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Some new characterizations of Sobolev spaces

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Abstract

In this paper, we present some new characterizations of Sobolev spaces. Here is a typical result. Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$; we prove that $g \in W^{1,p}(\mathbb{R}^N)$ if and only if

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Moreover,

$$\lim_{\delta\to 0} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int\limits_{\mathbb{R}^N} |\nabla g(x)|^p\,dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N),$$

where $K_{N,p}$ is defined by (12).

This result is somewhat related to a characterization of Sobolev spaces due to J. Bourgain, H. Brezis, P. Mironescu (see [J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations, A Volume in Honour of A. Bensoussan's 60th Birthday, IOS Press, 2001, pp. 439–455]). However, the precise connection is not transparent.

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1. Introduction

We first recall a result due to J. Bourgain, H. Brezis, P. Mironescu.

Theorem 1. (J. Bourgain, H. Brezis, P. Mironescu) *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy \leqslant C, \quad \forall n \geqslant 1,$$

for some constant $C > 0$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx, \quad \forall g \in L^p\big(\mathbb{R}^N\big),$$

where $K_{N,p}$ is defined by (12). *Here $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of functions satisfying*

$$\rho_n \geqslant 0, \qquad \rho_n(x) = \rho_n\big(|x|\big),$$

$$\lim_{n\to\infty}\int_{\tau}^{\infty}\rho_n(r)r^{N-1}\,dr = 0, \quad \forall \tau > 0,$$

and

$$\lim_{n\to\infty}\int_0^{+\infty}\rho_n(r)r^{N-1}\,dr = 1.$$

Here is a typical example.

Proposition 1. *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if*

$$\sup_{0<\delta<1}\frac{1}{|\ln\delta|}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ \delta<|x-y|<1}} \frac{|g(x)-g(y)|^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Moreover,

$$\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ \delta<|x-y|<1}} \frac{|g(x)-g(y)|^p}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx.$$

The reader can find many other interesting examples in [1,3].

In this paper, we present some new characterizations of Sobolev spaces. Our first result is the following.

Theorem 2. *Let $1 < p < +\infty$. Then*

(a) *There exists a constant $C_{N,p}$ depending only on N and p such that*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p}\int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx, \quad \forall \delta > 0,\ \forall g \in W^{1,p}\big(\mathbb{R}^N\big). \tag{1}$$

(b) *If $g \in L^p(\mathbb{R}^N)$ satisfies*

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < +\infty, \tag{2}$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

(c) *Moreover, for any $g \in W^{1,p}(\mathbb{R}^N)$,*

$$\lim_{\delta\to 0} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx, \tag{3}$$

where $K_{N,p}$ is defined by (12).

Remark 1. Assertions (a) and (c) are due to A. Ponce and J. Van Schaftingen [5]. Our proof of assertion (c) is slightly different from their original proof.

In the proof of Theorem 2 we will use the following theorem (Theorem 3) which is closely related to Theorem 1. However we do not know any simple statement unifying Theorems 1–3.

Theorem 3. *Let $1 < p < +\infty$. Then*

(a) *For every $g \in W^{1,p}(\mathbb{R}^N)$,*

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$
$$\leqslant C_{N,p}\int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx,$$

where $C_{N,p}$ is a positive constant depending only on N and p.

(b) *If $g \in L^p(\mathbb{R}^N)$ satisfies*

$$\sup_{0<\varepsilon<1} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

(c) *Moreover, for any $g \in W^{1,p}(\mathbb{R}^N)$,*

$$\lim_{\varepsilon \to 0} \iint\limits_{\substack{\mathbb{R}^N\,\mathbb{R}^N \\ |g(x)-g(y)| \leqslant 1}} \frac{\varepsilon |g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy = K_{N,p} \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ is defined by (12).

The remainder of this paper is organized as follows. In Section 2 we present the proofs of Theorems 2 and 3. In Section 3 we discuss some variants and generalizations. Finally, in Section 4, we discuss some partial results for the case $p = 1$ which seems to be delicate.

2. Proof of Theorems 2 and 3

2.1. Some useful lemmas

We first prove the following lemmas. They will be used in the proofs of Theorems 2 and 3. Here is the first lemma.

Lemma 1. *Let Ω be a measurable set in $\mathbb{R}^m$, Ψ and Φ be two measurable nonnegative functions on Ω, and $\alpha > -1$. Then*

$$\int_0^1 \int\limits_{\Phi(x) > \delta} \delta^\alpha \Psi(x)\, dx\, d\delta = \int\limits_{\Phi(x) \leqslant 1} \frac{1}{\alpha+1} \Phi^{\alpha+1}(x) \Psi(x)\, dx + \int\limits_{\Phi(x) > 1} \frac{1}{\alpha+1} \Psi(x)\, dx.$$

Proof. Applying Fubini's theorem, one has

$$\int_0^1 \int\limits_{\Phi(x) > \delta} \delta^\alpha \Psi(x)\, dx\, d\delta = \int\limits_{\Omega} \Psi(x) \int\limits_{\substack{0 \\ \delta < \Phi(x)}}^{1} \delta^\alpha\, d\delta\, dx.$$

A direct computation gives the conclusion of Lemma 1. □

The second lemma is as follows:

Lemma 2. *Let $g \in W^{1,p}(\mathbb{R}^N)$, $1 < p < +\infty$. One has*

$$\iint\limits_{\substack{\mathbb{R}^N\,\mathbb{R}^N \\ |g(x)-g(y)| > \delta}} \frac{\delta^p}{|x-y|^{N+p}} \leqslant C_{N,p} \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx, \quad \forall \delta > 0, \tag{4}$$

where $C_{N,p}$ is a positive constant depending only on N and p.

Proof. Using polar coordinates, one has

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ |g(x+h\sigma)-g(x)|>\delta}}^{\infty} \frac{\delta^p}{h^{p+1}}\,dh\,dx\,d\sigma. \tag{5}$$

Therefore, it suffices to show that there exists a constant C_p depending only on p such that for all $\sigma \in \mathbb{S}^{N-1}$,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ |g(x+h\sigma)-g(x)|>\delta}}^{\infty} \frac{\delta^p}{h^{p+1}}\,dh\,dx \leqslant C_p \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx. \tag{6}$$

Without loss of generality, one may assume that $\sigma = e_N = (0, \dots, 0, 1)$.

Note that

$$\left|g(x+he_N) - g(x)\right| \leqslant h \fint_{x_N}^{x_N+h} \left|\frac{\partial g}{\partial x_N}(x', s)\right| ds \leqslant h M_N\left(\frac{\partial g}{\partial x_N}\right)(x),$$

for almost everywhere $(x, h) \in \mathbb{R}^N \times (0, +\infty)$. Here $M_N(f)$ denotes the maximal function of f with respect to the variable x_N in the positive direction, i.e.,

$$M_N(f)(x', x_N) = \sup_{h>0} \fint_{x_N}^{x_N+h} \left|f(x', s)\right| ds. \tag{7}$$

Hence

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ |g(x+he_N)-g(x)|>\delta}}^{\infty} \frac{\delta^p}{h^{p+1}}\,dh\,dx \leqslant \int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ hM_N(\partial g/\partial x_N)(x)>\delta}}^{\infty} \frac{\delta^p}{h^{p+1}}\,dh\,dx.$$

Thus, by a direct computation,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ |g(x+he_N)-g(x)|>\delta}}^{\infty} \frac{\delta^p}{h^{p+1}}\,dh\,dx \leqslant \frac{1}{p}\int\limits_{\mathbb{R}^N} \left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx. \tag{8}$$

On the other hand, using the theory of maximal functions (see, e.g., [6, Chapter 1]), one finds

$$\int\limits_{\mathbb{R}^{N-1}}\int\limits_{\mathbb{R}} \left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx_N\,dx' \leqslant C_p \int\limits_{\mathbb{R}^{N-1}}\int\limits_{\mathbb{R}} \left|\frac{\partial g}{\partial x_N}(x)\right|^p dx_N\,dx',$$

which shows that

$$\int\limits_{\mathbb{R}^N} \left| M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx \leqslant C_p \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx. \tag{9}$$

Therefore, (6) follows immediately from (8) and (9). The proof is complete. □

Here is the third lemma.

