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GENERALIZED IMPEDANCE BOUNDARY CONDITIONS FOR SCATTERING PROBLEMS FROM STRONGLY ABSORBING OBSTACLES: THE CASE OF MAXWELL'S EQUATIONS

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This paper is dedicated to the construction and analysis of so-called Generalized Impedance Boundary Conditions (GIBCs) for electromagnetic scattering problems from imperfect conductors with smooth boundaries. These boundary conditions can be seen as higher order approximations of a perfect conductor condition. We consider here the 3-D case with Maxwell equations in a harmonic regime. The construction of GIBCs is based on a scaled asymptotic expansion with respect to the skin depth. The asymptotic expansion is theoretically justified at any order and we give explicit expressions till the third order. These expressions are used to derive the GIBCs. The associated boundary value problem is analyzed and error estimates are obtained in terms of the skin depth.

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1. Introduction

Generalized Impedance Boundary Conditions (GIBC) have become a rather classical notion in the mathematical modeling of wave propagation phenomena (see for instance, Refs. 13 and 16). They are used in electromagnetism for time harmonic scattering problems from obstacles that are partially or totally penetrable. The general idea is to replace the use of an "exact model" inside (the penetrable part of) the obstacle by approximate boundary conditions (also called equivalent or effective conditions). This idea is pertinent if the boundary condition can be easily handled numerically, for instance when it is local. The diffraction problem of electromagnetic waves by perfectly conducting obstacles coated with a thin layer of dielectric material is well suited for the notion of impedance conditions: due to the small (typically with respect to the wavelength) thickness of the coating, the effect of the layer on the exterior medium is, as a first approximation, local (see for instance, Refs. 16, 13, 7, 3 and 1).

The application we consider here is the diffraction of waves by highly conducting materials in electromagnetism. In such a case, it is the well-known skin effect that creates a "thin layer" phenomenon. The high conductivity limitates the penetration of the wave to a boundary layer whose depth is inversely proportional to the square root of its magnitude. Then, here again, the effect of the obstacle is, as a first approximation, local.

The first effective boundary conditions for highly absorbing obstacles was proposed by Leontovich. This condition "sees" only locally the tangent plane to the frontier. Later, Rytov,[15,16] proposed an extension of the Leontovitch condition, and his analysis was already based on the principle of asymptotic expansions with respect to the small parameter in the problem: the skin depth δ. More recently, Antoine–Barucq–Vernhet[2] proposed a new derivation of such conditions based on the technique of pseudo-differential operator expansions. However, in all these works, the rigorous mathematical justification of the resulting impedance conditions was not treated.

This paper is the continuation of the work in Ref. 11, in which we considered the case of the scalar wave equation. Our objective is to extend the results to the case of 3D Maxwell's equations by constructing and analyzing GIBCs of order 1, 2 and 3 (with respect to the skin depth, the small parameter of the problem). These conditions are of impedance type (or H-to-E nature): they relate the tangential traces of the electric and magnetic fields via a local impedance operator.

As in Ref. 11, the construction of the approximate conditions relies on an asymptotic expansion of the exact solution, based on a scaling technique and a boundary layer expansion in the neighborhood of the boundary of the scatterer. Though the organization of this paper contents is similar to Ref. 11, it is much more technical. Moving from the scalar wave equation to the Maxwell system increases considerably the complexity of the problem at two levels.

- The first one is linked with the algebra involved in the formal construction of the asymptotic expansion of the exact solution (see Sec. 5). This is essentially due to the vectorial nature of the unknowns and the expression of the curl operator in a parametric coordinates system (see Sec. 3). The latter is based on the formulas proposed in Ref. 10 with some simplifications.
- The second one is related to the mathematical analysis on the GIBCs. This is not only due to the fact that we have to deal with the usual functional analysis difficulties linked to Maxwell equations (in particular trace operators and compact embedding properties — see Secs. 6 and 7 and Appendix A) but also because we have to face some new difficulties in the case of the third-order condition.

The tangential differential operators that would naturally appear in the construction of the third-order condition have no good "sign properties" to be able to guarantee the existence of the approximate solution and the convergence (at optimal order) to the true solution. This leads us to apply various regularization procedures to construct the modified third-order conditions (see Sec. 4.2).

Our objectives in this work are essentially theoretical. The numerical pertinence of obtained conditions have already been demonstrated in Ref. 6 where, in particular, the interest of using a third-order condition rather than a first or a second order condition is clearly shown.

The outline of the paper is as follows. Section 2.1 contains a description of the physical and mathematical diffraction problem under study with some basic stability properties of the solutions and asymptotic estimates with respect to the conductivity. We state the main results of our paper in Sec. 4: the GIBCs are presented in Secs. 4.1 and 4.2 while the corresponding error estimates are given in Sec. 4.3. The formal construction of the asymptotic expansion is given in Sec. 5. This construction is rigorously justified in Sec. 6 by proving optimal error estimates at each order. The last section is dedicated to the study of the boundary value problems associated with the GIBCs as well as the proof of optimal error estimates between these solutions and truncated asymptotic expansions. The main result of our paper is obtained as a combination of the results of Secs. 7 and 6. Some nonstandard technical results related to the $H(\mathrm{curl})$ space (appropriate trace inequalities and special compact embedding properties) that may have their own interest have been gathered in Appendix A.

2. Description and Properties of the Mathematical Model

2.1. *The model problem*

Let Ω_i be an open bounded domain in $\mathbb{R}^3$ with connected complement, occupied by a homogeneous conducting medium. We denote by Γ the boundary of Ω_i and assume that this boundary is a C^∞ manifold. We are interested in computing the electromagnetic diffracted wave when the conductivity of the medium, denoted by σ^δ, is sufficiently high (δ denotes a small parameter). More precisely we assume that $\sigma^\delta \to \infty$ as $\delta \to 0$ and would like to study the asymptotic behavior of the diffracted electromagnetic field as $\delta \to 0$ in order to derive efficient approximate models to compute the diffracted waves.

We assume that the exterior domain is homogeneous and the time and space scales are chosen such that the wave speed is 1 in this medium. The electromagnetic wave propagation is therefore governed by the following Maxwell's equations:

$$\begin{cases} \varepsilon(x)\dfrac{\partial \mathbf{E}^\delta}{\partial t} + \sigma^\delta(x)\mathbf{E}^\delta - \operatorname{curl}\mathbf{H}^\delta = F, & \text{in } \Omega, \\ \dfrac{\partial \mathbf{H}^\delta}{\partial t} + \operatorname{curl}\mathbf{E}^\delta = 0, & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ denotes the propagative medium that we shall assume to be bounded, regular and simply connected with connected boundary (for instance an open ball), the functions $\sigma^\delta(x)$ and $\varepsilon(x)$ are defined by:

$$(\varepsilon, \sigma^\delta)(x) = \begin{cases} (1,0), & \text{in } \Omega_e, \\ (\varepsilon_r, \sigma^\delta), & \text{in } \Omega_i \end{cases}$$

where $\Omega_e = \Omega \backslash \overline{\Omega_i}$ and where $\varepsilon_r > 0$ denotes the relative electric permittivity of the conducting medium. The right-hand side F denotes some source term that we shall assume to be harmonic in time: $F(x,t) = \mathrm{Re}\{f(x)\exp(i\omega t)\}$, where $\omega > 0$ denotes a given frequency, and where $\mathrm{Re}(z)$ denotes the real part of z. Hence, the solutions are also time harmonic:

$$\mathbf{E}^\delta(x,t) = \mathrm{Re}\{E^\delta(x)\exp(i\omega t)\}, \quad \mathbf{H}^\delta(x,t) = \mathrm{Re}\{H^\delta(x)\exp(i\omega t)\},$$

where the field (E^δ, H^δ) is solution to the harmonic Maxwell system:

$$\begin{cases} \text{(i)} \quad (i\varepsilon\omega + \sigma^\delta)E^\delta - \operatorname{curl} H^\delta = f, & \text{in } \Omega, \\ \text{(ii)} \quad i\omega H^\delta + \operatorname{curl} E^\delta = 0, & \text{in } \Omega. \end{cases} \tag{2.1}$$

We assume that the support of the source term f does not touch Ω_i. The system of equations (2.1) has to be complemented with a boundary condition on the exterior boundary $\partial\Omega$, for instance we work with the following absorbing boundary condition

$$E_T^\delta - H^\delta \times n = g, \quad \text{on } \partial\Omega, \tag{2.2}$$

where $E_T := n \times (E \times n)$, n is a normal vector to $\partial\Omega$ directed to the exterior of Ω and g denoting some possible source term. As mentioned above we are interested in describing the asymptotic behavior of the solution for large σ^δ. As suggested by the expression of the analytic solution when Ω_i is a half space, the appropriate small length parameter is

$$\delta := 1/\sqrt{\omega\sigma^\delta} \Leftrightarrow \sigma^\delta = 1/(\omega\delta^2).$$

This small parameter defines the so-called skin depth: the "width" of the penetrable region inside the conducting medium is proportional to δ.

For the construction of approximate models in the exterior domain Ω_e, it is useful to rewrite the problem (2.1)–(2.2) as a transmission problem between $(E_i^\delta, H_i^\delta) := (E^\delta, H^\delta)_{|\Omega_i}$ and $(E_e^\delta, H_e^\delta) = (E^\delta, H^\delta)_{|\Omega_e}$ as follows:

$$\begin{cases} i\omega E_e^\delta - \operatorname{curl} H_e^\delta = f, & \text{in } \Omega_e, \\ i\omega H_e^\delta + \operatorname{curl} E_e^\delta = 0, & \text{in } \Omega_e, \\ E_{e,T}^\delta - H_e^\delta \times n = g, & \text{on } \partial\Omega \\ E_e^\delta \times n = E_i^\delta \times n, & \text{on } \Gamma, \end{cases} \tag{2.3}$$

$$\begin{cases} \left(i\varepsilon_r\omega + \dfrac{1}{\omega\delta^2}\right) E_i^\delta - \operatorname{curl} H_i^\delta = 0, & \text{in } \Omega_i, \\ i\omega H_i^\delta + \operatorname{curl} E_i^\delta = 0, & \text{in } \Omega_i, \\ H_i^\delta \times n = H_e^\delta \times n, & \text{on } \Gamma. \end{cases} \tag{2.4}$$

We have chosen to split the two transmission conditions (namely the continuity of the tangential electric and magnetic fields) in such a way that the first one appears as a boundary condition in (2.3) for the interior field while the second one appears as a boundary condition in (2.3) for the interior field. Roughly speaking, the approximate models are then obtained from replacing in system (2.4) the exact boundary condition on Γ by an approximate one, whose expression is derived from seeking appropriate asymptotic expansion of the solution in the boundary layer inside Ω_i.

2.2. *Stability and interior decay*

With $H(\operatorname{curl}, \mathcal{O})$ denoting the space of functions $V \in L^2(\mathcal{O})^3$ such that $\operatorname{curl} V \in L^2(U)^3$, where $\mathcal{O}$ is an open domain of $\mathbb{R}^3$, we define

$$\tilde{H}(\operatorname{curl}, \mathcal{O}) = \{V \in H(\operatorname{curl}, \mathcal{O}); V_T \in L_t^2(\partial\mathcal{O})\}, \tag{2.5}$$

where V_T is the tangential trace of V (cf. Sec. 3 for more details), $L_t^2(\partial\mathcal{O})$ denotes the space of functions $V \in L^2(\partial\mathcal{O})^3$ such that $V \cdot n = 0$ on $\partial\mathcal{O}$, where n denotes a normal to $\partial\mathcal{O}$. We recall that $\tilde{H}(\operatorname{curl}, \mathcal{O})$ is a Hilbert space with scalar product

$$(U, V)_{\tilde{H}(\operatorname{curl},\mathcal{O})} = (U, V)_{L^2(\mathcal{O})} + (\operatorname{curl} U, \operatorname{curl} V)_{L^2(\mathcal{O})} + (U_T, V_T)_{L_t^2(\partial\mathcal{O})}.$$

Theorem 2.1. *For given $f \in L^2(\Omega)^3$ and $g \in L_t^2(\partial\Omega)$ there exists a unique solution $(E^\delta, H^\delta) \in \tilde{H}(\operatorname{curl}, \Omega) \times \tilde{H}(\operatorname{curl}, \Omega)$ satisfying* (2.1)–(2.2). *Moreover, there exists a positive constant C independent of δ such that,*

$$\|E^\delta\|_{\tilde{H}(\operatorname{curl},\Omega)} + \frac{1}{\delta}\|E^\delta\|_{L^2(\Omega_i)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L_t^2(\partial\Omega)}). \tag{2.6}$$

For any $\bar{\nu} > 0$ small enough so that $\Omega_i \backslash \Omega_i^{\bar{\nu}}$ is a non-empty set, where $\Omega_i^{\bar{\nu}} := \{x \in \Omega_i; \operatorname{dist}(x, \partial\Omega_i) < \bar{\nu}\}$, there exist two positive constants $C_{\bar{\nu}}$ and $c_{\bar{\nu}}$ independent of δ such that

$$\|E_i^\delta\|_{H(\operatorname{curl},\Omega_i\backslash\Omega_i^{\bar{\nu}})} + \|H_i^\delta\|_{H(\operatorname{curl},\Omega_i\backslash\Omega_i^{\bar{\nu}})} \leq C_{\bar{\nu}}\,(e^{-\frac{c_{\bar{\nu}}}{\delta}}\,\|f\|_{L^2(\Omega)} + \|g\|_{L_t^2(\partial\Omega)}). \tag{2.7}$$

Proof. The proof of existence and uniqueness can be found in Ref. 14 (Theorem 4.17). The proof of stability is rather standard, based on a contradiction argument, the Helmholtz decomposition and a compactness argument. See Theorem 2.1 of Ref. 12 for more details. The proof of estimate (2.7) follows the same lines as the scalar case treated in Ref. 11 and the details are also provided in Ref. 12, Theorem 2.2. $\square$

Estimate (2.7) indicates how the interior solution concentrates near the boundary Γ. This result is also a consequence of the asymptotic analysis performed in the following sections, but the methodology is more complex than the direct proof given in Ref. 12.

3. Preliminary Material and Notation

We recall in this section some well-known facts on differential geometry and introduce some notation in connection with surface operators and functional spaces.

Local coordinates. Let n be the normal field defined on Γ and directed to the interior of Ω_i. For a sufficiently small positive constant $\bar{\nu}$ (see condition (3.4) below) we define

$$\Omega_i^{\bar{\nu}} = \{x \in \Omega_i; \mathrm{dist}(x, \partial\Omega_i) < \bar{\nu}\}.$$

To any $x \in \Omega_i^{\bar{\nu}}$ we uniquely associate the local parametric coordinates $(x_\Gamma, \nu) \in \Gamma \times (0, \bar{\nu})$ through

$$x = x_\Gamma + \nu\, n, \quad x \in \Omega_i^{\bar{\nu}}. \tag{3.1}$$

Tangential (*or surface*) *differential operators.* In what follows we deal with various fields defined on Γ: scalar fields φ (with values in $\mathbb{C}$), vector fields V (with values in $\mathbb{C}^3$) and matrix (or tensor) fields $\mathbf{A}$ (with values in $\mathcal{L}(\mathbb{C}^3)$). By definition:

- A vector field V is tangential iff $V \cdot n = 0$ (as a scalar field along Γ).
- A matrix field $\mathbf{A}$ is tangential iff $\mathbf{A}\ n = 0$ (as a vector field along Γ).

For simplicity, we assume that these fields have at least C^1 regularity, but this can be removed by interpreting the derivatives in the sense of distributions.

We recall that the surface gradient operator ∇_Γ is defined by:

$$\nabla_\Gamma \varphi(x_\Gamma) = \nabla\hat{\varphi}(x_\Gamma), \quad \forall\, \varphi : \Gamma \to \mathbb{R},$$

where $\hat{\varphi}$ is the 3-D vector field defined locally in $\Omega_i^{\bar{\nu}}$ by $\hat{\varphi}(x_\Gamma + \nu\, n) = \varphi(x_\Gamma)$. Note that $\nabla_\Gamma\varphi$ is a tangential vector field. We can define in the same way the surface gradient of a vector field as a tangential matrix field whose columns are the surface gradients of each component of the vector field.

We denote by $-\mathrm{div}_\Gamma$ the $L^2(\Gamma)$-adjoint of ∇_Γ : $-\mathrm{div}_\Gamma$ maps a tangential vector field into a scalar field. More generally, if $\mathbf{A}(x_\Gamma)$ is a tangential matrix field on Γ, we define the operator $\mathbf{A}\nabla_\Gamma$ for a scalar field $\varphi(x_\Gamma)$ by

$$(\mathbf{A}\nabla_\Gamma)u := \mathbf{A}(\nabla_\Gamma u).$$

We also define the operator $(\mathbf{A}\nabla_\Gamma)\cdot$ acting on a tangential vector field $V(x_\Gamma)$ as:

$$(\mathbf{A}\nabla_\Gamma) \cdot V := \sum_{i=1}^{3} (\mathbf{A}\nabla_\Gamma V_i)_i,$$

where the subscript i denotes the ith component of a vector in $\mathbb{R}^3$.

We then define the surface curl of a tangential vector filed $V(x_\Gamma)$ and the surface vector curl of a scalar function $\varphi(x_\Gamma)$ as

$$\mathrm{curl}_\Gamma V := \mathrm{div}_\Gamma(V \times n) \quad \text{and} \quad \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi := (\nabla_\Gamma \varphi) \times n.$$

Geometrical tools. In what follows, and for the sake of the notation conciseness, we shall most of time not explicitly indicate the dependence on x_Γ of the functions, except when we feel it necessary. We shall be more precise in mentioning the possible dependence with respect to the normal coordinate ν.

A particularly fundamental tensor field is the curvature tensor $\mathcal{C}$, defined by $\mathcal{C} := \nabla_\Gamma n$. We recall that $\mathcal{C}$ is symmetric and $\mathcal{C}\, n = 0$. We denote c_1, c_2 the other two eigenvalues of $\mathcal{C}$ (namely the *principal curvatures*) associated with tangential eigenvectors τ_1, τ_2 ($\tau_1 \cdot n = \tau_2 \cdot n = 0$). We also introduce

$$g := c_1 c_2 \quad \text{and} \quad h := \frac{1}{2}(c_1 + c_2) \tag{3.2}$$

which are respectively the *Gaussian* and *mean curvatures* of Γ, and also introduce the associated matrix fields:

$$\mathcal{H} = h I_\Gamma \quad \text{and} \quad \mathcal{G} = g\, I_\Gamma, \tag{3.3}$$

where $I_\Gamma(x_\Gamma)$ denotes the projection operator on the tangent plane to Γ at x_Γ.

