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Shimura Curves and Special Values of p-adic L-functions

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We construct "generalized Heegner cycles" on a variety fibered over a Shimura curve, defined over a number field. We show that their images under the p-adic Abel–Jacobi map coincide with the values (outside the range of interpolation) of a p-adic L-function $L_p(f, \chi)$ which interpolates special values of the Rankin–Selberg convolution of a fixed newform f and a theta-series θ_χ attached to an unramified Hecke character of an imaginary quadratic field K. This generalizes previous work of Bertolini, Darmon, and Prasanna, which demonstrated a similar result in the case of modular curves. Our main tool is the theory of Serre–Tate coordinates, which yields p-adic expansions of modular forms at CM points, replacing the role of q-expansions in computations on modular curves.

1 Introduction

The aim of this article is to prove a p-adic Gross–Zagier-type formula for Shimura curves over $\mathbb{Q}$, generalizing a result [3] of Bertolini, Darmon, and Prasanna.

Let N be a positive integer and f a modular form of weight k on $\Gamma_0(N)$. An imaginary quadratic field K is said to satisfy the Heegner hypothesis with respect to f if all primes dividing N are split in K. Under this assumption, and in the case $k=2$, the seminal work of Gross and Zagier [19] established a precise formula relating the

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derivatives

$$L'(f, \chi, 1)$$

to the heights of Heegner points on the modular curves $X_0(N)$. Here, χ is a finite-order unramified character of K and $L(f, \chi, s)$ denotes the usual Rankin–Selberg L-function. This result has been generalized in many different directions, for example to the case of even $k > 2$ [37] and to the setting of Shimura curves over totally real fields [36, 38]. There are also similar formulas relating the cyclotomic derivative of a two-variable p-adic L-function to p-adic heights of Heegner points and Heegner cycles due to Perrin-Riou [30] ($k = 2$) and Nekovář [28] ($k > 2$) as well as recent work by Disegni [18] in the setting of Shimura curves over totally real fields. Here, $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ is an odd prime splitting in K.

The formula of [3] is of a different nature and involves studying not the heights (or p-adic heights) of algebraic cycles but rather their p-adic logarithms, which are defined by taking the image of a (homologically trivial) cycle under the étale Abel–Jacobi map and then applying the inverse of the Bloch–Kato exponential. The range of characters considered is more general and includes Hecke characters of weight strictly smaller than the weight of f. The main result of [3] is then, assuming a weaker version of the Heegner hypothesis, that there exist certain homologically trivial cycles (*generalized Heegner cycles*) corresponding to the vanishing of the Rankin–Selberg L-function $L(f, \chi, s)$ at the center and that the logarithms of these cycles can be explicitly related to the values (rather than the derivatives) of a p-adic L-function at a point outside its range of interpolation.

This article drops the Heegner hypothesis from [3]. We assume that N is square-free and prime to the discriminant of K. Factor $N = N^+N^-$ where primes dividing N^+ split in K and primes dividing N^- remain inert. Assuming that the sign of the functional equation for $L(f, \chi, s)$ is negative, a simple computation of epsilon factors shows that the number of primes dividing N^- is even. Let B be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant N^- and choose a maximal order $\mathcal{O}_B$ in B. There is a Shimura curve X, defined over $\mathbb{Q}$, which, for $N^+ > 3$, is a fine moduli space for principally polarized abelian surfaces A together with an embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ ("false elliptic curves") and a certain type of level structure that depends on N^+ (described in Section 2.2).

Assume that the weight k of f is even and positive, and write $k = 2r + 2$. If $k > 2$, we assume also that $N^+ > 3$. We will work over the ray class field F of K mod N^+. Writing $\mathcal{A}$ for the universal abelian surface over the Shimura curve $X_{/F}$ and $\mathcal{A}_r$ for its r-fold fiber product over $X_{/F}$, we study generalized Heegner cycles on $\mathcal{A}_r \times A^r$, where A is a fixed false elliptic curve over F with CM by $\mathcal{O}_K$. (Here, the $\mathcal{O}_K$ action is required to commute with the $\mathcal{O}_B$-action, which implies that A is isogenous to the self-product of an elliptic curve with CM by $\mathcal{O}_K$.)

Given an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we construct a p-adic L-function, $L_p(f,-)$ in Section 8.4, whose domain is a space of Hecke characters, which is characterized by an interpolation law of the form

$$\frac{L_p(f,\psi)}{\Omega_p^{2(\ell_1'-\ell_2')}} = C\, E_p(f,\psi)^2 \frac{L(f,\psi^{-1},0)}{\Omega^{2(\ell_1'-\ell_2')}},$$

where

(1) ψ ranges over the space of Hecke characters of type (ℓ_1',ℓ_2') with trivial central character such that $\ell_1'+\ell_2' = k$ and $\ell_2' \leq 0$.
(2) $E_p(f,\chi)$ is the Euler factor of $L(f,\chi^{-1},s)$ at $\bar{\mathfrak{p}}$ evaluated at 0, where $\mathfrak{p}$ is the prime of K distinguished by the given embedding.
(3) Ω is a complex period attached to A and Ω_p is a p-adic period attached to A.
(4) C is an explicit nonzero constant.

