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GENERALIZED IMPEDANCE BOUNDARY CONDITIONS FOR SCATTERING BY STRONGLY ABSORBING OBSTACLES: THE SCALAR CASE

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We derive different classes of generalized impedance boundary conditions for the scattering problem from strongly absorbing obstacles. Compared to existing works, our construction is based on an asymptotic development of the solution with respect to the medium absorption. Error estimates are derived to validate the accuracy of each condition.

Keywords: Asymptotic models; general impedance boundary conditions; strongly absorbing media.

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1. Introduction

The concept of Generalized Impedance Boundary Condition (GIBC) is now a rather classical notion in the mathematical modeling of wave propagation phenomena (see for instance, Refs. 13 and 16), and is used particularly in electromagnetism for time harmonic scattering problems from obstacles that are partially or totally penetrable. The general idea is to replace the use of an "exact model" inside (the penetrable part of) the obstacle by approximate boundary conditions (also called equivalent or effective conditions). This idea is pertinent if the boundary condition can be easily handled numerically, for instance when it is local. The same type of idea led to the construction of local absorbing boundary conditions for the wave equation,[8,10] or more recently to the construction of On Surface Radiation Conditions.[5,4]

The diffraction problem of electromagnetic waves by perfectly conducting obstacles coated with a thin layer of dielectric material is well suited for the notion of impedance conditions: due to the small (typically with respect to the wavelength) thickness of the coating, the effect of the layer on the exterior medium is, as a first approximation, local (see for instance, Refs. 2, 7, 9, 13 and 16).

Another application, the one we have in mind here, is the diffraction of waves by strongly absorbing obstacles, typically highly conducting materials in electromagnetism. In such a case, it is the well-known skin effect that creates a "thin layer" phenomenon. The high conductivity limits the penetration of the wave to a boundary layer whose depth is inversely proportional to its magnitude. Then, here again, the effect of the obstacle is, as a first approximation, local.

The research on effective boundary conditions for highly absorbing obstacles began with Leontovich before the apparition of computers and the development of numerical methods. He proposed an impedance boundary condition, known as the Leontovitch boundary condition. This condition "sees" only locally the tangent plane to the frontier. Later, Rytov,[15,16] proposed an extension of the Leontovitch condition which was already based on the principle of an asymptotic expansion. More recently, Antoine–Barucq–Vernhet[6] proposed a new derivation of such conditions based on the technique of pseudo-differential operators (following the original ideas of Engquist–Majda[10] for absorbing boundary conditions).

Our purpose in this paper is to revisit the question of GIBCs for the scattering of waves by highly absorbing obstacles with a double objective:

- Propose a new construction of GIBCs which is based, as Rytov's contruction, on an ansatz for the asymptotic expansion of the exact solution but which is technically different: we use a scaling technique and a boundary layer expansion in the neighborhood of the boundary while Rytov uses an ansatz similar to the one for high frequency asymptotics.
- Develop a complete mathematical analysis (existence and uniqueness of the solution, stability and error estimates) for the approximate problems with respect to the medium's absorption.

The second point is probably the main contribution of the present work. It permits to make precise the notion of *order* of a given GIBC, whose meaning is not always clear in the literature (it is sometimes related to the order of the differential operators involved in the condition, sometimes linked to the truncation order of some Taylor expansion,...): a GIBC will be of order k if it provides an error in $O(\varepsilon^{k+1})$. A point deserves to be emphasized in this introduction: for a given order k there is not uniqueness of possible GIBCs. We shall present here several GIBCs of order 2 and 3; for the same order, different GIBCs may be distinguished by other features, such as their adaptation to a given numerical method.

Not surprisingly, a large amount of work has been devoted in the mathematical literature to the analysis or the study of numerical methods for wave propagation models with GIBCs (see for instance, Refs. 1 and 17). Curiously, concerning a rigorous asymptotic analysis of GIBCs for highly absorbing media, it seems that, although some of the works done by Artola–Cessenat[3] goes in this direction (for different problems than ours, however), there are few works on highly absorbing media, as compared with the case of thin coatings.

In this first paper on the subject, we investigate in detail the question of GIBCs for strongly absorbing media in the context of time harmonic acoustic waves in 3 dimensions. The case of Maxwell's equations will be the object of a second paper (note however that, in the degenerate 2D case, we get with this work GIBCs for 2D electromagnetic waves, at least in the case of TE polarization). The outline of the paper is as follows. In Sec. 2, we present our model problem and give the basic mathematical results for this problem (Theorems 2.1 and 2.2 and Corollary 2.1)). We state the main results of our paper in Sec. 3: the so-called NtD (Sec. 3.1), DtN (Sec. 3.2) and robust (in a sense defined in Sec. 3.2) GIBCs and the approximation Theorems 3.1. Section 4 is devoted to the construction of GIBCs (see Sec. 4.4) through the use of a standard scaling technique (cf. Sec. 4.2) that allows an analytic description of the boundary layer (Sec. 4.3) using local coordinates (Sec. 4.1). The central section of the paper is Sec. 5 where we prove error estimates for NtD GIBCs. The analysis is split into two steps: a justification (Sec. 5.1) of the asymptotic expansion of Sec. 4.2 (Lemma 5.1 and Corollary 5.1) and the study (Sec. 5.2) of the GIBC itself (Lemmas 5.4 and 5.6). Finally we explain in Sec. 6 how to modify the analysis for DtN and robust GIBCs.

2. Model Settings

Let Ω, Ω_i and Ω_e be open domains of $\mathbb{R}^3$ such that $\bar{\Omega} = \bar{\Omega}_e \cup \bar{\Omega}_i$ and $\Omega_i \cap \Omega_e = \emptyset$. Ω_i is supposed to be simply connected and $\partial\Omega \cap \partial\Omega_i = \emptyset$. In the sequel, we set $\Gamma = \partial\Omega_i$ and, for the simplicity of the exposition, we shall assume that Γ is a C^∞ manifold (see Fig. 1). We are interested in the acoustic wave propagation inside the domain Ω. We assume that the time and space scales are chosen in such a way that the speed of waves is 1 and we assume that the medium inside Ω_i is an absorbing

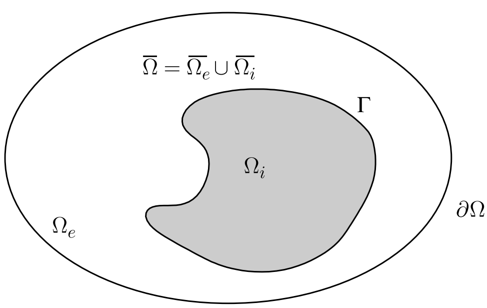

Fig. 1. Geometry of the medium.

medium. In other words, the wave propagation is governed inside Ω, by:

$$\frac{\partial^2 U^\varepsilon}{\partial t^2} + \sigma^\varepsilon(x)\frac{\partial U^\varepsilon}{\partial t} - \Delta U^\varepsilon = F, \tag{2.1}$$

where $\sigma^\varepsilon(x)$ is the function that characterizes the absorption of the medium and ε a small parameter defined later:

$$\sigma^\varepsilon(x) = \begin{cases} 0, & \text{in } \Omega_e, \\ \sigma^\varepsilon > 0, & \text{in } \Omega_i. \end{cases} \tag{2.2}$$

Considering a time harmonic source $F(x,t) = f(x)\sin\omega t$, where $\omega > 0$ denotes a given frequency, one looks for time harmonic solutions:

$$U^\varepsilon(x,t) = \mathcal{R}e\left(u^\varepsilon(x)\exp i\omega t\right).$$

Then, the function $u^\varepsilon(x)$ is governed by the Helmholtz equation:

$$-\Delta u^\varepsilon - \omega^2 u^\varepsilon + i\omega\, \sigma^\varepsilon(x)u^\varepsilon = f, \quad \text{in } \Omega, \tag{2.3}$$

where we assume that the support of the function f is contained in Ω_e. Equation (2.3) has to be complemented with a boundary condition on the exterior boundary $\partial\Omega$, for instance an absorbing boundary condition (see Remark 2.1)

$$\partial_n u^\varepsilon + i\omega u^\varepsilon = 0, \quad \text{on } \partial\Omega. \tag{2.4}$$

Remark 2.1. According to (2.4), the boundary $\partial\Omega$ can be seen as a physical absorbing boundary where a standard impedance condition is applied. The problem (2.3), (2.4) can also be seen as a (low order) approximation of the outgoing radiation condition at infinity for the exterior scattering problem in $\mathbb{R}^3\backslash\Omega_i$.

We are interested in describing the solution behavior for large σ^ε. For this, it is useful to introduce as a small parameter the quantity:

$$\varepsilon = \sqrt{1/\omega\sigma^\varepsilon} \Longleftrightarrow \sigma^\varepsilon = 1/(\omega\varepsilon^2). \tag{2.5}$$

It is easy to see that ε has the same dimension as a length. It represents in fact the width of the penetrable boundary layer inside Ω_i (also called the skin depth).

Our goal in this paper is to characterize, in an approximate way, the restriction u_e^ε of u^ε to the exterior domain Ω_e. In order to do so, it is useful to rewrite problem ((2.3), (2.4)) as a transmission problem between $u_i^\varepsilon = u^\varepsilon_{|\Omega_i}$ and $u_e^\varepsilon = u^\varepsilon_{|\Omega_e}$:

$$\begin{cases} \text{(i)} \ -\Delta u_e^\varepsilon - \omega^2 u_e^\varepsilon = f, & \text{in } \Omega_e, \\ \text{(ii)} \ -\Delta u_i^\varepsilon - \omega^2 u_i^\varepsilon + \dfrac{i}{\varepsilon^2}u_i^\varepsilon = 0, & \text{in } \Omega_i, \\ \text{(iii)} \ \partial_n u_e^\varepsilon + i\omega u_e^\varepsilon = 0, & \text{on } \partial\Omega, \\ \text{(iv)} \ u_i^\varepsilon = u_e^\varepsilon, & \text{on } \Gamma, \\ \text{(v)} \ \partial_n u_i^\varepsilon = \partial_n u_e^\varepsilon, & \text{on } \Gamma. \end{cases} \tag{2.6}$$

2.1. *Existence-uniqueness-stability*

Some basic theoretical results related to problem (2.3) are presented in this section. They constitute a preliminary step towards the forthcoming asymptotic analysis.

Theorem 2.1. *There exists a unique solution $u^\varepsilon \in H^1(\Omega)$ to problem* ((2.3), (2.4)). *Moreover, there exits a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{L^2(\Omega)} \le C \|f\|_{L^2(\Omega)} . \tag{2.7}$$

Proof. The existence and uniqueness proof is a classical exercise on the use of Fredholm's alternative. Let us simply recall that the uniqueness result rely on the following identity:

$$\int_\Omega |\nabla u^\varepsilon|^2 - \omega^2 |u^\varepsilon|^2 dx + i\left(\int_{\partial\Omega} \omega |u^\varepsilon|^2 ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |u^\varepsilon|^2 dx\right) = 0$$

that is valid for any solution u^ε of the homogeneous boundary value problem associated with (2.3), (2.4) (simply multiply Eq. (2.3) by $\bar{u}^\varepsilon$ and integrate by parts over Ω). In particular, $u^\varepsilon = 0$ in Ω_i and by unique continuation $u^\varepsilon = 0$.

