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Partial Differential Equations

Inequalities related to liftings and applications

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Abstract

We present two inequalities for liftings of smooth maps from the torus $\mathbb{T}^d$ into the unit circle $\mathbb{S}^1$. Using these inequalities, we answer a question of J. Bourgain, H. Brezis, and P. Mironescu in [J. Bourgain, H. Brezis, P. Mironescu, Lifting, degree, and distributional Jacobian revisited, Comm. Pure Appl. Math. 58 (2005) 529–551] and establish an estimate of liftings in the spirit of R.R. Coifman and Y. Meyer [R.R. Coifman, Y. Meyer, Une généralisation du théorème de Calderon sur l'intégrale de Cauchy, in: Fourier Analysis, in: Proc. Sem., El Escorial, vol. 1, Asoc. Mat. Espa nola, Madrid, 1980, pp. 87–116]. ***To cite this article: H.-M. Nguyen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Inégalités relatives aux relèvements et applications. Nous présentons deux inégalités pour des relèvements des applications régulières du tore $\mathbb{T}^d$ dans le cercle unité $\mathbb{S}^1$. Ces inégalités nous permettent de répondre à une question de J. Bourgain, H. Brezis, et P. Mironescu dans [J. Bourgain, H. Brezis, P. Mironescu, Lifting, degree, and distributional Jacobian revisited, Comm. Pure Appl. Math. 58 (2005) 529–551] et d'établir une estimation des relèvements dans l'esprit de R.R. Coifman et Y. Meyer [R.R. Coifman, Y. Meyer, Une généralisation du théorème de Calderon sur l'intégrale de Cauchy, in : Fourier Analysis, in : Proc. Sem., El Escorial, vol. 1, Asoc. Mat. Espa nola, Madrid, 1980, pp. 87–116]. ***Pour citer cet article : H.-M. Nguyen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

J. Bourgain et H. Brezis [1] (voir aussi [2]) ont prouvé

Théorème 1. *Soient $d \geqslant 1$ un entier, $\mathbb{T}^d$ le tore de dimension d, $\psi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, et $g = e^{i\psi}$. Alors il existe $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$ telles que $\psi = \psi_1 + \psi_2$,*

$$|\psi_1|_{W^{1,1}} \leqslant C|g|^2_{H^{\frac{1}{2}}} \quad \textit{et} \quad |\psi_2|_{H^{\frac{1}{2}}} \leqslant C|g|_{H^{\frac{1}{2}}},$$

pour une certaine constante $C > 0$ qui ne dépend que de d.

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Dans cette Note, on considère la semi-norme suivante de l'espace $W^{s,p}(\mathbb{T}^d)$ ($0 < s < 1$ et $p > 1$) défini par (1) et on note $H^{\frac{1}{2}} = W^{\frac{1}{2},2}$.

Question. Soient $d \geqslant 1$ un entier, $\mathbb{T}^d$ le tore de dimension d, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, et $g = e^{i\psi}$. Est-ce qu'il existe $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$ telles que $\psi = \psi_1 + \psi_2$,

$$|\psi_1|_{W^{1,1}} \leqslant C|g|^p_{W^{\frac{1}{p},p}} \quad \text{et} \quad |\psi_2|_{W^{\frac{1}{p},p}} \leqslant C|g|_{W^{\frac{1}{q},q}},$$

pour une certaine constante $C > 0$ qui ne dépend que de d et de p ?

Le but principal de la Note est de donner une réponse positive à cette question au cas $d = 1$. En fait, on prouve :

Théorème 2. *Soient $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^1, \mathbb{R})$, et $g = e^{i\psi}$. Alors il existe $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^1, \mathbb{R})$ telles que $\psi = \psi_1 + \psi_2$,*

$$|\psi_1|_{W^{1,1}} \leqslant C\mathbf{T}_{\sqrt{3}}(g) \quad \text{et} \quad |\psi_2|_{W^{\frac{1}{p},p}} \leqslant C|g|_{W^{\frac{1}{q},q}},$$

pour une certaine constante $C > 0$ qui ne dépend que de d et de p. Ici T_g est défini par (2).