There is an idempotent e in $B \otimes K$ selected in Section 2.1, and we state our results in terms of the cohomology group $eH^1_{\mathrm{dR}}(A/F)$.

Theorem 1.1. Suppose that f has even weight $2r+2$, with $r \geq 0$, and χ is an unramified Hecke character of K with trivial central character and infinity type (ℓ_1,ℓ_2) with $\ell_1+\ell_2 = 2r+2$ and $\ell_1,\ell_2 \geq 1$, so that $(\ell_1,\ell_2) = (2r+1-j, 1+j)$ with $0 \leq j \leq 2r$. Then there is, for each $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O}_K)$, an algebraic cycle $\Delta_r(\mathfrak{a})$ on

$$X_r := \mathcal{A}_r \times A^r$$

that is homologically trivial and defined over F, such that

$$\frac{L_p(f,\chi)}{\Omega_p^{4r-4j}} = E_p(f,\chi)^2 \cdot \left\{ \frac{1}{j!} \sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a}) \cdot \mathrm{AJ}_p(\Delta_r(\mathfrak{a}))(\omega_{f_B} \wedge \omega_A^j \eta_A^{2r-j}) \right\}^2, \qquad (1)$$

where

(1) AJ_p is the p-adic Abel–Jacobi map, viewed (see Section 6.3) as a map

$$\mathrm{CH}_0^{2r+1}(X_{r/F_p}) \to (S_{2r+2}(F_p) \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A_{/F_p}))^\vee$$

with F_p being the completion of F at the chosen prime above p and $S_{2r+2}(F_p)$ the space of weight k modular forms over F_p of level N^+.

(2) f_B is the Jacquet–Langlands lift of f to X, normalized as in Section 2.7, ω_{f_B} is the associated differential form on $\mathcal{A}_r$, and $\{\omega_A, \eta_A\}$ are a basis for $eH^1_{\mathrm{dR}}(A/f)$, with ω_A holomorphic on $A(\mathbb{C})$ and η_A antiholomorphic on $A(\mathbb{C})$, normalized such that the cup product $\langle \omega_A, \eta_A \rangle = 1$. □

Under the hypothesis that $p \mid N^-$ exactly once (in which case p is inert in K), Masdeu [26] has proved a similar result by using a p-adic analytic uniformization of the corresponding Shimura curve, which has bad reduction at p. Such a uniformization is not available in the case of good reduction. Conversely, our techniques rely on the good reduction of the Shimura curve and on p being split, and thus do not recover Masdeu's results.

Before outlining our methods, we summarize the proof of the main theorem of [3]. For this paragraph only, $r = k - 2$. Write $\mathcal{E}_r$ for the r-fold self-fiber product of the universal generalized elliptic curve over $X_1(N)$ with itself, and fix an elliptic curve E over F with complex multiplication. In [3], a generalized Heegner cycle Υ is built as a graph of an isogeny, modified by an algebraic projector, due to Scholl, which projects the cohomology of the variety $\mathcal{E}_r \times E^r$ onto the subspace

$$S_{r+2}(\Gamma_1(N)) \otimes \mathrm{Sym}^r H^1_{\mathrm{dR}}(E).$$

The image of Υ under the p-adic Abel–Jacobi map is computed in two steps. The first step is to relate this image to a "Coleman primitive" for the section of a line bundle on $X_1(N)$ attached to f. The second is to express the Coleman primitive of f in terms of $\theta^{-1-j} f$, for $0 \le j \le k-2$, where $\theta = q\frac{\mathrm{d}}{\mathrm{d}q}$ is the Atkin–Serre p-adic differential operator which maps the space of p-adic modular forms of weight k to the space of p-adic modular forms of weight $k+2$. The values of θf coincide with the values of $\Theta_\infty f$ at CM points, where Θ_∞ denotes the Maass–Shimura operator

$$\frac{1}{2\pi i}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{k}{\tau - \bar{\tau}}\right).$$

The p-adic L-function is then computed using a Waldspurger-type result expressing values of the classical Rankin–Selberg L-function in terms of values of $\Theta_\infty^j f$ at CM points.

Our proof follows [3], but replaces q-expansions, which are unavailable in the Shimura curve case, by "Serre–Tate" expansions. In Section 2, we review the arithmetic of Shimura curves and Shimura's reciprocity law. In Section 3, we discuss the theory of modular forms on Shimura curves, defining the Shimura curve analogs of the p-adic differential operator θ and the Maass–Shimura operator Θ_∞ discussed above, as well as some Hecke operators.

In Section 4, we compute the differential operators of Section 3. Serre–Tate theory gives both an explicit uniformizer in the ring of rigid analytic functions on an ordinary residue disk of the Shimura curve and an explicit trivialization of the bundle of p-adic modular forms of weight k over this disk. We can thus use it to express p-adic modular forms locally in coordinates. The main results of this section, which uses work of Brakočević and Mori, are formulas for Hecke operators in these coordinates and a proof that the Atkin–Serre operator is invertible on the space of "prime to p" p-adic modular forms. (A modular form is prime to p if it is fixed by the idempotent p-adic Hecke operator $1 - UV$, as defined in Section 3.6. A classical p-adic modular form is "prime to p" if and only if all Fourier coefficients of the form a_{np} vanish.)