The stability estimate (2.7) is proved by contradiction. Assume the existence of a sequence f^ε with $\|f^\varepsilon\|_{L^2(\Omega)} = 1$ such that the corresponding solution of ((2.3), (2.4)), denoted u^ε, satisfies $\|u^\varepsilon\|_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$. Let us set

$$v^\varepsilon = u^\varepsilon / \|u^\varepsilon\|_{L^2(\Omega)} \quad \text{and} \quad g^\varepsilon = f^\varepsilon / \|u^\varepsilon\|_{L^2(\Omega)} .$$

Then $\|v^\varepsilon\|_{L^2(\Omega)} = 1$ and $\|g^\varepsilon\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$. One gets from (2.3)

$$\begin{cases} -\Delta v^\varepsilon - \omega^2 v^\varepsilon + i\omega\sigma^\varepsilon v^\varepsilon = g^\varepsilon, & \text{in } \Omega, \\ \partial_n v^\varepsilon + i\omega v^\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Consequently (again, multiply the previous equation by $\bar{v}^\varepsilon$ and integrate over Ω)

$$\int_\Omega \left(|\nabla v^\varepsilon|^2 - \omega^2 |v^\varepsilon|^2\right) dx + i\left(\omega \int_{\partial\Omega} |v^\varepsilon|^2 ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |v^\varepsilon|^2 dx\right) = \int_{\Omega_e} g^\varepsilon \bar{v}^\varepsilon \, dx. \tag{2.9}$$

Taking the real part of (2.9) yields

$$\int_\Omega |\nabla v^\varepsilon|^2 dx = -\omega^2 \int_\Omega |v^\varepsilon|^2 dx + \mathcal{R}e \int_{\Omega_e} g^\varepsilon \bar{v}^\varepsilon dx.$$

Therefore one deduces that v^ε is bounded in $H^1(\Omega)$. Hence one can assume that, up to the extraction of a subsequence, $v^\varepsilon \to v$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. First we have $\|v\|_{L^2(\Omega)} = 1$. Taking the limit in (2.8), restricted to Ω_e, yields

$$\begin{cases} -\Delta v - \omega^2 v = 0, & \text{in } \Omega_e, \\ \partial_n v + i\omega v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

On the other hand, taking the imaginary part in (2.9) shows in particular that

$$\|v^\varepsilon\|^2_{L^2(\Omega_i)} \le \varepsilon^2 \|g^\varepsilon\|_{L^2(\Omega)} \|v^\varepsilon\|_{L^2(\Omega)} .$$

Thus $v^\varepsilon \to 0$ in $L^2(\Omega_i)$, hence $v = 0$ in Ω_i. In particular, $v = 0$ on $\partial\Omega_i$. Combined with (2.10), this condition shows that $v = 0$ in Ω_e. We then get $v = 0$ in Ω which is in contradiction with $\|v\|_{L^2(\Omega)} = 1$. □

Corollary 2.1. *There exists a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)} \quad \textit{and} \quad \|u^\varepsilon\|_{L^2(\Omega_i)} \le C\varepsilon \|f\|_{L^2(\Omega)}. \tag{2.11}$$

Proof. This corollary is a direct consequence of energy identity

$$\int_\Omega \left(|\nabla u^\varepsilon|^2 - \omega^2 |u^\varepsilon|^2\right) dx + i\left(\int_{\partial\Omega} \omega |u^\varepsilon|^2 ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |u^\varepsilon|^2 dx\right) = \int_{\Omega_e} f\bar{u}^\varepsilon\, dx$$

and the stability result of Theorem 2.1. □

Corollary 2.1 shows in particular that the solution converges to 0 like $O(\varepsilon)$ inside Ω_i, at least in the L^2 sense. This result is not optimal. A sharper one will be given in Lemma 5.1, where we show that $\|u^\varepsilon\|_{L^2(\Omega_i)}$ is $O(\varepsilon^{3/2})$ (see Remark 5.2).

2.2. *Exponential interior decay of the solution*

It is shown that the norm of the solution in a domain strictly interior to Ω_i goes to 0 more rapidly than power of ε. This is a first way to express that the main part of the interior solution is concentrated near the boundary Γ. The precise result is the following (we omit here the proof and refer the reader to Ref. 12, where some numerical examples are also shown to illustrate this so-called skin effect):

Theorem 2.2. *For any $\delta > 0$ small enough such that $\Omega_i^\delta = \{x \in \Omega_i; B(x,\delta) \subset \Omega_i\}$ is not empty, where $B(x,\delta)$ denotes the closed ball of center x and radius δ, there exist two positive constants C^δ and γ^δ independent of ε such that*

$$\|u_i^\varepsilon\|_{H^1(\Omega_i^\delta)} \le C^\delta \exp(-\gamma^\delta/\varepsilon)\,\|f\|_{L^2}\,.$$

3. Statement of the Main Results

We shall present various exterior boundary value problems that define different approximations of the "exact" solution u_e^ε in the exterior domain. Each of these approximate problems is made of the standard Helmholtz equation in the exterior domain Ω_e, the outgoing impedance condition on $\partial\Omega$,

$$\begin{cases} -\Delta u^{\varepsilon,k} - \omega^2 u^{\varepsilon,k} = f & \text{in } \Omega_e, \\ \partial_n u^{\varepsilon,k} + i\omega u^{\varepsilon,k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and an appropriate GIBC on the interior boundary Γ. We shall denote by $u^{\varepsilon,k}$ the approximate solution, where the integer index k refers to the order of the GIBC. The precise mathematical meaning of this order will be clarified with the error estimates (see Theorem 3.1). Let us simply say here that a GIBC of order k is a boundary condition that provides a (sharp) $O(\varepsilon^{k+1})$ error (in a sense to be given later).

All the GIBCs that will be dealt with are of the form of a linear relationship between the Dirichlet and Neumann boundary values, $u^{\varepsilon,k}$ and $\partial_n u^{\varepsilon,k}$, involving local (differential) operators along the boundary Γ. The method that we shall use for deriving these GIBCs naturally lead to Neumann-to-Dirichlet (NtD) GIBCs. These are the ones that we choose to present first in Sec. 3.1. Although it is possible to derive, at least formally, a GIBC of any order, the algebra becomes more involved as k increases, and it is difficult to write a general theory (existence, stability and error analysis). That is why we shall restrict ourselves, in this paper, to GIBCs of order $k = 0, 1, 2$ and 3.

In Sec. 3.2, we shall show how to easily derive, from (NtD) GIBCs, some modified GIBCs that can be of Dirichlet-to-Neumann (DtN) nature (as more commonly presented in the literature) or of mixed type.

We do not discuss in this paper the better choice for a GIBC of a given order. Several criteria can guide such a choice: the suitability of a particular numerical method, the robustness of the GIBC (this question will be slightly discussed later) or more importantly, its actual accuracy. It appears that a valuable comparison between the accuracy of GIBCs of the same order will rely on numerical computations. Also, it is not clear from the subsequent convergence theorems whether, at a given ε, a GIBC of order $k+1$ is more accurate than a GIBC of order k or not. The results only concern the asymptotic behavior as ε goes to zero. Finally, one can easily be convinced that the asymptotic models are still meaningful for Lipschitz interface Γ, even though the convergence study requires additional regularity. It would be interesting to numerically check the accuracy of GIBCs with respect to the scatterer regularity.

All these points are delayed to a forthcoming work of more numerical nature.

3.1. *Neumann-to-Dirichlet GIBCs*

Neumann-to-Dirichlet GIBC can be seen as a (local) approximation of the exact Neumann-to-Dirichlet condition that would characterize u_e^ε, namely:

$$u_e^\varepsilon + \mathcal{D}^\varepsilon \partial_n u_e^\varepsilon = 0, \quad \text{on } \Gamma, \tag{3.2}$$

where $\mathcal{D}^\varepsilon \in \mathcal{L}(H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))$ is the boundary operator defined by:

$$\mathcal{D}^\varepsilon \varphi = u_i^\varepsilon(\varphi),$$

and u_i^ε is the unique solution of the interior boundary value problem:

$$\begin{cases} -\Delta u_i^\varepsilon(\varphi) - \omega^2 u_i^\varepsilon(\varphi) + \dfrac{i}{\varepsilon^2} u_i^\varepsilon(\varphi) = 0, & \text{in } \Omega_i, \\ -\partial_n u_i^\varepsilon(\varphi) = \varphi, & \text{on } \Gamma. \end{cases} \tag{3.3}$$

The absorbing nature of the interior medium is equivalent to the following absorption property of the operator $\mathcal{D}^\varepsilon$ (this follows from Green's formula):

$$\forall\, \varphi \in H^{-\frac{1}{2}}(\Gamma), \quad \mathcal{I}m\, \langle \mathcal{D}^\varepsilon \varphi, \varphi \rangle_\Gamma = -\frac{1}{\varepsilon^2} \int_{\Omega_i} |u_i^\varepsilon(\varphi)|^2 dx \le 0. \tag{3.4}$$

It is well known that the operator $\mathcal{D}^\varepsilon$ is a nonlocal pseudo-differential operator whose explicit expression is not known in general. Nevertheless as $\varepsilon \to 0$, this operator becomes "almost local" (even differential), which is more or less intuitive according to the strong interior exponential decay of the solution for small ε.

In the following, $\alpha := \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ denotes the complex square root of i with positive real part, $\mathcal{H}$ and G are the mean and Gaussian curvatures of Γ (see Sec. 4.1), and Δ_Γ denotes the Laplace–Beltrami operator along Γ. We claim that a Neumann-to-Dirichlet GIBC of order $k = 1, 2, 3$ is given by:

$$u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\partial_n u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{3.5}$$

$$\text{for } k = 1,\ \mathcal{D}^{\varepsilon,1} = \frac{\varepsilon}{\alpha}, \tag{3.6}$$

$$\text{for } k = 2,\ \mathcal{D}^{\varepsilon,2} = \frac{\varepsilon}{\alpha} + i\mathcal{H}\varepsilon^2, \tag{3.7}$$

$$\text{for } k = 3,\ \mathcal{D}^{\varepsilon,3} = \frac{\varepsilon}{\alpha} + i\mathcal{H}\varepsilon^2 - \frac{\alpha\varepsilon^3}{2}\left(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma\right). \tag{3.8}$$

The main results of our paper are summarized in the following theorem:

Theorem 3.1. *Let $k = 0, 1, 2$ or 3, then, for sufficiently small ε, the boundary value problem ((3.1), (3.5)) has a unique solution $u^{\varepsilon,k} \in H^1(\Omega_e)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(\Omega_e)} \le C_k \varepsilon^{k+1}. \tag{3.9}$$

3.2. *Modified GIBCs*

Dirichlet to Neumann GIBCs. If we introduce $\mathcal{N}^\varepsilon := (\mathcal{D}^\varepsilon)^{-1}$, then the exact boundary condition for u_e^ε can be rewritten as:

$$\partial_n u_e^\varepsilon + \mathcal{N}^\varepsilon u_e^\varepsilon = 0, \quad \text{on } \Gamma. \tag{3.10}$$

In our terminology, a DtN GIBC will be of the form:

$$\partial_n u^{\varepsilon,k} + \mathcal{N}^{\varepsilon,k} u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{3.11}$$

where $\mathcal{N}^{\varepsilon,k}$ denotes some local approximation of $\mathcal{N}^\varepsilon$. They can be directly obtained from $\mathcal{D}^{\varepsilon,k}$ by seeking local operators $\mathcal{N}^{\varepsilon,k}$ that formally satisfy:

$$\mathcal{D}^{\varepsilon,k} = \left(\mathcal{N}^{\varepsilon,k}\right)^{-1} + O(\varepsilon^{k+1}). \tag{3.12}$$

The expression of $\mathcal{N}^{\varepsilon,k}$ is given by a formal Taylor expansion of $(\mathcal{D}^{\varepsilon,k})^{-1}$. One gets

$$\text{for } k = 2,\ \mathcal{N}^{\varepsilon,2} = \frac{\alpha}{\varepsilon} + \mathcal{H}, \tag{3.13}$$

$$\text{for } k = 3,\ \mathcal{N}^{\varepsilon,3} = \frac{\alpha}{\varepsilon} + \mathcal{H} - \frac{\varepsilon}{2\alpha}(\Delta_\Gamma + \mathcal{H}^2 - G + \omega^2). \tag{3.14}$$

The important point here is that the results (existence, uniqueness and error estimates) stated in Theorem 3.1 for problem (3.1), (3.5) still hold for problem (3.1), (3.11). We refer to Sec. 6.

Robust GIBCs. As mentioned earlier, an important property of the "exact" impedance condition is what we shall refer to as *absorption property*. It can be formally formulated for $\mathcal{D}^\varepsilon$ (resp. $\mathcal{N}^\varepsilon$) as:

$$\mathcal{I}m \int_\Gamma \mathcal{D}^\varepsilon \varphi \cdot \bar{\varphi}\, ds \leq 0, \quad \left(\text{resp. } \mathcal{I}m \int_\Gamma \varphi \cdot \overline{\mathcal{N}^\varepsilon \varphi}\, ds \leq 0\right), \tag{3.15}$$

for all $\varphi \in C^\infty(\Gamma)$ and all $\varepsilon > 0$.