Let us introduce (this is the Jacobian of the transformation $(x_\Gamma, \nu) \to x$ — see (3.1))

$$J(\nu)\big(= J(\nu, x_\Gamma)\big) := \det(I + \nu\, \mathcal{C}) = 1 + 2\nu h + \nu^2 g,$$

and we choose $\bar{\nu}$ sufficiently small in such a way that

$$\forall\, \nu < \bar{\nu}, \quad \forall\, x_\Gamma \in \Gamma, \quad J(\nu, x_\Gamma) = 1 + 2\nu h(x_\Gamma) + \nu^2 g(x_\Gamma) > 0. \tag{3.4}$$

Thus, for each $\nu < \bar{\nu}$, there exists a tangential matrix field $x_\Gamma \to \mathcal{R}_\nu(x_\Gamma)$ such that

$$(I + \nu\, \mathcal{C}(x_\Gamma))\mathcal{R}_\nu(x_\Gamma) = I_\Gamma(x_\Gamma).$$

More precisely, there exists a tangential matrix field on Γ, $\mathcal{M}(x_\Gamma)$, such that:

$$I_\Gamma + \nu \mathcal{M} := J(\nu)\mathcal{R}_\nu, \quad \forall x_\Gamma \in \Gamma, \quad \forall \nu < \bar{\nu}.$$

One easily sees (using for instance the eigenbasis $(\tau_1,\, \tau_2,\, n)$ of $\mathcal{C}$) that

$$\mathcal{M} = 2\mathcal{H} - \mathcal{C} \quad \text{and} \quad \mathcal{M}\,\mathcal{C} = \mathcal{G}.$$

The curl operator in local coordinates. The basic step of our forthcoming calculations will be to rewrite the Maxwell equations in the domain $\Omega_i^{\bar{\nu}}$, by using the local coordinates. For this, we need the expression of the curl operator in the variables (x_Γ, ν). It is shown in Ref. 10 that the curl of a 3-D vector field V: $\Omega_i^{\bar{\nu}} \to \mathbb{R}^3$ is given in parametric coordinates by:

$$\mathrm{curl}\, V = [(\mathcal{R}_\nu \nabla_\Gamma) \cdot (\widehat{V} \times n)]n + [\mathcal{R}_\nu \nabla_\Gamma(\widehat{V} \cdot n)] \times n - (\mathcal{R}_\nu \mathcal{C} \widehat{V}) \times n - \partial_\nu(\widehat{V} \times n),$$

where V and $\widehat{V}$ (defined on $\Gamma \times (0, \bar{\nu})$) are related by

$$\widehat{V}(x_\Gamma, \nu) = V(x_\Gamma + \nu\, n).$$

This formula can be written in a more convenient form, after multiplication by $J(\nu)$:

$$\begin{aligned} J(\nu)\mathrm{curl}\, V = {} & [\big((I + \nu\mathcal{M})\nabla_\Gamma\big) \cdot (\widehat{V} \times n)]n + [(I + \nu\mathcal{M})\nabla_\Gamma(\widehat{V} \cdot n)] \times n \\ & - [(\mathcal{C} + \nu\mathcal{G})\widehat{V}] \times n - J(\nu)\partial_\nu(\widehat{V} \times n), \end{aligned}$$

or, in an equivalent form,

$$J(\nu)\mathrm{curl}\, V = \left(C_\Gamma + \nu C_\Gamma^M\right) \widehat{V} - J(\nu)\partial_\nu(\widehat{V} \times n), \tag{3.5}$$

where we have introduced the notation

$$\begin{cases} C_\Gamma \widehat{V} = (\mathrm{curl}_\Gamma\, \widehat{V})n + \overrightarrow{\mathrm{curl}}_\Gamma(\widehat{V} \cdot n) - \mathcal{C}\widehat{V} \times n, \\ C_\Gamma^M \widehat{V} = \left(\mathrm{curl}_\Gamma^M\, \widehat{V}\right)n + \overrightarrow{\mathrm{curl}}_\Gamma^M(\widehat{V} \cdot n) - \mathcal{G}\widehat{V} \times n \\ \overrightarrow{\mathrm{curl}}_\Gamma^M\, u := (\mathcal{M}\nabla_\Gamma u) \times n \quad \text{and} \quad \mathrm{curl}_\Gamma^M\, \widehat{V} = (\mathcal{M}\nabla_\Gamma) \cdot (\widehat{V} \times n). \end{cases} \tag{3.6}$$

This expression is convenient for the asymptotic matching procedure, described hereafter, because we made explicit the (polynomial) dependence of the operators with respect to ν.

Functional spaces on Γ *and trace spaces.* We shall denote by $H^s(\Gamma)$ the usual Sobolev space on Γ for s real and denote by $(\cdot,\cdot)_\Gamma$ and $\langle\cdot,\cdot\rangle_\Gamma$, respectively the inner product in $L^2(\Gamma)^3$ and the duality bracket $\mathcal{D}'(\Gamma)^3 - \mathcal{D}(\Gamma)^3$.

Next, we introduce some notation for spaces of tangent vector fields on Γ. For any $s \geq 0$, we set:

$$\begin{aligned} H_t^s(\Gamma) &= \{V \in H^s(\Gamma)^3 / V \cdot n = 0 \text{ on } \Gamma\} \quad H_t^0(\Gamma) = L_t^2(\Gamma) \\ H_t^{-s}(\Gamma) &= \{V \in H^{-s}(\Gamma)^3 / \langle V, \varphi\, n\rangle_\Gamma = 0,\ \forall \varphi \in H^s(\Gamma)\} \quad \left(\equiv \left(H_t^s(\Gamma)\right)'\right., \end{aligned}$$

as well as

$$\begin{aligned} H^s(\mathrm{div}_\Gamma, \Gamma) &= \{V \in H_t^s(\Gamma)^3 / \mathrm{div}_\Gamma\, V \in H^s(\Gamma)\}, \\ H^s(\mathrm{curl}_\Gamma, \Gamma) &= \{V \in H_t^s(\Gamma)^3 / \mathrm{curl}_\Gamma\, V \in H^s(\Gamma)\}, \end{aligned}$$

equipped with their natural graph norms (we notice that $H^0(\mathrm{div}_\Gamma, \Gamma)$ and $H^0(\mathrm{curl}_\Gamma, \Gamma)$ are often denoted by respectively $H(\mathrm{div}_\Gamma, \Gamma)$ and $H(\mathrm{curl}_\Gamma, \Gamma)$). Finally, we recall the well-known trace theorems stating that the two mappings

$$\begin{cases} u \in C^\infty(\Omega_e)^3 \mapsto u \times n|_\Gamma \\ u \in C^\infty(\Omega_e)^3 \mapsto u_T := u - (u \cdot n)n \quad (\equiv n \times (u \times n)) \end{cases}$$

can be extended as continuous and surjective linear applications from $H(\mathrm{curl}, \Omega_e)$ onto $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ respectively. Moreover, $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is the dual of $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ and one has the Green's formula:

$$\int_\Omega (\mathrm{curl}\, u \cdot v - u \cdot \mathrm{curl}\, v)\, dx = \langle u \times n, v_T \rangle_\Gamma = -\langle v \times n, u_T \rangle_\Gamma,$$

$$\forall (u, v) \in H(\mathrm{curl}, \Omega_e)^2.$$

4. Statement of the Main Results

We shall denote by $(E_e^{\delta,k}, H_e^{\delta,k})$, the approximate solutions in the exterior domain Ω_e, the presence of the integer k meaning that these fields will provide an approximation of order $O(\delta^{k+1})$ of the exact exterior electromagnetic field (E_e^δ, H_e^δ), in a sense that will be made precise by the error estimates (see Theorem 4.1). They are obtained by solving the standard Maxwell equations in the exterior domain Ω_e

$$\begin{cases} i\omega E_e^{\delta,k} - \mathrm{curl}\, H_e^{\delta,k} = f & \text{in } \Omega_e, \\ i\omega H_e^{\delta,k} + \mathrm{curl}\, E_e^{\delta,k} = 0 & \text{in } \Omega_e, \\ E_{e,T}^{\delta,k} - H_e^{\delta,k} \times n = g & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where n denotes the normal to $\partial\Omega$ directed to the exterior of Ω, coupled with an appropriate GIBC on the interior boundary Γ of the form

$$E_e^{\delta,k} \times n + \omega \mathcal{D}^{\delta,k}(H_{e,T}^{\delta,k}) = 0, \tag{4.2}$$

where n denotes the normal to Γ directed to the exterior of Ω_e, $H_{e,T}^{\delta,k}$ is the tangential trace of $H_e^{\delta,k}$, and where $\mathcal{D}^{\delta,k}$ is an adequate local approximation of the *H-to-E* map for the Maxwell equations inside Ω_i, namely the operator:

$$\mathcal{D}^\delta : H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \to H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

defined by

$$\mathcal{D}^\delta \varphi = -\frac{1}{\omega} E_i^\delta \times n|_\Gamma,$$

where $\big(E_i^\delta(\varphi), H_i^\delta(\varphi)\big)$ is the solution of the *interior* boundary value problem

$$\begin{cases} \left(i\varepsilon_r \omega + \dfrac{1}{\omega\delta^2} \right) E_i^\delta(\varphi) - \mathrm{curl}\, H_i^\delta(\varphi) = 0, & \text{in } \Omega_i, \\ i\omega H_i^\delta(\varphi) + \mathrm{curl}\, E_i^\delta(\varphi) = 0, & \text{in } \Omega_i, \\ H_{i,T}^\delta(\varphi) = \varphi, & \text{on } \Gamma. \end{cases}$$

4.1. *The "natural" GIBCs for* $k = 0, 1, 2$

The approach that we shall use in Sec. 5 for the formal derivation of the GIBCs leads to the following expressions of $\mathcal{D}^{\delta,k}$ (for $k = 0, 1, 2, 3$),

$$\begin{cases} \mathcal{D}^{\delta,0} = 0, \\ \mathcal{D}^{\delta,1} = \delta\sqrt{i}, \\ \mathcal{D}^{\delta,2} = \delta\sqrt{i} + \delta^2(\mathcal{H} - \mathcal{C}), \end{cases} \tag{4.3}$$

where $\sqrt{i} := \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ denotes the complex square root of i with positive real part, and $\mathcal{C}$ and $\mathcal{H}$ are the curvature and mean curvature tensors of Γ (we refer to Sec. 3 for more details). Note that the condition of order 0 simply expresses the fact that the limit exterior problem when δ goes to 0 corresponds to the perfectly conducting boundary condition.

4.2. *The modified third-order GIBC*

The same approach extended to $k = 3$ would suggest to take:

$$\mathcal{D}^{\delta,3} = \mathcal{D}_0^{\delta,3}, \tag{4.4}$$

where by definition (we refer to Sec. 3 for the definition of the surface operators ∇_Γ, div_Γ, curl_Γ and $\overrightarrow{\mathrm{curl}}_\Gamma$)

$$\mathcal{D}_0^{\delta,3} := \delta\sqrt{i} + \delta^2(\mathcal{H} - \mathcal{C}) + \frac{\delta^3}{2\sqrt{i}}(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2 + \nabla_\Gamma \,\mathrm{div}_\Gamma + \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma). \tag{4.5}$$

However, we did not succeed in proving that such a choice was mathematically sound due to the presence of the second order surface operator $\nabla_\Gamma \,\mathrm{div}_\Gamma + \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma$. As a self-adjoint operator in $L_t^2(\Gamma)$, this operator (more precisely the associated quadratic form) has no fix sign. This induces difficulties in the study of the forward problem via variational techniques and, as a consequence, the well-posedness of the corresponding boundary value problem is not clear: this is a new difficulty with respect to the scalar wave equation.

This is why we propose hereafter another third-order condition, that (formally) gives the same order of accuracy as the one in (4.3) but admits good mathematical properties with respect to stability and error estimates. The reader will easily notice that the proposed modifications are not the only possible ones (see Remarks 3.1 and 3.2 of Refs. 12 for more details), we exhibit only one particular choice. We shall hereafter present the intuitive reasons that led us to introduce these modifications, postponing the rational justification to the error analysis of Sec. 7.3.

The first desirable (and probably necessary) property is the absorption property:

$$\mathcal{R}e \int_\Gamma \mathcal{D}^{\delta,3}\varphi \cdot \bar{\varphi} d\sigma \geq 0,$$

for any smooth tangential vector field φ on Γ. Such a property is satisfied by the exact DtN operator and expresses the absorbing nature of the conductive medium:

$$\mathcal{R}e \int_\Gamma \mathcal{D}^{\delta,3}\varphi \cdot \bar{\varphi} d\sigma = \frac{1}{\omega\delta^2} \int_{\Omega_i} |E_i^\delta(\varphi)|^2 dx.$$

It will play an essential role in proving the *uniqueness* of solutions. One can observe that this condition is satisfied by $\mathcal{D}^{\delta,1}$ and $\mathcal{D}^{\delta,2}$. For $\mathcal{D}^{\delta,3}$, we see that

$$\mathcal{R}e\mathcal{D}_0^{\delta,3} = \delta\frac{\sqrt{2}}{2} + \delta^2(\mathcal{H} - \mathcal{C}) + \frac{\delta^3}{2\sqrt{2}}(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2) + \frac{\delta^3}{2\sqrt{2}}\nabla_\Gamma \,\mathrm{div}_\Gamma + \frac{\delta^3}{2\sqrt{2}}\overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma.$$

The problem comes from the operator $\nabla_\Gamma \operatorname{div}_\Gamma$ which is negative in the L^2 sense. However, we can write formally

$$\begin{aligned}\frac{\delta\sqrt{2}}{2} + \frac{\delta^3}{2\sqrt{2}}\nabla_\Gamma \operatorname{div}_\Gamma &= \frac{\delta}{2\sqrt{2}} + \frac{\delta}{2\sqrt{2}} + \frac{\delta^3}{2\sqrt{2}}\nabla_\Gamma \operatorname{div}_\Gamma \\ &= \frac{\delta}{2\sqrt{2}}(1-\delta^2\nabla_\Gamma \operatorname{div}_\Gamma)^{-1} + O(\delta^5),\end{aligned} \tag{4.6}$$

which suggests to define the real part of $\mathcal{D}^{\delta,3}$ as

$$\begin{aligned}\mathcal{R}e\mathcal{D}^{\delta,3} = {} & \frac{\delta}{2\sqrt{2}} + \delta^2(\mathcal{H}-\mathcal{C}) + \frac{\delta^3}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) + \frac{\delta^3}{2\sqrt{2}}\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma \\ & + \frac{\delta}{2\sqrt{2}}(1-\delta^2\nabla_\Gamma \operatorname{div}_\Gamma)^{-1}.\end{aligned} \tag{4.7}$$

Remark 4.1. The approximation process (4.6) is analogous to the process used in the construction of absorbing boundary conditions for the wave equations, see Refs. 8 and 4 for instance, where the Padé approximations are preferred to Taylor approximations in order to enforce the stability of the resulting approximate problem.

The second modification was guided by the *existence* proof for the boundary value problem associated to the boundary condition (4.2). We realized that it was useful that the imaginary part of $\mathcal{D}^{\delta,3}$ satisfies a "Garding type" inequality, namely that the principal part of this operator be positive in the L^2 sense. This property is not satisfied by the imaginary part of $\mathcal{D}_0^{\delta,3}$:

$$\mathcal{I}m\mathcal{D}_0^{\delta,3} = \delta\frac{\sqrt{2}}{2} - \frac{\delta^3}{2\sqrt{2}}(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2) - \frac{\delta^3}{2\sqrt{2}}\nabla_\Gamma \operatorname{div}_\Gamma - \frac{\delta^3}{2\sqrt{2}}\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma.$$

This time, the problem is due to the negative operator $-\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma$. The same manipulations as before suggest to define the imaginary part of $\mathcal{D}_r^{\delta,3}$ as

$$\begin{aligned}\mathcal{I}m\mathcal{D}_r^{\delta,3} = {} & \frac{\delta}{2\sqrt{2}} + \frac{\delta^3}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) - \frac{\delta^3}{2\sqrt{2}}\nabla_\Gamma \operatorname{div}_\Gamma \\ & + \frac{\delta}{2\sqrt{2}}(1+\delta^2\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma)^{-1}.\end{aligned} \tag{4.8}$$

Modifications (4.7) and (4.8) lead us to introduce the operator

$$\begin{aligned}\tilde{\mathcal{D}}^{\delta,3} = {} & \delta\frac{\sqrt{i}}{2} + \delta^2(\mathcal{H}-\mathcal{C}) + \frac{\delta^3}{2\sqrt{i}}(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2) \\ & + \frac{\sqrt{2}}{4}\delta((1-\delta^2\nabla_\Gamma \operatorname{div}_\Gamma)^{-1} + \delta^2\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma) \\ & + i\frac{\sqrt{2}}{4}\delta((1+\delta^2\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma)^{-1} - \delta^2\nabla_\Gamma \operatorname{div}_\Gamma),\end{aligned} \tag{4.9}$$

which formally satisfies $\tilde{\mathcal{D}}^{\delta,3} = \mathcal{D}_0^{\delta,3} + O(\delta^5)$.