In Sections 5 and 6, we review Coleman's p-adic methods for computing residues on vector bundles with flat connection, then produce the cycles $\Delta_r(\mathfrak{a})$ of Theorem 1.1. When the weight of f is larger than 2, the cycles are constructed in a manner similar to [3], but using a theorem of Besser to construct an algebraic correspondence that projects the cohomology of the Kuga–Sato variety onto a subspace generated by quaternionic modular forms. When the weight of f is 2, a special construction is needed, since the usual construction involves subtracting the cusp at infinity to make the cycle homologically trivial, and this is unavailable. The construction used in [38] to study p-adic heights, which involves subtracting a multiple of the Hodge bundle, seems less natural for studying p-adic logarithms; instead, we project the Heegner point onto its f-isotypic component. This gives a cohomologically trivial cycle whose p-adic logarithm can be computed easily.

In Section 7, we follow [3] closely in interpreting the p-adic Abel–Jacobi map as a Coleman integral. We then use the formulas from Section 4 to compute the image of our cycle under the p-adic Abel–Jacobi map. In Section 8, we use a Waldspurger-type result to build a p-adic L-function (the construction of which is originally due to Hida), and then establish Theorem 1.1.

There are potential applications to generalizations of [1, 2, 4] to the setting of Shimura curves. In addition, there is forthcoming work of Skinner [34] on the converse to the Gross–Zagier–Kolyvagin theorem for elliptic curves of rank one and of Bhargava and Skinner [6] on average ranks of elliptic curves, which make essential use not just of the results in [3] but also of the generalization given in this article.

2 Shimura Curves

2.1 Initial setup

Fix an odd prime p, an isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$, and compatible embeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$. Fix also a newform f_{GL_2} of level N, with $p \nmid N$, of nebentypus ϵ_f, and of even weight $k = 2r + 2$ (with $r \geq 0$).

Let K be an imaginary quadratic field in which $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ splits. Factor $N = N^+ N^-$, where primes dividing N^- are inert in K, and primes dividing N^+ are split or ramified in K. If a prime divides both N and the discriminant of K, assume also that it divides N exactly once (in other words, K satisfies the Heegner hypothesis with respect to the level N^+). Assume also that N^- is square-free and divisible by an even, nonzero number of primes.

Write B for the (necessarily indefinite) quaternion algebra over $\mathbb{Q}$ of discriminant N^-. As in [21], fix an auxiliary prime p_0 with the following properties:

(1) For all ℓ, the Hilbert symbol $(p_0, N^-)_\ell$ satisfies

$$(p_0, N^-)_\ell = -1 \quad \text{if and only if } \ell \mid N^-.$$

(2) All primes dividing N^+ split in the real quadratic field

$$M = \mathbb{Q}(\sqrt{p_0}).$$

Such primes exist by Dirichlet's theorem on arithmetic progressions. This choice of p_0 determines a *Hashimoto model* for B: the algebra B is generated as a vector space by the basis $\{1, s, j, sj\}$ with $s^2 = -N^-$, $j^2 = p_0$, and $sj = -js$. (We reserve the symbol i for a complex square root of -1.) The $\mathbb{Z}$-span of this basis is contained in a unique maximal order $\mathcal{O}_B$.

By definition, a *false elliptic curve* over a base $\mathbb{Z}[\frac{1}{N}]$-scheme S is a relative abelian surface A/S, together with an embedding

$$\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A).$$

We typically denote the pair (A, ι) as A. A *false isogeny* of false elliptic curves is an isogeny commuting with the $\mathcal{O}_B$ action.

There is an involution $\dagger$ on B given by the rule

$$b^\dagger = s^{-1}\bar{b}s,$$

where $b \mapsto \bar{b}$ denotes the main involution. For any false elliptic curve A over any $\mathbb{Z}[\frac{1}{N}]$-scheme S, there is a unique principal polarization whose associated Rosati involution on $\mathrm{End}_S(A)$ restricts to $\dagger$ on $\mathcal{O}_B$ (this is a theorem of Milne over a characteristic zero field; over an arbitrary base $\mathbb{Z}[\frac{1}{N}]$-scheme, see the discussion in Section 1 of [10]).

Following [27], we consider the element $e \in \mathcal{O}_B \otimes \mathcal{O}_M[\frac{1}{2p_0}]$, given by the formula

$$e = \frac{1}{2}\left(1 \otimes 1 + \frac{1}{p_0} j \otimes \sqrt{p_0}\right).$$

Then, $e = e^\dagger$ is a nontrivial idempotent. There is an isomorphism

$$\iota_M : B \otimes M \to M_2(M)$$

given by

$$\iota_M(j) = \begin{pmatrix} \sqrt{p_0} & 0 \\ 0 & -\sqrt{p_0} \end{pmatrix} \quad \text{and} \quad \iota_M(s) = \begin{pmatrix} 0 & 1 \\ -N^- & 0 \end{pmatrix},$$

which satisfies

$$\iota_M(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Under ι_M, the involution $\dagger$ on $B \otimes M$ is carried to the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \frac{c}{N^-} \\ bN^- & d \end{pmatrix}$$

of $M_2(M)$. The map ι_M extends along the given embedding $M \hookrightarrow \mathbb{R}$ to an isomorphism

$$\iota_\infty : B \otimes \mathbb{R} \to M_2(\mathbb{R}).$$

For each place v of M, the map ι_M gives a map

$$\iota_v : B \otimes M_v \to M_2(M_v),$$

where M_v denotes the completion of M at v.