Definition 3.1. We shall say that a NtD GIBC of the form (3.2) (resp. a DtN *GIBC* of the form (3.10)) is *robust* if the absorption property (3.15) still holds for all $\varepsilon > 0$, when $\mathcal{D}^\varepsilon$ is replaced by $\mathcal{D}^{\varepsilon,k}$ (resp. $\mathcal{N}^\varepsilon$ is replaced by $\mathcal{N}^{\varepsilon,k}$).

In particular, establishing robustness implies the well-posedness of the approximate problem for any $\varepsilon > 0$. With this respect, the second-order NtD GIBC (3.7) is not robust since (3.15) is guaranteed only if $\varepsilon\mathcal{H} \leq \frac{\sqrt{2}}{2}$, a.e. on Γ, which is a constraint for non-convex Ω_i. However, the second order DtN GIBC (3.13) is robust (thus can be seen as a robust version of (3.7)). Concerning the third-order conditions, neither the NtD GIBC (3.8) nor the DtN GIBC (3.14) is robust. Indeed, one has the identities:

$$\int_\Gamma \mathcal{D}^{\varepsilon,3}\varphi \cdot \bar{\varphi}\, ds = \frac{\alpha\varepsilon^3}{2}\int_\Gamma |\nabla_\Gamma \varphi|^2 ds + \varepsilon\bar{\alpha}\int_\Gamma \left[1 + \frac{\varepsilon\mathcal{H}}{\alpha} - i\,\frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right]|\varphi|^2 ds,$$

$$\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\alpha\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi|^2 ds + \frac{\bar{\alpha}}{\varepsilon}\int_\Gamma \left[1 + \frac{\varepsilon\mathcal{H}}{\bar{\alpha}} + i\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2)\right]|\varphi|^2 ds,$$

from which one easily computes that $\left(\text{remember } \alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$:

$$\begin{cases} \mathcal{I}m \displaystyle\int_\Gamma \mathcal{D}^{\varepsilon,3}\varphi \cdot \bar{\varphi}\, ds = \frac{\sqrt{2}\varepsilon^3}{4}\int_\Gamma |\nabla_\Gamma \varphi|^2 ds - \frac{\varepsilon\sqrt{2}}{2}\int_\Gamma \rho_1^\varepsilon |\varphi|^2 ds, \\ \mathcal{I}m \displaystyle\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\sqrt{2}\varepsilon}{4}\int_\Gamma |\nabla_\Gamma \varphi|^2 ds - \frac{\sqrt{2}}{2\varepsilon}\int_\Gamma \rho_2^\varepsilon |\varphi|^2 ds, \end{cases}$$

where the functions ρ_j^ε converge (uniformly on Γ) to 1 as ε goes to 0 (and are thus positive for ε small enough). The main problem is that the integrals in $|\nabla_\Gamma \varphi|^2$ come with the wrong sign.

As we shall explain, it is possible to construct robust GIBCs of order 3. The idea is to use some appropriate Padé approximation of the imaginary part of the boundary operators that formally gives the same order of approximation but restore absorption property. Consider for instance the NtD GIBC of order 3. Indeed

$$\mathcal{I}m\, \mathcal{D}^{\varepsilon,3} = -\varepsilon\frac{\sqrt{2}}{2}\left(1 - \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right).$$

One can therefore formally write

$$\mathcal{I}m\, \mathcal{D}^{\varepsilon,3} = -\varepsilon\,\frac{\sqrt{2}}{2}\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right)\left(1 - \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma\right) + O(\varepsilon^4).$$

Note that, as $\mathcal{H}^2 - G = \frac{1}{4}(c_1 - c_2)^2$, where c_1 and c_2 are the two principal curvatures along Γ (see Sec. 4.1), we have

$$1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2) > 0.$$

It is then sufficient to seek a positive approximation of $(1 - \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma)$ which can be obtained by considering the formal inverse, namely

$$1 - \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma = \left\{1 + \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}(4\mathcal{H}^2 - \Delta_\Gamma)\right\}^{-1} + O(\varepsilon^3).$$

Therefore,

$$\begin{aligned}\mathcal{I}m\,\mathcal{D}^{\varepsilon,3} &= -\varepsilon\frac{\sqrt{2}}{2}\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right)\\ &\quad\times\left(1 + \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}(4\mathcal{H}^2 - \Delta_\Gamma)\right)^{-1} + O(\varepsilon^4).\end{aligned}$$

A robust NtD-like GIBC of order 3 is obtained by replacing $\mathcal{D}^{\varepsilon,3}$ by

$$\begin{aligned}\mathcal{D}_r^{\varepsilon,3} &:= \varepsilon\frac{\sqrt{2}}{2}\left(1 - \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right)\\ &\quad - i\varepsilon\frac{\sqrt{2}}{2}\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right)\left(1 + \varepsilon\sqrt{2}\,\mathcal{H} + \frac{\varepsilon^2}{2}(4\mathcal{H}^2 - \Delta_\Gamma)\right)^{-1}. \end{aligned} \tag{3.16}$$

This expression will be used in practice in the following form:

$$\begin{aligned}\mathcal{D}_r^{\varepsilon,3}\varphi &:= \varepsilon\frac{\sqrt{2}}{2}\left[\left(1 - \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right)\varphi\right.\\ &\quad\left. - i\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right)\psi\right],\end{aligned} \tag{3.17}$$

where ψ is a solution of

$$-\frac{\varepsilon^2}{2}\Delta_\Gamma\psi + (1 + \varepsilon\sqrt{2}\,\mathcal{H} + 2\varepsilon^2\mathcal{H}^2)\psi = \varphi. \tag{3.18}$$

One can easily verify, with $a = (1 + \varepsilon\sqrt{2}\mathcal{H} + 2\varepsilon^2\mathcal{H}^2) > 0$, that

$$\begin{aligned}\int_\Gamma \mathcal{I}m\,\mathcal{D}_r^{\varepsilon,3}\varphi \cdot \bar{\varphi}\,ds &= -\varepsilon\int_\Gamma \frac{\sqrt{2}}{2}\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right)\\ &\quad\times\left(a|\psi|^2 + \frac{\varepsilon^2}{2}|\nabla_\Gamma\,\psi|^2\right)ds.\end{aligned} \tag{3.19}$$

The right-hand side is non-positive for all ε whence the absorption property for $\mathcal{D}_r^{\varepsilon,3}$. Of course one can follow a similar procedure to derive robust DtN third-order GIBC. The expression of this condition is based on the approximation:

$$\mathcal{I}m\,\mathcal{N}^{\varepsilon,3} = \frac{\sqrt{2}}{2\varepsilon}\left(1 + \frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2)\right)\left\{1 - \frac{\varepsilon^2}{2}\Delta_\Gamma\right\}^{-1} + O(\varepsilon^2). \tag{3.20}$$

Hence, replacing $\mathcal{N}^{\varepsilon,3}$ by

$$\mathcal{N}_r^{\varepsilon,3} := \frac{\sqrt{2}}{2\varepsilon}\left(1+\varepsilon\sqrt{2}\,\mathcal{H} - \frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right) + i\frac{\sqrt{2}}{2\varepsilon}\left(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G+\omega^2)\right)\left\{1-\frac{\varepsilon^2}{2}\Delta_\Gamma\right\}^{-1}, \tag{3.21}$$

in (3.14) gives another third-order DtN GIBC. This condition is robust in view of the following identity, where the right-hand side is nonpositive for all ε,

$$\mathcal{I}m\int_\Gamma \varphi\cdot\overline{\mathcal{N}^{\varepsilon,3}\varphi}\,ds = -\frac{\sqrt{2}}{2\varepsilon}\int_\Gamma\left(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2-G+\omega^2)\right)\left(|\psi|^2+\frac{\varepsilon^2}{2}|\nabla_\Gamma\,\psi|^2\right)ds, \tag{3.22}$$

where ψ is solution to $-\frac{\varepsilon^2}{2}\Delta_\Gamma\psi + \psi = \varphi$.

4. Formal Derivation of the GIBC

4.1. *Preliminary material*

Geometrical tools. Let n be the inward normal field defined on $\partial\Omega_i$ and let δ be a given positive constant chosen to be sufficiently small so that

$$\Omega_i^\delta = \{x\in\Omega_i;\ \mathrm{dist}(x,\partial\Omega_i) < \delta\},$$

can be uniquely parametrized by the tangential coordinate x_Γ on Γ and the normal coordinate $\nu\in(0,\delta)$ through

$$x = x_\Gamma + \nu n,\quad x\in\Omega_i^\delta. \tag{4.1}$$

Let us recall some concepts and identities from differential geometry (the notion of surface differential operator is supposed to be known — see Ref. 11). Let $\mathcal{C} := \nabla_\Gamma n$ be the curvature tensor on Γ. We recall that $\mathcal{C}$ is symmetric and $\mathcal{C}\,n = 0$. We denote c_1, c_2 the eigenvalues of $\mathcal{C}$ (namely the *principal curvatures*) associated with tangential eigenvectors τ_1, τ_2. $G := c_1c_2$ and $\mathcal{H} := \frac{1}{2}(c_1+c_2)$ are respectively the *Gaussian* and *mean curvatures* of Γ. Let us define the tangential operator $\mathcal{R}_\nu$ on Γ by

$$(I+\nu\mathcal{C}(x_\Gamma))\mathcal{R}_\nu(x_\Gamma) = I_\Gamma(x_\Gamma),$$

where $I_\Gamma(x_\Gamma)$ is the projection on the tangent plane to Γ at x_Γ. Then one has (see Ref. 11)

$$\nabla = \mathcal{R}_\nu\nabla_\Gamma + \partial_\nu n, \tag{4.2}$$

where ∇_Γ is the surface gradient on Γ. If one sets

$$J_\nu := \det(I+\nu\,\mathcal{C}) = 1 + 2\nu\mathcal{H} + \nu^2 G,$$

then from integration by part formulas and (4.2), one gets

$$\Delta = \frac{1}{J_\nu}\mathrm{div}_\Gamma(\mathcal{R}_\nu J_\nu \mathcal{R}_\nu)\nabla_\Gamma + \frac{1}{J_\nu}\partial_\nu J_\nu \partial_\nu, \tag{4.3}$$

where div_Γ is the surface divergence on Γ. Define the tangential operator $\mathcal{M}$ as

$$I_\Gamma + \nu\mathcal{M} = J_\nu \mathcal{R}_\nu,$$

then $\mathcal{M}$ is independent of ν and one has

$$\mathcal{C}\,\mathcal{M} = G I_\Gamma.$$

Therefore, identity (4.3) can be transformed into

$$\Delta = \frac{1}{J_\nu}\mathrm{div}_\Gamma\left(\frac{1}{J_\nu}(I_\Gamma + \nu\mathcal{M})^2\right)\nabla_\Gamma + \frac{1}{J_\nu}\partial_\nu J_\nu \partial_\nu,$$

or, in an equivalent form

$$\begin{aligned} J_\nu^3\Delta = {} & J_\nu \mathrm{div}_\Gamma\,(I_\Gamma + \nu\mathcal{M})^2\nabla_\Gamma - \nabla_\Gamma J_\nu \cdot (I_\Gamma + \nu\mathcal{M})^2\nabla_\Gamma \\ & + J_\nu^3\partial_{\nu\nu}^2 + 2J_\nu^2(\mathcal{H} + \nu G)\partial_\nu. \end{aligned} \tag{4.4}$$

This latter expression is more convenient for the asymptotic matching procedure, that we shall describe later, because we made the dependence of the operators coefficients polynomial with respect to ν.