Lemma 3. *Let $g \in W^{1,p}(\mathbb{R}^N)$, $1 < p < +\infty$. Then*

$$\lim_{\delta \to 0} \int\limits_{\mathbb{R}^N} \int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = \frac{1}{p} K_{N,p} \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ is defined by (12).

Proof. First, we claim that there exists a constant C_p depending only on p such that for every $\sigma \in \mathbb{S}^N$,

$$\int\limits_{\mathbb{R}^N} \int\limits_{\substack{0 \\ |\frac{g(x+\delta h\sigma)-g(x)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}}\, dh\, dx \leqslant C_p \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx, \quad \forall \delta > 0, \tag{10}$$

and

$$\lim_{\delta \to 0} \int\limits_{\mathbb{R}^N} \int\limits_{\substack{0 \\ |\frac{g(x+\delta h\sigma)-g(x)}{\delta h}|h>1}}^{\infty} \frac{1}{h^{p+1}}\, dh\, dx = \frac{1}{p} \int\limits_{\mathbb{R}^N} \left|\nabla g(x) \cdot \sigma\right|^p dx. \tag{11}$$

Without loss of generality, we assume that $\sigma = e_N = (0, \dots, 0, 1)$. Since $g(x', \cdot) \in W^{1,p}(\mathbb{R})$ for almost everywhere $x' \in \mathbb{R}^{N-1}$, we can assume in addition that

$$g(x + he_N) - g(x) = \int\limits_{x_N}^{x_N+h} \frac{\partial g}{\partial x_N}(x', s)\, ds,$$

for all $(x_N, h) \in \mathbb{R} \times (0, +\infty)$ and for almost everywhere $x' \in \mathbb{R}^{N-1}$.

For $K \subset \mathbb{R} \times [0, +\infty)$, let χ_K denote the characteristic function of the set K, i.e.,

$$\chi_K(x_N, h) = \begin{cases} 1 & \text{if } (x_N, h) \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$A(x',\delta)=\left\{(x_N,h);\ h>0 \text{ and } \left|\frac{g(x+\delta h e_N)-g(x)}{\delta h}\right|h>1\right\},$$

$$A(x')=\left\{(x_N,h);\ h>0 \text{ and } \left|\frac{\partial g}{\partial x_N}(x)\right|h>1\right\},$$

$$B(x')=\left\{(x_N,h);\ h>0 \text{ and } \left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|h>1\right\}.$$

Then

$$\chi_{A(x',\delta)}(x_N,h)\leqslant \chi_{B(x')}(x_N,h),$$

where $M_N(f)$ is defined in (7); and

$$\int\limits_{\mathbb{R}^N}\int\limits_0^\infty \frac{1}{h^{p+1}}\chi_{B(x')}(x_N,h)\,dh\,dx_N\,dx'=\frac{1}{p}\int\limits_{\mathbb{R}^N}\left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx.$$

On the other hand, we have (see (9))

$$\int\limits_{\mathbb{R}^N}\left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx\leqslant C_p\int\limits_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx.$$

Thus (10) is proved.

Consequently, (11) follows since

$$\lim_{\delta\to 0}\chi_{A(x',\delta)}(x_N,h)=\chi_{A(x')}(x_N,h),\quad \text{for a.e. } (x',x_N,h)\in\mathbb{R}^{N-1}\times\mathbb{R}\times[0,+\infty).$$

We are ready to prove the lemma. By a change of variables,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}}\frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\substack{0\\ |\frac{g(x+\delta h\sigma)-g(x)}{\delta h}|h>1}}^\infty\frac{1}{h^{p+1}}\,dh\,d\sigma\,dx.$$

Thus, using (10), (11) and applying Lebesgue's dominated convergence theorem, one finds

$$\lim_{\delta\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}}\frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy=\lim_{\delta\to 0}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{0\\ \frac{|g(x+\delta h\sigma)-g(x)|}{\delta h}h>1}}^\infty\frac{1}{h^{p+1}}\,dh\,dx\,d\sigma$$

$$=\frac{1}{p}\int\limits_{\mathbb{S}^{N-1}}\int\limits_{\mathbb{R}^N}\left|\nabla g(x)\cdot\sigma\right|^p dx\,d\sigma.$$

Next we recall that (see [1])

$$\int_{\mathbb{S}^{N-1}} |V\cdot\sigma|^p\,d\sigma = K_{N,p}|V|^p, \quad \forall V\in\mathbb{R}^N,\ \forall p\geqslant 1,$$

where

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p\,d\sigma, \tag{12}$$

for any $e\in\mathbb{S}^{N-1}$.

Therefore,

$$\lim_{\delta\to 0} \int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx. \qquad \square$$

Here is the fourth lemma. The method used in the proof of Lemma 4 was introduced by J. Bourgain, H. Brezis, P. Mironescu, see [1].

Lemma 4. *Assume that $h\in L^p(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N)$ such that*

$$C(h) := \sup_{0<\varepsilon<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon|h(x)-h(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy < +\infty. \tag{13}$$

Then $h\in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N}\big|\nabla h(x)\big|^p\,dx \leqslant \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon|h(x)-h(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy. \tag{14}$$

Proof. Rewriting (13) in polar coordinates, we obtain

$$\sup_{0<\varepsilon<1}\int_{\mathbb{S}^{N-1}}\int_{B_A}\int_0^1 \frac{\varepsilon|h(x+r\sigma)-h(x)|^{p+\varepsilon}}{r^{p+1}}\,dr\,dx\,d\sigma \leqslant C(h),$$

where B_A denotes the ball centered at the origin of radius $A>0$.

In this proof, C will denote a constant independent of x, r, σ, and ε. Since $h\in C^\infty(\mathbb{R}^N)$,

$$\big|Dh(x)\cdot r\sigma\big| \leqslant \big|h(x+r\sigma)-h(x)\big| + Cr^2, \quad \forall(\sigma,x,r)\in\mathbb{S}^{N-1}\times B_A\times(0,1).$$

In other words, since $|h(x+r\sigma)-h(x)|\leqslant Cr$, for $(\sigma,x,r)\in\mathbb{S}^{N-1}\times B_A\times(0,1)$,

$$\big(\big|h(x+r\sigma)-h(x)\big| + Cr^2\big)^{p+\varepsilon} \leqslant \big|h(x+r\sigma)-h(x)\big|^{p+\varepsilon} + Cr^{p+\varepsilon+1},$$

for all $(\sigma,x,r)\in\mathbb{S}^{N-1}\times B_A\times(0,1)$.