Lemma 2.1. For $v \mid pN^+$, one has

$$\iota_v(\mathcal{O}_B \otimes \mathcal{O}_{M,v}) = M_2(\mathcal{O}_{M,v}).$$ □

Proof. By maximality, one must only check that

$$\iota_v(\mathcal{O}_B \otimes \mathcal{O}_{M,v}) \subset M_2(\mathcal{O}_{M,v}).$$

The order $\mathcal{O}_B$ has a $\mathbb{Z}$-basis given by

$$s, \frac{1+j}{2}, \frac{i+sj}{2}, \frac{aj+sj}{p_0}$$

for some rational integer a (this is the case $D = D_0$ of [21, Theorem 2.2]), so the lemma is obvious for $v \nmid 2$. If 2 divides N^+, then $p_0 \equiv 1 \mod 8$ and the claim follows from the explicit description of $\iota_M(j)$. ∎

For each prime $\ell \mid pN^+$, choose a prime v_ℓ of M with $\mathbb{Q}_\ell = M_{v_\ell}$ (for $\ell = p$ such a choice has already been made) to get embeddings $\iota_\ell : B \otimes \mathbb{Q}_\ell \to M_2(\mathbb{Q}_\ell)$. By the lemma, these embeddings have the property that $\iota_\ell(\mathcal{O}_B \otimes \mathbb{Z}_\ell) = M_2(\mathbb{Z}_\ell)$. For $\ell \mid N^+$, these maps give rise to a trivialization

$$\iota_{N^+} : \mathcal{O}_B \otimes \mathbb{Z}/N^+\mathbb{Z} \to M_2(\mathbb{Z}/N^+\mathbb{Z}).$$

Write $\mathcal{O}_{B,N^+}$ for the standard Eichler order of level N^+ in $\mathcal{O}_B$; write Γ for the group of norm one units of $\mathcal{O}_B$ and Γ_{0,N^+} for that of $\mathcal{O}_{N^+}$. The group Γ_{0,N^+} admits a canonical map to $\frac{\mathbb{Z}}{N^+\mathbb{Z}}$ with the property that

$$\iota_{N^+}^{-1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto d;$$

let Γ_{1,N^+} denote the kernel of this map.

2.2 Arithmetic model and level structures

For S a $\mathbb{Z}[1/N]$-scheme and A/S a false elliptic curve, a *full level* N^+ structure on A is an isomorphism of group schemes

$$A[N^+] \to \mathcal{O}_B \otimes (\mathbb{Z}/N^+\mathbb{Z})_{/S}$$

commuting with the action of $\mathcal{O}_B$. A *level structure of type* $V_1(N^+)$ is an equivalence class of full level N^+ structures under the (right) action of the group

$$\left\{\begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix}\right\} \subset M_2(\mathbb{Z}/N^+\mathbb{Z}),$$

which is viewed as a subgroup of $\mathcal{O}_B \otimes \frac{\mathbb{Z}}{N^+\mathbb{Z}}$ via ι_N^+. Note that $V_1(N^+)$ level structure behaves covariantly under false isogenies of false elliptic curves. Then, we have the following fundamental theorem (see [17, Section 4]):

Theorem 2.2 (Morita)**.** For $N^+ > 3$, the moduli problem attaching to a $\mathbb{Z}[1/N]$-scheme S the set of isomorphism classes of false elliptic curves over S together with $V_1(N^+)$ level structure is representable by a smooth proper $\mathbb{Z}[\frac{1}{N}]$ scheme $\mathcal{C}$. □

If A is a false elliptic curve over an algebraically closed field k, we may view a $V_1(N^+)$ level structure as an $M_2(\mathbb{Z}/N^+\mathbb{Z})$-equivariant map

$$(\mathbb{Z}/N^+\mathbb{Z})^2 \to A[N^+](k)$$

(this is the definition in [22]). Explicitly, a full level structure induces, via ι_M, an isomorphism

$$A[N^+](k) \xrightarrow{\sim} M_2(\mathbb{Z}/N^+\mathbb{Z}).$$

The latter is isomorphic to $(\mathbb{Z}/N^+\mathbb{Z})^2 \times (\mathbb{Z}/N^+\mathbb{Z})^2$ as a left $M_2(\mathbb{Z}/N^+\mathbb{Z})$-module, and the map including $(\mathbb{Z}/N^+\mathbb{Z})^2$ onto the second factor only depends on the $V_1(N^+)$-level structure induced by the chosen full level structure. Note that if P is an N^+-torsion point satisfying $eP = P$, then there is a unique $V_1(N^+)$ level structure (in the sense of [22]) mapping $\binom{1}{0}$ to P.

2.3 CM points on Shimura curves

The complex points of $\mathcal{C}$ are naturally identified with the compact Riemann surface

$$\mathcal{H}/\Gamma_{1,N^+},$$

as we now explain.