It turns out that even if this condition is suitable for variational study of existence and uniqueness of the resulting boundary value problem, it did not enable us to have a direct proof of optimal error estimates (although we think it can be achieved by constructing the full asymptotic expansion associated with the associated boundary value problem). We realized that the difficulties encountered in the analysis are related to the fact that the operator $\tilde{\mathcal{D}}^{\delta,3}$ is a pseudo-differential operator of order 2, while the exact impedance operator which maps continuously $H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ into $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is something between an operator of order -1 and an operator of order 1. This gave us the idea to force our approximate operator to be of order 0 by applying a *regularization* process (the Yosida regularization) to the operators $\overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma$ and $\nabla_\Gamma \,\mathrm{div}_\Gamma$

$$\begin{cases} \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma \simeq \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma (1 + \delta^2 \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma)^{-1} & \text{in } O(\delta^2), \\ \nabla_\Gamma \,\mathrm{div}_\Gamma \simeq \nabla_\Gamma \,\mathrm{div}_\Gamma \,(1 - \delta^2 \nabla_\Gamma \,\mathrm{div}_\Gamma)^{-1}\big) & \text{in } O(\delta^2). \end{cases} \tag{4.10}$$

Such an approximation is consistent with the $O(\delta^5)$ accuracy provided by $\tilde{\mathcal{D}}^{\delta,3}$ since $\overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma$ and $\nabla_\Gamma \,\mathrm{div}_\Gamma$ are multiplied by δ^3. Moreover, it does not affect the good sign properties of the real and imaginary parts of the operator since we "divide" by positive operators. Therefore, we propose as 3rd order condition:

$$\begin{aligned} \mathcal{D}^{\delta,3} := {}& \delta \frac{\sqrt{i}}{2} + \delta^2 (\mathcal{H} - \mathcal{C}) + \frac{\delta^3}{2\sqrt{i}} \big(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r \omega^2 \big) \\ & + \frac{\sqrt{2}}{4} \delta \big((1 - \delta^2 \nabla_\Gamma \,\mathrm{div}_\Gamma)^{-1} + \delta^2 \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma \,(1 + \delta^2 \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma)^{-1} \big) \\ & + i \frac{\sqrt{2}}{4} \delta \big((1 + \delta^2 \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma)^{-1} - \delta^2 \nabla_\Gamma \,\mathrm{div}_\Gamma \,(1 - \delta^2 \nabla_\Gamma \,\mathrm{div}_\Gamma)^{-1} \big). \end{aligned} \tag{4.11}$$

It happens that this operator has the good *consistency*, *coercivity* and *continuity* properties that lead to optimal error estimates. More precisely:

- One can check that

$$\mathcal{D}^{\delta,3} = \mathcal{D}_0^{\delta,3} + \delta^5 \mathcal{R}^{\delta,3}, \tag{4.12}$$

  where the operator $\mathcal{R}^{\delta,3}$, given by

$$\begin{aligned} \mathcal{R}^{\delta,3} = {}& \frac{\sqrt{2}}{4} (1 + i) \big[\big(1 - \delta^2 \nabla_\Gamma \,\mathrm{div}_\Gamma\big)^{-1} (\nabla_\Gamma \,\mathrm{div}_\Gamma)^2 \\ & + (1 + \delta^2 \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma \big)^{-1} (\overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma)^2 \big], \end{aligned}$$

  maps continuously $H_t^{s+4}(\Gamma)$ into $H_t^s(\Gamma)$ and satisfies the uniform bound

$$\|\mathcal{R}^{\delta,3}\|_{\mathcal{L}\left(H_t^{s+4}(\Gamma); H_t^s(\Gamma)\right)} \le 1. \tag{4.13}$$

- One can prove (see Lemma 7.1) that $\mathcal{D}^{\delta,3}$ is a pseudo-differential operator of order 0 that has the following fundamental properties (obviously satisfied by

$\mathcal{D}^{\delta,1}$ and $\mathcal{D}^{\delta,2}$)

$$\forall\, \varphi \in L^2_t(\Gamma), \quad \|\mathcal{D}^{\delta,k}\varphi\|_\Gamma \le C_1\delta\|\varphi\|_\Gamma, \quad \mathcal{R}e\,(\mathcal{D}^{\delta,k}\varphi,\varphi)_\Gamma \ge C_2\delta\|\varphi\|^2_\Gamma,$$

with C_1 and C_2 strictly positive constants. These appear to be sufficient properties to transform the consistency properties $\mathcal{D}^{\delta,3}$ into optimal error estimates (see the proof of Lemma 7.2).

4.3. *Existence, uniqueness and error estimates*

The natural functional spaces for the solutions of the approximate problems vary according to the regularity of their traces on Γ. We shall distinguish the case $k = 0$ for which we set

$$\mathcal{V}^0_H = \{H \in H(\mathrm{curl}, \Omega_e); (H \times n)|_{\partial\Omega} \in L^2_t(\partial\Omega)\}, \quad \mathcal{V}^0_E = \{E \in \mathcal{V}^0_H; (E \times n)|_\Gamma = 0\},$$

from the case $k = 1$, 2 or 3 for which we set

$$\mathcal{V}^k_H = \mathcal{V}^k_E = \tilde{H}(\mathrm{curl}, \Omega_e)$$

(see (2.5) for the definition of $\tilde{H}(\mathrm{curl}, \Omega_e)$). Then we have the following central theorem, that uses and combines the partial results of Secs. 4–6.

Theorem 4.1. *For $k = 0, 1, 2$ or 3, there exists δ_k such that for $\delta \le \delta_k$, the boundary value problem ((4.1), (4.2)) has a unique solution $(E^{\delta,k}_e, H^{\delta,k}_e) \in \mathcal{V}^k_E \times \mathcal{V}^k_H$. Moreover, there exists a constant C_k, independent of δ, such that*

$$\|E^\delta_e - E^{\delta,k}_e\|_{H(\mathrm{curl},\Omega_e)} \le C_k\delta^{k+1}.$$

Remark 4.2. For $k = 0, 1$, the above theorem holds for all δ.

5. Formal Derivation of the GIBCs

5.1. *The asymptotic ansatz*

To formulate our ansatz, it is useful to introduce a cutoff function $\chi \in C^\infty(\Omega_i)$ such that $\chi = 1$ in $\Omega^{\bar{\nu}}_i$ and $\chi = 0$ in $\Omega_i \backslash \Omega^{2\,\bar{\nu}}_i$ for a sufficiently small $\bar{\nu} > 0$. For this ansatz we are not interested in the part of the solution inside the support of $(1 - \chi)$, since we already know that the norm of the solution in this part exponentially decay to 0 as δ goes to 0 (is Theorem 2.1). For the remaining part of the solution, we postulate the following expansions:

$$\begin{cases} E^\delta_e(x) = E^0_e(x) + \delta E^1_e(x) + \delta^2 E^2_e(x) + \cdots \quad \text{for } x \in \Omega_e, \\ H^\delta_e(x) = H^0_e(x) + \delta H^1_e(x) + \delta^2 H^2_e(x) + \cdots \quad \text{for } x \in \Omega_e, \end{cases} \tag{5.1}$$

where E^ℓ_e, H^ℓ_e, $\ell = 0, 1, \ldots$ are functions defined on Ω_e and

$$\begin{cases} \chi(x)E^\delta_i(x) = E^0_i(x_\Gamma, \nu/\delta) + \delta E^1_i(x_\Gamma, \nu/\delta) + \delta^2 E^2_i(x_\Gamma, \nu/\delta) + \cdots \quad \text{for } x \in \Omega^{\bar{\nu}}_i, \\ \chi(x)H^\delta_i(x) = H^0_i(x_\Gamma, \nu/\delta) + \delta H^1_i(x_\Gamma, \nu/\delta) + \delta^2 H^2_i(x_\Gamma, \nu/\delta) + \cdots \quad \text{for } x \in \Omega^{\bar{\nu}}_i, \end{cases} \tag{5.2}$$

where x, x_Γ and ν are as in (3.1) and where $E_i^\ell(x_\Gamma,\eta)$, $H_i^\ell(x_\Gamma,\eta) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{C}$ satisfy

$$\begin{cases} \text{For a.e. } x_\Gamma \in \Gamma, \quad \displaystyle\int_0^{+\infty} |E_i^\ell(x_\Gamma,\eta)|^2 d\eta < +\infty = 0, \\ \text{For a.e. } x_\Gamma \in \Gamma, \quad \displaystyle\int_0^{+\infty} |H_i^\ell(x_\Gamma,\eta)|^2 d\eta < +\infty. \end{cases} \tag{5.3}$$

Remark 5.1. The condition (5.3) will imply that E_i^ℓ and H_i^ℓ are exponentially decreasing with respect to η.

$$\begin{cases} \text{For a.e. } x_\Gamma \in \Gamma, \quad \lim\limits_{\eta\to\infty} E_i^\ell(x_\Gamma,\eta) = 0, \quad \text{(exponentially fast)} \\ \text{For a.e. } x_\Gamma \in \Gamma, \quad \lim\limits_{\eta\to\infty} H_i^\ell(x_\Gamma,\eta) = 0, \quad \text{(exponentially fast)} \end{cases}$$

which corroborates the existence of a boundary layer as suggested by Theorem 2.1.

Remark 5.2. Expansion (5.2) makes sense since the local coordinates (x_Γ,ν) can be used inside the support of χ.

Next we shall identify the equations satisfied by (E_e^ℓ, H_e^ℓ) and (E_i^ℓ, H_i^ℓ), $\ell \geq 0$ by writing, formally, that we want to solve the transmission problem (2.3)–(2.4).

In the sequel, it is useful to introduce the notation

$$\begin{cases} \widetilde{E}_i^\delta(x_\Gamma,\eta) := E_i^0(x_\Gamma,\eta) + \delta E_i^1(x_\Gamma,\eta) + \delta^2 E_i^2(x_\Gamma,\eta) + \cdots \quad (x_\Gamma,\eta) \in \Gamma\times\mathbb{R}^+, \\ \widetilde{H}_i^\delta(x_\Gamma,\eta) := H_i^0(x_\Gamma,\eta) + \delta H_i^1(x_\Gamma,\eta) + \delta^2 H_i^2(x_\Gamma,\eta) + \cdots \quad (x_\Gamma,\eta) \in \Gamma\times\mathbb{R}^+, \end{cases} \tag{5.4}$$

so that ansatz (5.2) has to be understood as

$$\begin{cases} \chi(x) E_i^\delta(x) = \widetilde{E}_i^\delta(x_\Gamma,\nu/\delta) + O(\delta^\infty) \quad \text{for } x \in \Omega_i^{\bar\nu}, \\ \chi(x) H_i^\delta(x) = \widetilde{H}_i^\delta(x_\Gamma,\nu/\delta) + O(\delta^\infty) \quad \text{for } x \in \Omega_i^{\bar\nu}. \end{cases} \tag{5.5}$$

5.2. *The equations for the exterior fields*

This is the easy part of the job. The equations are directly derived from (2.3) and we obtain that (E_e^k, H_e^k) satisfy

$$\begin{cases} i\omega E_e^k - \operatorname{curl} H_e^k = f_k, & \text{in } \Omega_e, \\ i\omega H_e^k + \operatorname{curl} E_e^k = 0, & \text{in } \Omega_e, \\ E^k_{e\,|T} - H_e^k \times n = g_k, & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

where we have set $f_0 = f$, $g_0 = g$ and $f_k = 0$, $g_k = 0$, for $k \geq 1$, (5.6) being complemented with the interface condition

$$E_e^k|_\Gamma(x_\Gamma) \times n = E_i^k(x_\Gamma, 0) \times n, \quad \text{for } x_\Gamma \in \Gamma, \tag{5.7}$$

which completely defines (E_e^k, H_e^k) if $E_i^k(x_\Gamma,0)\times n$ is known.

5.3. *The equations for the interior fields*

As indicated above, we need to compute the interior fields E_i^k. The principle consists of expressing this field in terms of the tangential boundary values of (H_e^ℓ), $\ell \leq k$ by solving the interior equations. More precisely, we now substitute the expansion (5.4), (5.5) into the system (2.4) and assume that the quantity:

$$H_e^\delta \times n = \sum_{k=0}^{+\infty} \delta^k H_e^k \times n$$

is known on Γ. We need of course to rewrite the equations of (2.4) in the local "scaled" coordinates

$$(x_\Gamma, \eta = \nu/\delta).$$

Using formula (3.5) with $\nu = \delta\eta$ we obtain the following equalities in $\Gamma \times [0, \frac{\bar{\nu}}{\delta})$,

$$\begin{cases} J(\delta\eta)\left(i\varepsilon_r\omega + \dfrac{1}{\omega\delta^2}\right)\widetilde{E}_i^\delta - (\mathbf{C}_\Gamma + \delta\eta\mathbf{C}_\Gamma^M)\widetilde{H}_i^\delta + \dfrac{J(\delta\eta)}{\delta}\partial_\eta \widetilde{H}_i^\delta \times n = O(\delta^\infty), \\ iJ(\delta\eta)\omega\,\widetilde{H}_i^\delta + (\mathbf{C}_\Gamma + \delta\eta\mathbf{C}_\Gamma^M)\widetilde{E}_i^\delta - \dfrac{J(\delta\eta)}{\delta}\,\partial_\eta \widetilde{E}_i^\delta \times n = O(\delta^\infty). \end{cases} \tag{5.8}$$

These equations are complemented by the boundary condition

$$\widetilde{H}_i^\delta(x_\Gamma, 0) \times n + O(\delta^\infty) = H_e^\delta \times n(x_\Gamma), \quad x_\Gamma \in \Gamma. \tag{5.9}$$

The substitution of (5.4), (5.5) into (5.8), (5.9) leads to a sequence of problems that enable us to inductively determine the fields (E_i^k, H_i^k). The computations are relatively delicate but straightforward. The most difficult task is to explain the recurrence properly, which is the aim of this section. In Sec. 5.4, we shall compute explicitly the first terms of the expansions.

It turns out to be very useful to make a change of unknown concerning the electric field. This is motivated by the observation that

$$E_i^0 = 0. \tag{5.10}$$

This fact can be explained along the following lines: indeed from (5.4) and (5.5) one deduces (at least formally) that:

$$\|E_i^\delta\|^2_{L^2(\Omega_i)} \sim \delta \int_\Gamma \int_0^{+\infty} |E_i^0(x_\Gamma, \eta)|^2 d\eta d\sigma.$$

Therefore, estimate (2.6), which says that $\|E_i^\delta\|^2_{L^2(\Omega_i)} = O(\delta^2)$, implies $E_i^0 = 0$.

The expansion for the electric field therefore starts with δE_i^1 while for the magnetic field $H_i^0 \neq 0$. In some sense there is a natural shift of one power of δ between the expansions of the electric and magnetic fields. This is why we introduce the "normalized" electric field:

$$\widehat{\mathbb{E}}_i^\delta = \frac{1}{\delta}\widetilde{E}_i^\delta, \tag{5.11}$$

and we seek an expansion of the form

$$\widehat{\mathbb{E}}_i^\delta(x_\Gamma,\eta) := \widehat{\mathbb{E}}_i^0(x_\Gamma,\eta)+\delta\widehat{\mathbb{E}}_i^1(x_\Gamma,\eta)+\delta^2\widehat{\mathbb{E}}_i^2(x_\Gamma,\eta)+\cdots \quad (x_\Gamma,\eta)\in\Gamma\times\mathbb{R}^+, \tag{5.12}$$

with the correspondence

$$E_i^{k+1} = \widehat{\mathbb{E}}_i^k, \quad k\ge 1.$$

We then rewrite (5.8) as a system of equations for $(\widehat{\mathbb{E}}_i^\delta, \widetilde{H}_i^\delta)$ (we have multiplied the first equation by δ) in $\Gamma\times[0,\frac{\bar\nu}{\delta})$,

$$\begin{cases} J(\delta\eta)\left(i\varepsilon_r\delta\omega+\dfrac{1}{\omega}\right)\widehat{\mathbb{E}}_i^\delta - \left(\delta\mathbf{C}_\Gamma + \delta^2\eta\mathbf{C}_\Gamma^M\right)\widetilde{H}_i^\delta + J(\delta\eta)\partial_\eta\widetilde{H}_i^\delta\times n = 0,\\ iJ(\delta\eta)\omega\widetilde{H}_i^\delta + (\delta\mathbf{C}_\Gamma + \delta^2\eta\mathbf{C}_\Gamma^M)\widehat{\mathbb{E}}_i^\delta - J(\delta\eta)\partial_\eta\widehat{\mathbb{E}}_i^\delta\times n = 0. \end{cases}$$

We rewrite the previous system as follows, by separating the "δ-independent" part, kept on the left-hand side, from the remaining terms, put in the right-hand side,

$$\begin{cases} \partial_\eta\widetilde{H}_i^\delta\times n + \dfrac{1}{\omega}\widehat{\mathbb{E}}_i^\delta = \displaystyle\sum_{\ell=1}^{4}\delta^\ell A_H^{(\ell)}(\widehat{\mathbb{E}}_i^\delta,\widetilde{H}_i^\delta) & \text{in } \Gamma\times\mathbb{R}^+,\\ -\partial_\eta\widehat{\mathbb{E}}_i^\delta\times n + i\omega\,\widetilde{H}_i^\delta = \displaystyle\sum_{\ell=1}^{2}\delta^\ell A_E^{(\ell)}(\widehat{\mathbb{E}}_i^\delta,\widetilde{H}_i^\delta), & \text{in } \Gamma\times\mathbb{R}^+. \end{cases} \tag{5.13}$$

The linear operators $\{A_H^{(\ell)}, \ell=1,2,3\}$ are given by:

$$\begin{cases} A_H^{(1)}(u,v) = \mathbf{C}_\Gamma\, v - 2h\eta\left(\partial_\eta v\times n + \dfrac{1}{\omega}u\right),\\ A_H^{(2)}(u,v) = -i\varepsilon_r\omega u + \eta\mathbf{C}_\Gamma^M\, v - g\eta^2\left(\partial_\eta v\times n + \dfrac{1}{\omega}u\right),\\ A_H^{(3)}(u,v) = -2\eta h i\varepsilon_r\omega u,\\ A_H^{(4)}(u,v) = -2\eta^2 g i\varepsilon_r\omega u, \end{cases}$$

and the linear operators $\{A_E^{(\ell)}, \ell=1,2\}$ are given by:

$$\begin{cases} A_E^{(1)}(u,v) = -\mathbf{C}_\Gamma\, u + 2h\eta\big(\partial_\eta u\times n - i\omega v\big),\\ A_E^{(2)}(u,v) = -\eta\mathbf{C}_\Gamma^M\, u + g\eta^2\big(\partial_\eta u\times n - i\omega v\big). \end{cases}$$

Substituting (5.4) and (5.12) into (5.13) then equating the same powers of δ leads to the following systems:

$$\begin{cases} \partial_\eta H_i^k\times n + \dfrac{1}{\omega}\widehat{\mathbb{E}}_i^k = \displaystyle\sum_{\ell=1}^{4} A_H^{(\ell)}(\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}), & \text{in } \Gamma\times\mathbb{R}^+,\\ -\partial_\eta\widehat{\mathbb{E}}_i^k\times n + i\omega\, H_i^k = \displaystyle\sum_{\ell=1}^{2} A_E^{(\ell)}(\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}), & \text{in } \Gamma\times\mathbb{R}^+, \end{cases} \tag{5.14}$$

for $k=0,1,2,\ldots$, with the convention $\widehat{\mathbb{E}}_i^\ell = H_i^\ell = 0$ for $\ell<0$.