Hence

$$\left|Dh(x)\cdot r\sigma\right|^{p+\varepsilon} \leqslant \left|h(x+r\sigma)-h(x)\right|^{p+\varepsilon} + Cr^{p+\varepsilon+1}, \tag{15}$$

for all $(\sigma, x, r) \in \mathbb{S}^{N-1} \times B_A \times (0,1)$.

Moreover,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{S}^{N-1}} \int_{B_A} \int_0^1 \frac{\varepsilon r^{p+\varepsilon+1}}{r^{p+1}}\, dr\, dx\, d\sigma = \lim_{\varepsilon\to 0} \int_{\mathbb{S}^{N-1}} \int_{B_A} \int_0^1 \varepsilon r^{\varepsilon} dr\, dx\, d\sigma = 0.$$

Thus it follows from (15) that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{\mathbb{S}^{N-1}} \int_{B_A} \int_0^1 \frac{\varepsilon|Dh(x)\cdot r\sigma|^{p+\varepsilon}}{r^{p+1}}\, dr\, dx\, d\sigma \\
&\quad \leqslant \liminf_{\varepsilon\to 0} \int_{\mathbb{S}^{N-1}} \int_{B_A} \int_0^1 \frac{\varepsilon|h(x+r\sigma)-h(x)|^{p+\varepsilon}}{r^{1+p}}\, dr\, dx\, d\sigma.
\end{aligned}$$

Consequently,

$$K_{N,p} \int_{B_A} \left|Dh(x)\right|^p dx \leqslant \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon|h(x)-h(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy.$$

Therefore, $h \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p} \int_{\mathbb{R}^N} \left|Dh(x)\right|^p dx \leqslant \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon|h(x)-h(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy. \qquad \square$$

2.2. *Proof of Theorem 3*

Step 1. Proof of assertion (a).

Let $g \in W^{1,p}(\mathbb{R}^N)$. By Lemma 2,

$$\int_{\mathbb{R}^N} \int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leqslant C_{N,p} \int_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx, \quad \forall \delta > 0. \tag{16}$$

Hereafter $C_{N,p}$ denotes a positive constant which can change from line to line but depends only on N and p.

As a consequence of (16),

$$\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx.$$

This proves part of statement (a).

Next, multiplying (16) by $\varepsilon\delta^{\varepsilon-1}$, $0<\varepsilon<1$, and integrating the expression obtained with respect to δ over $(0,1)$, one finds

$$\int_0^1\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\varepsilon\delta^{p+\varepsilon-1}}{|x-y|^{N+p}}\,dx\,dy\,d\delta \leqslant C_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx.$$

By Lemma 1,

$$\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{\varepsilon}{|x-y|^{N+p}}\,dx\,dy$$
$$\leqslant C_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx.$$

Hence

$$\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx.$$

The proof of assertion (a) of Theorem 3 is complete.

Step 2. Proof of assertion (c).

Assume that $g\in W^{1,p}(\mathbb{R}^N)$, $1<p<+\infty$. By Lemma 3,

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx$$

and

$$\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx, \quad \forall\delta>0.$$

It follows that

$$\lim_{\varepsilon\to 0}\int_0^1 (p+\varepsilon)\varepsilon\delta^{\varepsilon-1}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy\,d\delta = K_{N,p}\int_{\mathbb{R}^N}\bigl|\nabla g(x)\bigr|^p\,dx.$$

By Lemma 1, this implies

$$\lim_{\varepsilon\to 0}\Biggl(\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{\varepsilon}{|x-y|^{N+p}}\,dx\,dy\Biggr)$$
$$= K_{N,p}\int_{\mathbb{R}^N}\bigl|\nabla g(x)\bigr|^p\,dx.$$

Consequently,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\bigl|\nabla g(x)\bigr|^p\,dx.$$

Step 3. Proof of assertion (b).

We split the proof of Step 3 in two parts.

Case 1. Assume, in addition, that $g\in L^\infty(\mathbb{R}^N)$. Then, since

$$\sup_{0<\varepsilon<1}\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty,$$

one has

$$C(g) := \sup_{0<\varepsilon<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

We will use the method introduced by J. Bourgain, H. Brezis, P. Mironescu and the suggestion of E. Stein (see [3]).

Let (γ_r) be an any sequence of smooth mollifiers.

Set

$$g_r = g*\gamma_r.$$

From the convexity of the function $t^{p+\varepsilon}$ on the interval $[0,+\infty)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon|g_r(x)-g_r(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy.$$

Applying Lemma 4, one has $g_r \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N} \left|\nabla g_r(x)\right|^p dx \leqslant C(g).$$

Therefore, $g \in W^{1,p}(\mathbb{R}^N)$.

Case 2. The general case.

Define g_A, for $A>0$, as follows:

$$g_A(x) = \begin{cases} g(x) & \text{if } |g(x)| < A, \\ Ag(x)/|g(x)| & \text{otherwise.} \end{cases} \tag{17}$$

Then

$$\left|g_A(x)-g_A(y)\right| \leqslant \left|g(x)-g(y)\right| \quad \text{for all } x, y \in \mathbb{R}^N. \tag{18}$$

It is clear that

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g_A(x)-g_A(y)|\leqslant 1} \frac{\varepsilon|g_A(x)-g_A(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \\ &\quad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{\substack{|g(x)-g(y)|\leqslant 1\\ |g_A(x)-g_A(y)|\leqslant 1}} \frac{\varepsilon|g_A(x)-g_A(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{\substack{|g(x)-g(y)|>1\\ |g_A(x)-g_A(y)|\leqslant 1}} \frac{\varepsilon|g_A(x)-g_A(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy. \end{aligned}$$

Thus it follows from (18) that

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g_A(x)-g_A(y)|\leqslant 1} \frac{\varepsilon|g_A(x)-g_A(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \\ &\quad \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)|\leqslant 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>1} \frac{\varepsilon}{|x-y|^{N+p}}\,dx\,dy. \end{aligned}$$

Also, from (18),

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g_A(x)-g_A(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy \leqslant \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy.$$

Applying the previous case, one has $g_A \in W^{1,p}(\mathbb{R}^N)$.

As a consequence of Step 2,

$$\begin{aligned} &K_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla g_A(x)\right|^p dx \\ &\leqslant \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|<1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\geqslant 1}} \frac{\varepsilon}{|x-y|^{N+p}}\,dx\,dy. \end{aligned}$$

Therefore,

$$K_{N,p}\int\limits_{\mathbb{R}^N}\left|\nabla g_A(x)\right|^p dx \leqslant \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|<1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy.$$

Since $A > 0$ is arbitrary, it follows that $g \in W^{1,p}(\mathbb{R}^N)$.

2.3. *Proof of Theorem 2*

Step 1. Proof of assertion (a).

This is the conclusion of Lemma 2.

Step 2. Proof of assertion (c).

This is the conclusion of Lemma 3.

Step 3. Proof of assertion (b).

Let $g \in L^p(\mathbb{R}^N)$ be such that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leqslant C, \quad \forall 0<\delta<1, \tag{19}$$

for some positive constant C. We will prove that $g \in W^{1,p}(\mathbb{R}^N)$.