Un autre résultat de la Note est

Théorème 3. *Soient $d \geqslant 1$, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^d; \mathbb{R})$, et $g = e^{i\psi}$. Supposons que $|g|_{\mathrm{BMO}} = \beta < 1$. Alors il existe une constante $C > 0$ qui ne depend que de d et de p telle que*

$$|\psi|_{W^{\frac{1}{p},p}} \leqslant \frac{C}{1-\beta}|g|_{W^{\frac{1}{p},p}}.$$

Ici la BMO-semi-norme est définie par (4).

1. Introduction and main results

J. Bourgain and H. Brezis [1] (see also [2]) proved

Theorem 1. *Let $d \geqslant 1$, $\mathbb{T}^d$ be the d-dimensional torus, $\psi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, and $g = e^{i\psi}$. Then there exist $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$ such that $\psi = \psi_1 + \psi_2$,*

$$|\psi_1|_{W^{1,1}} \leqslant C|g|^2_{H^{\frac{1}{2}}} \quad \text{and} \quad |\psi_2|_{H^{\frac{1}{2}}} \leqslant C|g|_{H^{\frac{1}{2}}},$$

for some constant $C > 0$ depending only on d.

In this Note, one considers the following semi-norm of $W^{s,p}(\mathbb{T}^d)$, for $0 < s < 1$ and $p > 1$,

$$|g|_{W^{s,p}} = \left(\iint\limits_{\mathbb{T}^d \times \mathbb{T}^d} \frac{|g(x) - g(y)|^p}{|x-y|^{d+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}, \tag{1}$$

and one notes $H^{\frac{1}{2}} = W^{\frac{1}{2},2}$.

Their proof is delicate. Using the inequality (see e.g. [7])

$$\left| \int\limits_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \zeta \, \mathrm{d}x \right| \leqslant C\big(\|\zeta\|_{L^\infty} + |\zeta|_{H^{\frac{1}{2}}}\big)\big(|g|^2_{H^{\frac{1}{2}}} + |g|_{H^{\frac{1}{2}}}\big)$$

where $g = e^{i\psi}$ and $\psi \in C^1(\mathbb{T}^d, \mathbb{R})$ and a dual argument one can prove that there exist ψ_1 and ψ_2 such that $\psi = \psi_1 + \psi_2$, $|\psi_1|_{BV} + |\psi_2|_{H^{\frac{1}{2}}} \leqslant C(|g|^2_{H^{\frac{1}{2}}} + |g|_{H^{\frac{1}{2}}})$ in the case $d = 1$ (argument communicated to us by H. Brezis and P. Mironescu). From these facts, J. Bourgain, H. Brezis, and P. Mironescu suggested the following (see [3, Open Problem 1])

Question. Let $d \geqslant 1$ be an integer, $\mathbb{T}^d$ be the d-dimensional torus, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, and $g = e^{i\psi}$. Does there exist $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$ such that $\psi = \psi_1 + \psi_2$,

$$|\psi_1|_{W^{1,1}} \leqslant C|g|^p_{W^{\frac{1}{p},p}} \quad \text{and} \quad |\psi_2|_{W^{\frac{1}{p},p}} \leqslant C|g|_{W^{\frac{1}{q},q}},$$

for some constant $C > 0$ depending only on d and p?

The main goal of this Note is to give an affirmative answer to this question in the case $d = 1$. We first introduce

Definition 1. Let $d \geqslant 1$ be an integer, $\mathbb{T}^d$ be the d-dimensional torus, $g \in L^\infty(\mathbb{T}^d, \mathbb{S}^1)$ and $\delta > 0$. Define

$$\mathbf{T}_\delta(g) = \iint\limits_{\substack{\mathbb{T}^d \times \mathbb{T}^d \\ |g(x)-g(y)| \geqslant \delta}} \frac{1}{|x-y|^{d+1}} \, \mathrm{d}x \, \mathrm{d}y. \tag{2}$$

We next prove two inequalities related to liftings.