Of course, these equations have to be complemented with the conditions (see (5.9) and (5.3))

$$\begin{cases} H_i^k(x_\Gamma, 0) \times n = H_e^k(x_\Gamma, 0), \quad \forall x_\Gamma \in \Gamma, \\ \displaystyle\int_0^{+\infty} |H_i^k(x_\Gamma, \eta)|^2 d\eta < +\infty \quad \text{and} \quad \int_0^{+\infty} |\widehat{\mathbb{E}}_i^k(x_\Gamma, \eta)|^2 d\eta < +\infty. \end{cases} \tag{5.15}$$

The reader can already notice how the roles of the variables η and x_Γ have been separated. The variable x_Γ appears as parameter for determining $(\widehat{\mathbb{E}}_i^k, H_i^k)$ from the previous $(\widehat{\mathbb{E}}_i^\ell, H_i^\ell)$'s since, for each x_Γ, one simply has to solve an ordinary differential system in the variable η. The solutions to this inductive system of equations can be expressed in a general way using the result of the following technical lemma. For that purpose it is useful to introduce

$$\mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3) := \left\{ u(x_\Gamma, \eta) = \sum_{j=1}^{k} a_j(x_\Gamma)\eta^j, a_j \in C^\infty(\Gamma; \mathbb{C}^3) \right\}.$$

Lemma 5.1. *Let* $(f, g) \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3)^2$ *and* $\varphi \in C^\infty(\Gamma; \mathbb{R}^3)$, *Then the problem, Find* $(u, v) \in C^\infty(\Gamma; C^\infty(\mathbb{R}^+))^2$ *such that,*

$$\begin{cases} \partial_\eta v \times n + \dfrac{1}{\omega} u = e^{-\sqrt{i}\,\eta} f(\eta, \cdot), \qquad \text{in } \Gamma \times \mathbb{R}^+, \\ -\partial_\eta u \times n + i\omega\, v = e^{-\sqrt{i}\,\eta} g(\eta, \cdot), \quad \text{in } \Gamma \times \mathbb{R}^+, \end{cases} \tag{5.16}$$

with the conditions:

$$\begin{cases} \forall\, x_\Gamma \in \Gamma, \quad u(x_\Gamma, 0) \times n = \varphi(x_\Gamma), \\ \forall\, x_\Gamma \in \Gamma, \quad \displaystyle\int_0^{+\infty} |u(x_\Gamma, \eta)|^2 d\eta < +\infty, \quad \int_0^{+\infty} |v(x_\Gamma, \eta)|^2 d\eta < +\infty, \end{cases} \tag{5.17}$$

has a unique solution, which is of the form

$$u(x_\Gamma, \eta) = e^{-\sqrt{i}\eta} p(x_\Gamma, \eta) \quad \text{and} \quad v(x_\Gamma, \eta) = e^{-\sqrt{i}\,\eta} q(x_\Gamma, \eta) \tag{5.18}$$

with $(p, q) \in \mathbf{P}_{k+1}(\Gamma, \mathbb{R}^+; \mathbb{C}^3)^2$ *and with the square root definition* $\sqrt{i} := \frac{\sqrt{2}}{2}(1 + i)$.

Proof. This is a simple exercise on ordinary differential equations. We refer to Ref. 12, Lemma 4.1 for the details of the proof. □

As an application of this lemma we obtain the following result.

Theorem 5.1. *The fields* $H_e^k \times n \in C^\infty(\Gamma; \mathbb{C}^3)$ *being given, there exists a unique sequence*

$$\{(\widehat{\mathbb{E}}_i^k, H_i^k) \in C^\infty(\Gamma; \mathbb{C}^3)^2, k = 0, 1, 2, \ldots\}$$

satisfying the sequence of problems (5.14)–(5.15). *Moreover,*

$$(e^{\sqrt{i}\,\eta}\widehat{\mathbb{E}}_i^k, e^{\sqrt{i}\,\eta}H_i^k) \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3)^2. \tag{5.19}$$

Proof. This theorem can be proved by using an induction on k and by carefully exploiting the polynomial structure with respect to η of the operators $A_E^{(\ell)}$ and $A_H^{(\ell)}$ (we refer again to Ref. 12, Lemma 4.2 for the technical details). □

5.4. *Explicit computation of the interior fields for* $k = 1, 2, 3$

This section is devoted to the presentation of the technical details related to the computation of the asymptotic terms up to the order $k = 3$. In the sequel, we shall systematically use the following formulas, deduced from (3.6),

$$(\mathbf{C}_\Gamma\, V) \cdot n = \mathrm{curl}_\Gamma V_T, \quad \mathbf{C}_\Gamma\, V \times n = \overrightarrow{\mathrm{curl}}_\Gamma (V \cdot n) \times n - \big(\mathcal{C} V \times n\big) \times n, \tag{5.20}$$

$$(\mathbf{C}_\Gamma^M\, V) \cdot n = \mathrm{curl}_\Gamma^M V_T, \quad \mathbf{C}_\Gamma^M\, V \times n = \overrightarrow{\mathrm{curl}}_\Gamma^M (V \cdot n) \times n - \big(\mathcal{G} V \times n\big) \times n. \tag{5.21}$$

Computation of $(\widehat{\mathbb{E}}_i^0 \equiv E_i^1, H_i^0)$. For $k = 0$, (5.14) gives

$$\begin{cases} \partial_\eta H_i^0 \times n + \dfrac{1}{\omega}\widehat{\mathbb{E}}_i^0 = 0, & \text{in } \Gamma \times \mathbb{R}^+, \\ -\partial_\eta \widehat{\mathbb{E}}_i^0 \times n + i\omega H_i^0 = 0 & \text{in } \Gamma \times \mathbb{R}^+, \end{cases} \tag{5.22}$$

whose unique L^2 solution satisfying $H_{i,T}^0(x_\Gamma, \eta) = H_{e,T}^0(x_\Gamma)$ is given by:

$$\begin{cases} \widehat{\mathbb{E}}_i^0(x_\Gamma, \eta) \equiv E_i^1(x_\Gamma, \eta) = \sqrt{i}\omega\big(H_e^0 \times n\big)(x_\Gamma)e^{-\sqrt{i}\,\eta}, \\ H_i^0(x_\Gamma, \eta) = H_{e,T}^0(x_\Gamma)e^{-\sqrt{i}\,\eta}, \end{cases} \tag{5.23}$$

from which we deduce the useful information for the construction of the GIBCs, namely:

$$E_i^1 \times n(x_\Gamma, 0) = -\sqrt{i}\omega H_{e,T}^0(x_\Gamma). \tag{5.24}$$

Computation of $(\widehat{\mathbb{E}}_i^1 \equiv E_i^2, H_i^1)$. For $k = 1$, (5.14) gives, using (5.22)

$$\begin{cases} \partial_\eta H_i^1 \times n + \dfrac{1}{\omega}\widehat{\mathbb{E}}_i^1 = \mathbf{C}_\Gamma H_i^0, & \text{in } \Gamma \times \mathbb{R}^+, \\ -\partial_\eta \widehat{\mathbb{E}}_i^1 \times n + i\omega H_i^1 = -\mathbf{C}_\Gamma \widehat{\mathbb{E}}_i^0, & \text{in } \Gamma \times \mathbb{R}^+. \end{cases} \tag{5.25}$$

We project (5.25) on n, use (5.20) and (5.23) for $\widehat{\mathbb{E}}_i^0$ and H_i^0, to obtain:

$$\begin{cases} \widehat{\mathbb{E}}_i^1 \cdot n = \omega \mathbf{C}_\Gamma H_i^0 \cdot n = \omega\big[\mathrm{curl}_\Gamma H_{e,T}^0\big](x_\Gamma)e^{-\sqrt{i}\,\eta}, & \text{in } \Gamma \times \mathbb{R}^+, \\ H_i^1 \cdot n = \dfrac{i}{\omega}\mathbf{C}_\Gamma\, \widehat{\mathbb{E}}_i^0 \cdot n = -\dfrac{1}{\sqrt{i}}\big[\mathrm{curl}_\Gamma\big(H_e^0 \times n\big)\big](x_\Gamma)e^{-\sqrt{i}\,\eta} & \text{in } \Gamma \times \mathbb{R}^+. \end{cases} \tag{5.26}$$

Next, we eliminate $\widehat{\mathbb{E}}_i^1$ in (5.25) and get the following equation in $H_{i,T}^1$

$$\big(\partial_{\eta\eta}^2 - i\big)H_{i,T}^1 = n \times \partial_\eta\big[\mathbf{C}_\Gamma H_i^0\big] - \frac{1}{\omega} n \times (\mathbf{C}_\Gamma \widehat{\mathbb{E}}_i^0 \times n).$$

We use again (5.20) and (5.23) to transform the right-hand side. Using the following identity, that can easily be deduced from the definitions (3.2) and (3.3)

$$\big(\mathcal{C}(V \times n)\big) \times n - \mathcal{C}V = -2\mathcal{H}V \quad \text{for all } V \in \mathbb{R}^3,$$

we finally get after some easy manipulations

$$(\partial^2_{\eta\eta} - i)H^1_{i,T} = 2\sqrt{i}\mathcal{H}H^0_{e,T}(x_\Gamma)e^{-\sqrt{i}\,\eta},$$

whose unique L^2 solution satisfying $H^0_i(x_\Gamma, \eta) = H^0_{e,T}(x_\Gamma)$ is given by:

$$H^1_{i,T}(x_\Gamma, \eta) = (H^1_{e,T}(x_\Gamma) - \eta\mathcal{H}H^0_{e,T}(x_\Gamma))e^{-\sqrt{i}\,\eta}. \tag{5.27}$$

Coming back to the first equation of (5.25), we get

$$\mathbb{E}^1_i \times n(x_\Gamma, \eta) = \omega(-\sqrt{i}\,H^1_{e,T}(x_\Gamma) + (\mathcal{C} - \mathcal{H})\,H^0_{e,T}(x_\Gamma) + \eta\sqrt{i}\mathcal{H}H^0_{e,T}(x_\Gamma))e^{-\sqrt{i}\,\eta}. \tag{5.28}$$

In particular,

$$E^2_i \times n(x_\Gamma, 0) = \omega(-\sqrt{i}H^1_{e,T}(x_\Gamma) + (\mathcal{C} - \mathcal{H})\,H^0_{e,T}(x_\Gamma)). \tag{5.29}$$

Remark 5.3. Notice that (5.26)–(5.28) prove Theorem 5.1 for $k = 1$.

Computation of $(\widehat{\mathbb{E}}^2_i \equiv E^3_i, H^2_i)$. The calculations are much harder and tedious than for the two previous cases. That is why we shall restrict ourselves to the main steps. Also, for the sake of simplicity, we shall often omit to mention the dependence of the various quantities we manipulate with respect to x_Γ.

For $k = 2$, (5.14) gives, using (5.25)

$$\begin{cases} \partial_\eta H^2_i \times n + \dfrac{1}{\omega}\widehat{\mathbb{E}}^2_i = r^2_H, & \text{in } \Gamma \times \mathbb{R}^+ \\ -\partial_\eta \widehat{\mathbb{E}}^2_i \times n + i\omega\, H^2_i = r^2_E, & \text{in } \Gamma \times \mathbb{R}^+, \end{cases} \tag{5.30}$$

where we have set

$$\begin{cases} r^2_H = \mathbf{C}_\Gamma H^1_i - i\varepsilon_r\omega\mathbb{E}^0_i + \eta\big(\mathbf{C}^M_\Gamma - 2h\mathbf{C}_\Gamma\big)H^0_i, \\ r^2_E = -\mathbf{C}_\Gamma\widehat{\mathbb{E}}^1_i - \eta\big(\mathbf{C}^M_\Gamma - 2h\mathbf{C}_\Gamma\big)\mathbb{E}^0_i. \end{cases}$$

We can go directly to the evaluation of $H^2_{i,T}$ which satisfies (apply $n \times \partial_\eta$ to the first equation of (5.30), divide the second equation by ω and add the two results)

$$\big(\partial^2_{\eta\eta} - i\big)\,H^2_{i,T} = n \times \partial_\eta\, r^2_H - \frac{1}{\omega}r^2_{E,T}. \tag{5.31}$$

The next step consists of expressing the right-hand side of (5.31) in terms of the previous $(\mathbb{E}^\ell_i, H^\ell_i)$'s. Using (5.20), (5.21) and the fact that $H^0_i \cdot n = 0$, we first compute that

$$n \times r^2_H = n \times \overrightarrow{\mathrm{curl}}_\Gamma\big(H^1_i \cdot n\big) - \mathcal{C}H^1_{i,T} - \eta\big(\mathcal{G} - 2h\mathcal{C}\big)H^0_{i,T} - i\varepsilon_r\omega\,\big(n \times \mathbb{E}^0_i\big),$$

Next, we use the expressions (5.23), (5.26) and (5.27) and the identity

$$n \times \overrightarrow{\mathrm{curl}}_\Gamma \big(\mathrm{curl}_\Gamma (V \times n)\big) = -\nabla_\Gamma \left(\mathrm{div}_\Gamma V\right)$$

to obtain

$$n \times r_H^2 = \left[\frac{1}{\sqrt{i}} \big(\nabla_\Gamma \,\mathrm{div}_\Gamma + \varepsilon_r \omega^2\big) H_{e,T}^0 - \mathcal{C} H_{e,T}^1 \right] e^{-\sqrt{i}\eta} + \eta \big(3h\mathcal{C} - \mathcal{G}\big) H_{e,T}^0 e^{-\sqrt{i}\eta}.$$

After differentiation, we get

$$\begin{aligned} n \times \partial_\eta r_H^2 = {} & [-(\nabla_\Gamma \,\mathrm{div}_\Gamma + \varepsilon_r \omega^2) H_{e,T}^0 + (3h\mathcal{C} - \mathcal{G}) H_{e,T}^0 + \sqrt{i}\mathcal{C} H_{e,T}^1] e^{-\sqrt{i}\eta} \\ & - \eta \sqrt{i} \big(3h\mathcal{C} - \mathcal{G}\big) H_{e,T}^0 e^{-\sqrt{i}\eta}. \end{aligned} \tag{5.32}$$

In the same way, using again (5.20), (5.21) and the fact that $\mathbb{E}_i^0 \cdot n = 0$, we calculate

$$r_{E,T}^2 = n \times (r_E^2 \times n) = -\mathrm{curl}_\Gamma \big(\mathbb{E}_i^1 \cdot n\big) + \big(\mathcal{C}\mathbb{E}_i^1\big) \times n + \eta \big[\big(\mathcal{G} - 2h\mathcal{C}\big)\mathbb{E}_i^0\big] \times n.$$

Next, we notice that

$$\mathcal{C} V = -\mathcal{C}\big[(V \times n) \times n\big] \quad \text{(and the same with } \mathcal{G})$$

and use the expressions (5.23), (5.26) and (5.28) respectively for $\mathbb{E}_i^0 \times n$, $\mathbb{E}_i^1 \cdot n$ and $\mathbb{E}_i^1 \times n$ to obtain

$$\begin{aligned} -\frac{1}{\omega} r_{E,T}^2 = {} & [\overrightarrow{\mathrm{curl}}_\Gamma (\mathrm{curl}_\Gamma \, H_{e,T}^0) + \eta \sqrt{i} ((\mathcal{G} - 3h\,\mathcal{C}) H_{e,T}^0 \times n) \times n] e^{-\sqrt{i}\eta} \\ & + [\mathcal{C}((\mathcal{H} - \mathcal{C}) H_{e,T}^0 \times n) \times n - \sqrt{i} (\mathcal{C}\big(H_{e,T}^1 \times n\big)) \times n] e^{-\sqrt{i}\eta}. \end{aligned}$$

This can be written in a simplified form, using the following identities that hold for all $V \in \mathbb{R}^3$ and that are easily deduced from (3.2) and (3.3)

$$\begin{cases} \{\mathcal{C}((\mathcal{H} - \mathcal{C})V) \times n)\} \times n = (3h\mathcal{C} - \mathcal{C}^2 - 2\mathcal{H}^2) V, \\ (\mathcal{C}(V \times n)) \times n = (\mathcal{C} - 2\mathcal{H}) V, \\ \{(3\mathcal{H}\mathcal{C} - \mathcal{G})(V \times n)\} \times n = (3\mathcal{H}\mathcal{C} + \mathcal{G} - 6\mathcal{H}^2) V. \end{cases}$$

We obtain

$$\begin{aligned} -\frac{1}{\omega} r_{E,T}^2 = {} & [\overrightarrow{\mathrm{curl}}_\Gamma (\mathrm{curl}_\Gamma H_{e,T}^0) + (3\mathcal{H}\mathcal{C} - \mathcal{C}^2 - 2\mathcal{H}^2) H_{e,T}^0] e^{-\sqrt{i}\eta} \\ & - \sqrt{i} (\mathcal{C} - 2\mathcal{H}) H_{e,T}^1 e^{-\sqrt{i}\eta} + \eta \sqrt{i} (3\mathcal{H}\mathcal{C} + \mathcal{G} - 6\mathcal{H}^2) H_{e,T}^0 e^{-\sqrt{i}\eta}. \end{aligned} \tag{5.33}$$

Substituting (5.32) and (5.33) in (5.31) leads to the following equation

$$\begin{aligned} (\partial_{\eta\eta}^2 - i) H_{i,T}^2 = e^{-\sqrt{i}\,\eta} \{ & 2\sqrt{i}\mathcal{H} H_{e,T}^1 + (\mathcal{C}^2 + 2\mathcal{H}^2 - \mathcal{G})\, H_{e,T}^0 \\ & - (\vec{\Delta}_\Gamma + \varepsilon_r \omega^2) H_{e,T}^0 - \eta \sqrt{i} (6\mathcal{H}^2 - 2\mathcal{G}) H_{e,T}^0 \}, \end{aligned}$$

where $\vec{\Delta}_\Gamma := \nabla_\Gamma \,\mathrm{div}_\Gamma - \overrightarrow{\mathrm{curl}}_\Gamma \,\mathrm{curl}_\Gamma$ is the vectorial Laplace Beltrami operator.