Because $\mathcal{O}_B$ is an order in B, there is a four-dimensional real torus

$$A_{\mathbb{R}} = \frac{B \otimes \mathbb{R}}{\mathcal{O}_B} = \frac{M_2(\mathbb{R})}{\iota_\infty(\mathcal{O}_B)}$$

endowed with endomorphisms by $\mathcal{O}_B$ via left-multiplication.

For $\tau \in \mathcal{H}$, write $J_\tau \in M_2(\mathbb{R})$ for the unique real matrix with

$$J_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = i \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Then, the action of right-multiplication by J_τ on $M_2(\mathbb{R})$ endows $A_\mathbb{R}$ with a complex structure for which the endomorphisms coming from $\mathcal{O}_B$ are holomorphic.

Write A_τ for the corresponding false elliptic curve. Then there is an isomorphism of false elliptic curves

$$A_\tau \xrightarrow{\sim} \frac{\mathbb{C}^2}{\iota_\infty(\mathcal{O}_B)\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)},$$

given on the universal cover $M_2(\mathbb{R})$ by the rule

$$M \mapsto M\begin{pmatrix}\tau\\1\end{pmatrix}.$$

There is an alternating form E on $M_2(\mathbb{R})$ given by

$$E(x, y) = \frac{1}{N^-}\operatorname{Tr}(sx\bar{y}) = \operatorname{Tr}(xsy^\dagger).$$

Proposition 2.3. The form E gives a principal polarization on A_τ for which the Rosati involution restricts to $b \mapsto b^\dagger$. □

Proof. To see that E gives a polarization, one must check that $E(ix, iy) = E(x, y)$ and that $E(ix, x) > 0$ for $x \neq 0$ (both claims with respect to the complex structure given by right-multiplication by J_τ on $M_2(\mathbb{R})$). The first condition is straightforward. For the second, writing $x = (x_{ij})$, one uses the explicit description of $\iota_\infty(s)$ and the formula

$$J_\tau = \frac{1}{\operatorname{Im}(\tau)}\begin{pmatrix}\operatorname{Re}(\tau) & -\|\tau\|^2\\ 1 & -\operatorname{Re}(\tau)\end{pmatrix}$$

to deduce

$$\begin{aligned} E(ix, x) &= \frac{1}{N^-\cdot\operatorname{Im}(\tau)}(N^-(x_{12}^2 + 2\operatorname{Re}(\tau)x_{11}x_{12} + \|\tau\|^2 x_{11}^2) + (x_{22}^2 + 2\operatorname{Re}(\tau)x_{21}x_{22} + \|\tau\|^2 x_{21}^2))\\ &> \frac{1}{N^-\cdot\operatorname{Im}(\tau)}(N^-(\operatorname{Re}(\tau)x_{11} + x_{12})^2 + (\operatorname{Re}(\tau)x_{22} + x_{11})^2). \end{aligned}$$

It is clear that $E(bx, y) = E(x, b^\dagger y)$ and that the polarization is principal. ■

The false elliptic curve A_τ is equipped with the full level structure

$$t_\tau : \frac{\mathbb{Z}}{N^+} \otimes \mathcal{O}_B \xrightarrow{\sim} \frac{\frac{1}{N^+}\mathcal{O}_B}{\mathcal{O}_B} \to A_\tau[N^+].$$

Then, A_τ is isomorphic to $A_{\gamma\tau}$ as a false elliptic curve if and only if $\gamma \in \Gamma$, in which case the isomorphism

$$A_\tau \to A_{\gamma\tau}$$

is given on the universal cover $M_2(\mathbb{R})$ of A_τ and $A_{\gamma\tau}$ as right-multiplication by $\iota_\infty(\gamma^{-1})$ (equivalently, is given on $\mathbb{C}^2$ as multiplication by the scalar $j(\iota_\infty(\gamma), \tau)^{-1}$). This isomorphism is compatible with the $V_1(N^+)$ level structure determined by t_τ if and only if $\gamma \in \Gamma_{1,N^+}$.

2.4 CM Points and Heegner points

Given any embedding

$$\iota : K \hookrightarrow B,$$

there is a unique $\tau \in \mathcal{H}$ with

$$\iota_\infty(\iota(K^\times))(\tau) = \tau.$$

It follows that the additive map $K \to \mathbb{C}$ given by

$$\alpha \mapsto j(\iota_\infty(\iota(\alpha)), \tau)$$

is also multiplicative and hence an embedding of fields. The map ι is said to be *normalized* if the induced field embedding $K \hookrightarrow \mathbb{C}$ is the identity on K. We say $\tau \in \mathcal{H}$ is a *CM point* if there exists an embedding ι with τ as its fixed point. The set of CM points is then in bijective correspondence with the set of normalized embeddings ι.

Write ι_τ for the normalized embedding $K \hookrightarrow B$ fixing τ. The group Γ acts by conjugation on the set of such embeddings, and this action satisfies

$$\gamma\iota_\tau = \iota_{\gamma\tau}.$$

Suppose that τ is a CM point. Then, the false elliptic curve A_τ has false endomorphisms via right-multiplication by $\iota_\tau(\mathcal{O}_K)$ (these endomorphisms commute with the complex structure J_τ, and $\alpha \in \mathcal{O}_K$ induces the scalar $j(\iota_\infty(\iota_\tau(\alpha)), \tau) = \alpha$ on the universal cover of A_τ).