The asymptotic ansatz. As it is quite standard, the derivation of the approximate boundary conditions will be based on an ansatz, i.e. a particular expansion of the solution in terms of ε. To formulate this ansatz, it is useful to introduce a cutoff function $\chi \in C^\infty(\Omega_i)$ such that $\chi = 1$ in $\Omega_i^{\delta/2}$ and $\chi = 0$ in $\Omega_i \backslash \Omega_i^\delta$. We do not consider $(1-\chi)u_i^\varepsilon$ since this term converges exponentially to 0 with ε (this is Theorem 2.2). For the remaining part of the solution, we postulate the following expansions:

$$u_e^\varepsilon(x) = u_e^0(x) + \varepsilon u_e^1(x) + \varepsilon^2 u_e^2(x) + \cdots, \quad \text{for } x \in \Omega_e, \tag{4.5}$$

where u_e^ℓ, $\ell = 0, 1, \ldots$ are functions defined on Ω_e and

$$\chi(x)u_i^\varepsilon(x) = u_i^0(x_\Gamma, \nu/\varepsilon) + \varepsilon u_i^1(x_\Gamma, \nu/\varepsilon) + \varepsilon^2 u_i^2(x_\Gamma, \nu/\varepsilon) + \cdots, \quad \text{for } x \in \Omega_i^\delta, \tag{4.6}$$

where x, x_Γ and ν are as in (4.1) and where $u_i^\ell(x_\Gamma, \eta) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{C}$ are functions such that

$$\lim_{\eta\to\infty} u_i^\ell(x_\Gamma, \eta) = 0 \quad \text{for a.e. } x_\Gamma \in \Gamma. \tag{4.7}$$

The latter condition will ensure that the u_i^ℓ's are exponentially decreasing with respect to η. In the next section, we shall identify the set of equations satisfied by (u_e^ℓ) and (u_i^ℓ) and the formal expansions (4.5) and (4.6) will be justified in Sec. 5. It will be useful to introduce the notation

$$\tilde{u}_i^\varepsilon(x_\Gamma, \eta) := u_i^0(x_\Gamma, \eta) + \varepsilon u_i^1(x_\Gamma, \eta) + \varepsilon^2 u_i^2(x_\Gamma, \eta) + \cdots, \ (x_\Gamma, \eta) \in \Gamma \times \mathbb{R}^+, \tag{4.8}$$

so that ansatz (4.6) has to be understood as

$$\chi(x)u_i^\varepsilon(x) = \tilde{u}_i^\varepsilon(x_\Gamma, \nu/\varepsilon) + O(\varepsilon^\infty) \quad \text{for } x \in \Omega_i^{\varepsilon/2}. \tag{4.9}$$

4.2. *Asymptotic formal matching*

Let us first consider the exterior field u_e^ε. It is clear that each u_e^k in the expansion (4.5) satisfies (simply substitute (4.5) into (2.6)(i) and (2.6)(iii)):

$$\begin{cases} -\Delta u_e^k - \omega^2 u_e^k = 0 & \text{in } \Omega_e, \\ \partial_n u_e^k + i\omega u_e^k = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

Concerning the interior field, from (2.6)(ii), (4.9) and the substitution $\nu = \varepsilon\eta$ in (4.4), we obtain the following equation, after some rearrangements:

$$\begin{aligned} (-\partial_{\eta\eta}^2 + i)\tilde{u}_i^\varepsilon &= (1 - J_{\varepsilon\eta}^3)(-\partial_{\eta\eta}^2 + i)\tilde{u}_i^\varepsilon + 2\varepsilon J_{\varepsilon\eta}^2(\mathcal{H} + \varepsilon\eta G)\partial_\eta \tilde{u}_i^\varepsilon + \varepsilon^2\omega^2 J_{\varepsilon\eta}^3 \tilde{u}_i^\varepsilon \\ &\quad + \varepsilon^2 J_{\varepsilon\eta} \mathrm{div}_\Gamma\, (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \nabla_\Gamma\, - \varepsilon^2 \nabla_\Gamma\, J_{\varepsilon\eta} \cdot (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \nabla_\Gamma\, \tilde{u}_i^\varepsilon. \end{aligned} \tag{4.11}$$

Considering that J_ν is a polynomial of degree 2 in ν, (4.11) can be rewritten as:

$$(-\partial_{\eta\eta}^2 + i)\tilde{u}_i^\varepsilon = \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell \tilde{u}_i^\varepsilon, \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{4.12}$$

where $\mathcal{A}_\ell$ are some partial differential operators in (x_Γ, η) that are independent of ε. Formal identification gives, after rather lengthy than complicated calculations,

$$\mathcal{A}_1 = 2\mathcal{H}\partial_\eta - 6\eta\mathcal{H}(-\partial_{\eta\eta}^2 + i), \tag{4.13}$$

$$\mathcal{A}_2 = \Delta_\Gamma + \omega^2 + 2\eta(G + 4\mathcal{H}^2)\partial_\eta - 3\eta^2(G + 4\mathcal{H}^2)(-\partial_{\eta\eta}^2 + i), \tag{4.14}$$

$$\begin{aligned} \mathcal{A}_3 &= 2\eta\big[\mathcal{H}\Delta_\Gamma + \mathrm{div}_\Gamma(\mathcal{M}\nabla_\Gamma\,) - \nabla_\Gamma\,\mathcal{H} \cdot \nabla_\Gamma\, + 3\omega^2\mathcal{H}\big] \\ &\quad + 4\eta^2\mathcal{H}\left[(3G + 2\mathcal{H}^2)\partial_\eta\right] - 4\eta^3\mathcal{H}(3G + 2\mathcal{H}^2)(-\partial_{\eta\eta}^2 + i), \end{aligned} \tag{4.15}$$

$$\begin{aligned} \mathcal{A}_4 &= \eta^2\big[G\Delta_\Gamma + 4\mathcal{H}\ \mathrm{div}_\Gamma(\mathcal{M}\nabla_\Gamma\,) + \mathrm{div}_\Gamma(\mathcal{M}^2\nabla_\Gamma\,)\big] \\ &\quad - \eta^2\big[\nabla_\Gamma\, G \cdot \nabla_\Gamma\, + 4\nabla_\Gamma\,\mathcal{H} \cdot (\mathcal{M}\nabla_\Gamma\,) - 3\omega^2(G + 4\mathcal{H}^2)\big] \\ &\quad + 4\eta^3 G(G + 4\mathcal{H}^2)\partial_\eta - 3\eta^4 G(G + 4\mathcal{H}^2)(-\partial_{\eta\eta}^2 + i), \end{aligned} \tag{4.16}$$

$$\begin{aligned} \mathcal{A}_5 &= 2\eta^3\left[G\ \mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) + \mathcal{H}\ \mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,)\right] \\ &\quad - 2\eta^3\left[\nabla_\Gamma\, G \cdot (\mathcal{M}\nabla_\Gamma\,) + \nabla_\Gamma\,\mathcal{H} \cdot (\mathcal{M}^2\nabla_\Gamma\,) - 2\omega^2\mathcal{H}(3G + 2\mathcal{H}^2)\right] \\ &\quad + 10\eta^4 G^2\mathcal{H}\partial_\eta - 6\eta^5 G^2\mathcal{H}\ (-\partial_{\eta\eta}^2 + i), \end{aligned} \tag{4.17}$$

$$\begin{aligned} \mathcal{A}_6 &= \eta^4\left[G\ \mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,) - \nabla_\Gamma\, G \cdot (\mathcal{M}^2\nabla_\Gamma\,) + 3\omega^2 G(G + 4\mathcal{H}^2)\right] \\ &\quad + 2\eta^5\ G^3\partial_\eta - \eta^6\ G^3(-\partial_{\eta\eta}^2 + i), \end{aligned} \tag{4.18}$$

$$\mathcal{A}_7 = 6\eta^5\omega^2 G^2\mathcal{H}, \tag{4.19}$$

$$\mathcal{A}_8 = \eta^6\omega^2 G^3. \tag{4.20}$$

Therefore, making the substitution (4.8) in Eq. (4.12) and equating the terms of the same order in ε, we obtain an induction on k that allows us to recursively determine the u_i^k's as functions of η. With the convention $u_i^k \equiv 0$ for $k < 0$, one can write it in the form

$$(-\partial^2_{\eta\eta} + i)u_i^k = \sum_{\ell=1}^{8} \mathcal{A}_\ell u_i^{k-\ell}, \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{4.21}$$

for all $k \geq 0$. For any $k \geq 0$, if one assumes that the fields u_i^l and u_e^l are known for $l < k$, then Eq. (4.21) can be seen as an ordinary differential equation in η for $\eta \in [0, +\infty[$ whose unknown $\eta \mapsto u_i^k(x_\Gamma, \eta)$ (the variable x_Γ is only a parameter). As this equation is of order 2, in addition to the decay condition at infinity (4.7), the solution of (4.21) with respect to η requires one initial condition at $\eta = 0$. This condition will be provided by one of the two interface conditions (2.6)(vi) and (2.6)(v). Here we choose to use condition (2.6)(v) which gives us a nonhomogeneous Neumann condition at $\eta = 0$ whose right-hand side will be given by the exterior field u_e^{k-1}, namely (substitute (4.5) and (4.6) into (2.6)(v) and identify the series after the change of variable $\nu = \varepsilon\eta$)

$$\partial_\eta u_i^k(x_\Gamma, 0) = \partial_n u_e^{k-1}|_\Gamma(x_\Gamma), \quad x_\Gamma \in \Gamma. \tag{4.22}$$

With such a choice, the other condition (2.6)(vi) will serve as a nonhomogeneous Dirichlet boundary condition for the exterior field u_e^k, to complete (4.10):

$$u_e^k|_\Gamma(x_\Gamma) = u_i^k(x_\Gamma, 0), \quad x_\Gamma \in \Gamma. \tag{4.23}$$

Remark 4.1. Choosing (4.22) as the boundary condition for (4.21) will naturally lead to NtD GIBCs. The alternative choice (4.23) would naturally lead to DtN GIBCs. Our choice seems to be more natural because, thanks to the shift of index in (4.22), the right-hand side naturally appears as something known from previous steps. Condition (4.23) plays the role of a coupling condition!

4.3. *Description of the interior field inside the boundary layer*

We are interested in getting analytic expression for the "interior fields" u_i^k by solving the boundary problem (in the variable η) made of (4.21), (4.22) and (4.7). To simplify the notation, we shall set:

$$du_e^k(x_\Gamma) := \partial_n u_e^k|_\Gamma(x_\Gamma), \quad x_\Gamma \in \Gamma. \tag{4.24}$$

Using standard techniques for linear differential equations, it is easy to prove that the solution u_i^k is of the form:

$$u_i^k(x_\Gamma, \eta) = P_{x_\Gamma}^k(\eta) e^{-\alpha\eta}, \tag{4.25}$$

for all $k \geq 0$, where $P_{x_\Gamma}^k$ is a polynomial with respect to η of degree k whose coefficients depend on $du_e^0, \ldots, du_e^{k-1}$ $\left(\text{note that } \alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$. More precisely, these polynomials satisfy an (affine) induction of order 8, of the form:

$$P_{x_\Gamma}^k(\eta) = -\frac{1}{\alpha} du_e^{k-1}(x_\Gamma) + \mathcal{L}_k\left(P_{x_\Gamma}^{k-1}(\eta), \ldots, P_{x_\Gamma}^{k-7}(\eta)\right),$$

where $\mathcal{L}_k$ is a linear form on $\mathbb{C}^7$ whose coefficients are linear in the $du_e^l(x_\Gamma)$'s. We shall not give here the expression of $\mathcal{L}_k$ for any k but restrict ourselves to the first four functions u_i^k (this is sufficient for GIBCs up to order 3)

$$u_i^0(x_\Gamma,\eta) = 0, \tag{4.26}$$

$$u_i^1(x_\Gamma,\eta) = -\frac{1}{\alpha} du_e^0(x_\Gamma) e^{-\alpha\eta}, \tag{4.27}$$

$$u_i^2(x_\Gamma,\eta) = \left\{ \left(-\frac{1}{\alpha} du_e^1(x_\Gamma) + \frac{\mathcal{H}}{\alpha^2} du_e^0(x_\Gamma) \right) + \eta \frac{\mathcal{H}}{\alpha} du_e^0(x_\Gamma) \right\} e^{-\alpha\eta}, \tag{4.28}$$

$$\begin{aligned} u_i^3(x_\Gamma,\eta) = \Bigg\{ & -\frac{1}{\alpha} du_e^2(x_\Gamma) + \frac{\mathcal{H}}{\alpha^2} du_e^1(x_\Gamma) \\ & - \frac{1}{2\alpha^3}(3\mathcal{H}^2 - G + \omega^2) du_e^0(x_\Gamma) - \frac{1}{2\alpha^3} \Delta_\Gamma [du_e^0](x_\Gamma) \\ & + \eta \left[\frac{\mathcal{H}}{\alpha} du_e^1(x_\Gamma) - \frac{1}{2\alpha^2}(\Delta_\Gamma - G + 3\mathcal{H}^2 + \omega^2) du_e^0(x_\Gamma) \right] \\ & + \eta^2 \frac{1}{2\alpha}(G - 3\mathcal{H}^2) du_e^0(x_\Gamma) \Bigg\} e^{-\alpha\eta}. \end{aligned} \tag{4.29}$$