Lemma 1. *Let $d \geqslant 1$, $p > 1$, $q = \frac{p}{p-1}$, $\zeta \in C^1(\mathbb{T}^d; \mathbb{R})$, $\psi \in C^1(\mathbb{T}^d; \mathbb{R})$, and $g = e^{i\psi}$. Then*

$$\left| \int\limits_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \zeta \, \mathrm{d}x \right| \leqslant C\big(\|\zeta\|_{L^\infty} \mathbf{T}_{\sqrt{3}}(g) + |\zeta|_{W^{\frac{1}{q},q}} |g|_{W^{\frac{1}{p},p}}\big), \tag{3}$$

for some constant $C > 0$ depending only on d and p.

Lemma 2. *Let $d \geqslant 1$, $p > 1$, $q = \frac{p}{p-1}$, $\zeta \in C^1(\mathbb{T}^d; \mathbb{R})$, $\psi \in C^1(\mathbb{T}^d; \mathbb{R})$, and $g = e^{i\psi}$. Assume that $|g|_{\mathrm{BMO}} = \beta < 1$. Then there exists a constant $C > 0$ depending only on d and p such that*

$$\left| \int\limits_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \zeta \, \mathrm{d}x \right| \leqslant \frac{C}{1-\beta} |\zeta|_{W^{\frac{1}{q},q}} |g|_{W^{\frac{1}{p},p}}.$$

Hereafter, we use the following BMO-semi-norm:

$$|f|_{\mathrm{BMO}(\Omega)} := \sup_{B(x,r) \Subset \Omega} \ \fint\limits_{B(x,r)} \left| f(\xi) - \fint\limits_{B(x,r)} f(\eta) \, \mathrm{d}\eta \right| \mathrm{d}\xi, \quad \forall f \in \mathrm{BMO}(\Omega). \tag{4}$$

Here $B(x,r)$ denotes the ball in Ω of radius r centered at x.

Using Lemma 1 and the dual argument, we can prove

Theorem 2. *Let $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^1, \mathbb{R})$, and $g = e^{i\psi}$. Then there exist $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^1, \mathbb{R})$ such that $\psi = \psi_1 + \psi_2$,*

$$|\psi_1|_{W^{1,1}} \leqslant C\mathbf{T}_{\sqrt{3}}(g) \quad \textit{and} \quad |\psi_2|_{W^{\frac{1}{p},p}} \leqslant C|g|_{W^{\frac{1}{q},q}},$$

for some constant $C > 0$ depending only on p.

This theorem answers positively [3, Open Problem 1], which corresponds to the question in the case $d = 1$ and $p > 2$. Our method does not seem to be generalized to the case $d \geqslant 2$. After our work was finished, P. Mironescu [8] informs us that the answer to the question is also true in higher dimensional cases. The reader can find many interesting questions related to liftings of $\mathbb{S}^1$-valued maps in [7].

We next give a useful consequence of Lemma 2.

Theorem 3. *Let $d \geqslant 1$, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^d; \mathbb{R})$, and $g = e^{i\psi}$. Assume that $|g|_{\mathrm{BMO}} = \beta < 1$. Then there exists a constant $C > 0$ depending only on d and p such that*

$$|\psi|_{W^{\frac{1}{p},p}} \leqslant \frac{C}{1-\beta} |g|_{W^{\frac{1}{p},p}}.$$

This result is inspired by the one of R.R. Coifman and Y. Meyer [6] (see also [5]) who proved that there exists a constant $c > 0$ such that if $|g|_{\mathrm{BMO}} < c$ then $|\psi|_{\mathrm{BMO}} \leqslant 4|g|_{\mathrm{BMO}}$ ($g = e^{i\psi}$).

2. Proofs

2.1. Proof of Lemma 1

Step 1: Proof of Lemma 1 in the one-dimensional case.