Multiplying inequality (19) by $\varepsilon\delta^{\varepsilon-1}$, $0<\varepsilon<1$, and integrating with respect to δ over $(0,1)$, by Lemma 1 one gets

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant C(p+1).$$

On the other hand, (19) gives

$$\int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N\\|g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Applying Theorem 3, one obtains $g \in W^{1,p}(\mathbb{R}^N)$.

Remark 2. Using the theory of maximal function (see [6, Chapter 1]), one knows that $\|Mf\|_{L^p(\mathbb{R})} \leqslant C\frac{2^p p}{p-1}\|f\|_{L^p(\mathbb{R})}$, where Mf denotes the maximal function of f and C is a universal constant. Therefore,

$$C_{N,p} \leqslant \frac{C_N}{p-1}, \quad \forall p \in (1,2), \tag{20}$$

where $C_{N,p}$ is the constant in Theorems 2 and 3, and C_N is a constant depending only on N. In fact, the bound for $C_{N,p}$ given in (20) is optimal for p near 1 in both Theorems 2 and 3.

Here is an example communicated to us by A. Ponce.

Let $g_p \in W^{1,p}(\mathbb{R})$ be defined as follows:

$$g_p(x) = \begin{cases} 0 & \text{if } x < 1-\tau, \\ \frac{1}{\tau}(x+\tau-1) & \text{if } 1-\tau \leqslant x < 1, \\ 1 & \text{if } 1 \leqslant x < 3-\tau, \\ 1+\frac{3-\tau-x}{\tau} & \text{if } 3-\tau \leqslant x < 3, \\ 0 & \text{if } x \geqslant 3, \end{cases}$$

where $\tau > 0$ depending only on p will be chosen later on.

Then

$$\int\limits_{\mathbb{R}}\int\limits_{\substack{\mathbb{R}\\|g_p(x)-g_p(y)|>1/2}} \frac{1}{|x-y|^{p+1}}\,dx\,dy \geqslant \int_0^{1-\tau}\int_1^2 \frac{1}{|x-y|^{p+1}}\,dx\,dy.$$

A direct computation yields

$$\int\limits_{\mathbb{R}}\int\limits_{\substack{\mathbb{R}\\|g_p(x)-g_p(y)|>1/2}} \frac{1}{|x-y|^{p+1}}\,dx\,dy \geqslant \frac{1}{p(p-1)}\big(\tau^{1-p}+2^{1-p}-(1+\tau)^{1-p}-1\big).$$

Now let $\tau = 3^{\frac{1}{1-p}}$. Then we have

$$\int\limits_{\mathbb{R}}\int\limits_{\substack{\mathbb{R}\\|g_p(x)-g_p(y)|>1/2}} \frac{1}{|x-y|^{p+1}} \geqslant \frac{1}{p(p-1)},$$

and

$$\int_{\mathbb{R}} \left|\nabla g_p(x)\right|^p = 2\tau^{1-p} = 6.$$

This gives the optimality of bound $C_{N,p}$ in the proof of Theorem 2 (see (20)).

On the other hand,

$$\int_{\mathbb{R}} \int_{\substack{\mathbb{R} \\ |g_p(x)-g_p(y)|\leqslant 1}} \frac{|g_p(x)-g_p(y)|^{p+1}}{|x-y|^{p+1}}\,dx\,dy \geqslant \int_{\mathbb{R}} \int_{\substack{\mathbb{R} \\ |g_p(x)-g_p(y)|>1/2}} \frac{1}{2^{p+1}|x-y|^{p+1}}\,dx\,dy$$
$$\geqslant \frac{1}{2^{p+1}p(p-1)}.$$

This implies the optimality of bound $C_{N,p}$ in the proof of Theorem 3 (see (20)).

Remark 3. A slightly stronger version of assertion (b) in Theorem 3 is true with the same proof: if $g \in L^p(\mathbb{R}^N)$ satisfies

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon_n |g(x)-g(y)|^{p+\varepsilon_n}}{|x-y|^{N+p}}\,dx\,dy + \int_{\mathbb{R}^N} \int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty,$$

for some sequence ε_n tending to 0, then $g \in W^{1,p}(\mathbb{R}^N)$.

A natural question in the same spirit is as follows. Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, and $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Does g belong to $W^{1,p}(\mathbb{R}^N)$?

The answer is positive but the argument is completely different and much more delicate (see [2]).

On the other hand, there is a natural question related to Γ-convergence. Let (g_n) be a sequence in $L^p(\mathbb{R})$ with $g_n \to g$ in $L^p(\mathbb{R})$, $1 < p < +\infty$. Assume that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} \int_{\substack{\mathbb{R} \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{p+1}}\,dx\,dy < +\infty,$$

for some sequence $\delta_n \to 0$.

Then one can show (see [4]) that $g \in W^{1,p}(\mathbb{R})$ and

$$c_p \int_{\mathbb{R}} \left|\nabla g(x)\right|^p dx \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}}\int_{\mathbb{R}}_{|g_n(x)-g_n(y)|>\delta_n} \frac{\delta_n^p}{|x-y|^{p+1}}\,dx\,dy, \tag{21}$$

for some constant $c_p > 0$ depending only on p. However, we have

Open question 1. *Can one replace c_p by $\frac{1}{p}K_{1,p}$ in* (21)?

One can raise similar questions in dimension $N \geqslant 2$.

3. Some variants and generalizations

3.1. Analogues for bounded domains

We first give an analogue of Lemma 3 for smooth bounded domains.

Lemma 5. *Let $g \in W^{1,p}(\Omega)$, $1 < p < +\infty$, Ω be an open set of $\mathbb{R}^N$. We have*

$$\liminf_{\delta\to 0} \int_{\Omega}\int_{\Omega}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geqslant \frac{1}{p}K_{N,p}\int_{\Omega} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ is defined by (12).

Moreover, if Ω is a smooth bounded domain then

$$\lim_{\delta\to 0} \int_{\Omega}\int_{\Omega}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int_{\Omega} \left|\nabla g(x)\right|^p dx.$$

Proof. Set, for $r > 0$ small,

$$\Omega_r = \left\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \geqslant r\right\}.$$

Applying the same method as in the proof of Lemma 3, one has

$$\lim_{\delta\to 0} \int_{\Omega_r}\int_{B_{r/2}}_{|g(x+h)-g(x)|>\delta} \frac{\delta^p}{|h|^{N+p}}\,dh\,dx = \frac{1}{p}K_{N,p}\int_{\Omega_r} \left|\nabla g(x)\right|^p dx. \tag{22}$$

Consequently,

$$\liminf_{\delta\to 0} \int_{\Omega}\int_{\Omega}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geqslant \frac{1}{p}K_{N,p}\int_{\Omega} \left|\nabla g(x)\right|^p dx.$$

Suppose now that Ω is a smooth bounded domain. Then there exists an extension $\tilde{g} \in W^{1,p}(\mathbb{R}^N)$ of g, i.e.,

$$\tilde{g}(x) = g(x), \quad \forall x \in \Omega.$$

Set, for $r > 0$,

$$\boldsymbol{\Omega}_r = \left\{x \in \mathbb{R}^N;\ \operatorname{dist}(x, \Omega) \leqslant r\right\}.$$

Applying the same method as in the proof of Lemma 3, one finds

$$\lim_{\delta \to 0} \int\limits_{\boldsymbol{\Omega}_r} \int\limits_{\substack{\boldsymbol{\Omega}_r \\ |\tilde{g}(x)-\tilde{g}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = \frac{1}{p} K_{N,p} \int\limits_{\boldsymbol{\Omega}_r} \left|\nabla \tilde{g}(x)\right|^p dx. \tag{23}$$

Combining (22) and (23) yields

$$\lim_{\delta \to 0} \int\limits_{\Omega} \int\limits_{\substack{\Omega \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = \frac{1}{p} K_{N,p} \int\limits_{\Omega} \left|\nabla g(x)\right|^p dx. \qquad \square$$

We present an analogue of Theorem 3 for smooth bounded domains.

