le \int_0^{3\pi/4} \frac{1}{|t^2 - st\cos\theta_1|^{p-1} t}\, d\theta_1 \le \frac{C_p}{t^p s^p}.$$

We obtain the conclusion. □

A.2. **Proof of Lemma 6.** We first claim that, for $k \in \mathbb{R}^3$ with $|k| \ge 2$,

$$\text{(A6)}\qquad \frac{1}{R}\int_{R/2}^{2R}\int_{\sigma_1\in\mathbb{S}^2}\int_{\substack{\sigma_2\in\mathbb{S}^2_{\sigma_1}\\ \operatorname{dist}(k,\Gamma_\xi)\ge|k|/|\xi|}} |\hat{K}_{s\sigma_2 - is\sigma_1}(k)|^2\, d\sigma_2\, d\sigma_1\, ds \le C\min\Big\{\frac{\ln R}{R^2|k|^2}, \frac{1}{|k|^4}\Big\}.$$

Here $\xi = \xi(s,\sigma_1,\sigma_2) = s\sigma_2 - is\sigma_1$ and

$$\Gamma_\xi := \{k \in \mathbb{R}^3;\ -|k|^2 + i\xi\cdot k = 0\}.$$

Indeed, since

$$|\hat{K}_{s\sigma_2 - is\sigma_1}(k)|^2 \le \frac{C}{|k|^4} \text{ for } |k| > 2s,$$

it suffices to prove that

$$\text{(A7)}\qquad \frac{1}{R}\int_{R/2}^{2R}\int_{\sigma_1\in\mathbb{S}^2}\int_{\substack{\sigma_2\in\mathbb{S}^2_{\sigma_1}\\ \operatorname{dist}(k,\Gamma)\ge|k|/|\xi|}} |\hat{K}_{s\sigma_2 - is\sigma_1}(k))|^2\, d\sigma_2\, d\sigma_1\, ds \le \frac{C\ln R}{|k|^2 R^2} \quad \text{for } |k| \le 2s.$$

Without loss of generality, one may assume that $k = te_1 = (t,0,0)$. As in the proof of Lemma 4, we have

$$\text{(A8)}\quad \int_{\sigma_1\in\mathbb{S}^{d-1}}\int_{\sigma_2\in\mathbb{S}^{d-1}_{\sigma_1}} \frac{1}{\big|-|k|^2 + ik\cdot\xi\big|^2 + (t/s)^2}\, d\sigma_2\, d\sigma_1$$

$$\le \frac{C}{(st)^2}\int_0^{\pi/2}\int_0^\pi \frac{\theta_1}{|\frac{t}{s} - \cos\theta_1|^2 + |\sin\theta_1\cos\theta_2|^2 + s^{-4}}\, d\theta_2\, d\theta_1.$$

A computation yields
(A9)

$$\int_0^{\pi/2}\int_0^\pi \frac{\theta_1}{|\frac{t}{s} - \cos\theta_1|^2 + |\sin\theta_1\cos\theta_2|^2 + s^{-4}}\, d\theta_2\, d\theta_1 \le C\int_0^{\pi/2}\frac{1}{\big|\frac{t}{s} - \cos\theta_1\big| + s^{-2}}\, d\theta_1.$$

and

$$\text{(A10)}\qquad \int_0^{\pi/2}\frac{1}{\big|\frac{t}{s} - \cos\theta_1\big| + s^{-2}}\, d\theta_1 \le C\ln s.$$

A combination of (A8), (A9), and (A10) yields (A7); hence (A6) is established.

In the rest, we only give the proof of (5.24). The proof of (5.25) follows similarly. Applying (A6), we have

$$\frac{1}{R}\int_{R/2}^{2R}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2_{\sigma_1}}\int_{\mathbb{R}^3}|\hat{q}(\eta)|^2\int_{4|\xi|\ge \operatorname{dist}(k,\Gamma_{s\sigma_2-is\sigma_1})\ge|k|/|\xi|}\frac{|k|^2|\hat{K}_{s\sigma_2-is\sigma_1}(k)|^2}{|k-\eta|^2}\,dk\,d\eta\,d\sigma_2\,d\sigma_1\,ds \le C\int_{\mathbb{R}^3}|\hat{q}(\eta)|^2\int_{10R\ge|k|}\frac{\ln R}{R^2|k-\eta|^2}\,dk\,d\eta.$$