Since $\int_{\mathbb{T}^1} \psi' \,\mathrm{d}s = 0$, without loss of generality, one may assume that $⨍_{\mathbb{T}^1} \zeta \,\mathrm{d}s = 0$. Set $\tilde{g}(s) = g(e^{is})$ and $\tilde{\zeta}(s) = \zeta(e^{is})$ for $s \in \mathbb{R}$. Let B_1 be the unit ball of $\mathbb{R}^2$ and $\chi \in C^1(B_1)$ be such that $\chi(x) = 1$ for $|x| \geqslant \frac{3}{4}$, $\chi(x) = 0$ for $|x| \leqslant \frac{1}{2}$, $0 \leqslant \chi(x) \leqslant 1$ and $|D\chi(x)| \lesssim 1$, for $x \in B_1$. Define $u : B_1 \mapsto \mathbb{R}^2$ and $\eta : B_1 \mapsto \mathbb{R}$ as follows

$$u\big[(1-h)e^{i\theta}\big] = ⨍_{\theta}^{\theta+2\pi h} \tilde{g}(s)\,\mathrm{d}s \quad \text{and} \quad \eta\big[(1-h)e^{i\theta}\big] = ⨍_{\theta}^{\theta+2\pi h} \tilde{\zeta}(s)\,\mathrm{d}s,$$

for $h \in [0, 1]$ and $\theta \in [0, 2\pi]$. Set $\tilde{\eta} = \chi\eta$ and define

$$\tilde{u}(X) = \begin{cases} u(X)/|u(X)| & \text{if } \big|u(X)\big| \geqslant \alpha, \\ u(X)/\alpha & \text{otherwise,} \end{cases} \qquad \forall X \in B_1,$$

for some small positive regular value α of $|u|$. Then $\tilde{u} \in W^{1,\infty}(B_1)$ and $\tilde{\eta} \in W^{1,\infty}(B_1)$. Since $|\tilde{u}| \leqslant 1$ and $|D\tilde{u}| \lesssim |Du|$ on B_1, integrating by part, one has

$$\left| \int_{\mathbb{T}^1} \psi'\zeta \,\mathrm{d}\sigma \right| \lesssim \|\eta\|_{L^\infty} \int_{B_1 \setminus B_{\frac{1}{2}}} |\det D\tilde{u}|\,\mathrm{d}X + \int_{B_1} |D\tilde{\eta}||Du|\,\mathrm{d}X. \tag{5}$$

Hereafter $B_{\frac{1}{2}} := B(0, \frac{1}{2})$. Applying the method used in [9] (see also [4] in the case $\sqrt{3}$ is replaced by δ for $\delta < \sqrt{2}$), one obtains

$$\int_{B_1 \setminus B_{\frac{1}{2}}} |\det D\tilde{u}|\,\mathrm{d}X \lesssim \mathbf{T}_{\sqrt{3}}(g). \tag{6}$$

On the other hand, since $\frac{1}{p} + \frac{1}{q} = 1$, by Holder's inequality, since $\tilde{\eta} = 0$ in $B_{\frac{1}{2}}$, one has

$$\int_{B_1} |D\tilde{\eta}||Du|\,\mathrm{d}X \leqslant \left(\int_{B_1 \setminus B_{\frac{1}{2}}} h^{p-2}|Du|^p \,\mathrm{d}X \right)^{\frac{1}{p}} \left(\int_{B_1 \setminus B_{\frac{1}{2}}} h^{q-2}|D\eta|^q \,\mathrm{d}X \right)^{\frac{1}{q}}. \tag{7}$$

However, for $h < \frac{1}{2}$,

$$|Du|\big[(1-h)e^{i\theta}\big] \lesssim \frac{1}{h}\big|\tilde{g}(\theta + 2\pi h) - \tilde{g}(\theta)\big| + \frac{1}{h} ⨍_{\theta}^{\theta+2\pi h} \big|\tilde{g}(s) - \tilde{g}(\theta + 2\pi h)\big|\,\mathrm{d}s \tag{8}$$

and

$$\int_0^{2\pi}\int_0^1 \frac{1-h}{h^2}\big|\tilde g(\theta+2\pi h)-\tilde g(\theta)\big|^p \,\mathrm{d}h\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^1 \frac{1-h}{h^2} \int_\theta^{\theta+2\pi h} \big|\tilde g(s)-\tilde g(\theta+2\pi h)\big|^p \,\mathrm{d}s\,\mathrm{d}h\,\mathrm{d}\theta \lesssim |g|^p_{W^{\frac{1}{p},p}}.$$