Theorem 4. *Let $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Then*

(a) *For every $g \in W^{1,p}(\Omega)$,*

$$\sup_{0<\varepsilon<1} \int\limits_{\Omega} \int\limits_{\substack{\Omega \\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon |g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy + \int\limits_{\Omega} \int\limits_{\substack{\Omega \\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\, dx\, dy$$
$$\leqslant C \int\limits_{\Omega} \left|\nabla g(x)\right|^p dx,$$

where $C = C_{N,p,\Omega}$ is a positive constant depending only on N, p and Ω.

(b) *If $g \in L^p(\Omega)$ satisfies*

$$\sup_{0<\varepsilon<1} \int\limits_{\Omega} \int\limits_{\substack{\Omega \\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon |g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy + \int\limits_{\Omega} \int\limits_{\substack{\Omega \\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\, dx\, dy < +\infty,$$

then $g \in W^{1,p}(\Omega)$.

(c) *Moreover, for any $g \in W^{1,p}(\Omega)$,*

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega} \int\limits_{\substack{\Omega \\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon |g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy = K_{N,p} \int\limits_{\Omega} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ is defined by (12).

Proof.

Step 1. Proof of assertion (a).

Set

$$\hat{g}(x) = g(x) - \fint_{\Omega} g(y)\,dy, \quad \forall x \in \Omega.$$

Since $\hat{g} \in W^{1,p}(\Omega)$ and Ω is a smooth bounded domain, there exists $\tilde{g} \in W^{1,p}(\mathbb{R}^N)$ such that $\tilde{g}(x) = \hat{g}(x)$ for all $x \in \Omega$, and

$$\|\tilde{g}\|_{W^{1,p}(\mathbb{R}^N)} \leqslant C_\Omega \|\hat{g}\|_{W^{1,p}(\Omega)}.$$

Using Poincaré's inequality, one has

$$\|\hat{g}\|_{W^{1,p}(\Omega)} \leqslant C_\Omega \|\nabla \hat{g}\|_{L^p(\Omega)} = C_\Omega \|\nabla g\|_{L^p(\Omega)}.$$

Thus

$$\|\tilde{g}\|_{W^{1,p}(\mathbb{R}^N)} \leqslant C_\Omega \|\nabla g\|_{L^p(\Omega)}. \tag{24}$$

Clearly,

$$\int_{\Omega}\int_{\Omega}\limits_{|g(x)-g(y)|\leqslant 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|\tilde{g}(x)-\tilde{g}(y)|\leqslant 1} \frac{\varepsilon|\tilde{g}(x)-\tilde{g}(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy$$

and

$$\int_{\Omega}\int_{\Omega}\limits_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}}\,dx\,dy \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|\tilde{g}(x)-\tilde{g}(y)|>1} \frac{1}{|x-y|^{N+p}}\,dx\,dy.$$

On the other hand, from assertion (a) of Theorem 3 and (24),

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|\tilde{g}(x)-\tilde{g}(y)|\leqslant 1} \frac{\varepsilon|\tilde{g}(x)-\tilde{g}(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p,\Omega} \int_{\Omega} \left|\nabla g(x)\right|^p dx.$$

Hence

$$\int_{\Omega}\int_{\Omega}\limits_{|g(x)-g(y)|\leqslant 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p,\Omega} \int_{\Omega} \left|\nabla g(x)\right|^p dx. \tag{25}$$

Applying the same method used to obtain (25), one finds

$$\int_{\Omega}\int_{\Omega}_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p,\Omega}\int_{\Omega}\left|\nabla g(x)\right|^p\,dx.$$

Step 2. Proof of assertion (c).

Applying the same method as in the proof of Theorem 3, Step 2, the conclusion of Step 2 follows from Lemma 5.

Step 3. Proof of assertion (b).

Case 1. Assume, in addition, that $g \in L^{\infty}(\Omega)$.

Since $g \in L^{\infty}(\Omega)$ and

$$\sup_{0<\varepsilon<1} \int_{\Omega}\int_{\Omega}_{|g(x)-g(y)|\leqslant 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int_{\Omega}\int_{\Omega}_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty,$$

one has

$$C(g) := \sup_{0<\varepsilon<1}\int_{\Omega}\int_{\Omega}\frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Set

$$\Omega_\tau = \left\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \geqslant \tau\right\}.$$

Let (γ_r) be an any sequence of radial mollifiers such that $\operatorname{supp}\gamma_r \subset B_r$, where B_r denotes the ball with center at 0 and radius r.

For any $0 < r < \tau/2$, set

$$g_r(x) = g * \gamma_r(x), \quad \text{for all } x \in \Omega_{\tau/2}.$$

From the convexity of function $t^{p+\varepsilon}$,

$$\int_{\Omega_{\tau/2}}\int_{\Omega_{\tau/2}}\frac{|g_r(x)-g_r(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant \int_{\Omega}\int_{\Omega}\frac{|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy.$$

It follows that

$$\int_{\Omega_\tau}\int_{B_{\tau/2}}\frac{\varepsilon|g_r(x+h)-g_r(x)|^{p+\varepsilon}}{|h|^{N+p}}\,dh\,dx \leqslant C(g).$$

Using the same method as in the proof of Lemma 4, one has

$$K_{N,p}\int\limits_{\Omega_\tau}\left|\nabla g_r(x)\right|^p dx \leqslant C(g).$$

Let r tend to 0, one deduces that $g \in W^{1,p}(\Omega_\tau)$ and

$$K_{N,p}\int\limits_{\Omega_\tau}\left|\nabla g(x)\right|^p dx \leqslant C(g), \quad \forall\tau > 0.$$

Consequently, $g \in W^{1,p}(\Omega)$.

Case 2. The general case.

For each $A > 0$, define g_A as in (17). By the same method as in the proof of Theorem 3 (see Case 2 of Step 3), one has $g_A \in W^{1,p}(\Omega)$ and

$$\begin{aligned}&\int\limits_{\Omega}\int\limits_{\substack{\Omega\\ |g_A(x)-g_A(y)|\leqslant 1}} \frac{\varepsilon|g_A(x)-g_A(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy\\ &\quad\leqslant \int\limits_{\Omega}\int\limits_{\substack{\Omega\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy + \int\limits_{\Omega}\int\limits_{\substack{\Omega\\ |g(x)-g(y)|> 1}} \frac{\varepsilon}{|x-y|^{N+p}}\,dx\,dy.\end{aligned}$$

Using the result of Step 2, one has

$$K_{N,p}\int\limits_{\Omega}\left|\nabla g_A(x)\right|^p dx \leqslant \liminf_{\varepsilon\to 0}\int\limits_{\Omega}\int\limits_{\substack{\Omega\\ |g(x)-g(y)|\leqslant 1}} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy \leqslant C(g).$$

Since $A > 0$ is arbitrary, one has $g \in W^{1,p}(\Omega)$. □

We now establish an analogue of Theorem 2 for smooth bounded domains.