2.5 The action of $\mathrm{Cl}(K)$ and Shimura's reciprocity law

Suppose that τ is a CM point, and let $\mathfrak{a}$ be an (integral) ideal of $\mathcal{O}_K$. Then, there is a left-ideal of $\mathcal{O}_B$ in B given by

$$\mathfrak{a}_B = \mathcal{O}_B(\iota_\tau(\mathfrak{a})).$$

Because B is an indefinite rational quaternion algebra, it has class number one and $\mathfrak{a}_B$ is *principal*, generated by some $\alpha \in B$.

Right-multiplication by α gives a false isogeny

$$A_\tau \to A_{\alpha^{-1}\tau}$$

with kernel $A_\tau[\mathfrak{a}]$, the subgroup of A_τ killed by all endomorphisms in the ideal $\mathfrak{a}$. If $(\mathfrak{a}, N) = 1$ and t is a level-N^+ structure on A_τ, this false isogeny induces a level-N^+ structure t_α on A_τ.

The image of $\alpha\tau$ under the uniformization map $\rho : \mathcal{H} \to \mathcal{H}/\Gamma$ does not depend on the choice of α. As a consequence, it makes sense to write $A_{\mathfrak{a}\star\tau}$ for the corresponding false elliptic curve. Alternatively, one may view $A_{\mathfrak{a}\star\tau}$ as the false elliptic curve $B \otimes \mathbb{R}/\mathfrak{a}_B$ (with underlying complex structure coming from J_τ as above). In these coordinates, the isogeny given by right-multiplication by α^{-1} is identified with the natural projection.

Shimura's reciprocity law states that the point $\rho(\tau)$ is defined over the Hilbert class field H of K, and, moreover, for $\mathfrak{a} \in \mathrm{Cl}(K)$, one has

$$\rho(\tau)^{(\mathfrak{a}^{-1}, H/K)} = \rho(\mathfrak{a} \star \tau).$$

(Note that if one replaces $\mathfrak{a}$ by $\lambda\mathfrak{a}$ for some $\lambda \in K$, then the corresponding $\alpha \in B$ is replaced by $\alpha\iota_\tau(\lambda)$ and thus, $A_{\mathfrak{a}\star\tau}$ does not change.) The set of isomorphism classes of CM false elliptic curves over H (or any field containing H) is thus a torsor for $\mathrm{Cl}(K)$ under the action $\star$.

If the embedding ι has the property that $\iota(\mathcal{O}_K) \subset \mathcal{O}_{B,N^+}$, then one refers to a CM point τ for ι as a *Heegner point*. Because the pair (K, N^+) satisfies the Heegner hypothesis, there is an ideal $\mathfrak{N}^+$ of K whose norm is N^+. By [33, Theorem 3.2], the image of τ under the uniformization map $\mathcal{H} \to \mathcal{H}/\Gamma_{1,N^+}$ is defined over the ray class field of K mod $\mathfrak{N}^+$. The false elliptic curve corresponding to a Heegner point comes with a level structure t, defined over the same field, induced by choosing $P = eP$ a point of exact order cN^+ in the kernel of the false isogeny

$$A_\tau \to A_{N^+\star\tau}.$$

We choose such a point once and for all, and we call t the Heegner level structure on A_τ. The existence of the Heegner level structure follows from our assumptions on the splitting behavior in K of primes dividing N (see [15, Lemma 4.17]).

2.6 Generalized Kuga-Sato varieties

Fix a number field F containing the real quadratic field M, the ray class field of K mod N^+, and the Hecke eigenvalues of f. Write C for $\mathcal{C}_F$, $\mathcal{A}$ for the universal false elliptic curve over C, and $\mathcal{A}_r$ for the r-fold fiber product of $\mathcal{A}$ with itself over C. Fix an embedding $\iota: K \hookrightarrow B$ with $\iota(\mathcal{O}_K) \subset \mathcal{O}_{B,N^+}$. Write τ for its fixed point in $\mathcal{H}$ and A for the corresponding false elliptic curve over $F \supset H$. Because p splits in K, the surface A is ordinary at p (which means that it has good reduction at the chosen prime above p and that the p-divisible group of its reduction is isomorphic to the self-product of the p-divisible group of an ordinary elliptic curve). Write

$$W_r = \mathcal{A}_r \times A^r.$$

This "enlarged Kuga–Sato variety" is the home of the arithmetic cycles which will be constructed in Section 6.