4.4. *Construction of the GIBCs*

Let us first check inductively that, starting from $u_i^0 = 0$ and u_e^0 solution of the exterior Dirichlet problem, the fields u_e^k and u_i^k are well defined. Assume that u_e^ℓ and u_i^ℓ are known for $\ell \leq k-1$. The du_e^ℓ's are known by (4.24), and u_i^k is given by expression (4.25) (more precisely (4.26) to (4.29) for $k = 0,1,2,3$). Then, u_e^k is determined as the solution of (we set $f^0 = f$ and $f^k = 0$ for $k \geq 1$):

$$\begin{cases} -\Delta u_e^k - \omega^2 u_e^k = f^k, & \text{in } \Omega_e, \\ \partial_n u_e^k + i\omega u_e^k = 0, & \text{on } \partial\Omega, \\ u_e^k = u^k_{i\,|\eta=0}, & \text{on } \Gamma. \end{cases} \tag{4.30}$$

Remark 4.2. Since f is compactly supported in Ω_e, by induction we deduce, using standard elliptic regularity results, that u_e^k is a smooth function in a neighborhood of Γ and $u_e^k(x_\Gamma, 0)$ is also a smooth function.

The GIBC of order k is obtained by considering the truncated expansion:

$$\tilde{u}^{\varepsilon,k} := \sum_{\ell=0}^{k} \varepsilon^\ell u_e^\ell, \tag{4.31}$$

as an approximation of order k of u_e^ε. For example, for $k = 0$, we have $\tilde{u}^{\varepsilon,0} = u_e^0$ and from Eqs. (4.23) and (4.26), we deduce that $\tilde{u}^{\varepsilon,0} = 0$ on Γ. In this case, we set $u^{\varepsilon,0} = \tilde{u}^{\varepsilon,0}$ and the homogeneous Dirichlet boundary condition is the GIBC of order 0.

For larger k, another approximation is needed. The principle of the calculation is the following. Using the second interface condition, namely (2.6)(iv), one has

$$\tilde{u}^{\varepsilon,k}|_\Gamma(x_\Gamma) = \sum_{\ell=0}^{k} \varepsilon^\ell u_i^\ell(x_\Gamma, 0) \quad \text{for } x_\Gamma \in \Gamma. \tag{4.32}$$

Substituting (4.26)–(4.29) into (4.32) leads to a boundary condition of the form

$$\tilde{u}^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\partial_n \tilde{u}^{\varepsilon,k} = \varepsilon^{k+1} g_k^\varepsilon \quad \text{on } \Gamma, \quad \text{with } g_k^\varepsilon = O(1), \tag{4.33}$$

where $\mathcal{D}^{\varepsilon,k}$ is some boundary operator. The GIBC of order k that defines $u^{\varepsilon,k}$ (not $\tilde{u}^{\varepsilon,k}$) is then obtained by neglecting the right-hand side of (4.33).

Obtaining (4.33) is purely algebraic and we shall not give the details of the computations that are rather straightforward and could be automatized. Notice however that their complexity increases rapidly with k! For $k \leq 3$, the reader easily checks that the operators $\mathcal{D}^{\varepsilon,k}$'s are the ones mentioned in Sec. 3.1 and that the g_k^ε's are given by:

$$\begin{cases} g_1^\varepsilon = \dfrac{1}{\alpha}\partial_n u_e^1, \quad g_2^\varepsilon = \dfrac{1}{\alpha}\partial_n u_e^2 + i\mathcal{H}\partial_n(u_e^1 + \varepsilon u_e^2), \\ g_3^\varepsilon = \dfrac{1}{\alpha}\partial_n u_e^3 + i\mathcal{H}\partial_n(u_e^2 + \varepsilon u_e^3) \\ \qquad - \dfrac{1}{2}[\Delta_\Gamma \partial_n + (3\mathcal{H}^2 - G + \omega^2)\partial_n]\left(u_e^1 + \varepsilon u_e^2 + \varepsilon^2 u_e^3\right). \end{cases} \tag{4.34}$$

5. Error Analysis of NtD GIBCs

Our goal in this section is to estimate the difference

$$u_e^\varepsilon - u^{\varepsilon,k}, \tag{5.1}$$

where $u^{\varepsilon,k}$ is the solution of the approximate problem ((3.1), (3.5)), whose well-posedness will be shown in Sec. 5.2 (Lemma 5.4). It appears nontrivial to work directly with the difference $u_e^\varepsilon - u^{\varepsilon,k}$, we shall use the truncated series $\tilde{u}^{\varepsilon,k}$ introduced in Sec. 4.4 as an intermediate quantity. Therefore, the error analysis is split into two steps:

(1) Estimate the difference $u_e^\varepsilon - \tilde{u}^{\varepsilon,k}$; this is done in Sec. 5.1, Lemma 5.1 and Corollary 5.1.
(2) Estimate the difference $\tilde{u}^{\varepsilon,k} - u^{\varepsilon,k}$; see Sec. 5.2, Lemma 5.6.

Estimates of Theorem 3.1 are a direct consequence of Corollary 5.1 and Lemma 5.6.

Remark 5.1. Note that step 1 of the proof is independent of the GIBC and will be valid for any k. Also, for $k = 0$, the second step is useless since $\tilde{u}^{\varepsilon,0} = u^{\varepsilon,0}$.

5.1. *Error analysis of the truncated expansions*

Let us introduce the function $\widetilde{u}_\chi^{\varepsilon,k}(x) : \Omega \mapsto \mathbb{C}$ such that

$$\widetilde{u}_\chi^{\varepsilon,k}(x) = \begin{cases} \sum_{\ell=0}^k \varepsilon^\ell u_e^\ell(x), & \text{for } x \in \Omega_e, \\ \chi(x)\sum_{\ell=0}^k \varepsilon^\ell u_i^\ell(x_\Gamma, \nu/\varepsilon), & \text{for } x \in \Omega_i, \end{cases} \tag{5.2}$$

where χ, x_Γ and ν are as in Sec. 4.1. The main result of this section is:

Lemma 5.1. *For any k, there exists a constant C_k independent of ε such that*

$$\begin{aligned}\|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{H^1(\Omega)} &\le C_k \varepsilon^{k+\frac{1}{2}},\\ \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{L^2(\Omega_i)} &\le C_k \varepsilon^{k+\frac{3}{2}},\\ \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{L^2(\Gamma)} &\le C_k \varepsilon^{k+1}.\end{aligned} \tag{5.3}$$

Note that this gives an $O(\varepsilon^{k+1})H^1(\Omega_e)$-error estimate for the "exterior field":

Corollary 5.1. *For any k, there exists a constant $\tilde{C}_k$ independent of ε such that*:

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \le \tilde{C}_k \varepsilon^{k+1}. \tag{5.4}$$

Proof. Simply write

$$u^\varepsilon - \widetilde{u}^{\varepsilon,k} = u^\varepsilon - \tilde{u}^{\varepsilon,k+1} + \varepsilon^{k+1} u_e^{k+1},$$

which yields, since $u^{\varepsilon,k+1} = u_\chi^{\varepsilon,k+1}$ in Ω_e,

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \le \|u^\varepsilon - \tilde{u}_\chi^{\varepsilon,k+1}\|_{H^1(\Omega_e)} + \varepsilon^{k+1}\|u_e^{k+1}\|_{H^1(\Omega_e)},$$

that is to say, thanks to the first estimate of Lemma 5.1:

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \le C_k \varepsilon^{k+\frac{3}{2}} + \varepsilon^{k+1}\|u_e^{k+1}\|_{H^1(\Omega_e)} \le \tilde{C}_k \varepsilon^{k+1}. \qquad \square$$

Remark 5.2. For $k = 0$, since $\tilde{u}_\chi^{\varepsilon,0} = 0$ inside Ω_i (cf. (4.27)), one deduces from the second estimate of (5.3) that $\|u^\varepsilon\|_{L^2(\Omega_e)} \le C\varepsilon^{\frac{3}{2}}$.

Next we state a trace lemma (Lemma 5.2 whose proof — essentially a modification of the standard trace theorem — is omitted here, see Ref. 12) and a stability estimate (Lemma 5.3) that constitute the basic ingredients to the proof of Lemma 5.1.

Lemma 5.2. *Let O be a bounded open set of $\mathbb{R}^n$ with C^1 boundary, then there exists a constant C only depending on O such that*

$$\|u\|^2_{L^2(\partial O)} \le C\left(\|\nabla u\|_{L^2(O)}\|u\|_{L^2(O)} + \|u\|^2_{L^2(O)}\right), \quad \textit{for all } u \in H^1(O). \tag{5.5}$$

Lemma 5.3. *Let $v^\varepsilon \in H^1(\Omega)$ satisfying*

$$\begin{cases} -\Delta v^\varepsilon - \omega^2 v^\varepsilon = 0, & \textit{in } \Omega_e,\\[2mm] \partial_n v^\varepsilon + i\omega v^\varepsilon = 0, & \textit{on } \partial\Omega,\end{cases} \tag{5.6}$$

and the a priori estimate

$$\begin{aligned}&\left|\int_\Omega \left(|\nabla v^\varepsilon|^2 - \omega^2 |v^\varepsilon|^2\right) dx + i\left(\int_{\partial\Omega} \omega |v^\varepsilon|^2\, ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |v^\varepsilon|^2\, dx\right)\right|\\ &\quad \le A\left(\varepsilon^{s+\frac{1}{2}}\|v^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^s \|v^\varepsilon\|_{L^2(\Omega_i)}\right),\end{aligned} \tag{5.7}$$

for some non-negative constants A and s independent of ε. Then there exists a constant C independent of ε such that

$$\|v^\varepsilon\|_{H^1(\Omega)} \le C\,\varepsilon^{s+1}, \quad \|v^\varepsilon\|_{L^2(\Omega_i)} \le C\,\varepsilon^{s+2}, \quad \|v^\varepsilon\|_{L^2(\Gamma)} \le C\,\varepsilon^{s+\frac{3}{2}}, \tag{5.8}$$

for sufficiently small ε.