Theorem 5. *Let $g \in L^p(\Omega)$, $1 < p < +\infty$, and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We have*:

(a) *If $g \in W^{1,p}(\Omega)$, then there exists a constant $C = C_{N,p,\Omega}$, independent of g, such that*

$$\int\limits_{\Omega}\int\limits_{\substack{\Omega\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leqslant C\int\limits_{\Omega}\left|\nabla g(x)\right|^p dx, \quad \forall\delta > 0.$$

(b) *If*

$$\sup_{0<\delta<1} \iint\limits_{\substack{\Omega\ \Omega \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < +\infty,$$

then $g \in W^{1,p}(\Omega)$.

(c) *Moreover, for all* $g \in W^{1,p}(\Omega)$,

$$\lim_{\delta\to 0} \iint\limits_{\substack{\Omega\ \Omega \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{1}{p}K_{N,p}\int\limits_{\Omega} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ *is defined in* (12).

Proof.

Step 1. Proof of assertion (a).

Applying the same approach as in the proof of Theorem 4, Step 1, the conclusion of assertion (a) follows from Theorem 2.

Step 2. Proof of assertion (b).

By the same method as in the proof Theorem 2, Step 2, the conclusion of assertion (b) is a consequence of Theorem 4.

Step 3. Proof of assertion (c).

This is the conclusion of Lemma 5. □

3.2. A generalized version of Theorem 2

We present here a generalized form of Theorem 2.

Theorem 6. *Let* $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, *D be a countable closed subset of* $(0,+\infty)$, *and* $\varphi : [0,+\infty) \to [0,+\infty)$ *be such that* φ *is continuous on* $[0,+\infty) \setminus D$ *and*

$$\int\limits_0^\infty \varphi(t)t^{-(p+1)}\,dt = 1. \tag{26}$$

Set

$$\varphi_\delta(t) = \delta^p\varphi(t/\delta), \quad \forall\delta > 0. \tag{27}$$

We have

(a) *If*

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy < +\infty \tag{28}$$

and

$$\int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N\\ |g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty, \quad \forall\delta > 0, \tag{29}$$

then $g \in W^{1,p}(\mathbb{R}^N)$ *and*

$$K_{N,p}\int_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx \leqslant \liminf_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy. \tag{30}$$

(b) *If* $g \in W^{1,p}(\mathbb{R}^N)$ *and* $\tilde{\varphi}$, *defined by*

$$\tilde{\varphi}(t) = \sup_{0\leqslant s\leqslant t} \varphi(s),$$

satisfies $\int_0^\infty \tilde{\varphi}(t)t^{-(p+1)}\,dt < +\infty$, *then*

(i) $$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy \leqslant C_{N,p}\int_0^\infty \tilde{\varphi}(t)t^{-(p+1)}\,dt \int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx, \quad \forall\delta > 0,$$

(ii) $$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx, \tag{31}$$

where $K_{N,p}$ *is defined by* (12) *and* $C_{N,p}$ *is a positive constant depending only on* N *and* p.

Proof.

Step 1. Proof of assertion (a).

We first prove that $g \in W^{1,p}(\mathbb{R}^N)$.

Since φ is nonnegative and

$$\int_0^\infty \varphi(t)t^{-(p+1)}\,dt = 1,$$

we claim that there exist four positive constants m, M, λ, and σ, $m < M$, such that

$$\operatorname{meas}\left\{t \in [m, M];\ \varphi(t) \geqslant \lambda\right\} \geqslant \sigma. \tag{32}$$

In fact, since

$$\int_0^\infty \varphi(t)t^{-(p+1)}\,dt = 1,$$

there exist two positive constants m, M, $m < M$, such that

$$\int_m^M \varphi(t)t^{-(p+1)}\,dt \geqslant \frac{1}{2}.$$

Thus

$$\operatorname{meas}\bigl\{t \in [m, M];\ \varphi(t) > 0\bigr\} > 0.$$

Hence there exist two positive numbers λ and σ such that

$$\operatorname{meas}\bigl\{t \in [m, M];\ \varphi(t) > \lambda\bigr\} \geqslant \sigma.$$

Therefore, (32) is proved.

Since φ is continuous on $[0, +\infty) \setminus D$ and D is a countable closed subset of $(0, +\infty)$, there exists an interval $A \neq \emptyset$ such that

$$A \subset \bigl\{t \in [m, M];\ \varphi(t) > \lambda\bigr\}.$$

Let χ_A denote the characteristic function of the set A, i.e.,

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, from (28),

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\delta^p \chi_A(|g(x) - g(y)|/\delta)}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

This implies

$$\sup_{0<\varepsilon<1} \int_0^1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varepsilon\delta^{p+\varepsilon-1} \chi_A(|g(x) - g(y)|/\delta)}{|x-y|^{N+p}}\,dx\,dy\,d\delta < +\infty.$$

By Fubini's theorem, it follows that

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|\leqslant m}} \frac{1}{|x-y|^{N+p}} \int_0^1 \varepsilon\delta^{p+\varepsilon-1} \chi_A\bigl(\bigl|g(x) - g(y)\bigr|/\delta\bigr)\,d\delta\,dx\,dy < +\infty.$$

Noting that $\delta^{p+\varepsilon-1} \geqslant M^{-p-\varepsilon+1}|g(x) - g(y)|^{p+\varepsilon-1}$ whenever $M \geqslant |g(x) - g(y)|/\delta$, we infer

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|\leqslant m}} \frac{\varepsilon|g(x) - g(y)|^{p+\varepsilon-1}}{|x-y|^{N+p}} \int_0^1 \chi_A\bigl(\bigl|g(x) - g(y)\bigr|/\delta\bigr)\,d\delta\,dx\,dy < +\infty.$$

However, since $A \subset [m, M]$,

$$\int_0^1 \chi_A(t/\delta)\,d\delta = \int_0^\infty \chi_A(t/\delta)\,d\delta = t\int_0^\infty \chi_A(1/\delta)\,d\delta = C(A)t, \quad \forall t \leqslant m,$$

where

$$C(A) := \int_0^\infty \chi_A(1/\delta)\,d\delta > 0.$$

Combining the latter two estimates yields

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|\leqslant m} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

On the other hand, it follows from (29) that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|>m} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Thus g_m defined by $g_m(x) = g(x)/m$ for all $x \in \mathbb{R}^N$ verifies the hypotheses of part (b) of Theorem 3. Hence $g_m \in W^{1,p}(\mathbb{R}^N)$. Consequently, $g \in W^{1,p}(\mathbb{R}^N)$.

It remains to prove (30). From the change of variables formula and the definition of φ_δ,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = \int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty \frac{\delta^p\varphi(|g(x+h\sigma)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx\,d\sigma.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty &\frac{\delta^p\varphi(|g(x+h\sigma)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx\,d\sigma \\ &= \int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty \frac{\varphi(|g(x+\delta h\sigma)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx\,d\sigma.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = \int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty \frac{\varphi(|g(x+\delta h\sigma)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx\,d\sigma. \quad (33)$$

Therefore, (30) follows from (33), the continuity of φ on $[0,+\infty)\setminus D$, and Fatou's lemma.