Since the L^2 solution to

$$(\partial_{\eta\eta}^2 - i) u = (a + b\,\eta)\, e^{-\sqrt{i}\,\eta} \quad \text{in } \mathbb{R}^+,$$

is given by

$$u(\eta) = \left(u(0) + \left(\frac{a}{2\sqrt{i}} - \frac{b}{4i} \right) \eta + \frac{b}{4\sqrt{i}} \eta^2 \right) e^{-\sqrt{i}\,\eta},$$

we deduce that H_T^2 is given by the expression

$$H_{i,T}^2(x_\Gamma, \eta) = e^{-\sqrt{i}\,\eta} \left\{ H_{e,T}^2 - \eta \mathcal{H} H_{e,T}^1 - \frac{\eta}{2\sqrt{i}} \big(\mathcal{C}^2 - \mathcal{H}^2 \big) H_{e,T}^2 \right. \\ \left. + \frac{\eta}{2\sqrt{i}} \big(\vec{\Delta}_\Gamma + \varepsilon_r \omega^2 \big) H_{e,T}^2 + \frac{\eta^2}{2} \big(3\mathcal{H}^2 - \mathcal{G} \big) H_{e,T}^1 \right\}.$$

Finally we go back to the first equation of obtaining, after lengthy calculations that we do not detail here,

$$\mathbb{E}_{i,T}^2 \times n = \omega e^{-\sqrt{i}\,\eta} \left\{ -\sqrt{i} H_{e,T}^2 + (\mathcal{C} - \mathcal{H}) H_{e,T}^1 - \frac{1}{2\sqrt{i}} \big(\mathcal{C}^2 - \mathcal{H}^2 \big) H_{e,T}^0 \right. \\ - \frac{1}{2\sqrt{i}} \big(\varepsilon_r \omega^2 + \nabla_\Gamma \operatorname{div}_\Gamma + \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma \big) H_{e,T}^0 \\ + \eta \left(\sqrt{i} \mathcal{H} H_{e,T}^1 + \frac{1}{2} \big(5\mathcal{H}^2 - 6\mathcal{H}\mathcal{C} + \mathcal{C}^2 - \vec{\Delta}_\Gamma - \varepsilon_r \omega^2 \big) H_{e,T}^0 \right) \\ \left. - \eta^2 \frac{\sqrt{i}}{2} \big(3\mathcal{H}^2 - \mathcal{G} \big) H_{e,T}^0 \right\}.$$

In particular, for $\eta = 0$,

$$E_{i,T}^3 \times n = \omega e^{-\sqrt{i}\,\eta} \left\{ -\sqrt{i} H_{e,T}^2 + (\mathcal{C} - \mathcal{H}) H_{e,T}^1 - \frac{1}{2\sqrt{i}} \big(\mathcal{C}^2 - \mathcal{H}^2 \big) H_{e,T}^0 \right. \\ \left. - \frac{1}{2\sqrt{i}} \big(\varepsilon_r \omega^2 + \nabla_\Gamma \operatorname{div}_\Gamma + \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma \big) H_{e,T}^0 \right\}. \tag{5.34}$$

5.5. *Construction of the GIBCs*

The GIBCS of order k is obtained by considering the truncated expansions in Ω_e

$$E_{e,k}^\delta := \sum_{\ell=0}^{k} \delta^\ell \, E_e^\ell \quad \text{and} \quad H_{e,k}^\delta := \sum_{\ell=0}^{k} \delta^\ell \, H_e^\ell$$

as (formal) approximations of order $k+1$ of E_e^δ and H_e^δ respectively (notice that k appears here as a subscript while it appears as an exponent in the notation of the solution of the approximate problem (4.1), (4.2)). Using the "second" interface condition, namely (5.7), one has

$$E_{e,k}^\delta|_\Gamma(x_\Gamma) \times n = \sum_{\ell=0}^{k} \delta^\ell \, E_i^\ell(x_\Gamma, 0) \times n \quad \text{for } x_\Gamma \in \Gamma. \tag{5.35}$$

Substituting into (5.35) the computed expressions of $E_i^\ell(x_\Gamma, 0)$ for $\ell = 1, 2$ and 3, respectively given by (5.24), (5.29) and (5.34) leads to an identity of the form

$$E_{e,k}^\delta \times n + \omega\, \mathcal{D}^{\delta,k}\big[(H_{e,k}^\delta)_T\big] = \delta^{k+1}\, \varphi_k^\delta \quad \text{on } \Gamma, \qquad \text{for } k = 0, 1, 2, \tag{5.36}$$

and

$$E_{e,3}^\delta \times n + \omega\, \mathcal{D}_0^{\delta,3}\big[(H_{e,3}^\delta)_T\big] = \delta^4\, \varphi_{3,0}^\delta \quad \text{on } \Gamma, \tag{5.37}$$

where $\mathcal{D}^{\delta,k}$, $k = 0, 1, 2$ are given by (4.3) and $\mathcal{D}_0^{\delta,3}$ is given by (4.5) and where $\varphi_k^\delta \in C^\infty(\Gamma)^3$, $k = 0, 1, 2$ are tangential vector fields given by

$$\begin{cases} \varphi_0^\delta = 0, \\ \varphi_1^\delta = \sqrt{i}\omega H_{e,T}^1, \\ \varphi_2^\delta = \sqrt{i}\omega H_{e,T}^2 + \omega(\mathcal{C} - \mathcal{H})\big(H_{e,T}^1 + \delta H_{e,T}^2\big), \end{cases} \tag{5.38}$$

and obviously satisfy the estimates (for δ small enough)

$$\|\varphi_k^\delta\|_{H_t^s(\Gamma)} \leq C_k(s), \quad k = 0, 1, 2 \tag{5.39}$$

where $C_k(s)$ is independent of δ, while $\varphi_{3,0}^\delta \in C^\infty(\Gamma)^3$ is given by

$$\begin{aligned} \varphi_{3,0}^\delta = {} & \sqrt{i}\omega H_{e,T}^3 + \omega(\mathcal{C} - \mathcal{H})\big(H_{e,T}^3 + \delta H_{e,T}^2\big) \\ & + \frac{1}{2\sqrt{i}}\big(\mathcal{C}^2 - \mathcal{H}^2\big)\big(H_{e,T}^1 + \delta H_{e,T}^2 + \delta^3 H_{e,T}^3\big) \\ & + \frac{1}{2\sqrt{i}}\big(\,\varepsilon_r\omega^2 + \nabla_\Gamma \operatorname{div}_\Gamma + \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma\big)\big(H_{e,T}^1 + \delta H_{e,T}^2 + \delta^3 H_{e,T}^3\big). \end{aligned} \tag{5.40}$$

The GIBC (4.2) is obtained for $k = 0, 1, 2$ by neglecting the right-hand side of (5.36). For $k = 3$, the same process leads to the condition (4.4) that is modified according to the process explained in Sec. 4.2. Notice that according to that construction, we have

$$E_{e,3}^\delta \times n + \omega\, \mathcal{D}^{\delta,k}\big[(H_{e,3}^\delta)_T\big] = \delta^4\, \varphi_3^\delta \text{on } \Gamma, \quad \text{where } \varphi_3^\delta = \varphi_{3,0}^\delta + \delta\mathcal{R}^{\delta,3}\big[(H_{e,3}^\delta)_T\big], \tag{5.41}$$

and using the property (4.13) of $\mathcal{R}^{\delta,3}$,

$$\|\varphi_3^\delta\|_{H_t^s(\Gamma)}, \leq C_3(s), \tag{5.42}$$

where $C_3(s)$ is independent of δ.

5.6. *Towards the theoretical justification of the GIBCs*

Our goal in the next two sections is to justify the GIBCs (4.2) by estimating

$$E_e^\delta - E_e^{\delta,k} \quad \text{and} \quad H_e^\delta - H_e^{\delta,k},$$

where $(E_e^{\delta,k}, H_e^{\delta,k})$ is the solution of the approximate problem ((4.1), (4.2)), whose well-posedness will be shown in Sec. 7.1 (see Theorem 7.1). It appears nontrivial to work directly with the differences $E_e^{\delta} - E_e^{\delta,k}$ and $H_e^{\delta} - H_e^{\delta,k}$, we shall use the truncated series $(E_{e,k}^{\delta}, H_{e,k}^{\delta})$ introduced in Sec. 5.5 as intermediate quantities. Therefore, the error analysis is split into two steps:

(1) Estimate the differences $E_e^{\delta} - E_{e,k}^{\delta}$ and $H_e^{\delta} - H_{e,k}^{\delta}$; this is done in Sec. 6, and more precisely in Lemma 6.1 and Corollary 6.1.
(2) Estimate the difference $E_{e,k}^{\delta} - E_e^{\delta,k}$ and $H_{e,k}^{\delta} - H_e^{\delta,k}$; this is done in Sec. 7.2 and more precisely in Theorem 7.7.

Remark 5.4. Notice that step 1 of the proof is completely independent of GIBC and will be valid for any integer k. Also, for $k = 0$, the second step is useless since $\widetilde{E}^{\delta,0} = E^{\delta,0}$.

6. Error Estimates for the Truncated Expansions

6.1. *Main results*

Let us introduce the fields $E_\chi^{\delta,k}(x), H_\chi^{\delta,k}(x)$: $\Omega \mapsto \mathbb{C}^3$ such that

$$E_\chi^{\delta,k}(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{k} \delta^\ell E_e^\ell(x) = E_{e,k}^{\delta}, & \text{for } x \in \Omega_e, \\ \chi(x) \displaystyle\sum_{\ell=0}^{k} \delta^\ell E_i^\ell(x_\Gamma, \nu/\delta) & \text{for } x \in \Omega_i, \end{cases}$$

$$H_\chi^{\delta,k}(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{k} \delta^\ell H_e^\ell(x) = H_{e,k}^{\delta}, & \text{for } x \in \Omega_e, \\ \chi(x) \displaystyle\sum_{\ell=0}^{k} \delta^\ell H_i^\ell(x_\Gamma, \nu/\delta) & \text{for } x \in \Omega_i, \end{cases}$$

where the local coordinates x_Γ and ν are defined as in Sec. 3 and the cutoff function χ is defined as in Sec. 5.1. These fields are good candidates to be good approximations of the exact fields (E^δ, H^δ). The main result of this section is:

Lemma 6.1. *For any k, there exists a constant C_k independent of δ such that*

$$\begin{cases} \text{(i)} \quad \|E^\delta - E_\chi^{\delta,k}\|_{H(\mathrm{curl},\Omega)} \le C_k \delta^{k+\frac{1}{2}}, \\ \text{(ii)} \quad \|E^\delta - E_\chi^{\delta,k}\|_{L^2(\Omega_i)} \le C_k \delta^{k+\frac{3}{2}}, \\ \text{(iii)} \quad \|E^\delta \times n - E_\chi^{\delta,k} \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \le C_k \delta^{k+1}. \end{cases} \tag{6.1}$$

The proof of Lemma 6.1, postponed to Sec. 6.3, rely on a fundamental *a priori* estimates that we shall state and prove in Sec. 6.2. We first give a straightforward corollary of Lemma 6.1.

Corollary 6.1. *For any k, there exists a constant $\widetilde{C}_k$ independent of δ such that*:

$$\begin{cases} \|E_e^\delta - E_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le \widetilde{C}_k \delta^{k+1}, \\ \|H_e^\delta - H_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le \widetilde{C}_k \delta^{k+1}. \end{cases}$$

Proof. Simply write

$$E_e^\delta - E_{e,k}^\delta = E_e^\delta - E_{e,k+1}^\delta + \delta^{k+1} E_e^{k+1}$$

which yields, since $E_{e,k}^\delta = E_\chi^{\delta,k+1}$ in Ω_e,

$$\|E_e^\delta - E_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le \|E_e^\delta - E_\chi^{\delta,k+1}\|_{H(\mathrm{curl},\Omega_e)} + \delta^{k+1}\|E_e^{k+1}\|_{H(\mathrm{curl},\Omega_e)}.$$

Using the estimate (6.1(i)) of Lemma 6.1, we get

$$\|E_e^\delta - E_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le C_k \delta^{k+\frac{3}{2}} + \delta^{k+1}\|E_e^{k+1}\|_{H(\mathrm{curl},\Omega_e)} \le \widetilde{C}_k \delta^{k+1}.$$

The estimates for $H_e^\delta - H_{e,k}^\delta$ is an immediate consequence of

$$\begin{cases} -i\omega(H_e^\delta - H_{e,k}^\delta) + \mathrm{curl}(E_e^\delta - E_{e,k}^\delta) = 0 & \text{in } \Omega_e, \\ i\omega(E_e^\delta - E_{e,k}^\delta) + \mathrm{curl}(H_e^\delta - H_{e,k}^\delta) = 0 & \text{in } \Omega_e. \end{cases}$$

□

6.2. *A fundamental a priori estimate*

The proof of Lemma 6.1 relies on the following fundamental technical lemma.

Lemma 6.2. *Assume that $\mathbf{E}^\delta \in H(\mathrm{curl},\Omega)$ satisfies*

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}^\delta - \omega^2 \mathbf{E}^\delta = 0, & \text{in } \Omega_e, \\ i\omega \mathbf{E}_T^\delta - \mathrm{curl}\,\mathbf{E}^\delta \times n = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

together with the following inequality

$$\begin{aligned} &\left| \int_\Omega (|\mathrm{curl}\,\mathbf{E}^\delta|^2 - \omega^2 |\mathbf{E}^\delta|^2) dx + i\omega \left(\int_{\partial\Omega} |\mathbf{E}^\delta \times n|^2 ds + \frac{1}{\delta^2} \int_{\Omega_i} |\mathbf{E}^\delta|^2 dx \right) \right| \\ &\quad \le A(\delta^{s+\frac{1}{2}} \|\mathbf{E}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} + \delta^s \|\mathbf{E}^\delta\|_{L^2(\Omega_i)}), \end{aligned} \tag{6.3}$$

for some non-negative constants A and s independent of δ. Then there exists a constant C independent of δ such that

$$\|\mathbf{E}^\delta\|_{H(\mathrm{curl},\Omega)} \le C\,\delta^{s+1}, \quad \|\mathbf{E}^\delta\|_{L^2(\Omega_i)} \le C\,\delta^{s+2}, \quad \|\mathbf{E}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \le C\,\delta^{s+\frac{3}{2}}, \tag{6.4}$$

for sufficiently small δ.

Proof. For convenience, we shall denote by C a positive constant whose value may change from one line to another but remains independent of δ. We divide the proof into two steps.

Step 1. We first prove by contradiction that $\|\mathbf{E}^\delta\|_{L^2(\Omega)} \le C\,\delta^{s+1}$. This is the main step of the proof which will use two important technical lemmas that are proven in the Appendix.

Assume that the positive quantity

$$\lambda^\delta := \delta^{-(s+1)} \|\mathbf{E}^\delta\|_{L^2(\Omega)}$$

is unbounded as $\delta \to 0$. After extraction of a subsequence, still denoted $\mathbf{E}^\delta$ with $\delta \to 0$, we can assume that $\lambda^\delta \to +\infty$. Let $\tilde{\mathbf{E}}^\delta = \mathbf{E}^\delta / \|\mathbf{E}^\delta\|_{L^2(\Omega)}$ (so that $\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)} = 1$).

Our goal is to show that, up to the extraction of another subsequence, $\tilde{\mathbf{E}}^\delta$ converges strongly in $L^2(\Omega_e)$ and to obtain a contradiction by looking at the limit field $\tilde{\mathbf{E}}$.

To show this, we wish to apply to $\tilde{\mathbf{E}}^\delta$ the compactness result of Appendix A.5 with $\mathcal{O} = \Omega_e$. Since $\operatorname{div} \mathbf{E}^\delta = 0$ and $\tilde{\mathbf{E}}^\delta$ is bounded in $L^2(\Omega_e)$, we only need to show that:

$$\begin{aligned} &\text{(i)} \quad \operatorname{curl} \tilde{\mathbf{E}}^\delta \text{ is bounded in } L^2(\Omega_e), \\ &\text{(ii)} \quad \tilde{\mathbf{E}}^\delta \times n|_{\partial\Omega} \text{ converges in } H^{-\frac{1}{2}}(\partial\Omega), \\ &\text{(iii)} \quad \tilde{\mathbf{E}}^\delta \times n|_{\Gamma} \text{ converges in } H^{-\frac{1}{2}}(\Gamma). \end{aligned} \tag{6.5}$$

We first notice that after division by $\|\mathbf{E}^\delta\|_{L^2(\Omega)}$, the inequality (6.3) yields

$$\begin{aligned} &\left| \int_\Omega (|\operatorname{curl} \tilde{\mathbf{E}}^\delta|^2 - \omega^2 |\tilde{\mathbf{E}}^\delta|^2) dx + i\omega \left(\int_{\partial\Omega} |\tilde{\mathbf{E}}^\delta \times n|^2 ds + \frac{1}{\delta^2} \int_{\Omega_i} |\tilde{\mathbf{E}}^\delta|^2 dx \right) \right| \\ &\quad \le \frac{A}{\lambda^\delta} (\delta^{-1} \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} + \delta^{-\frac{1}{2}} \|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)}). \end{aligned} \tag{6.6}$$

We shall now establish estimates on the two terms on the right-hand side of (6.6) in terms of $\|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)}$ (namely inequalities (6.10) and (6.12)).

Considering the imaginary part of the left-hand side of (6.6), we observe that since $1/\lambda^\delta$ is bounded,

$$\|\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega_i)} \le C\delta^{\frac{3}{2}} \|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} + C\delta \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}.$$

Next, we use the trace inequality (A.1) of Appendix A.1 with $\mathcal{O} = \Omega_i$ to get

$$\|\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega_i)} \le C\delta^{\frac{3}{2}} \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} (\|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}) + C\delta \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)},$$

which yields, after division by $\|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}$,

$$\|\tilde{\mathbf{E}}^\delta\|^{\frac{3}{2}}_{L^2(\Omega_i)} \le C_1 \delta \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + C_2 \delta^{\frac{3}{2}} \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}. \tag{6.7}$$

Let K be a positive constant to be fixed later. Using Young's inequality $ab \le 2/3 a^{3/2} + 1/3 b^3$ with $a = K^{-1}\delta$ and $b = K \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}$, we get

$$\delta \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} \le \frac{2}{3} K^{-\frac{3}{2}} \delta^{\frac{3}{2}} + \frac{K^3}{3} \|\tilde{\mathbf{E}}^\delta\|^{\frac{3}{2}}_{L^2(\Omega_i)}. \tag{6.8}$$

Choosing $C_1 K^3 = 3/2$ and substituting (6.7) into (6.8), one deduces

$$\|\tilde{\mathbf{E}}^\delta\|^{\frac{3}{2}}_{L^2(\Omega_i)} \le C\delta^{\frac{3}{2}} (1 + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}), \tag{6.9}$$

which yields

$$\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} \le C(1 + \|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|^{\frac{1}{3}}_{L^2(\Omega_i)}). \tag{6.10}$$

Now considering the real part of the left-hand side of (6.6) and using the fact that $\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)} = 1$, we observe that

$$\|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega)} \le C(1 + \delta^{-\frac{1}{2}}\|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} + \delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}). \tag{6.11}$$

On the other hand, after multiplication by $\delta^{-\frac{1}{2}}$, the trace inequality (A.1) applied to $\tilde{\mathbf{E}}^\delta$ is equivalent to

$$\delta^{-\frac{1}{2}}\|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \le C\delta^{\frac{1}{2}}\{\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\} + C\{\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\}^{\frac{1}{2}}\|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}.$$

After applying the Cauchy–Schwarz inequality to the second term of the right-hand side of the above inequality, we easily get, since δ is bounded

$$\begin{aligned}\delta^{-\frac{1}{2}}\|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} &\le C\{\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} + \|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\},\\ &\le C\{1 + \|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|^{\frac{1}{3}}_{L^2(\Omega_i)} + \|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\},\end{aligned} \tag{6.12}$$

where we used (6.10) for the second inequality. Substituting (6.12) into (6.11) gives

$$\|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega)} \le C(1 + \|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|^{\frac{1}{3}}_{L^2(\Omega_i)} + \|\operatorname{curl}\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}).$$

This proves (6.5)(i). We also deduce, thanks to (6.10) and (6.12), that

$$\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} \quad \text{and} \quad \delta^{-\frac{1}{2}}\|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \text{are bounded,} \tag{6.13}$$

which proves in particular (6.5)(ii) ($\tilde{\mathbf{E}}^\delta \times n$ converges to 0 in $H^{-\frac{1}{2}}(\Gamma)$). This also means that the right-hand side of (6.6) remains bounded. Thus, going back to (6.6) shows that $\|\tilde{\mathbf{E}}^\delta \times n\|_{L^2(\partial\Omega)}$ is bounded, which proves (6.5)(iii) by the compactness of the $L^2(\partial\Omega)$ embedding into $H^{-\frac{1}{2}}(\partial\Omega)$.