Proof. We first prove by contradiction that $\|v^\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon^{s+1}$. This is the main step of the proof. Let $w^\varepsilon = v^\varepsilon/\|v^\varepsilon\|_{L^2(\Omega)}$ and assume that $\lambda^\varepsilon := \varepsilon^{-s-1}\|v^\varepsilon\|_{L^2(\Omega)}$ is unbounded as $\varepsilon \to 0$. Estimate (5.7) (note that it is not homogeneous in v^ε) yields

$$\begin{aligned}&\left|\int_\Omega \left(|\nabla w^\varepsilon|^2 - \omega^2 |w^\varepsilon|^2\right) dx + i\left(\int_{\partial\Omega} \omega |w^\varepsilon|^2\, ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |w^\varepsilon|^2\, dx\right)\right| \\ &\quad \le \frac{A}{\lambda^\varepsilon}\left(\varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right).\end{aligned} \tag{5.9}$$

For the sake of conciseness, we will denote by C a positive constant whose value may change from one line to another but remains independent of ε. For instance, since $1/\lambda^\varepsilon$ is bounded, (5.9) yields in particular,

$$\|w^\varepsilon\|^2_{L^2(\Omega_i)} \le C\varepsilon^{\frac{3}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} + C\varepsilon\|w^\varepsilon\|_{L^2(\Omega_i)}.$$

Next, we use Lemma 5.2 with $\mathcal{O} = \Omega_i$ to get

$$\|w^\varepsilon\|^2_{L^2(\Omega_i)} \le C\varepsilon^{\frac{3}{2}}\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\left(\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\right) + C\varepsilon\|w^\varepsilon\|_{L^2(\Omega_i)},$$

which yields, after division by $\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}$,

$$\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega_i)} \le C_1\varepsilon\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + C_2\varepsilon^{\frac{3}{2}}\|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}. \tag{5.10}$$

Using Young's inequality $ab \le 2/3 a^{3/2} + 1/3 b^3$ with $a = K^{-1}\varepsilon$ and $b = K\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}$ (where K is a positive constant to be fixed later) we can write

$$\varepsilon\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} \le \frac{2}{3}K^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}} + \frac{K^3}{3}\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega_i)}. \tag{5.11}$$

Choosing $C_1K^3 = 3/2$ and substituting (5.10) into (5.11), we deduce a first main inequality,

$$\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega_i)} \le C\varepsilon^{\frac{3}{2}}\left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\right). \tag{5.12}$$

Now, observe that another consequence of (5.9) is, since $\|w^\varepsilon\|_{L^2(\Omega)} = 1$,

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \le C\left(1 + \varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right). \tag{5.13}$$

On the other hand, using Lemma 5.2 once again, we have

$$\varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} \le C\varepsilon^{\frac{1}{2}}\left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right\} + C\left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right\}^{\frac{1}{2}}\|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)},$$

which, using (5.13), implies

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq C + C\left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right\}\left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\right) \tag{5.14}$$

for ε bounded.

Coming back to (5.12), we deduce that

$$\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)} \leq C\Big(1 + \|\nabla w^\varepsilon\|^{\frac{1}{3}}_{L^2(\Omega_i)}\Big), \tag{5.15}$$

that we use in (5.14) to obtain

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq C\Big(1 + \|\nabla w^\varepsilon\|^{\frac{2}{3}}_{L^2(\Omega_i)}\Big).$$

This implies in particular that $\|\nabla w^\varepsilon\|_{L^2(\Omega)}$ is uniformly bounded with respect to ε and therefore w^ε is a bounded sequence of $H^1(\Omega)$. Up to an extracted subsequence, one can therefore assume that w^ε converges weakly in $H^1(\Omega)$ and strongly $L^2(\Omega)$ to some w with $\|w\|_{L^2(\Omega)} = 1$.

From (5.12), we deduce that $w = 0$ in Ω_i. On the other hand, taking the weak limit in the equations satisfied by w^ε in Ω_e and on $\partial\Omega$, then using that $w \in H^1(\Omega)$ one gets

$$\begin{cases} -\Delta w - \omega^2 w = 0, & \text{in } \Omega_e, \\ \partial_n w + i\omega w = 0 & \text{on } \partial\Omega, \\ w = 0 & \text{on } \Gamma. \end{cases} \tag{5.16}$$

Therefore $w = 0$ in Ω_e, hence $w = 0$ in Ω which contradicts $\|w\|_{L^2(\Omega)} = 1$. Consequently

$$\|v^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{s+1}. \tag{5.17}$$

Estimate (5.7) and Lemma 5.2 yields

$$\|v^\varepsilon\|^2_{L^2(\Omega_i)} \leq C\left(\varepsilon^{s+\frac{5}{2}}\|\nabla v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\|v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \varepsilon^{s+2}\|v^\varepsilon\|_{L^2(\Omega_i)}\right) \tag{5.18}$$

and using (5.17)

$$\|\nabla v^\varepsilon\|^2_{L^2(\Omega)} \leq C\Big(\varepsilon^{2s+2} + \varepsilon^{s+\frac{1}{2}}\|\nabla v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\|v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \varepsilon^{s}\|v^\varepsilon\|_{L^2(\Omega_i)}\Big). \tag{5.19}$$

Therefore, combining these two estimates, it is not difficult to obtain

$$\|v^\varepsilon\|^2_{L^2(\Omega_i)} + \varepsilon^2\|\nabla v^\varepsilon\|^2_{L^2(\Omega)} \leq C\big(\varepsilon^{2s+4} + \varepsilon^{s+2}\left(\|v^\varepsilon\|_{L^2(\Omega_i)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(\Omega)}\right)\big),$$

which yields

$$\|v^\varepsilon\|_{L^2(\Omega_i)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{s+2}.$$

This corresponds to the first two estimates of (5.8). The third one is a direct consequence of these two estimates by the application of Lemma 5.2 to Ω_i. □

Proof of Lemma 5.1. Let us set $e_k^\varepsilon = u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}$. The idea of the proof is to show that e_k^ε satisfies an *a priori* estimate of the type (5.7) and then to use the stability

Lemma 5.3. To prove such an estimate, we shall use the equations satisfied by e^ε_k, respectively in Ω_i and Ω_e as well as transmission conditions across Γ.

The exterior equation. By construction, $\widetilde{u}^{\varepsilon,k}_\chi$ satisfies in Ω_e the nonhomogeneous Helmholtz equation with the radiation boundary condition on $\partial\Omega$ and right-hand side f (this is a direct consequence of (4.10) for each k). Hence, $e^\varepsilon_{e,k} = e^\varepsilon_k|_{\Omega_e}$ satisfies the homogeneous equation:

$$\begin{cases} -\Delta e^\varepsilon_{e,k} - \omega^2 e^\varepsilon_{e,k} = 0, & \text{in } \Omega_e, \\ \partial_n e^\varepsilon_{e,k} + i\omega e^\varepsilon_{e,k} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.20}$$

The interior equation. The truncated series $\widetilde{u}^{\varepsilon,k}_\chi$ does not exactly satisfy the damped Helmholtz equation inside Ω_i. They verify this equation with a small right-hand side. To see that, let us set:

$$\tilde{u}^{\varepsilon,k}_i = \sum_{\ell=0}^{k} \varepsilon^\ell u^\ell_i, \quad \text{so that } \widetilde{u}^{\varepsilon,k}_\chi = \chi \tilde{u}^{\varepsilon,k}_i \quad \text{in } \Omega_i. \tag{5.21}$$

Indeed

$$\Delta\widetilde{u}^{\varepsilon,k}_\chi + \omega^2\widetilde{u}^{\varepsilon,k}_\chi - \frac{i}{\varepsilon^2}\widetilde{u}^{\varepsilon,k}_\chi = \chi\left\{\Delta\tilde{u}^{\varepsilon,k}_i + \omega^2\tilde{u}^{\varepsilon,k}_i - \frac{i}{\varepsilon^2}\tilde{u}^{\varepsilon,k}_i\right\} + 2\nabla\chi\cdot\nabla\tilde{u}^{\varepsilon,k}_i + \Delta\chi\tilde{u}^{\varepsilon,k}_i.$$

Inside the support of χ the local coordinates $(x_\Gamma, \nu = \varepsilon\eta)$ can be used to make the identification [cf. (4.12)]

$$\Delta + \omega^2 - \frac{i}{\varepsilon^2} \equiv \frac{1}{J^3_\nu \varepsilon^2}\left(-\partial^2_{\eta\eta} + i - \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell\right). \tag{5.22}$$

From Eq. (4.21), after multiplication by the correct power of ε and addition, it is not difficult to see (after some lengthy calculations) that

$$\left(-\partial^2_{\eta\eta} + i - \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell\right)\tilde{u}^{\varepsilon,k}_i = -\varepsilon^{k+1}\sum_{\ell=1}^{8}\sum_{p=0}^{\ell-1} \varepsilon^p \mathcal{A}_{\ell-p-1} u^{k+p+1-\ell}_i. \tag{5.23}$$

Therefore, thanks to (5.22) and (5.23),

$$\Delta\widetilde{u}^{\varepsilon,k}_\chi + \omega^2\widetilde{u}^{\varepsilon,k}_\chi - \frac{i}{\varepsilon^2}\widetilde{u}^{\varepsilon,k}_\chi = g^\varepsilon_{k,i}, \quad \text{in } \Omega_i, \tag{5.24}$$

where the function g^ε_i is given, with obvious notation, by

$$g^\varepsilon_{k,i} = -\varepsilon^{k-1}\chi\sum_{\ell=1}^{8}\sum_{p=0}^{\ell-1} \varepsilon^p \mathcal{A}_{\ell-p-1} u^{k+p+1-\ell}_i(., \nu/\varepsilon) + 2\nabla\chi\cdot\nabla\tilde{u}^{\varepsilon,k}_i + \Delta\chi\tilde{u}^{\varepsilon,k}_i. \tag{5.25}$$

From expression (4.25) and the identity

$$\int_0^{+\infty} \left(\frac{\nu}{\varepsilon}\right)^n e^{-\frac{\nu}{\sqrt{2}\varepsilon}}\, d\nu = C_n\varepsilon, \quad \forall\, n \in \mathbb{N},$$

it is not difficult to deduce the following estimate for each u_i^q:

$$\left(\int_{\Omega_i^\delta} |u_i^q(.,\nu/\varepsilon)|^2\,dx\right)^{\frac{1}{2}} \le C_q(\delta)\varepsilon^{\frac{1}{2}}. \tag{5.26}$$

In the same way, one easily shows that:

$$\left(\int_{\Omega_i^\delta\setminus\Omega_i^{\frac{\delta}{2}}} \{|\tilde{u}_i^{\varepsilon,k}|^2 + |\nabla\tilde{u}_i^{\varepsilon,k}|^2\}dx\right)^{\frac{1}{2}} \le C(\delta)\varepsilon^{-\frac{1}{2}}\exp(-\delta/(\sqrt{2}\,\varepsilon)). \tag{5.27}$$

Regrouping estimates (5.26) and (5.27) into (5.25), yields

$$\|g_{k,i}^\varepsilon\|_{L^2(\Omega_i)} \le C\varepsilon^{k-\frac{1}{2}}. \tag{5.28}$$

Taking the difference between (5.24) and (2.6)(ii), we see that $e_{i,k}^\varepsilon := e_{k|\Omega_i}^\varepsilon$ satisfies

$$-\Delta e_{i,k}^\varepsilon + \left(-\omega^2 + \frac{i}{\varepsilon^2}\right)e_{i,k}^\varepsilon = g_{k,i}^\varepsilon, \quad \text{in } \Omega_i. \tag{5.29}$$

The transmission equations. From interface condition (4.23) it is clear that $\widetilde{u}_\chi^{\varepsilon,k}$, and thus e_k^ε, is continuous across Γ. However, from (4.22), due to the shift of index between left- and right-hand sides, the normal derivative of $\widetilde{u}_\chi^{\varepsilon,k}$, and thus of e_k^ε, is discontinuous across Γ. More precisely, straightforward calculations lead to the following transmission conditions

$$\begin{cases} e_{e,k}^\varepsilon - e_{i,k}^\varepsilon = 0, & \text{on } \Gamma, \\ \partial_n e_{e,k}^\varepsilon - \partial_n\, e_{i,k}^\varepsilon = \varepsilon^k \partial_n u_e^k, & \text{on } \Gamma. \end{cases} \tag{5.30}$$

Error estimates. We can now proceed to the final step of the proof. Multiplying Eq. (5.20) by $\overline{e_{e,k}^\varepsilon}$ and integrating over Ω_e, we obtain by using Green's formula,

$$\int_{\Omega_e} |\nabla e_{e,k}^\varepsilon|^2\,dx - \omega^2\int_{\Omega_e} |e_{e,k}^\varepsilon|^2\,dx + i\omega\int_{\partial\Omega} |e_{e,k}^\varepsilon|^2\,d\sigma = \int_\Gamma \partial_n e_{e,k}^\varepsilon \overline{e_{e,k}^\varepsilon}\,d\sigma. \tag{5.31}$$

Multiplying Eq. (5.29) by $\overline{e_{i,k}^\varepsilon}$ and integrating over Ω_i, one gets

$$\begin{aligned} &\int_{\Omega_i} |\nabla e_{i,k}^\varepsilon|^2\,dx - \omega^2\int_{\Omega_e} |e_{i,k}^\varepsilon|^2\,dx + \frac{i}{\varepsilon^2}\int_{\Omega_i} |e_{i,k}^\varepsilon|^2\,dx \\ &\quad = -\int_\Gamma \partial_n e_{i,k}^\varepsilon \overline{e_{i,k}^\varepsilon} d\sigma + \int_{\Omega_i} g_{k,i}^\varepsilon \overline{e_{i,k}^\varepsilon}\,dx. \end{aligned} \tag{5.32}$$

Adding together (5.32) and (5.31) and using (5.30) and (5.28), gives

$$\begin{aligned} &\left|\int_\Omega |\nabla e_k^\varepsilon|^2 - \omega^2\int_\Omega |e_k^\varepsilon|^2 + i\omega\int_{\partial\Omega} |e_k^\varepsilon| + \frac{i}{\varepsilon^2}\int_{\Omega_i} |e_k^\varepsilon|^2\right| \\ &\quad \le C_k\left(\varepsilon^k\|e_k^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{k-\frac{1}{2}}\|e_k^\varepsilon\|_{L^2(\Omega_i)}\right), \end{aligned} \tag{5.33}$$

where C_k is a constant independent of ε. Ones deduces the desired estimates by applying Lemma 5.3. $\square$

5.2. *Error estimates for the GIBCs*

Existence and uniqueness results for the approximate problems. We shall check here that the $u^{\varepsilon,k}$'s are well defined. This is our next result.