Let $\mathcal{H}^i$ the ith relative de Rham cohomology bundle on C attached to the map $\mathcal{A} \to C$. Write $\underline{\omega}$ for the bundle $e\Omega_{\mathcal{A}/C}$ and $\mathcal{L}_n$ for $\mathrm{Sym}^n e\mathcal{H}^1$. Note that $\mathcal{L}_{2r}$ is naturally a sub-bundle of the relative de Rham cohomology bundle $\mathcal{H}^{2r}(\mathcal{A}_r/C)$ of the rth Kuga–Sato variety over C. The bundle $\mathcal{L}_1$ admits a canonical self-duality

$$\langle\, , \rangle : \mathcal{L}_1^{\otimes 2} \to e\mathcal{H}^2 = \mathcal{O}_C.$$

We normalize the isomorphism $e\mathcal{H}^2 \to \mathcal{O}_C$ using the trivialization induced from the *opposite* of the nowhere vanishing section of $e\mathcal{H}^2$ attached to the class of the universal principal polarization. This choice, which is consistent with the choice in the GL_2 case, is motivated below. The pairing $\langle\, , \rangle$ extends to the bundles $\mathcal{L}_{2r}$.

There is a Hodge sequence

$$0 \to \underline{\omega} \to \mathcal{L}_1 \to \underline{\omega}^{-1} \to 0.$$

When we write $\underline{\omega}^{-1}$ here, we use the following fact: the standard identification of $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ with the relative tangent bundle of the dual abelian scheme $\hat{\mathcal{A}} \to C$, combined with the universal principal polarization $\hat{\mathcal{A}} = \mathcal{A}$, gives rise to a (cotangent–tangent) pairing

$$\Omega_{\mathcal{A}/C} \times R^1\pi_*\mathcal{O}_{\mathcal{A}} \to \mathcal{O}_C,$$

and because e is fixed by the Rosati involution, this pairing restricts to a perfect pairing $\underline{\omega} \otimes eR^1\pi_*\mathcal{O}_{\mathcal{A}} \to \mathcal{O}$. Finally, there is a bundle $\mathcal{L}_{n,n}$ on C given by $\mathcal{L}_{n,n} = \mathcal{L}_n \otimes \mathrm{Sym}^n eH^1_{\mathrm{dR}}(A)$.

As it is alternating, the pairing $\langle\,,\rangle$ on $\mathcal{L}_1$ induces a pairing

$$\underline{\omega} \otimes eR^1\pi_*\mathcal{O}_{\mathcal{A}} \to \mathcal{O}.$$

As can be checked on fibers after base changing to $\mathbb{C}$, under the normalization chosen above, this pairing coincides with the cotangent–tangent pairing rather than its opposite.

2.7 The transfer

The *Jacquet–Langlands correspondence* implies the existence of a holomorphic function f on the upper half plane, called the transfer of f_{GL_2}, with the following properties:

(1) f is a modular form for $\Gamma_{1,N^+} \subset B$ with Nebentypus ϵ_f for the action of Γ_{0,N^+}.
(2) f has weight k.
(3) For $(n, N^-) = 1$, f is an eigenform for the operator T_n with the same eigenvalue as f_{GL_2}.

These properties determine f as a holomorphic function on the upper half plane only up to a scalar multiple. However, one can normalize f further. The function f gives rise canonically to a section of $\underline{\omega}_{\mathbb{C}}$ in the following manner: the universal false elliptic curve $\mathcal{A}_{\mathcal{H}}$ over $\mathcal{H}$ is the quotient of $\mathcal{H} \times \mathbb{C}^2$ by the action of $\mathcal{O}_B$ given by

$$b\left(\tau, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \left(\tau, \iota_\infty(b) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right).$$

Because f is modular for Γ_{1,N^+}, the relative one-form

$$\omega_f = (2\pi i)^k f(\tau)\, \mathrm{d}z_1^{\otimes k} \in e\pi_*\Omega_{\mathcal{A}_{\mathcal{H}}}^{\otimes k}$$

for the universal false elliptic curve descends to a section of $\underline{\omega}_{\mathbb{C}}$. Because C and $\underline{\omega}$ both admit canonical models over $\mathcal{O}_F[1/N]$ and the Hecke eigenvalues of f lie in this ring, we may assume that our section is defined over this ring. Thus, the choice of transfer is ambiguous up to multiplication by a unit in this ring.

2.8 Standard cohomology classes

Consider the Hodge exact sequence for A:

$$0 \to \Omega_{A/F} \to H^1_{\mathrm{dR}}(A/F) \to H^1(A, \mathcal{O}_A) \to 0.$$

Because A has CM, this sequence canonically splits, with $H^1(A, \mathcal{O}_A)$ identified as the subspace of $H^1_{\mathrm{dR}}(A/F)$ on which $\mathcal{O}_K$ acts via complex conjugation. In fact:

(1) Over $\mathbb{C}$, this splitting coincides with the complex-analytic splitting of the Hodge sequence, that is, the space $H^1(A_{\mathbb{C}}, \mathcal{O}_{A_{\mathbb{C}}})$ is identified with the subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})$ spanned by anti-holomorphic one-forms on $A(\mathbb{C})$.

(2) Over any p-adic field L containing F, this splitting coincides with the "unit-root" splitting, that is, after tensoring the Hodge sequence with L, the space $H^1(A_L, \mathcal{O}_{A_L})$ is identified with the subspace of $H^1_{\mathrm{dR}}(A/L)$ on which the semi-linear Frobenius map ϕ acts via a unit.