Now we shall conclude the proof of Step 1. From (6.5)(i) one deduces that $\tilde{\mathbf{E}}^\delta$ is a bounded sequence in $H(\operatorname{curl},\Omega)$, therefore, up to extracted subsequence, we can assume that $\tilde{\mathbf{E}}^\delta$ weakly converges in $H(\operatorname{curl},\Omega)$ to some $\tilde{\mathbf{E}}$. Considering the restriction to Ω_e, thanks to (6.5) we can apply the compactness result of Lemma A.5 and deduce that an extracted subsequence of $\tilde{\mathbf{E}}^\delta$, denoted again by $\tilde{\mathbf{E}}^\delta$ for simplicity, strongly converges to $\tilde{\mathbf{E}}$ in $L^2(\Omega_e)$. On the other hand, we observe from (6.13) that $\tilde{\mathbf{E}}^\delta$ strongly converges to 0 in $L^2(\Omega_i)$, hence $\tilde{\mathbf{E}} = 0$ in Ω_i, which implies in particular

$$\tilde{\mathbf{E}} \times n = 0 \quad \text{on } \Gamma. \tag{6.14}$$

Passing to the weak limit in equations (6.2) one easily verify that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\tilde{\mathbf{E}} - \omega^2\tilde{\mathbf{E}} = 0, & \text{in } \Omega_e,\\ i\omega\tilde{\mathbf{E}}_T - \operatorname{curl}\tilde{\mathbf{E}} \times n = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.15}$$

The uniqueness of solutions to (6.14)–(6.15) in $H(\operatorname{curl},\Omega_e)$ implies that also $\tilde{\mathbf{E}} = 0$ in Ω_e. We therefore obtain that $\tilde{\mathbf{E}}^\delta$ converges to 0 in $L^2(\Omega)$ which is a contradiction with $\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)} = 1$. Consequently λ^δ is bounded, that is to say

$$\|\mathbf{E}^\delta\|_{L^2(\Omega)} \le C\delta^{s+1}. \tag{6.16}$$

Step 2. We shall now proceed with the proof of estimates (6.1). Considering the imaginary part of the left-hand side of estimate (6.3) and applying Lemma A.1 (with $\mathcal{O} = \Omega_i$) yields

$$\|\mathbf{E}^\delta\|^2_{L^2(\Omega_i)} \le C(\delta^{s+\frac{5}{2}}\|\mathrm{curl}\,\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\|\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \delta^{s+2}\|\mathbf{E}^\delta\|_{L^2(\Omega_i)}).$$

Using two times the Young inequality $ab \le 1/2(a^2+b^2)$, the first time with

$$a = \delta^{\frac{1}{2}}\|\mathrm{curl}\,\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} \quad \text{and} \quad b = \|\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)},$$

and the second time with

$$a = \|\mathbf{E}^\delta\|_{L^2(\Omega_i)} \quad \text{and} \quad b = \delta^{s+2},$$

leads to (we also use $\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega_i)} \le \|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)}$)

$$\|\mathbf{E}^\delta\|^2_{L^2(\Omega_i)} \le C(\delta^{2s+4} + \delta^{s+2}(\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)})). \tag{6.17}$$

On the other hand, considering this the real part of the left-hand side of estimate (6.3) and using (6.16), we get

$$\|\mathrm{curl}\,\mathbf{E}^\delta\|^2_{L^2(\Omega)} \le C(\delta^{2s+2} + \delta^{s+\frac{1}{2}}\|\mathrm{curl}\,\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\|\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \delta^{s}\|\mathbf{E}^\delta\|_{L^2(\Omega_i)}),$$

which gives, using Young's inequality once again,

$$\|\mathrm{curl}\,\mathbf{E}^\delta\|^2_{L^2(\Omega)} \le C(\delta^{2s+2} + \delta^{s}(\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)})). \tag{6.18}$$

Combining (6.17) and (6.18) leads to

$$\|\mathbf{E}^\delta\|^2_{L^2(\Omega_i)} + \delta^2\|\mathrm{curl}\,\mathbf{E}^\delta\|^2_{L^2(\Omega)} \le C(\delta^{2s+4} + \delta^{s+2}(\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)})),$$

which yields

$$\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)} \le C\delta^{s+2}$$

and in particular the second inequality of (6.4). The third inequality of (6.4) is a consequence of the first two and the application of Appendix A.1 in Ω_i. □

Remark 6.1. Notice that since we simply used in the first step of the proof the fact that $1/\lambda^\delta$ is bounded, we have proved in fact that

$$\lim_{\delta\to 0} \delta^{-(s+1)}\|E^\delta\|_{L^2(\Omega)} = 0.$$

6.3. *The proof of Lemma* 6.1

Let us introduce, for each integer k, the *error fields*

$$\mathcal{E}^{\delta,k} = E^\delta_e - E^{\delta,k}_\chi, \quad \mathcal{H}^{\delta,k} = H^\delta_e - H^{\delta,k}_\chi. \tag{6.19}$$

The idea of the proof is to show that $\mathcal{E}^{\delta,k}$ satisfies an *a priori* estimate of the type (6.3) and then to use the stability Lemma 6.2. To prove such an estimate, we shall use the equations satisfied by $(\mathcal{E}^{\delta,k}, \mathcal{H}^{\delta,k})$, respectively in Ω_i and Ω_e.

The equations in Ω_e. It is straightforward to check that in the exterior domain Ω_e, the errors

$$(\mathcal{E}_e^{\delta,k}, \mathcal{H}_e^{\delta,k}) := (\mathcal{E}^{\delta,k}|_{\Omega_e}, \mathcal{H}^{\delta,k}|_{\Omega_e})$$

satisfies the homogeneous equation:

$$\begin{cases} \text{(i)} \quad \operatorname{curl} \mathcal{H}_e^{\delta,k} + i\omega \mathcal{E}_e^{\delta,k} = 0, \quad \text{in } \Omega_e, \\ \text{(ii)} \quad \operatorname{curl} \mathcal{E}_e^{\delta,k} - i\omega \mathcal{H}_e^{\delta,k} = 0, \quad \text{in } \Omega_e, \end{cases} \tag{6.20}$$

and

$$(\mathcal{E}_e^{\delta,k})_T - \mathcal{H}_e^{\delta,k} \times n = 0, \quad \text{on } \partial\Omega. \tag{6.21}$$

Eliminating $\mathcal{H}_e^{\delta,k}$ in (6.20), we get

$$\begin{cases} \operatorname{curl}(\operatorname{curl} \mathcal{E}_e^{\delta,k}) - \omega^2 \mathcal{E}_e^{\delta,k} = 0, \quad \text{in } \Omega_e, \\ \operatorname{curl} \mathcal{E}_e^{\delta,k} \times n + i\omega \big(\mathcal{E}_e^{\delta,k}\big)_T = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{6.22}$$

The equations in Ω_i. Now consider the restrictions to Ω_i and set

$$(\mathcal{E}_i^{\delta,k}, \mathcal{H}_i^{\delta,k}) := (\mathcal{E}^{\delta,k}|_{\Omega_i}, \mathcal{H}^{\delta,k}|_{\Omega_i}).$$

It is also useful to introduce the fields

$$E_{i,k}^{\delta}(x_\Gamma, \nu) := \sum_{\ell=0}^{k} \delta^\ell E_i^\ell \left(x_\Gamma, \frac{\nu}{\delta}\right), \quad H_{i,k}^{\delta}(x_\Gamma, \eta) := \sum_{\ell=0}^{k} \delta^\ell H_i^\ell \left(x_\Gamma, \frac{\nu}{\delta}\right),$$

so that using the local coordinates, we can write

$$E_\chi^{\delta,k}(x) = \chi E_{i,k}^{\delta}(x_\Gamma, \eta), \quad H_\chi^{\delta,k}(x) = \chi H_{i,k}^{\delta}(x_\Gamma, \eta) \quad \text{in } \Omega_i.$$

Our goal is to show that $(E_\chi^{\delta,k}, H_\chi^{\delta,k})$ satisfy the "interior equations" except that two *small* source terms appear at the right-hand side, respectively due to the cutoff function χ and the truncation of the series at order k. We first compute that

$$\begin{cases} \operatorname{curl} H_\chi^{\delta,k} + i\omega E_\chi^{\delta,k} - \dfrac{1}{\omega\,\delta^2} E_\chi^{\delta,k} = \chi\left(i\omega E_{i,k}^{\delta} + \operatorname{curl} H_{i,k}^{\delta} - \dfrac{1}{\omega\,\delta^2} E_{i,k}^{\delta}\right) + \nabla\chi \times H_{i,k}^{\delta}, \\ \operatorname{curl} E_\chi^{\delta,k} - i\omega H_\chi^{\delta,k} = \chi(\operatorname{curl} E_{i,k}^{\delta} - i\omega H_{i,k}^{\delta}) + \nabla\chi \times E_{i,k}^{\delta}. \end{cases} \tag{6.23}$$

Thanks to the exponential decay of $E_{i,k}^{\delta}(x_\Gamma, \eta)$ and $H_{i,k}^{\delta}(x_\Gamma, \eta)$ with respect to η (cf. Theorem 5.1), the terms in factor of $\nabla\chi$ are exponentially small in δ.

It remains to compute the terms in factor of χ. These calculations are tedious, but the idea is simple and consists — in some sense — to do the same calculations as in Sec. 5.3 but in the reverse sense. According to (5.11) we define

$$\mathbb{E}_{i,k}^{\delta} := \frac{E_{i,k}^{\delta}}{\delta} = \sum_{p=0}^{k-1} \delta^p \mathbb{E}_i^p.$$

With the notation of Sec. 5.3, we have

$$\operatorname{curl} E_{i,k}^{\delta} - i\omega H_{i,k}^{\delta} = r_i^{\delta,k}(x_\Gamma, \nu/\delta), \tag{6.24}$$

where the function $r_i^{\delta,k}(x,\eta)$ is given by

$$r_i^{\delta,k} = \partial_\eta \mathbb{E}_{i,k}^{\delta} \times n - i\omega\, H_{i,k}^{\delta} + \sum_{\ell=1}^{2} \delta^\ell A_E^{(\ell)}(\mathbb{E}_{i,k}^{\delta}, H_{i,k}^{\delta}).$$

Replacing $\mathbb{E}_{i,k}^{\delta}$ and $H_{i,k}^{\delta}$ by their polynomial expansion in δ, we get

$$r_i^{\delta,k} = \sum_{p=0}^{k-1} \delta^p \big(\partial_\eta \mathbb{E}_i^p \times n - i\omega\, H_i^p\big) - i\omega\,\delta^k H_i^k + \sum_{\ell=1}^{2} \delta^\ell \sum_{p=0}^{k-1} \delta^p A_E^{(\ell)}(\mathbb{E}_i^p, H_i^p).$$

Using Eq. (5.14) satisfied by the $\mathbb{E}_i^p$'s and H_i^p's, we get

$$r_i^{\delta,k} = -i\omega\,\delta^k H_i^k + \sum_{\ell=1}^{2}\sum_{p=0}^{k-1} \delta^{p+\ell} A_E^{(\ell)}(\mathbb{E}_i^p, H_i^p) - \sum_{\ell=1}^{2}\sum_{p=0}^{k-1} \delta^p A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}).$$

Applying the change of index $p + l \to p$ in the first sum, we get

$$r_i^{\delta,k} = -i\omega\,\delta^k H_i^k + \sum_{\ell=1}^{2}\sum_{p=0}^{k-1+\ell} \delta^p A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}) - \sum_{\ell=1}^{2}\sum_{p=0}^{k-1} \delta^p A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}),$$

that is to say

$$r_i^{\delta,k} = -i\omega\,\delta^k H_i^k + \sum_{\ell=1}^{2}\sum_{p=k}^{k-1+\ell} \delta^p A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}).$$

Paying attention to the above expression and using the form of the functions $\mathbb{E}_i^p$ and H_i^p (cf. Theorem 5.1), we see that

$$\operatorname{curl} \mathbb{E}_{i,k}^{\delta} - i\omega H_{i,k}^{\delta} = \delta^k \big(g_{k,0}^{\delta} + \delta g_{k,1}^{\delta}\big) \quad \text{in} \;\; \operatorname{supp} \chi,$$

where the functions g_0^δ and g_1^δ are of the form

$$g_{k,q}^{\delta}(x) = p_{k,q}\left(x_\Gamma, \frac{\nu}{\delta}\right) e^{-\sqrt{i}\,\frac{\nu}{\delta}}, \quad p_{k,q} \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3), \quad q = 0, 1. \tag{6.25}$$

From (6.25), we easily deduce that

$$\|\chi g_{k,q}^{\delta}\|_{L^2(\Omega_i)} \le C_{k,q}\delta^{\frac{1}{2}}, \quad \|\chi \operatorname{curl} g_{k,q}^{\delta}\|_{L^2(\Omega_i)} \le C'_{k,q}\delta^{\frac{1}{2}}, \quad q = 0, 1. \tag{6.26}$$

In the same way, using again local coordinates, we have

$$i\omega E_{i,k}^{\delta} + \operatorname{curl} H_{i,k}^{\delta} - \frac{1}{\delta^2} E_{i,k}^{\delta} = \frac{1}{\delta} s_i^{\delta,k}(x, \nu/\delta),$$

with

$$s_i^{\delta,k} = \partial_\eta H_{i,k}^{\delta} \times n + \frac{1}{\omega}\mathbb{E}_{i,k}^{\delta} - \sum_{\ell=1}^{4} \delta^\ell A_H^{(\ell)}(\mathbb{E}_{i,k}^{\delta}, H_{i,k}^{\delta}).$$

Replacing $\mathbb{E}^\delta_{i,k}$ and $H^\delta_{i,k}$ by their polynomial expansion in δ we get

$$s_i^{\delta,k} = \sum_{p=0}^{k-1} \delta^p \left(\partial_\eta H_i^p \times n + \frac{1}{\omega}\,\mathbb{E}_i^p \right) - \sum_{\ell=1}^{4} \delta^\ell \sum_{p=0}^{k-1} \delta^p A_H^{(\ell)}(\mathbb{E}_i^p, H_i^p)$$

$$+ \delta^k \partial_\eta H_i^p \times n - \sum_{\ell=1}^{4} \delta^{\ell+k} A_H^{(\ell)}(0, H_i^k).$$

Using equations (5.14) satisfied by the $\mathbb{E}_i^p$'s and H_i^p's, we get

$$s_i^{\delta,k} = \sum_{\ell=1}^{4}\sum_{p=0}^{k-1} \delta^p A_H^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}) - \sum_{\ell=1}^{4}\sum_{p=0}^{k-1} \delta^{p+\ell} A_H^{(\ell)}(\mathbb{E}_i^p, H_i^p)$$

$$+ \delta^k \partial_\eta H_i^p \times n - \sum_{\ell=1}^{4} \delta^{\ell+k} A_H^{(\ell)}(0, H_i^k),$$

or equivalently

$$s_i^{\delta,k} = \sum_{p=k}^{k+\ell-1} \delta^p \sum_{\ell=1}^{4} A_H^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell})$$

$$+ \delta^k \partial_\eta H_i^p \times n - \sum_{\ell=1}^{4} \delta^{\ell+k} A_H^{(\ell)}(0, H_i^k).$$

This time, we see that we can write

$$i\omega E^\delta_{i,k} + \operatorname{curl} H^\delta_{i,k} - \frac{1}{\delta^2} E^\delta_{i,k} = \frac{1}{\delta}\sum_{q=0}^{3} \delta^q h^\delta_{k,q} \quad \text{in } \operatorname{supp} \chi,$$

where the expression of $h^\delta_{k,q}$ is similar to the $g_{k,q}$'s (see formula (6.25)) and implies in particular that

$$\|\chi h^\delta_{k,q}\|_{L^2(\Omega_i)} \le C_{k,q}\delta^{\frac{1}{2}}, \quad q = 0, 1, 2, 3. \tag{6.27}$$

In summary, taking the difference between (6.23) and (2.4) we have shown that

$$\begin{cases} \operatorname{curl} \mathcal{H}_i^{\delta,k} + i\omega \mathcal{E}_i^{\delta,k} - \dfrac{1}{\omega\delta^2}\mathcal{E}_i^{\delta,k} = \delta^{k-1}\chi \left(\displaystyle\sum_{q=0}^{3} \delta^q h^\delta_{k,q} \right) + \nabla\chi \times H^\delta_{i,k}, \\ \operatorname{curl} \mathcal{E}_i^{\delta,k} - i\omega \mathcal{H}_i^{\delta,k} = \delta^k \chi \left(\displaystyle\sum_{q=0}^{1} \delta^q g^\delta_{k,q} \right) + \nabla\chi \times E^\delta_{i,k}, \end{cases}$$

where eliminating $\mathcal{H}_i^{\delta,k}$ we get

$$\operatorname{curl}\operatorname{curl} \mathcal{E}_i^{\delta,k} - \omega^2 \mathcal{E}_i^{\delta,k} + \frac{i}{\delta^2}\mathcal{E}_i^{\delta,k} = f_k^\delta \tag{6.28}$$

with

$$f_k^\delta := \delta^k \chi \left(\sum_{q=0}^{1} \delta^q \operatorname{curl} g_{k,q}^\delta \right) + \nabla\chi \times \left(\sum_{q=0}^{1} \delta^q g_{k,q}^\delta \right) + \nabla\chi \times \operatorname{curl}\left(\nabla\chi \times E_{i,k}^\delta\right)$$
$$- i\omega\delta^{k-1}\chi\Big(\sum_{q=0}^{3} \delta^q h_{k,q}^\delta \Big) - i\omega\nabla\chi \times H_{i,k}^\delta.$$

Taking into account the form of the functions $g_{k,q}^\delta$ and the exponential decay of the the fields E_i^p and H_i^p (Theorem 5.1), and since the support of $\nabla\chi$ is separated from Γ, there exists a constant $\tau > 0$ such that:

$$\| \nabla\chi \times \operatorname{curl}\left(\nabla\chi \times E_{i,k}^\delta\right) \|_{L^2(\Omega_e)} \le C_1(k) e^{-\tau\,\delta},$$
$$\left\| \nabla\chi \times \left(\sum_{q=0}^{1} \delta^q g_{k,q}^\delta \right) \right\|_{L^2(\Omega_e)} \le C_2(k) e^{-\tau\,\delta},$$
$$\| \nabla\chi \times H_{i,k}^\delta \|_{L^2(\Omega_e)} \le C_3(k) e^{-\tau\,\delta}.$$