Since

$$\int_{\mathbb{R}^3}|\hat{q}(\eta)|^2\int_{10R\ge|k|}\frac{\ln R}{R^2|k-\eta|^2}\,dk\,d\eta \le C\int_{\mathbb{R}^3}|\hat{q}(\eta)|^2\min\left\{\frac{\ln R}{R},\frac{R\ln R}{|\eta|^2}\right\}d\eta,$$

it follows that

(A11)
$$\frac{1}{R}\int_{R/2}^{2R}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2_{\sigma_1}}\int_{\mathbb{R}^3}|\hat{q}(\eta)|^2\int_{4|\xi|\ge \operatorname{dist}(k,\Gamma_{s\sigma_2-is\sigma_1})\ge|k|/|\xi|}\frac{|k|^2|\hat{K}_{s\sigma_2-is\sigma_1}(k)|^2}{|k-\eta|^2}\,dk\,d\eta\,d\sigma_2\,d\sigma_1\,ds \le C\int_{\mathbb{R}^3}|\hat{q}(\eta)|^2\min\left\{\frac{\ln R}{R},\frac{R\ln R}{|\eta|^2}\right\}d\eta.$$

Define

$$\widetilde{q}(k) := \sup_{\eta\in B_4(k)}|\hat{q}(\eta)|.$$

We have

(A12)
$$\frac{1}{R}\int_{R/2}^{2R}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2_{\sigma_1}}\int_{\mathbb{R}^3}|\hat{q}(\gamma)|^2\int_{\operatorname{dist}(\eta,\Gamma_{s\sigma_2-is\sigma_1})\le|\eta|/|\xi|}\frac{1}{|\eta-\gamma|^2}\,d\eta\,d\gamma\,d\sigma_2\,d\sigma_1\,ds \le C\int_{\mathbb{R}^3}|\tilde{q}(\gamma)|^2\min\left\{\frac{R}{|\gamma|^2},\frac{1}{R}\right\}d\gamma.$$

Fix $q\in C^\infty_{\rm c}(\mathbb{R}^3)$ such that $\varphi=1$ in B_1 and $\operatorname{supp}\varphi\subset B_2$ and define

$$\widetilde{\varphi}(k) = \sup_{\eta\in B_4(k)}|\hat{\varphi}(\eta)|.$$

Using the fact that

(A13)
$$|\tilde{q}| \le \tilde{\varphi} * |\hat{q}|,$$

and applying Hölder's inequality, we have

$$\int_{\mathbb{R}^3}|\tilde{q}(\gamma)|^2\min\left\{\frac{R}{|\gamma|^2},\frac{1}{R}\right\}d\gamma \le C\int_{\mathbb{R}^3}|\hat{q}(\beta)|^2\int_{\mathbb{R}^3}\tilde{\varphi}(\gamma-\beta)\min\left\{\frac{R}{|\gamma|^2},\frac{1}{R}\right\}d\gamma\,d\beta.$$

It follows from (A13) that

(A14)
$$\int_{\mathbb{R}^3}|\tilde{q}(\gamma)|^2\min\left\{\frac{R}{|\gamma|^2},\frac{1}{R}\right\}d\gamma \le C\int_{\mathbb{R}^3}|\hat{q}(\beta)|^2\min\left\{\frac{R}{|\beta|^2},\frac{1}{R}\right\}d\beta.$$

A combination of (A12) and (A14) yields

$$\text{(A15)}\quad \frac{1}{R}\int_{R/2}^{2R}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2_{\sigma_1}}\int_{\mathbb{R}^3}|\tilde{q}(\gamma)|^2\int_{\mathrm{dist}(\eta,\Gamma_{s\sigma_2-is\sigma_1})\le|\eta|/|\xi|}\frac{1}{|\eta-\gamma|^2}\,d\eta\,d\gamma\,d\sigma_2\,d\sigma_1\,ds$$
$$\le C\int_{\mathbb{R}^3}|\hat{q}(\beta)|^2\min\left\{\frac{R}{|\beta|^2},\frac{1}{R}\right\}d\beta.$$