Lemma 5.4. *For $k = 0, 1, 2, 3$, the boundary value problem*:

$$\begin{cases} -\Delta u^{\varepsilon,k} - \omega^2 u^{\varepsilon,k} = f, & \textit{in } \Omega_e, \\ \partial_n u^{\varepsilon,k} + i\omega u^{\varepsilon,k} = 0, & \textit{on } \partial\Omega, \\ u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k} u^{\varepsilon,k} = 0, & \textit{on } \Gamma, \end{cases} \tag{5.34}$$

admits a unique solution in $H^1(\Omega_e)$ provided that $\varepsilon\mathcal{H} \le \sqrt{2}/2$ if $k = 2$ or ε is small enough if $k = 3$.

Proof. Since the proof for $k = 0, 1, 2$ is quite classical, we shall concentrate here on the case $k = 3$. We start by reformulation problem (5.34) as a system.

New formulation of the problem. Introducing $\varphi^\varepsilon = \partial_n u^{\varepsilon,3}|_\Gamma$ as a new unknown, problem (5.34) is equivalent, for $k = 3$, to find $(u^{\varepsilon,3}, \varphi^\varepsilon) \in H^1(\Omega_e) \times H^1(\Gamma)$ such that

$$\begin{cases} -\Delta u^{\varepsilon,3} - \omega^2 u^{\varepsilon,3} = f, & \text{in } \Omega_e, \\ \partial_n u^{\varepsilon,3} + i\omega u^{\varepsilon,3} = 0, & \text{on } \partial\Omega, \\ \partial_n u^{\varepsilon,3} = \varphi^\varepsilon, & \text{on } \Gamma, \\ -\Delta_\Gamma \varphi^\varepsilon - \dfrac{2i}{\varepsilon^2}\theta_3(\varepsilon)\varphi^\varepsilon = \dfrac{2i\alpha}{\varepsilon^3} u^{\varepsilon,3} & \text{on } \Gamma, \end{cases} \tag{5.35}$$

where we have set $\theta_3(\varepsilon) = 1 - \frac{\varepsilon\mathcal{H}}{\alpha} - i\,\frac{\varepsilon^2 A(\omega)}{2}$ with $A(\omega) = 3\mathcal{H}^2 - G + \omega^2$.

Next we show that problem (5.35) is of Fredholm type. For this, we first note that (5.35) is equivalent to the variational problem:

$$\begin{cases} \text{Find } (u^{\varepsilon,3}, \varphi^\varepsilon) \in H^1(\Omega_e) \times H^1(\Gamma) \text{ such that } \forall (v, \psi) \in H^1(\Omega_e) \times H^1(\Gamma), \\ a_1\big((u^{\varepsilon,3}, \varphi^\varepsilon), (v, \psi)\big) + a_2^\varepsilon\left((u^{\varepsilon,3}, \varphi^\varepsilon), (v, \psi)\right) = \displaystyle\int_{\Omega_e} f\bar{v}\,dx, \end{cases} \tag{5.36}$$

where we have set:

$$\begin{cases} a_1((u,\varphi),(v,\psi)) = \displaystyle\int_{\Omega_e} \nabla u\cdot\nabla\bar{v}\,dx + i\omega\int_{\partial\Omega} u\bar{v}\,dx + \int_\Gamma (\nabla_\Gamma\varphi\cdot\nabla_\Gamma\bar{\phi} + \varphi\bar{\psi})ds \\ a_2^\varepsilon((u,\varphi),(v,\psi)) = -\omega^2\displaystyle\int_{\Omega_e} u\bar{v}\,dx - \int_\Gamma\left[1 + \frac{2i}{\varepsilon^2}\theta_3(\varepsilon)\right]\varphi\bar{\psi}\,ds \\ \qquad\qquad - \dfrac{2i\alpha}{\varepsilon^3}\displaystyle\int_\Gamma u\bar{\psi}\,ds - \int_\Gamma \varphi\bar{v}\,ds. \end{cases}$$

One next remarks that $a_1(\cdot,\cdot)$ is coercive in $H^1(\Omega_e) \times H^1(\Gamma)$ while $a_2^\varepsilon(\cdot,\cdot)$ is weakly compact in $H^1(\Omega_e) \times H^1(\Gamma)$:

$$(u^n, \varphi^n) \rightharpoonup (u, \varphi) \text{ in } H^1(\Omega_e) \times H^1(\Gamma) \Rightarrow a_2^\varepsilon\left((u^n, \varphi^n)(u^n, \varphi^n)\right) \to a_2^\varepsilon\left((u, \varphi)(u, \varphi)\right).$$

Therefore, to prove the existence of the solution of (5.35) (or (5.36)), it is sufficient to prove uniqueness.

Uniqueness proof. We prove the uniqueness result for ε small enough by contradiction. If uniqueness fails then, up to the extraction of a sequence of values of ε tending to 0, one can assume that there exists a nontrivial solution $(u^{\varepsilon,3}, \varphi^{\varepsilon})$ of the homogeneous problem associated with (5.35), which we can normalize so that:

$$\|u^{\varepsilon,3}\|_{L^2(\Omega_e)} = 1. \tag{5.37}$$

We multiply the Helmholtz equation by the complex conjugate of $u^{\varepsilon,3}$ and after integration by parts, we replace, in the boundary term on Γ, the trace of $u^{\varepsilon,3}$ by its expression as a function of φ^{ε} from the last equation of (5.35). This leads to

$$\begin{aligned}\int_{\Omega_e} \left(|\nabla u^{\varepsilon,3}|^2 - \omega^2 |u^{\varepsilon,3}|^2\right) dx + \frac{\bar{\alpha}\varepsilon^3}{2} \int_{\Gamma} |\nabla_\Gamma \varphi^{\varepsilon}|^2 \, ds \\ + \varepsilon\alpha \int_{\Gamma} \theta_3(\varepsilon) |\varphi^{\varepsilon}|^2 \, ds + i\omega \int_{\partial\Omega} |u^{\varepsilon,3}|^2 \, ds = 0.\end{aligned}$$

We now take the real part of the last equality (contrary to what is more usual, taking the imaginary part does not provide the desired estimate since the term in $|\nabla_\Gamma \varphi^{\varepsilon}|^2$ comes with the wrong sign) and use (5.37) to get

$$\int_{\Omega_e} |\nabla u^{\varepsilon,3}|^2 \, dx + \frac{\varepsilon^3\sqrt{2}}{4} \int_{\Gamma} |\nabla_\Gamma \varphi^{\varepsilon}|^2 \, ds + \varepsilon \int_{\Gamma} \mathcal{R}e(\alpha\theta_3(\varepsilon)) |\varphi^{\varepsilon}|^2 \, ds \leq \omega^2. \tag{5.38}$$

Since $\mathcal{R}e(\alpha\theta_3(\varepsilon))$ tends to $\sqrt{2}/2$ as ε goes to 0, we deduce that $u^{\varepsilon,3}$ is bounded in $H^1(\Omega_e)$. Therefore, up to the extraction of a subsequence, we can assume that:

$$\begin{cases} u^{\varepsilon,3} \to u, & \text{weakly in } H^1(\Omega_e), \\ u^{\varepsilon,3} \to u, & \text{strongly in } L^2(\Omega_e), \\ \Delta u^{\varepsilon,3} \to \Delta u, & \text{weakly in } L^2(\Omega_e), \end{cases}$$

the latter property being deduced from the Helmholtz equation. By trace theorem, $\partial_n u^{\varepsilon,3}|_\Gamma$ (resp. $\partial_n u^{\varepsilon,3}|_{\partial\Omega}$) converges to $\partial_n u|_\Gamma$ (resp. $\partial_n u|_{\partial\Omega}$) in $H^{-\frac{1}{2}}(\Gamma)$ (resp. $H^{-\frac{1}{2}}(\partial\Omega)$). Of course, at the limit, we have:

$$\begin{cases} -\Delta u - \omega^2 u = 0, & \text{in } \Omega_e, \\ \partial_n u + i\omega u = 0, & \text{in } \partial\Omega, \end{cases} \tag{5.39}$$

while, passing to the (weak) limit in the last boundary equation of (5.35) after multiplication by ε^3, we obtain

$$u = 0, \quad \text{on } \Gamma. \tag{5.40}$$

From (5.39) and (5.40), we get $u = 0$ which contradicts $\|u\|_{L^2(\Omega_e)} = 1$. $\square$

Analysis of the difference $u^{\varepsilon,k} - \tilde{u}^{\varepsilon,k}$. From now on, we shall set for $k = 0, 1, 2, 3$,

$$\mathbf{e}^{\varepsilon,k} := u^{\varepsilon,k} - \tilde{u}^{\varepsilon,k}. \tag{5.41}$$

The starting point of the error analysis is to remark that $\mathbf{e}^{\varepsilon,k}$ is a solution of a homogeneous Helmholtz equation with outgoing absorbing condition on $\partial\Omega$,

$$\begin{cases} -\Delta \mathbf{e}^{\varepsilon,k} - \omega^2 \mathbf{e}^{\varepsilon,k} = 0 & \text{in } \Omega_e, \\ \partial_n \mathbf{e}^{\varepsilon,k} + i\omega \mathbf{e}^{\varepsilon,k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.42}$$

and satisfies a nonhomogeneous GIBC boundary condition on Γ with small right-hand side. This comes directly from the construction of the GIBC itself and is obtained by making the difference between (4.33) and (3.5). Let us formulate this as a lemma:

Lemma 5.5. *For* $k = 1, 2, 3$, *there exists a smooth function* g_k^ε *and* $C_k > 0$ *independent of* ε *such that*

$$\mathbf{e}^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k} \partial_n \mathbf{e}^{\varepsilon,k} = \varepsilon^{k+1} g_k^\varepsilon, \tag{5.43}$$

with the estimate

$$\|g_k^\varepsilon\|_{H^{\frac{1}{2}}(\Gamma)} \le C_k, \quad \textit{for } k = 1, 2, 3. \tag{5.44}$$

This result is a *consistency* result for the boundary condition. Combined with a *stability* argument, it is then possible to obtain the following estimates.