Combining these inequalities with estimates (6.26) and (6.27), we see that:

$$\|f_k^\delta\|_{L^2(\Omega_e)} \le C_k \delta^{k-\frac{1}{2}}. \tag{6.29}$$

Error estimates. We can now proceed with the final step of the proof. First, we multiply Eq. (6.22) by $\overline{\mathcal{E}_e^{\delta,k}}$ and integrate over Ω_e. Using the Stokes formula and the boundary condition in (6.22), we get

$$\int_{\Omega_e} |\operatorname{curl} \mathcal{E}_e^{\delta,k}|^2 dx - \omega^2 \int_{\Omega_e} |\mathcal{E}_e^{\delta,k}|^2 dx - i\omega \int_{\partial\Omega} |\mathcal{E}_e^{\delta,k} \times n|^2 + \langle \operatorname{curl} \mathcal{E}_e^{\delta,k} \times n, \big(\overline{\mathcal{E}_e^{\delta,k}}\big)_T \rangle_\Gamma = 0.$$

Next, we multiply the equation (6.28) by $\mathcal{E}_i^{\delta,k}$ and integrate over Ω_i. We get

$$\int_{\Omega_i} |\operatorname{curl} \mathcal{E}_i^{\delta,k}|^2 dx - \omega^2 \int_{\Omega_i} |\mathcal{E}_i^{\delta,k}|^2 dx - \frac{1}{\delta^2} \int_{\Omega_i} |\mathcal{E}_i^{\delta,k}|^2 - \langle \big(\operatorname{curl} \mathcal{E}_e^{\delta,k} \times n\big) \cdot \big(\overline{\mathcal{E}_i^{\delta,k}}\big)_T \rangle_\Gamma$$
$$= \int_{\Omega_i} f_k^\delta \cdot \overline{\mathcal{E}_i^{\delta,k}} dx.$$

Adding the last two equalities we get, since $\mathcal{E}^{\delta,k} \in H(\operatorname{curl};\Omega)$,

$$\int_{\Omega} |\operatorname{curl} \mathcal{E}^{\delta,k}|^2 - \omega^2 \int_{\Omega} |\mathcal{E}^{\delta,k}|^2 - i\omega \left(\int_{\partial\Omega} |\mathcal{E}^{\delta,k} \times n|^2 + \frac{1}{\delta^2} \int_{\Omega_i} |\mathcal{E}^{\delta,k}|^2 \right)$$
$$= \langle \operatorname{curl} \mathcal{E}_e^{\delta,k} \times n - \operatorname{curl} \mathcal{E}_i^{\delta,k} \times n, \big(\overline{\mathcal{E}^{\delta,k}}\big)_T \rangle_\Gamma + \int_{\Omega_i} f_k^\delta \cdot \overline{\mathcal{E}_i^{\delta,k}} dx. \tag{6.30}$$

It remains to compute the jump

$$\operatorname{curl} \mathcal{E}_e^{\delta,k} \times n - \operatorname{curl} \mathcal{E}_i^{\delta,k} \times n \equiv \operatorname{curl} E_{e,k}^\delta \times n - \operatorname{curl} E_{i,k}^\delta \times n$$

across Γ. Taking the trace on Γ of Eq. (6.24), we get, with $\rho_i^{\delta,k}(x_\Gamma) = r_i^{\delta,k}(x_\Gamma, 0)$,

$$\operatorname{curl} E_{i,k}^\delta \times n = i\omega H_{i,k}^\delta \times n + \rho_i^{\delta,k} \quad \text{on } \Gamma.$$

The function $\rho_i^{\delta,k}$ is not zero but small. In particular, according to the expression of $r_i^{\delta,k}$, we have

$$\|\rho_i^{\delta,k}\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_k \delta^k. \tag{6.31}$$

On the other hand, taking the trace on Γ of the first equation of (6.20) we get

$$\operatorname{curl} E_{e,k}^{\delta} \times n = i\omega H_{e,k}^{\delta} \times n, \quad \text{on } \Gamma.$$

The continuity conditions (5.15) imply $H_{i,k}^{\delta} \times n = H_{e,k}^{\delta} \times n$ on Γ so that

$$\operatorname{curl} \mathcal{E}_e^{\delta,k} \times n - \operatorname{curl} \mathcal{E}_i^{\delta,k} \times n \equiv \operatorname{curl} E_{e,k}^{\delta} \times n - \operatorname{curl} E_{i,k}^{\delta} \times n = \rho_i^{\delta,k}. \tag{6.32}$$

Substituting (6.32) into (6.30) and using the estimates (6.29) and (6.31), we get

$$\left| \int_{\Omega} |\operatorname{curl} \mathcal{E}^{\delta,k}|^2 - \omega^2 \int_{\Omega} |\mathcal{E}^{\delta,k}|^2 - i\omega \left(\int_{\partial\Omega} |\mathcal{E}^{\delta,k} \times n|^2 + \frac{1}{\delta^2} \int_{\Omega_i} |\mathcal{E}^{\delta,k}|^2 \right) \right|$$
$$\leq C_k \big(\delta^{k-\frac{1}{2}} \|\mathcal{E}^{\delta,k} \times n\|_{L^2(\Omega_i)} + \delta^k \|\mathcal{E}^{\delta,k} \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \big).$$

We can finally apply Lemma 6.2 with $\mathbf{E}^{\delta} = \mathcal{E}^{\delta,k}$ and $s = k - \frac{1}{2}$, which provides the desired estimates. □

7. Analysis of the GIBCs

7.1. *Well-posedness of the approximate problems*

We shall prove in this section that the approximate fields $(E^{\delta,k}, H^{\delta,k})$, solution of (4.1), (4.2) for $k = 0, 1, 2, 3$, are well defined. In fact, for $k \leq 2$ this result is an application (or an adaptation) of classical results about Maxwell equations with an impedance boundary condition of the form

$$E \times n + \omega\, Z H_T = 0 \quad \text{on } \Gamma,$$

where Z is a function with positive real part (see for instance Ref. 14). To include the case $k = 3$ it is sufficient to extend these results to the cases where Z is a continuous operator form $L_t^2(\Gamma)$ into $L_t^2(\Gamma)$ with positive definite real part. More precisely we shall assume that there exists two positive constants z_* and z^* such that

$$\begin{cases} \text{(i)} \quad \|Z\varphi\|_{\Gamma} \leq z^* \|\varphi\|_{\Gamma}, \\ \text{(ii)} \quad \mathcal{R}e\ (Z\varphi, \varphi)_{\Gamma} \geq z_* \|\varphi\|_{\Gamma}^2, \end{cases} \tag{7.1}$$

for all $\varphi \in L_t^2(\Gamma)$. These properties are satisfied by the operators $\mathcal{D}^{\delta,k}$, $k = 1, 2, 3$ for δ sufficiently small, and can be seen as a special consequence of Lemma 7.1 (stated and proved in the next section) where the dependence of the constants z_* and z^* in terms of δ is also given (which is important for error analysis). The functional space adapted to this type of boundary conditions is the same as for constant impedances, namely $\tilde{H}(\operatorname{curl}, \Omega_e)$ (see (2.5) for the definition of this space).

Theorem 7.1. *Let $f \in L^2(\Omega_e)$ be compactly supported in Ω_e and $g \in L^2_t(\partial\Omega_e)$. Then the boundary value problem*

$$\begin{cases} \operatorname{curl} H + i\omega E = 0, & \text{in } \Omega_e, \\ \operatorname{curl} E - i\omega H = 0, & \text{in } \Omega_e, \\ E \times n + H_T = g, & \text{on } \partial\Omega, \\ E \times n + \omega Z H_T = g, & \text{on } \Gamma, \end{cases}$$

has a unique solution (E, H) in $\tilde{H}(\operatorname{curl}, \Omega_e) \times \tilde{H}(\operatorname{curl}, \Omega_e)$.

Proof. The proof uses basically the same arguments as for classical impedance conditions (see Ref. 14) and the details are provided in Ref. 12, Theorem 6.1 of Ref. 12. □

7.2. *Error estimates for the GIBCs*

The error estimates rely on some key properties of the boundary operator $\mathcal{D}^{\delta,k}$ that we shall summarize in the following lemma. We recall that $\mathcal{D}^{\delta,k} = 0$, $\delta\sqrt{i}$ and $\delta\sqrt{i} + \delta^2(\mathcal{H} - \mathcal{C})$ for $k = 0, 1$ and 2, respectively. For $k = 3$, we denote by A^δ and B^δ the two operators

$$A^\delta := \left(1 - \delta^2 \nabla_\Gamma \operatorname{div}_\Gamma\right)^{-1}, \quad B^\delta = \left(1 + \delta^2 \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma\right)^{-1}.$$

By Lax–Milgram's Lemma these operators are well defined as continuous operators from $L^2_t(\Gamma)$ to respectively $H(\operatorname{div}_\Gamma, \Gamma)$ and $H(\operatorname{curl}_\Gamma, \Gamma)$. Setting

$$\alpha_\delta := \frac{1}{2\sqrt{2}} + \delta(\mathcal{H} - \mathcal{C}) + \frac{\delta^2}{2\sqrt{2}}(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r \omega^2),$$

$$\beta_\delta := \frac{1}{2\sqrt{2}} - \frac{\delta^2}{2\sqrt{2}}(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r \omega^2),$$

the expression (4.11) of $\mathcal{D}^{\delta,3}$ can be written in the form

$$\begin{cases} \mathcal{D}^{\delta,3}\varphi = \delta\, \alpha_\delta \varphi + \dfrac{\sqrt{2}}{4}\delta(A^\delta \varphi + \delta^2 \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi) \\ \qquad\qquad + i\delta\, \beta_\delta \varphi + i\dfrac{\sqrt{2}}{4}\delta\left(B^\delta \varphi - \delta^2 \nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi\right). \end{cases}$$

The main properties of the operators $\mathcal{D}^{\delta,k}$ are summarized in the following lemma.

Lemma 7.1. *Let $k = 1, 2$ or 3. There exist a constant $\delta_k > 0$ and two constants $C_1 > 0$ and $C_2 > 0$, independent of δ, such that*

$$\begin{aligned} &\text{(i)} \quad \|\mathcal{D}^{\delta,k}\varphi\|_\Gamma \le C_1 \delta \|\varphi\|_\Gamma, \\ &\text{(ii)} \quad \mathcal{R}e\, (\mathcal{D}^{\delta,k}\varphi, \varphi)_\Gamma \ge C_2 \delta \|\varphi\|^2_\Gamma, \end{aligned} \tag{7.2}$$

for all $\varphi \in L^2_t(\Gamma)$ and $\delta \le \delta_k$.

Proof. These properties are straightforward for $k = 1$ and 2. We shall concentrate on the case $k = 3$. We first observe that α_δ and β_δ are bounded functions on Γ, and if $\varphi \in L^2_t(\Gamma)$ then $\delta^2 \nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi = (A^\delta \varphi - \varphi) \in L^2_t(\Gamma)$ and $\delta^2 \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi = (-B^\delta \varphi + \varphi) \in L^2_t(\Gamma)$. Therefore $\mathcal{D}^{\delta,3}\varphi \in L^2_t(\Gamma)$ and one has

$$\begin{cases} (\mathcal{D}^{\delta,3}\varphi, \psi)_\Gamma = \delta(\alpha_\delta \varphi, \psi)_\Gamma + \dfrac{\sqrt{2}}{4}\delta((A^\delta \varphi, \psi)_\Gamma + \delta^2 (\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi, \psi)_\Gamma) \\ \qquad\qquad + i\delta(\beta_\delta \varphi, \psi)_\Gamma + i\dfrac{\sqrt{2}}{4}\delta\left((B^\delta \varphi, \psi)_\Gamma - \delta^2(\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi, \psi)_\Gamma\right) \end{cases} \tag{7.3}$$

for all $\varphi, \psi \in L^2_t(\Gamma)$. For δ sufficiently small, the functions α_δ and β_δ satisfy

$$0 < \alpha_* < |\alpha_\delta| < \alpha^* \quad \text{and} \quad 0 < \beta_* < |\beta_\delta| < \beta^* \tag{7.4}$$

for some positive constants α_*, α^*, β_* and β^* independent of δ. On the other hand, from the identities

$$(1 - \delta^2 \nabla_\Gamma \operatorname{div}_\Gamma) A^\delta \varphi = \varphi \quad \text{and} \quad (1 + \delta^2 \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma) B^\delta \varphi = \varphi$$

one respectively deduces

$$\begin{cases} (A^\delta \varphi, \varphi)_\Gamma = \|A^\delta \varphi\|^2_\Gamma + \delta^2 \|\operatorname{div}_\Gamma A^\delta \varphi\|^2_\Gamma, \\ -(\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi, \varphi)_\Gamma = \|\operatorname{div}_\Gamma A^\delta \varphi\|^2_\Gamma + \delta^2 \|\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi\|^2_\Gamma \end{cases} \tag{7.5}$$

and

$$\begin{cases} (B^\delta \varphi, \varphi)_\Gamma = \|B^\delta \varphi\|^2_\Gamma + \delta^2 \|\operatorname{curl}_\Gamma B^\delta \varphi\|^2_\Gamma, \\ (\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi, \varphi)_\Gamma = \|\operatorname{curl}_\Gamma B^\delta \varphi\|^2_\Gamma + \delta^2 \|\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi\|^2_\Gamma. \end{cases} \tag{7.6}$$

Property (ii) is obtained as an immediate consequence of (7.5), (7.6) and (7.4) when applied to (7.3) with $\psi = \varphi$. Identities (7.5) and (7.6) also respectively imply,

$$\|A^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma, \quad \delta^2 \|\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma,$$
$$\|B^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma \quad \text{and} \quad \delta^2 \|\overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma.$$

Property (i) is then easily obtained from (7.3) with $\psi = \mathcal{D}^{\delta,3}\varphi$ and using these estimates, as well as (7.4). $\square$

7.3. *Error estimates for the GIBCs*

We shall set for $k = 0, 1, 2, 3$,

$$\begin{cases} \widetilde{\mathcal{E}}^{\delta,k}_e = E^{\delta,k}_e - \displaystyle\sum_{\ell=0}^{k} E^\ell_e, \\ \widetilde{\mathcal{H}}^{\delta,k}_e = H^{\delta,k}_e - \displaystyle\sum_{\ell=0}^{k} H^\ell_e. \end{cases} \tag{7.7}$$

Using (5.36), together with (4.12) and (4.13) when $k=3$, we see that $(\widetilde{\mathcal{E}}_e^{\delta,k},\widetilde{\mathcal{H}}_e^{\delta,k})\in\mathcal{V}_E^k\times\mathcal{V}_H^k$ is solution of the boundary value problem:

$$\begin{cases}\operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}+i\omega\widetilde{\mathcal{E}}_e^{\delta,k}=0, & \text{in } \Omega_e,\\ \operatorname{curl}\widetilde{\mathcal{E}}_e^{\delta,k}-i\omega\widetilde{\mathcal{H}}_e^{\delta,k}=0, & \text{in } \Omega_e,\\ \left(\widetilde{\mathcal{E}}_e^{\delta,k}\right)_T-\widetilde{\mathcal{H}}_e^{\delta,k}\times n=0, & \text{on } \partial\Omega,\\ \widetilde{\mathcal{E}}_e^{\delta,k}\times n+\omega\,\mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T=\delta^{k+1}\,\varphi_k^\delta & \text{on } \Gamma,\end{cases}\tag{7.8}$$

where the vector fields φ_k^δ remain bounded with respect to δ in all spaces $H_t^s(\Gamma)^3$.

Eliminating $\widetilde{\mathcal{E}}_e^{\delta,k}$, we see that $\widetilde{\mathcal{H}}_e^{\delta,k}\in\mathcal{V}_H^k$ satisfies

$$\begin{cases}\operatorname{curl}\big(\operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}\big)-\omega^2\widetilde{\mathcal{H}}_e^{\delta,k}=0 & \text{in } \Omega_e,\\ \operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}\times n-i\omega^2\mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T=\delta^{k+1}\,\varphi_k^\delta, & \text{on } \Gamma,\\ \operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}\times n-i\omega(\widetilde{\mathcal{H}}_e^{\delta,k})_T=0, & \text{on } \partial\Omega.\end{cases}$$

The proof of error estimates is based on some key *a priori* estimates that we shall give hereafter. We multiply by $\overline{\widetilde{\mathcal{H}}_e^{\delta,k}}$ the equation satisfied by $\widetilde{\mathcal{H}}_e^{\delta,k}$ in Ω_e, integrate over Ω_e and use Green's formula to obtain, after having used the boundary conditions on $\partial\Omega$ and Γ:

$$\begin{aligned}\int_{\Omega_e}\big(|\operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}|^2-\omega^2|\widetilde{\mathcal{H}}_e^{\delta,k}|^2\big)dx-i\int_{\partial\Omega}|(\widetilde{\mathcal{H}}_e^{\delta,k})_T|^2d\sigma&\\ -i\omega^2\big(\mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T,(\widetilde{\mathcal{H}}_e^{\delta,k})_T\big)_\Gamma=\delta^{k+1}\langle\varphi_k^\delta,(\widetilde{\mathcal{H}}_e^{\delta,k})_T\rangle_\Gamma,&\end{aligned}\tag{7.9}$$

where $\langle,\rangle_\Gamma$ here denotes a duality pairing between $H^{-1/2}(\mathrm{div},\Gamma)$ and $H^{-1/2}(\mathrm{curl},\Gamma)$. Considering the imaginary part of (7.9) and using (7.2)(ii) together with trace theorems in $H(\mathrm{curl},\Omega_e)$, one obtains the existence of two non-negative constants C_1 and C_2 independent of δ such that

$$C_1\delta\|(\widetilde{\mathcal{H}}_e^{\delta,k})_T\|_\Gamma^2+\|(\widetilde{\mathcal{H}}_e^{\delta,k})_T\|_{\partial\Omega}^2\le C_2\delta^{k+1}\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)}.\tag{7.10}$$

More precisely we have $C_1=0$ for $k=0$ and $C_1>0$ for $k\neq0$. Using (7.2)(i) and (7.10) one also deduces that

$$|\mathcal{I}m\,\big(\mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T,(\widetilde{\mathcal{H}}_e^{\delta,k})_T\big)_\Gamma|\le C_3\delta^{k+1}\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)},\tag{7.11}$$

for some constant C_3 independent of δ. Now considering the real part of (7.9) and using (7.11) as well as the trace theorem in $H(\mathrm{curl},\Omega_e)$, one gets the existence of two positive constants C_4 and C_5 independent of δ such that

$$\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)}^2\le C_4\delta^{k+1}\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)}+C_5\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)}^2.\tag{7.12}$$

Based on these *a priori* estimates we are in a position to prove the following result.

Lemma 7.2. *For $k = 0, 1, 2$ or 3, there exist a constant C independent of δ and $\delta_0 > 0$ such that*

$$\|\widetilde{\mathcal{E}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)} + \|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)} \leq C_k \delta^{k+1}$$

for all $\delta \leq \delta_0$.