Hence, it follows that

$$\int_{B_1\setminus B_{\frac12}} h^{p-2}|Du|^p \,\mathrm{d}X \lesssim |g|^p_{W^{\frac{1}{p},p}}. \tag{9}$$

Similarly,

$$\int_{B_1\setminus B_{\frac12}} h^{q-2}|D\eta|^q \,\mathrm{d}X \lesssim |\zeta|^q_{W^{\frac{1}{q},q}}. \tag{10}$$

The conclusion in the case $N=1$ follows from (5), (6), (9), and (10), since $⨍_{\mathbb{T}^1} \zeta \,\mathrm{d}s = 0$.

Step 2: Proof of Lemma 1 in the general case.

Without loss of generality, one may assume that $i=1$. The proof in this case follows as in Step 1 by using the extension $u: B_1\times\mathbb{T}^{d-1}\mapsto\mathbb{R}$ defined as follows

$$u\big[(1-h)e^{i\theta}, x_2,\dots,x_d\big] = ⨍_{B(x,h)} \tilde g(s)\,\mathrm{d}s,$$

where $x=(e^{i\theta},x_2,\dots,x_d)\in\mathbb{T}^d$. The details are left to the reader. □

2.2. *Proof of Lemma 2*

Without loss of generality, one may assume that $i=1$. We use the same notation as in the proof of Lemma 1. The essential point in this proof is to choose $1-\beta<\alpha<1-\beta/2$ such that α is a regular value of $|u|$. Then since $|g|_{\mathrm{BMO}}=\beta$, it follows that $|u|>\alpha$ in B_1. Hence, from the definition of $\tilde u$, $|\tilde u|=1$. Thus (see (5) in the case $d=1$)

$$\left|\int_{\mathbb{T}^d} \frac{\partial\psi}{\partial x_1}\zeta\,\mathrm{d}x\right| \lesssim \frac{1}{1-\beta}\int_{B_1\times\mathbb{T}^{d-1}} |D\tilde\eta||Du|\,\mathrm{d}x.$$

Therefore, as in the proof of Lemma 1, one obtains

$$\left|\int_{\mathbb{T}^d} \frac{\partial\psi}{\partial x_1}\zeta\,\mathrm{d}x\right| \lesssim \frac{1}{1-\beta}|\zeta|_{W^{\frac{1}{q},q}}|g|_{W^{\frac{1}{p},p}}. \qquad \square$$