Lemma 5.6. *For* $k = 1, 2, 3$, *there exists* $C_k > 0$ *independent of* ε *such that*

$$\|u^{\varepsilon,k} - \tilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \le C_k \varepsilon^{k+1}. \tag{5.45}$$

Proof. From (5.42) and (5.43) and Green's formula,

$$\begin{aligned} &\int_{\Omega_e} (|\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2)\, dx + i\omega \int_{\partial\omega} |\mathbf{e}^{\varepsilon,k}|^2\, ds \\ &\quad + \int_\Gamma \mathcal{D}^{\varepsilon,k} \partial_n \mathbf{e}^{\varepsilon,k} \cdot \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds = \varepsilon^{k+1} \int_\Gamma g_k^\varepsilon \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds. \end{aligned} \tag{5.46}$$

Setting $\varphi_k^\varepsilon = \partial_n \mathbf{e}^{\varepsilon,k}|_\Gamma$, and introducing the functions $\theta_1(\varepsilon) = 1$, $\theta_2(\varepsilon) = 1 - \frac{\varepsilon\mathcal{H}}{\alpha}$, ($\theta_3(\varepsilon)$ has been defined in the proof of Lemma 5.4, one can derive the following general identity by using the explicit expressions of the $\mathcal{D}^{\varepsilon,k}$'s,

$$\begin{aligned} &\int_{\Omega_e} (|\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2) dx + i\omega \int_{\partial\omega} |\mathbf{e}^{\varepsilon,k}|^2\, ds \\ &\quad + \varepsilon\alpha \int_\Gamma \theta_k(\varepsilon) |\varphi_k^\varepsilon|^2\, ds + \nu_k \frac{\bar{\alpha}\varepsilon^3}{2} \int_\Gamma |\nabla_\Gamma \varphi_k^\varepsilon|^2\, ds = \varepsilon^{k+1} \int_\Gamma g_k^\varepsilon \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds, \end{aligned} \tag{5.47}$$

where $\nu_k = 0$ for $k = 0, 1, 2$ and $\nu_3 = 1$. Taking the real part,

$$\begin{aligned} &\int_{\Omega_e} (|\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2) dx + \varepsilon \int_\Gamma \mathcal{R}e(\alpha\theta_k(\varepsilon)) |\varphi_k^\varepsilon|^2\, ds \\ &\quad + \nu_k \frac{\sqrt{2}\varepsilon^3}{4} \int_\Gamma |\nabla_\Gamma \varphi_k^\varepsilon|^2\, ds = \varepsilon^{k+1} \mathcal{R}e \int_\Gamma g_k^\varepsilon \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds. \end{aligned} \tag{5.48}$$

In particular, since $\nu_k \geq 0$ and $\mathcal{R}e(\alpha\theta_k(\varepsilon))$ tends to $\sqrt{2}$ as ε tends to 0, we obtain the following estimate, for ε small enough,

$$\begin{aligned}\int_{\Omega_e} (|\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2|\mathbf{e}^{\varepsilon,k}|^2)\, dx &\leq \varepsilon^{k+1}\|g_k^\varepsilon\|_{H^{\frac{1}{2}}(\Gamma)}\|\partial_n \mathbf{e}^{\varepsilon,k}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq C_k\varepsilon^{k+1}\|\mathbf{e}^{\varepsilon,k}\|_{H^1(\Omega)}, \end{aligned} \tag{5.49}$$

where the latter inequality comes from (5.44) and the fact that $\mathbf{e}^{\varepsilon,k}$ is a solution of the Helmholtz equation inside Ω_e. The remaining part of the proof is then rather straightforward. We first prove by contradiction that

$$\|\mathbf{e}^{\varepsilon,k}\|_{L^2(\Omega_e)} \leq C_k\varepsilon^{k+1}. \tag{5.50}$$

If (5.50) were not true, then $\mu_k^\varepsilon = \varepsilon^{-(k+1)}\|\mathbf{e}^{\varepsilon,k}\|$ would blow up (for a subsequence) as ε goes to 0. Then, introducing

$$w^{\varepsilon,k} = \mathbf{e}^{\varepsilon,k}/\|\mathbf{e}^{\varepsilon,k}\|_{L^2(\Omega_e)},$$

ones derives from (5.49)

$$\int_{\Omega_e} |\nabla w^{\varepsilon,k}|^2\, dx \leq \omega^2 + C_k(\mu_k^\varepsilon)^{-1}\|w^{\varepsilon,k}\|_{H^1(\Omega)} \leq C_k(1 + \|w^{\varepsilon,k}\|_{H^1(\Omega)}). \tag{5.51}$$

Therefore, $w^{\varepsilon,k}$ is bounded in $H^1(\Omega)$ and thus, up to the extraction of a subsequence, converges weakly in $H^1(\Omega_e)$ but strongly in $L^2(\Omega_e)$ to some $w^k \in H^1(\Omega_e)$ that satisfies $\|w^k\|_{L^2(\Omega_e)} = 1$ as well as

$$\begin{cases} -\Delta w^k - \omega^2 w^k = 0, & \text{in } \Omega_e, \\ \partial_n w^k + i\omega w^k = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.52}$$

Finally, passing to the limit (in the weak sense) in the boundary condition

$$w^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\partial_n w^{\varepsilon,k} = g_k^\varepsilon/\|\mathbf{e}^{\varepsilon,k}\|_{L^2(\Omega_e)} = (\mu_k^\varepsilon)^{-1}(g_k^\varepsilon/\varepsilon^{k+1}), \tag{5.53}$$

we see ($g_k^\varepsilon/\varepsilon^{k+1}$ is bounded and $(\mu_k^\varepsilon)^{-1}$ tends to 0) that w^k also satisfies

$$w^k = 0, \quad \text{on } \Gamma. \tag{5.54}$$

System ((5.52), (5.54)) implies that $w^k = 0$, which contradicts $\|w^k\|_{L^2(\Omega)} = 1$. Therefore, (5.50) holds and the estimate is a direct consequence of (5.50) and (5.49). □

6. About the Analysis of Modified GIBCs

The error analysis of modified GIBCs can be done in a similar way as for the NtD GIBCs. We shall restrict ourselves to stating the results and indicating the needed modifications.

6.1. *Analysis of DtN GIBCs*

Theorem 6.1. *Let $k = 1, 2$ or 3, then assuming ε being sufficiently small when $k = 3$, the boundary value problem ((3.1), (3.11)) has a unique solution $u^{\varepsilon,k} \in H^1(\Omega_e)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(\Omega_e)} \le C_k\, \varepsilon^{k+1}. \tag{6.1}$$

Proof. We shall only treat here the case $k = 3$ (the others are easy) and directly go to the proof of estimate (6.1) assuming the existence and uniqueness of the solution. Of course, we only need to treat the difference $u^{\varepsilon,3} - \tilde{u}^{\varepsilon,3}$, namely to prove the equivalent to Lemma 5.6.

Rather curiously, it appears that treating the boundary condition directly in its DtN form (3.11) does not immediately give the optimal error estimate. This is why we shall rewrite it as an NtD condition by introducing the inverse of the operator $\mathcal{N}^{\varepsilon,3}$ (note that, by Lax–Milgram's lemma, $\mathcal{N}^{\varepsilon,3}$ is an isomorphism from $H^{s+2}(\Gamma)$ onto $H^s(\Gamma)$). We hereafter repeat the approach of Lemma 5.6. One first checks that the error $\mathbf{e}^{\varepsilon,3}$ satisfies the homogenenous Helmholtz equation in Ω_e together with the nonhomogeneous boundary condition (see Remark 6.1 below):

$$\mathbf{e}^{\varepsilon,3} + \left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \partial_n \mathbf{e}^{\varepsilon,3} = \varepsilon^4 g_3^\varepsilon, \tag{6.2}$$

where g_3^ε is a smooth function satisfying:

$$\|g_3^\varepsilon\|_{H^{\frac{1}{2}}(\Gamma)} \le C_k, \quad \text{for } k = 1, 2, 3. \tag{6.3}$$

Proceeding as in the proof of Lemma 5.6, we obviously get

$$\begin{aligned} &\int_{\Omega_e} (|\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,3}|^2)\, dx + i\omega \int_{\partial\omega} |\mathbf{e}^{\varepsilon,3}|^2\, ds \\ &\quad + \int_\Gamma \overline{\left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \partial_n \mathbf{e}^{\varepsilon,3}} \cdot \partial_n \mathbf{e}^{\varepsilon,3}\, ds = \varepsilon^{k+1} \int_\Gamma g_3^\varepsilon \overline{\partial_n \mathbf{e}^{\varepsilon,3}}\, ds. \end{aligned} \tag{6.4}$$

The key point is that, at least for ε small enough, and any ψ smooth enough,

$$\mathcal{R}e \int_\Gamma \overline{\left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \psi} \cdot \psi\, dx \le 0. \tag{6.5}$$

This is a consequence of

$$\mathcal{R}e \int_\Gamma \mathcal{N}^{\varepsilon,3} \varphi \cdot \overline{\varphi}\, dx \le 0, \quad \text{for any } \varphi \text{ smooth enough,}$$

that follows from the identity (proved in Sec. 3.2)

$$\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\alpha\varepsilon}{2} \int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \frac{\bar{\alpha}}{\varepsilon} \int_\Gamma \left[1 + \frac{\varepsilon\mathcal{H}}{\bar{\alpha}} + i\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2)\right] |\varphi|^2\, ds$$

and the observation that

$$\mathcal{R}e\alpha = \mathcal{R}e\bar{\alpha} = \sqrt{2}/2, \quad \lim_{\varepsilon \to 0} \left[1 + \frac{\varepsilon\mathcal{H}}{\bar{\alpha}} + i\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2)\right] = 1.$$

Therefore we have shown that, as soon as ε is small enough,

$$\int_{\Omega_e} (|\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,3}|^2)\, dx \leq \varepsilon^{k+1} \|g_k^\varepsilon\|_{H^{\frac{1}{2}}(\Gamma)} \|\partial_n \mathbf{e}^{\varepsilon,k}\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{6.6}$$

The conclusion of the proof is identical to the one of Lemma 5.6 (cf. (5.49)). □

Remark 6.1. Proceeding as in Sec. 4.4 (for formulas (4.34)), one first gets:

$$\partial_n \mathbf{e}^{\varepsilon,3} + \mathcal{N}^{\varepsilon,3} \mathbf{e}^{\varepsilon,3} = \varepsilon^3 h_3^\varepsilon,$$

where h_3^ε (as g_3^ε in formula (4.34)) is a polynomial of degree 3 in ε, whose coefficients are smooth functions of x_Γ, explicitly known in terms of u_e^1, u_e^2 and u_e^3. In particular $h_3^\varepsilon = O(1)$, in any Sobolev norm. One then deduces (6.2) with $g_3^\varepsilon = (\varepsilon \mathcal{N}^{\varepsilon,3})^{-1}\ h_3^\varepsilon$. One finally obtains (6.3) after having noticed that (cf. (3.14):

$$(\varepsilon \mathcal{N}^{\varepsilon,3})^{-1} = \frac{1}{\alpha} \left\{ 1 + \frac{\varepsilon}{\alpha} \mathcal{H} + i \frac{\varepsilon^2}{2} (\Delta_\Gamma + \mathcal{H}^2 - G + \omega^2) \right\}^{-1} = O(\varepsilon).$$

6.2. *Analysis of robust GIBCs*

Theorem 6.2. *For any $\varepsilon > 0$, the boundary value problem associated with* (3.1) *and the boundary condition*:

$$u^{\varepsilon,3} + \mathcal{D}_r^{\varepsilon,3} \partial_n u^{\varepsilon,3} = 0, \quad \text{on } \Gamma, \tag{6.7}$$

where $\mathcal{D}_r^{\varepsilon,3}$ is given by (3.16), *has a unique solution $u^{\varepsilon,3} \in H^1(\Omega_e)$. Moreover, there exists a constant C_3, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,3}\|_{H^1(\Omega_e)} \leq C_3 \varepsilon^4. \tag{6.8}$$

The same result holds if one replaces (6.7) *by*:

$$\partial_n u^{\varepsilon,3} + \mathcal{N}_r^{\varepsilon,3} u^{\varepsilon,3} = 0, \quad \text{on } \Gamma, \tag{6.9}$$

where $\mathcal{N}_r^{\varepsilon,3}$ is given by (3.21).

We shall note that the proof of this theorem is almost identical to that of Theorem 6.1 or Lemma 5.6. The main difference lies in the fact that the algebra to obtain equivalent to identities (5.5) and (6.2) is slightly more complicated and the calculations equivalent to property (6.5) are longer. The fact that existence and uniqueness results are valid for any positive ε is a consequence of robustness properties (3.19) and (3.22).

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