Proof. According to estimate (7.12) and the first two equations of (7.8), it is sufficient to prove the existence of a constant C independent of δ such that

$$\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)} \leq C\delta^{k+1}. \tag{7.13}$$

Let us assume that (7.13) does not hold, i.e. $\lambda_\delta := \delta^{k+1}/\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)}$ goes to 0 as $\delta \to 0$, and consider the scaled fields

$$h^\delta = \widetilde{\mathcal{H}}_e^{\delta,k}/\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)} \quad \text{and} \quad e^\delta = \widetilde{\mathcal{E}}_e^{\delta,k}/\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)}.$$

Dividing (7.12) by $\|\widetilde{\mathcal{H}}_e^{\delta,k}\|^2_{L^2(\Omega_e)}$ implies in particular that (h^δ) is a bounded sequence in $H(\mathrm{curl}, \Omega_e)$. Dividing (7.10) by the same quantity and using the latter result shows that

$$C_1\delta\|h_T^\delta\|^2_\Gamma + \|h_T^\delta\|^2_{\partial\Omega} \to 0 \text{ as } \delta \to 0. \tag{7.14}$$

The last boundary condition in (7.8) combined with property (7.2)(i) shows in particular that

$$\|e_T^\delta\|_\Gamma \leq C_6\delta\|h_T^\delta\|_\Gamma + \lambda_\delta \, \|\varphi_k^\delta\|_\Gamma$$

(with the alternative, C_6 and $C_1 > 0$ or $C_6 = C_1 = 0$). We therefore conclude that $\|e_T^\delta\|_\Gamma$ goes to 0 as $\delta \to 0$. The first three equations of (7.8) imply that e^δ is a bounded sequence in $\tilde{H}_0(\mathrm{curl}, \Omega_e)$ (see definition in the proof of Theorem 7.1). Therefore, up to extracted subsequence, one can assume that e^δ converges strongly in $L^2(\Omega_e)$ and weakly in $\tilde{H}_0(\mathrm{curl}, \Omega_e)$ to some $e \in \tilde{H}_0(\mathrm{curl}, \Omega_e)$. Passing to the limit as $\delta \to 0$ in (7.8), we observe that $e \in \tilde{H}_0(\mathrm{curl}, \Omega_e)$ is solution of

$$\begin{cases} \mathrm{curl}\ \mathrm{curl}\ e - \omega^2 e = 0, & \text{in } \Omega_e, \\ \mathrm{curl}\ e \times n - i\omega e_T \times n = 0, & \text{on } \partial\Omega, \\ e \times n = 0 & \text{on } \Gamma, \end{cases}$$

and therefore $e = 0$. We then deduce that $\mathrm{curl}\, h^\delta$ strongly converges to 0 in $L^2(\Omega_e)$. Coming back to identity (7.9) and considering the real part, one deduces after division by $\|\widetilde{\mathcal{H}}_e^{\delta,k}\|^2_{L^2(\Omega_e)}$ that

$$\|h^\delta\|^2_{H(\mathrm{curl},\Omega_e)} \leq \lambda_\delta\, \tilde{C}_0\|h^\delta\|_{H(\mathrm{curl},\Omega_e)} + \tilde{C}_1\|\mathrm{curl}\, h^\delta\|^2_{L^2(\Omega_e)} + \tilde{C}_2|\mathcal{I}m\ (\mathcal{D}^{\delta,k}h_T^\delta, h_T^\delta)|. \tag{7.15}$$

Property (7.2)(i) shows that

$$|\mathcal{I}m\ (\mathcal{D}^{\delta,k}h_T^\delta, h_T^\delta)| \leq \tilde{C}_3\delta|h_T^\delta|^2_\Gamma \to 0$$

according to (7.14). Therefore, considering the limit as $\delta \to 0$ in (7.15) implies that h^δ strongly converges to 0 in $H(\mathrm{curl}, \Omega_e)$, which contradicts $\|h^\delta\|_{L^2(\Omega_e)} = 1$. $\square$

Appendix A. Technical Lemmas

The first lemma is a slight variation of the classical trace lemma in $H(\mathrm{curl}, O)$.

Lemma A.1. *Let O be a bounded open subset of $\mathbb{R}^3$ of class C^2 (and which is locally from one side of its normal). Then there exists a constant C depending only on O such for all $u \in H(\mathrm{curl}, O)$, that*

$$\|u \times n\|^2_{H^{-\frac{1}{2}}(\partial O)} \leq C\|u\|_{L^2(O)} \left(\|\mathrm{curl}\, u\|_{L^2(O)} + \|u\|_{L^2(O)}\right). \tag{A.1}$$

Proof. (i) Let us consider first the case where there exist $a > 0, b > 0, \delta > 0$ and $h \in C^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$ such that, if $\varphi(y_1, y_2, y_3) := \big(y_1, y_2, y_3 + h(y_1, y_2)\big)$,

$$\mathrm{supp}\ u \subset \varphi(\,]-a, a[\times]-b, b[\times]0, \delta[\,) \subset O,$$

$$\{\partial O \cap\ \mathrm{supp}\ u\} \subset \Sigma := \{\varphi(y_1, y_2, 0); (y_1, y_2) \in]-a, a\,[\times]-b, b\,[\}.$$

Setting $\tilde{u} := u \circ \varphi$, $\tilde{n} := n \circ \varphi$, and $f := \left(1 + |Dh|^2\right)^{-\frac{1}{2}}$, one has for $y_3 = 0$,

$$\tilde{u} \times \tilde{n} = f\left(\tilde{u}_2 + \tilde{u}_3\frac{\partial h}{\partial y_2}, \tilde{u}_3\frac{\partial h}{\partial y_1} - \tilde{u}_1, -\tilde{u}_1\frac{\partial h}{\partial y_2} + \tilde{u}_2\frac{\partial}{\partial y_1}h\right). \tag{A.2}$$

Let us consider the first component of $\tilde{u} \times \tilde{n}$. Setting $u_2^* = \tilde{u}_2 + \tilde{u}_3\frac{\partial h}{\partial y_2}$, we have,

$$\frac{\partial u_2^*}{\partial y_3} = -r_1 + \frac{\partial \tilde{u}_3}{\partial y_2} + \tilde{u}_3\frac{\partial^2 h}{\partial y_3 \partial y_2}, \tag{A.3}$$

where

$$r_1 := (\mathrm{curl}\, u \circ \varphi) \cdot e_1 = \frac{\partial \tilde{u}_3}{\partial y_2} - \frac{\partial \tilde{u}_3}{\partial y_3}\frac{\partial h}{\partial y_2} - \frac{\partial \tilde{u}_2}{\partial y_3}.$$

Let $\mathcal{F}$ be the Fourier transform in (y_1, y_2), (ξ_1, ξ_2) the dual variables, $\hat{u}_i$ the Fourier transform of $\tilde{u}_i$ and $\hat{u}_2^*$ the Fourier transform of u_2^*. By definition of the norm in $H^{-\frac{1}{2}}(\mathbb{R}^2)$,

$$\|u_2^*(\cdot, \cdot, 0)\|^2_{H^{-\frac{1}{2}}} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{-\frac{1}{2}} |\hat{u}_2^*(\xi_1, \xi_2, 0)|^2 d\xi_1 d\xi_2.$$

Since

$$|\hat{u}_2^*(\xi_1, \xi_2, 0)|^2 = -2\mathcal{R}e \int_0^\delta \left[\frac{\partial \hat{u}_2^*}{\partial y_3}\overline{\hat{u}_2^*}\right](\xi_1, \xi_2, y_3) dy_3$$

using (A.3), we have, $\widehat{r}_1$ being the Fourier transform of r_1,

$$\begin{aligned} |\hat{u}_2^*(\xi_1, \xi_2, 0)|^2 = 2\mathcal{R}e &\int_0^\delta [\widehat{r}_1\overline{\hat{u}_2^*}](\xi_1, \xi_2, y_3) dy_3 \\ &- 2\mathcal{R}e \int_0^\delta i\xi_2[\tilde{u}_3\overline{\hat{u}_2^*}](\xi_1, \xi_2, y_3) dy_3 \\ &- 2\mathcal{R}e \int_0^\delta \left[\mathcal{F}\left(\tilde{u}_3\frac{\partial^2 h}{\partial y_3 \partial y_1}\right)\overline{\tilde{u}_2^*}\right](\xi_1, \xi_2, y_3) dy_3. \end{aligned}$$

We divide the above equality by $(1+|\xi|^2)^{-\frac{1}{2}}$ and integrate over ξ. Next we use $(1+|\xi|^2)^{-\frac{1}{2}} \leq 1$, $|\xi_2|(1+|\xi|^2)^{-\frac{1}{2}} \leq 1$ and Plancherel's theorem to obtain, since $h \in W^{2,\infty}(\mathbb{R}^2)$,

$$\|u_2^*(\cdot,\cdot,0)\|^2_{H^{-\frac{1}{2}}} \leq 2\int_0^\delta \int_{\mathbb{R}^2} |r_1||u_2^*|dy + C\int_0^\delta \int_{\mathbb{R}^2} |u_3||u_2^*|dy.$$

Coming back to the variable x through the change of variable $x = \varphi(y)$, we easily get, since $|u_2^*| \leq |u_2| + C|u_3|$,

$$\|u_2^*(\cdot,\cdot,0)\|^2_{H^{-\frac{1}{2}}} \leq C(\|u\|^2_{L^2(O)} + \|u\|_{L^2(O)}\|\mathrm{curl}\, u\|_{L^2(O)}),$$

where the constant C only depends on h.

Finally, using Lemma A.2 (note that f belongs to $W^{1,\infty}$), we get, since $(\tilde{u} \times \tilde{n})_1 = fu_2^*$

$$\|(\tilde{u} \times \tilde{n})_1\|^2_{H^{-\frac{1}{2}}} \leq C(\|u\|^2_{L^2(O)} + \|u\|_{L^2(O)}\|\mathrm{curl}\, u\|_{L^2(O)}). \tag{A.4}$$

We proceed in the same way for the other two components of $\tilde{u} \times \tilde{n}$.

(ii) Obtaining the same inequality in the general case can be deduced by using a partition of unity $(\varphi_i)_{i=1,\ldots,N}$ of O and noticing that

$$\begin{aligned}\|\mathrm{curl}\, \varphi_i u\|_{L^2(O)} &= \|\varphi_i \mathrm{curl}\, u + \nabla\varphi_i \times u\|_{L^2(O)} \\ &\leq \|\varphi_i\|_\infty \|\mathrm{curl}\, u\|_{L^2(O)} + \|\nabla\varphi_i\|_\infty \|u\|_{L^2(O)}.\end{aligned}$$

□

The proof of the previous lemma uses the following result whose proof can be found in Lemma A.2 of Ref. 12.

Lemma A.2. *Let $f \in W^{1,\infty}(R^n)$ and $g \in H^{-\frac{1}{2}}(R^n)$, then $f\,g \in H^{-\frac{1}{2}}(R^n)$ and one has,*

$$\|f\,g\|_{H^{-\frac{1}{2}}(R^n)} \leq 3^{\frac{1}{4}}\|f\|_{W^{1,\infty}(R^n)}\|g\|_{H^{-\frac{1}{2}}(R^n)}.$$

Lemma A.3. *Let $O \subset R^3$ be a bounded open set with a C^2 boundary Γ. There exists a constant C that depends only on Γ such that*

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} \leq C(\|\nabla_\Gamma u\|_{H^{-\frac{1}{2}}(\Gamma)} + \|u\|_{L^2(\Gamma)}) \quad \forall\, u \in H^{\frac{1}{2}}(\Gamma).$$

Proof. In the case $O = \{(x_1,x_2,x_3) \in \mathbb{R}^3/x_3 \geq 0\}$ one can check by using Fourier transform in the plane (x_1,x_2) that

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \|\nabla_\Gamma u\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|u\|^2_{L^2(\Gamma)}\,.$$

The inequality is therefore trivially verified in this case. The general case can be easily deduced by using local parameterizations of the boundary Γ. This is where the C^2-regularity of Γ is taken into account. □

The next lemma is a sharper version of classical compact-embedding theorem for spaces of L^2 functions with bounded divergence and curl into L^2. Let us define

$$H(\mathrm{curl}, \mathrm{div}, O) := \{u \in L^2(O)^3 / \mathrm{curl}\, u \in L^2(O)^3 \text{ and } \mathrm{div}\, u \in L^2(O)\}$$

equipped with the norm

$$\|u\|^2_{H(\mathrm{curl},\mathrm{div},O)} = \|u\|^2_{L^2(O)} + \|\mathrm{curl}\, u\|^2_{L^2(O)} + \|\mathrm{div}\, u\|^2_{L^2(O)}.$$

Lemma A.4. *Let $O \subset R^3$ be a bounded simply connected open set with C^2 boundary Γ. Then every bounded sequence $(u_k)_{k\in\mathbb{N}}$ of $H(\mathrm{curl}, \mathrm{div}, O)$ such that*

$$(u_{k|\Gamma} \times n)_{k\in\mathbb{N}} \text{ is convergent in } H_t^{-\frac{1}{2}}(\Gamma) \tag{A.5}$$

has a convergent subsequence $(u_{k'})$ in $L^2(O)^3$.

Proof. Our proof is an adaptation of the proof given by Costabel in the case where, instead of (A.5), one has an L^2 control of the boundary term of the sequence (see Theorem 2 of Ref. 5).

The idea is to make a Helmholtz decomposition of u_k of the form:

$$u_k = w_k + \nabla p_k, \quad (w_k, p_k) \in H^1(O) \times L^2(O), \quad \mathrm{div}\, w_k = 0, \tag{A.6}$$

constructed in such a way that:

(i) w_k is bounded in $H^1(O)$ (and thus admits a converging subsequence in $L^2(O)^3$): this uses the fact that $\mathrm{curl}\, u_k$ is bounded in $L^2(O)^3$,
(ii) ∇p_k admits a converging subsequence in $L^2(O)^3$: this uses the fact that $\mathrm{div}\, u_k$ is bounded in $L^2(O)^3$ and that $(u_k \times n)_{|\Gamma}$ converges according to (A.5).

The proof of (i) is the same as is Ref. 5 and is omitted here. One can also see Lemma A.4 of Ref. 12 for the details of the construction of w_k. For the remaining of the proof, we can therefore assume that (up to extracted subsequence) w_k converges in $L^2(O)$.

Since $\mathrm{curl}(u_k - w_k) = 0$ and O is simply connected, one can construct p_k (unique up to an additive constant) such that $\nabla p_k = u_k - w_k$ (use for instance Theorem 2.9 of Ref. 9). Fixing p_k by imposing that $\int_O p_k dx = 0$ gives raise to a bounded sequence p_k in $H^1(O)$ by the Poincaré–Wirtinger inequality. Since we further have that $(\mathrm{div}\, u_k)$ is bounded in $L^2(O)$ and (w_k) is bounded in $H^1(O)$, then, up to extracted subsequence, one can assume that $\mathrm{div}\, u_k$ is convergent in $H^{-1}(O)$, $w_k|_{\partial O}$ is convergent in $H^{-\frac{1}{2}}(\partial O)$ and $p_k|_{\partial O}$ is convergent in $L^2(\partial O)$. We shall deduce that p_k is strongly convergent in $H^1(O)$. We first observe that p_k satisfies

$$\begin{cases} -\Delta p_k = \mathrm{div}\, u_k, & \text{in } O, \\ \nabla p_k \times n = u_k \times n - w_k \times n, & \text{on } \partial O. \end{cases}$$

Let m and k be two indices. From

$$\begin{cases} \Delta(p_k - p_m) = \operatorname{div}(u_k - u_m) & \text{in } O, \\ \nabla(p_k - p_m) \times n = (u_k - u_m) \times n - (w_k - w_m) \times n & \text{on } \partial O, \end{cases} \tag{A.7}$$

and using the classical theory for elliptic equations one gets the existence of a constant C_1 such that

$$\|\nabla p_k - \nabla p_m\|_{L^2(\Omega)} \leq C_1(\|\operatorname{div}(u_k - u_m)\|_{H^{-1}(O)} + \|p_k - p_m\|_{H^{\frac{1}{2}}(\partial O)}). \tag{A.8}$$

On the other hand, using Lemma A.3 one has

$$\|p_k - p_m\|_{H^{\frac{1}{2}}(\partial O)} \leq C_2(\|\nabla(p_k - p_m) \times n\|_{H^{-\frac{1}{2}}(\partial O)} + \|p_k - p_m\|_{L^2(\partial O)}). \tag{A.9}$$

From the second equation of (A.7)–(A.9) it is easily seen that

$$\begin{aligned} \|\nabla p_k - \nabla p_m\|_{L^2(\Omega)} \leq C_3(&\|\operatorname{div}(u_k - u_m)\|_{H^{-1}(O)} + \|(u_k - u_m) \times n\|_{H^{-\frac{1}{2}}(\partial O)} \\ &+ \|w_k - w_m\|_{H^{-\frac{1}{2}}(\partial O)} + \|p_k - p_m\|_{L^2(\partial O)}). \end{aligned}$$

Using assumption (A.5) one concludes ∇p_k is a Cauchy sequence in $L^2(\partial O)$. The result of the lemma is then proved since $u_k = w_k + \nabla p_k$. □

Lemma A.4 also applies to domains Ω_i that are not simply connected. This is proved in the following lemma.

Lemma A.5. *The result of Lemma* A.4 *applies to bounded open domains $O \in R^3$ of class C^2.*

Proof. Let x be an arbitrary point in $\overline{O}$. If $x \in O$, one defines U_x as a ball centered at x such that $\overline{U}_x \subset O$. If not, one defines U_x as a neighborhood of x such that there exits a bijective map $\phi_x : Q \mapsto U_x$ such that

$$\phi_x \in C^1(\overline{Q}), \phi^{-1} \in C^1(\overline{U}_x), \phi(Q_+) = U_x \cap O \quad \text{and} \quad \phi(Q_0) = U \cap \partial O,$$

where Q denotes the unit cube of R^3, $Q_+ := \{x \in Q | x_3 > 0\}$, and $Q_0 = \{x \in Q | x_3 = 0\}$.

With this definition one observes that $U_x \cap O$ is a simply connected domain for all $x \in \overline{O}$. By the compactness of $\overline{O}$ one can extract a finite covering of $\overline{O}$ from $\{U_x; x \in \overline{O}\}$. Let us denote by $\{U_i, i \in I\}$ this finite covering and consider a partition of unity $(\theta_i)_{i \in I} \subset C^\infty(R^3)$ subordinated to this covering, i.e.

$$\text{supp } \theta_i \subset U_i, \sum_{i \in I} \theta_i = 1 \text{ on } \overline{O}.$$

Then define $u_n^i := \theta_i \, u_n$ for all $i \in I$. It is easy to see that for every i, the sequence (u_n^i) satisfies the hypotheses of Lemma A.4 with O replaced by U_i. Using a finite diagonal process, one can therefore assume that there exists a subsequence n_k such that

$$u_{n_k}^i \quad \text{converges in } L^2(U_i) \quad \text{for all } i \in I.$$

Consequently, the sequence $u_{n_k} = \sum_{i \in I} u_{n_k}^i$ is convergent in $L^2(O)$. □

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