2.3. *Proof of Theorems 2 and 3*

Proof of Theorem 2. Set $D(A)=\{(\zeta,\zeta);\ \zeta\in C^1(\mathbb{T}^1;\mathbb{R})\}$ and define $A: D(A)\mapsto\mathbb{R}$ by $A(\zeta,\zeta)=\int_{\mathbb{T}^1}\psi'\zeta$. By Hahn–Banach's theorem and Theorem 1, there exists $\mathbf{A}: L^\infty(\mathbb{T}^1)\times W^{\frac{1}{q},q}(\mathbb{T}^1;\mathbb{R})\mapsto\mathbb{R}$ such that $\mathbf{A}(\zeta_1,\zeta_2)=A(\zeta_1,\zeta_2)$ for $(\zeta_1,\zeta_2)\in D(A)$ and $|\mathbf{A}(\zeta_1,\zeta_2)|\lesssim \mathbf{T}_{\sqrt3}\|\zeta_1\|_{L^\infty}+|g|_{W^{\frac1p,p}}\|\zeta_2\|_{W^{\frac1q,q}}$. Define $\varphi_1(\zeta)=\mathbf{A}(\zeta,0)$ and $\varphi_2(\zeta)=\mathbf{A}(0,\zeta)$ for $\zeta\in C^1(\mathbb{T}^1)$. Then $\varphi_1\in\mathcal{M}(\mathbb{T}^1)$, the space of all Radon measures on $\mathbb{T}^1$, and $\varphi_2\in[W^{\frac1q,q}]^*$, the duality of $W^{\frac1q,q}$, $\|\varphi_1\|_{\mathcal{M}}\lesssim\mathbf{T}_{\sqrt3}(g)$ and $\|\varphi_2\|_{[W^{\frac1q,q}]^*}\lesssim|g|_{W^{\frac1q,q}}$. Without loss of generality, one may assume that $\langle\varphi_1,1\rangle=\langle\varphi_2,1\rangle=0$. Then there exist ψ_1 and ψ_2 such that $\psi_1'=\varphi_1$, $\psi_2'=\varphi_2$ and $\psi=\psi_1+\psi_2$. It is clear to see that $|\psi_1|_{BV}\lesssim\mathbf{T}_{\sqrt3}(g)$, $|\psi_2|_{W^{\frac1p,p}}\lesssim|g|_{W^{\frac1p,p}}$. Define $\psi_{1,n}=\psi_1*\rho_n$ and $\psi_{2,n}=\psi_1-\psi_{1,n}+\psi_2$. Since $\psi_1=\psi-\psi_2\in W^{\frac1p,p}$, when n is sufficiently big, one has $|\psi_{2,n}|_{W^{\frac1p,p}}\leqslant|\psi_1-\psi_{1,n}|_{W^{\frac1p,p}}+|\psi_2|_{W^{\frac1p,p}}\lesssim|g|_{W^{\frac1p,p}}$. On the other hand, it is clear to see that $|\psi_{1,n}|_{BV}\leqslant|\psi_1|\lesssim\mathbf{T}_{\sqrt3}(g)$. □

Proof of Theorem 3. For $\xi \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} \xi \, \mathrm{d}x = 0$, define ζ by $\Delta_{\mathbb{T}^d} \zeta = \xi$ and $\int_{\mathbb{T}^d} \zeta \, \mathrm{d}x = 0$. Here $\Delta_{\mathbb{T}^d}$ denotes the Laplace–Beltrami operator on $\mathbb{T}^d$. Applying Lemma 2, one has

$$\left| \int_{\mathbb{T}^d} \psi \xi \, \mathrm{d}x \right| = \left| \sum_{i=1}^d \int_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \frac{\partial \xi}{\partial x_i} \, \mathrm{d}x \right| \lesssim \sum_{i=1}^d \frac{1}{1-\beta} \left| \frac{\partial \zeta}{\partial x_i} \right|_{W^{\frac{1}{q},q}} |g|_{W^{\frac{1}{p},p}}.$$

This implies

$$\left| \int_{\mathbb{T}^d} \psi \xi \, \mathrm{d}x \right| \lesssim \frac{1}{1-\beta} \|\xi\|_{[W^{\frac{1}{p},p}]^*} |g|_{W^{\frac{1}{p},p}}.$$

Hence,

$$\left| \int_{\mathbb{T}^d} \left(\psi - \fint_{\mathbb{T}^d} \psi \right) \xi \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^d} \left(\xi - \fint_{\mathbb{T}^d} \xi \right) \psi \, \mathrm{d}x \right| \lesssim \frac{1}{1-\beta} \|\xi\|_{[W^{\frac{1}{p},p}]^*} |g|_{W^{\frac{1}{p},p}}, \quad \forall \xi \in C^\infty(\mathbb{T}^d).$$

It follows that

$$|\psi|_{W^{\frac{1}{p},p}} = \left| \psi - \fint_{\mathbb{T}^d} \psi \right|_{W^{\frac{1}{p},p}} \lesssim \frac{1}{1-\beta} |g|_{W^{\frac{1}{p},p}}. \qquad \square$$

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