Step 2. Proof of assertion (b).

We claim that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_0^\infty \frac{\varphi(|g(x+\delta h\sigma)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx=\int_{\mathbb{R}^N}\left|\nabla g(x)\cdot\sigma\right|^p dx \tag{34}$$

and

$$\int_{\mathbb{R}^N}\int_0^\infty \frac{\varphi(|g(x+\delta h\sigma)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx\leqslant C_p\int_0^\infty \tilde{\varphi}(t)t^{-(p+1)}\int_{\mathbb{R}^N}\left|\nabla g(x)\right|^p dx,\quad \forall\delta>0, \tag{35}$$

where C_p is a positive constant depending only on p.

From $g\in W^{1,p}(\mathbb{R}^N)$ we have $g(x',\cdot)\in W^{1,p}(\mathbb{R})$, for almost everywhere $x'=(x_1,\dots,x_{N-1})\in\mathbb{R}^{N-1}$.

Fix $x'\in\mathbb{R}^{N-1}$ such that $g(x',\cdot)\in W^{1,p}(\mathbb{R})$. Without loss of generality, suppose that

$$g(x+he_N)-g(x)=\int_{x_N}^{x_N+h}\frac{\partial g}{\partial x_N}(x',s)\,ds,\quad \forall(x_N,h)\in\mathbb{R}\times(0,+\infty).$$

Then

$$\lim_{\delta\to 0}\frac{g(x',x_N+\delta h)-g(x',x_N)}{\delta}=\lim_{\delta\to 0}h\fint_{x_N}^{x_N+\delta h}\frac{\partial g}{\partial x_N}(x',s)\,ds=h\frac{\partial g}{\partial x_N}(x',x_N),$$

for almost everywhere $x_N\in\mathbb{R}$.

Consequently,

$$\lim_{\delta\to 0}\varphi\big(\left|g(x',x_N+\delta h)-g(x',x_N)\right|/\delta\big)=\varphi\left(h\left|\frac{\partial g}{\partial x_N}(x',x_N)\right|\right)(x',x_N), \tag{36}$$

for almost everywhere $(x_N,h)\in\mathbb{R}\times(0,+\infty)$.

Here the continuity of φ on $[0,+\infty)\setminus D$ and $D\subset(0,+\infty)$ is used.

Note that

$$\frac{|g(x',x_N+\delta h)-g(x',x_N)|}{\delta}\leqslant h\fint_{x_N}^{x_N+\delta h}\left|\frac{\partial g}{\partial x_N}(x',s)\right|ds\leqslant hM_N\left(\frac{\partial g}{\partial x_N}\right)(x',x_N),$$

where $M_N(f)$ is defined in (7).

Then one deduces from the definition of $\tilde{\varphi}$ that

$$\varphi\big(\big|g(x', x_N + \delta h) - g(x', x_N)\big|/\delta\big) \leqslant \tilde{\varphi}\big(\big|g(x', x_N + \delta h) - g(x', x_N)\big|/\delta\big) \leqslant \tilde{\varphi}\bigg(hM_N\bigg(\frac{\partial g}{\partial x_N}\bigg)(x', x_N)\bigg). \tag{37}$$

On the other hand,

$$\int_{\mathbb{R}^N}\int_0^\infty \tilde{\varphi}\bigg(hM_N\bigg(\frac{\partial g}{\partial x_N}\bigg)(x', x_N)\bigg)h^{-(p+1)}\,dh\,dx = \int_{\mathbb{R}^N}\bigg|M_N\bigg(\frac{\partial g}{\partial x_N}\bigg)(x', x_N)\bigg|^p\,dx\int_0^\infty \tilde{\varphi}(t)t^{-(p+1)}\,dt. \tag{38}$$

Moreover, one has (see (9))

$$\int_{\mathbb{R}^N}\bigg|M_N\bigg(\frac{\partial g}{\partial x_N}\bigg)(x)\bigg|^p\,dx \leqslant C_p\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx. \tag{39}$$

Since

$$\int_0^\infty \tilde{\varphi}(t)t^{-(p+1)}\,dt < +\infty,$$

combining (36)–(39), after applying Lebesgue's dominated convergence theorem, one obtains (34) and (35) with $\sigma = e_N$.

As a consequence of (34), (35) and Lebesgue's dominated convergence theorem,

$$\lim_{\delta\to 0}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty \frac{\varphi(|g(x+\delta h\sigma) - g(x)|/\delta)}{h^{p+1}}\,dh\,dx\,d\sigma = \int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\big|\nabla g(x)\cdot\sigma\big|^p\,dx\,d\sigma. \tag{40}$$

We recall that (see [1]) that

$$\int_{\mathbb{S}^{N-1}} |V\cdot\sigma|^p\,d\sigma = K_{N,p}|V|^p.$$

Therefore, from (33) and (40),

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi_\delta(|g(x) - g(y)|)}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}\big|\nabla g(x)\big|^p\,dx.$$

Thus 44(ii) is proved.

On the other hand, the estimate 31(i) follows from (33) and (35).

The proof of Theorem 6 is complete. □

Example 1. Assume that $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, satisfies

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ \delta<|g(x)-g(y)|<2\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Then $g \in W^{1,p}(\mathbb{R}^N)$ and

$$\lim_{\delta\to 0} \int\limits_{\mathbb{R}^N}\int\limits_{\substack{\mathbb{R}^N \\ \delta<|g(x)-g(y)|<2\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{2^p-1}{p2^p} K_{N,p} \int\limits_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ is defined by (12).

Proof. We verify the hypotheses of Theorem 6 with

$$\varphi(t) = \begin{cases} \frac{p2^p}{2^p-1} & \text{if } 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$

A direct computation gives

$$\int_0^\infty \varphi(t) t^{-(p+1)}\,dt = 1.$$

Moreover,

$$\tilde{\varphi}(t) := \sup_{0\leqslant s\leqslant t} \varphi(s) = \begin{cases} \frac{p2^p}{2^p-1} & \text{if } t > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\int_0^\infty \tilde{\varphi}(t) t^{-(p+1)} = \frac{p2^p}{2^p-1} \int_1^\infty t^{-(p+1)} = \frac{2^p}{2^p-1} < +\infty. \qquad \square$$

Remark 4. If the assumption of $\tilde{\varphi}$ in Theorem 6 fails, i.e.,

$$\int_0^\infty \tilde{\varphi}(t) t^{-(p+1)}\,dt = +\infty,$$

then the conclusion in part (b) of Theorem 6 may fail.

Here is an example. Let $(t_n)_{n\geqslant 1}$, $(\varepsilon_n)_{n\geqslant 1}$ be two sequences of positive numbers to be defined later. Consider $\varphi:[0,+\infty)\to[0,+\infty)$ and $g\in W^{1,p}(\mathbb{R})$ are defined as follows:

$$\varphi(h)=\begin{cases} t_n & \text{if } |h-n|\leqslant\varepsilon_n \text{ for some } n\in\mathbb{Z}_+,\\ 0 & \text{otherwise,}\end{cases} \tag{41}$$

and

$$g(x)=\begin{cases} 0 & \text{if } x\leqslant 0 \text{ or } x>3,\\ x & \text{if } x\in(0,1],\\ 1 & \text{if } x\in(1,2],\\ 3-x & \text{if } x\in(2,3].\end{cases} \tag{42}$$

Proposition 2. *Let φ, g be the functions defined by* (41), (42), *and φ_δ be a function defined by* (27), *for all* $0<\delta<1$.