We derive (5.24) from (A11), and (A15). □

Appendix B. Boundary determination

In this appendix, we prove the following result

Proposition B1. *Let $d \ge 2$, Ω be an open subset of $\mathbb{R}^d$ of class C^1, and $\gamma_1, \gamma_2 \in W^{1,1}(\Omega)$. Assume $DtN_{\gamma_1} = DtN_{\gamma_2}$, then we have*

$$\gamma_1 = \gamma_2 \text{ on } \partial\Omega.$$

Proof. We give the proof in the case $d \ge 3$. The proof in the $2d$ case follows similarly. We prove this result by contradiction. Assume that the conclusion is not true. Hence there exists some z on $\partial\Omega$ such that

$$\gamma_1(z) \neq \gamma_2(z) \tag{B1}$$

$$\lim_{r\to 0} \fint_{B(z,r)\cap\Omega} |\gamma_1(x) - \gamma_1(z)| = 0, \tag{B2}$$

and

$$\lim_{r\to 0} \fint_{B(z,r)\cap\Omega} |\gamma_2(x) - \gamma_2(z)| = 0. \tag{B3}$$

These last two statement following from the fact that for $\mathcal{H}^{d-2}$ a.e. $y \in \partial\Omega$, we have (see e.g. [8, Theorem 2 on page 181])

$$\lim_{r\to 0} \fint_{B(y,r)\cap\Omega} |\gamma_1(x) - \gamma_1(y)| = 0,$$

and

$$\lim_{r\to 0} \fint_{B(y,r)\cap\Omega} |\gamma_2(x) - \gamma_2(y)| = 0.$$

Let z_n be a sequence in $\mathbb{R}^d \setminus \Omega$ such that

$$\mathrm{dist}(z_n, \Omega) = |z_n - z| \quad \text{and} \quad \lim_{n\to\infty} |z_n - z| = 0.$$

Set

$$v_n = \frac{1}{|x - z_n|^{d-2}} \text{ in } \mathbb{R}^d,$$

and let $u_{j,n} \in H^1(\Omega)$ $(j = 1, 2)$ be the unique solution to the system

$$\begin{cases} \mathrm{div}(\gamma_j \nabla u_{j,n}) = 0 & \text{in } \Omega, \\ u_{j,n} = v_n & \text{on } \partial\Omega. \end{cases}$$

Define

$$w_{j,n} = u_{j,n} - v_n \text{ in } \Omega.$$

It is clear that

$$\Delta v_n = 0 \quad \text{in } \Omega. \tag{B4}$$

We also have

$$-\operatorname{div}(\gamma_j \nabla w_{j,n}) = -\operatorname{div}(\gamma_j \nabla u_{j,n}) - \operatorname{div}(\gamma_j \nabla v_n) = -\operatorname{div}([\gamma_j - \gamma_j(z)]\nabla v_n) \text{ in } \Omega,$$

where in the last identity, we used (B4). This implies

$$\int_\Omega \gamma_j |\nabla w_{j,n}|^2 = \int_\Omega [\gamma_j - \gamma_j(z)] \nabla v_n \nabla w_{j,n}.$$

It follows from (B2) and (B3) that

$$\|\nabla w_{j,n}\|_{L^2} \le \|[\gamma_j - \gamma_j(z)]\nabla v_n\|_{L^2} = \frac{o(1)}{|z - z_n|^{(d-2)/2}}.$$

Here and in the following we let $o(1)$ denote a quantity going to 0 as $n \to \infty$; hence,

$$\nabla w_{j,n} = \nabla v_n + \frac{g_n}{|z - z_n|^{(d-2)/2}},$$

for some $\|g_n\|_{L^2} \to 0$ as $n \to \infty$. On the other hand,

$$\int_\Omega (\gamma_1 - \gamma_2) \nabla w_{1,n} \nabla w_{2,n} = 0$$

which implies

$$[\gamma_1(z) - \gamma_2(z)] \frac{1}{|z - z_n|^{d-2}} = o(1) \frac{1}{|z - z_n|^{d-2}}.$$

Hence

$$\gamma_1(z) = \gamma_2(z).$$

This contradicts (B1), and the conclusion follows. □

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