(a) *Let $t_n=an^p$, $\varepsilon_n=n^{-(p+2)}$, for all $n\geqslant 1$ where a is a positive constant such that $\int_0^\infty\varphi(h)h^{-(p+1)}\,dh=1$. Then φ and g verify the hypotheses of assertion* (a) *of Theorem* 6. *However,*

$$K_{1,p}\int_{\mathbb{R}}\left|\nabla g(x)\right|^p dx<\liminf_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\varphi_{1/n}(|g(x)-g(y)|)}{|x-y|^{p+1}}\,dx\,dy.$$

(b) *Let $t_n=bn^{p+1}$, $\varepsilon_n=n^{-(p+3)}$, for all $n\geqslant 1$ where b is a positive constant such that $\int_0^\infty\varphi(h)h^{-(p+1)}\,dh=1$. Then*

$$\limsup_{\delta\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{p+1}}\,dx\,dy=+\infty. \tag{43}$$

Proof.

Step 1. Proof of assertion (a).

A direct computation gives

$$\sup_{0<\delta<1}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\varphi_\delta(|g(x)-g(y)|)}{|x-y|^{p+1}}\,dx\,dy<+\infty.$$

On the other hand,

$$\int_{-\infty}^{0}\int_{0}^{\infty}\limits_{1<x+\delta h<2}\frac{\varphi(|g(x+\delta h)-g(x)|/\delta)}{h^{p+1}}\,dh\,dx\geqslant\delta^p\varphi(1/\delta)\int_{-\infty}^{0}\frac{1}{(2+|x|)^{p+1}}\,dh\,dx. \tag{44}$$

Thus the conclusion of assertion (a) is a consequence of (44), Fatou's lemma and the fact that

$$\int_{-\infty}^{0}\left|g'(x)\right|^p dx=0.$$

Step 2. Proof of assertion (b).

Take $\delta = 1/n$ in inequality (44); (43) follows from the choice of t_n ($t_n = bn^{p+1}$). □

The following result, whose proof is given in [4], is a natural generalization of Theorems 2 and 3.

Theorem 7. *Let* $1 < p < +\infty$ *and* $(F_\delta)_{0<\delta<1}$ *be a family of functions from* $[0, +\infty)$ *into* $[0, +\infty)$ *such that*:

(i) $F_\delta(t)$ *is non-decreasing function with respect to* t *on* $[0, +\infty)$, *for all* $0 < \delta < 1$.
(ii) $\int_0^1 F_\delta(t)t^{-(p+1)}\,dt = 1$, *for all* $0 < \delta < 1$.
(iii) $F_\delta(t)$ *converges uniformly to* 0 *on every compact subset of* $(0, +\infty)$ *when* δ *goes to* 0; *and*

$$\sup_{0<\delta<1} \int_0^\infty F_\delta(t)t^{-(p+1)}\,dt < +\infty.$$

Then

(a) *If* $g \in W^{1,p}(\mathbb{R}^N)$, *then*

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\delta(|g(x) - g(y)|)}{|x - y|^{N+p}}\,dx\,dy \leqslant C_{N,p} \sup_{0<\delta<1} \int_0^\infty F_\delta(t)t^{-(p+1)}\,dt \int_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx,$$

where $C_{N,p}$ *is a positive constant depending only on* N *and* p.

(b) *If* $g \in L^p(\mathbb{R}^N)$ *satisfies*

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\delta(|g(x) - g(y)|)}{|x - y|^{N+p}}\,dx\,dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

(c) *Moreover, for any* $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\delta\to 0} \underset{|x-y|<1}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{F_\delta(|g(x) - g(y)|)}{|x - y|^{N+p}}\,dx\,dy = K_{N,p} \int_{\mathbb{R}^N} \left|\nabla g(x)\right|^p dx,$$

where $K_{N,p}$ *is defined by* (12).

4. The case $p = 1$

We emphasize that in Theorem 2 we assumed that $1 < p < +\infty$. If (2) holds with $p = 1$, then one can still conclude that $g \in BV(\mathbb{R}^N)$ (see Theorem 8). However, (1) and (3) are no longer true. In fact, there exists a function $g \in W^{1,1}(\mathbb{R})$ such that (see Proposition 3)

$$\lim_{\delta \to 0} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^2} \, dx \, dy = +\infty.$$

The following property is obtained by the same method as in the proof of Theorem 2.

Theorem 8. *Let $g \in L^1(\mathbb{R}^N)$ be such that*

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^{N+1}} \, dx \, dy < +\infty.$$

Then $g \in BV(\mathbb{R}^N)$ and

$$K_{N,1} \|\nabla g\| \leqslant \limsup_{\delta \to 0} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^{N+1}} \, dx \, dy,$$

where $K_{N,1}$ is defined by (12) *with $p = 1$ and $\|\nabla g\|$ denotes the total mass of ∇g.*

Remark 5. Under the assumption of Theorem 8 we also have, when $N = 1$,

$$c \|\nabla g\| \leqslant \liminf_{\delta \to 0} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^2} \, dx \, dy, \tag{45}$$

for some universal constant $c > 0$ (the proof uses the ideas introduced in [2]). However, we have

Open question 2. *Can one replace c by $K_{1,1}$ in* (45)?

One can also ask similar questions for $N \geqslant 2$.

The following proposition is due to A. Ponce (personal communication).

Proposition 3. *There exists a function $g \in W^{1,1}(\mathbb{R})$ such that*

$$\sup_{0<\delta<1} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^2} \, dx \, dy = +\infty.$$

Proof. It suffices to construct a function $g \in W^{1,1}(0,1)$ such that

$$\lim_{\delta \to 0} \int_0^1 \int_0^1_{|g(x)-g(y)| \geqslant \delta} \frac{\delta}{|x-y|^2}\,dx\,dy = +\infty.$$

Let $a, b \in \mathbb{R}$, $a < b$, and c be the middle point of the interval $[a,b]$, $c = \frac{a+b}{2}$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to 0.

Then

$$\lim_{n \to \infty} \int_a^{c-\varepsilon_n} \int_{c+\varepsilon_n}^b \frac{1}{|x-y|^2}\,dx\,dy = +\infty. \tag{46}$$

Set

$$\delta_n = \frac{1}{2^n}, \qquad m_n = \frac{\delta_n + \delta_{n+1}}{2}.$$

In view of (46), it is possible to chose ε_n such that

$$\int_{\delta_{n+1}}^{m_n-\varepsilon_n} \int_{m_n+\varepsilon_n}^{\delta_n} \frac{\delta_n}{|x-y|^2}\,dx\,dy \geqslant n.$$

The desired function $g : [0,1] \to \mathbb{R}$ will be defined as follows:

$$g(x) = \begin{cases} \delta_n & \text{if } x \in [m_n + \varepsilon_n, \delta_n], \\ \delta_{n+1} & \text{if } x \in [\delta_{n+1}, m_n - \varepsilon_n], \end{cases}$$

and g is linear on $[m_n - \varepsilon_n, m_n + \varepsilon_n]$. $\square$

Open question 3. *Characterize the functions $g \in L^1(\mathbb{R}^N)$ such that*

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy < +\infty.$$

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