To see these facts, note that, as is explained on p. 919 of [31], we may find a false isogeny $\phi : A \to E_1 \times E_2$, defined over F and with degree prime to p, where E_1 and E_2 are elliptic curves with CM by $\mathcal{O}_K$. For CM elliptic curves, the coincidence of the splittings of the Hodge exact sequence follows concretely from the observation that on the Weierstrass model $y^2 = 4x^3 + ax + b$, the differential

$$\frac{\mathrm{d}x}{y}$$

lies in the subspace of $H^1_{\mathrm{dR}}(A/F)$ on which K acts via the identity embedding, the holomorphic subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})$, and the p-root subspace of $H^1_{\mathrm{dR}}(A/L)$, whereas the meromorphic differential form

$$x\frac{\mathrm{d}x}{y}$$

lies in the subspace of $H^1_{\mathrm{dR}}(A/F)$ on which K acts via the conjugate embedding, the anti-holomorphic subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})$, and the unit-root subspace of $H^1_{\mathrm{dR}}(A/L)$.

Fix a nonvanishing differential $\omega \in eH^0(A, \Omega_A)$. This determines a class $\eta \in eH^1(A, \mathcal{O}_A)$ dual to ω_A under the Serre duality pairing. We will view ω and η as classes in $eH^1_{\mathrm{dR}}(A/F)$ (using the canonical splitting of the Hodge sequence for η).

3 Modular Forms and p-adic Modular Forms on Shimura Curves

3.1 Modular forms on Shimura curves

There are several equivalent definitions of modular forms for Shimura curves. We will never need integrality conditions away from p, so we define them over algebras R over the localization $\mathcal{O}_{M,\mathfrak{p}}$ of $\mathcal{O}_M$ at $\mathfrak{p}_M$, where $\mathfrak{p}$ is the prime of M above p selected by the given embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

If $\pi : A \to S$ is a (relative) false elliptic curve, write $\underline{\omega}_{A/S}$ for $e\pi_* \Omega_{A/S}$; in the particular case $\mathcal{A}_R \to \mathcal{C}_R$ of the universal false elliptic curve over an $\mathcal{O}_{L,\mathfrak{p}_L}$-algebra R, write $\underline{\omega}_R$. The three definitions are:

Definition 3.1. A modular form of weight k over R is a global section of $\underline{\omega}_R^{\otimes k}$. □

Definition 3.2. Let R_0 be an R-algebra. A *test triple* is a triple $(A/R_0, t, \omega)$ consisting of a false elliptic curve A over R_0, a $V_1(N^+)$ level structure t on A, and a nonvanishing global section of $\underline{\omega}_{A/R_0}$. Two test-triples $(A/R_0, t, \omega)$ and $(A'/R_0, t', \omega')$ over R_0 are isomorphic if there is an isomorphism $f : A \to A'$ with

$$f(t) = t'$$

and

$$f^* \omega' = \omega.$$

A modular form of level N^+ and weight k over R is a rule F that assigns to every isomorphism class of test triple $(A/R_0, t, \omega)$ over every R-algebra R_0 an element of R_0, subject to the following axioms:

(1) Compatibility with base change: If $f : R_0 \to R_0'$ is a map of R-algebras and A/R_0 is the base change of A'/R_0' along f, one has

$$F(A, t, f^* \omega) = F(A', f(t), \omega).$$

(2) The weight condition: For any $\lambda \in R_0^\times$, one has $F(A/R_0, t, \lambda\omega) = \lambda^{-k} F(A/R_0, t, \omega)$. □

Definition 3.3. A *test pair* is a pair $(A/R_0, t)$ of a false elliptic curve $\pi : A \to \operatorname{Spec} R_0$ and a $V_1(N^+)$ level structure t. A modular form of weight k over R is a rule G that assigns a translation-invariant section of $\underline{\omega}_{A/R_0}^{\otimes k}$ to every isomorphism class of test pair $(A/R_0, t)$ over any R-algebra R_0, subject to the following base change axiom: if $f : R_0 \to R_0'$ is a map of R-algebras and A is the base change of A'/R_0' along f, one has

$$G(A, t) = f^* G(A', f(t)).$$ □

Given a modular form as in Definition 3.3, one gets a modular form as in Definition 3.1 by taking the section given by the universal false elliptic curve with level structure $(\mathcal{A}_R/\mathcal{C}_R, t_r)$; this process is bijective because $\mathcal{A}_R$ is universal. To go between Definitions 3.3 and 3.2, choose any translation-invariant global section ω and use the formula

$$G(A, t) = F(A, t, \omega)\omega^{\otimes k},$$

which is independent of this choice.

We now define p-adic modular forms on Shimura curves. Write L for the completion of the maximal unramified extension of $\mathbb{Q}_p$, W for the ring of integers of L, and k for the residue field $\bar{\mathbb{F}}_p$. By properness, there is a reduction map $\mathrm{red}: C(L) = \mathcal{C}(L) \to \mathcal{C}(k)$. A *residue disk* D is a subset of $C(L)$ of the form

$$\{P \in C(L) \mid \mathrm{red}(P) = x\}$$

for some fixed point $x \in C(k)$. A residue disk is not Zariski open, but is a (nonaffinoid) open subset of C^{rig}, the rigid analytic space associated with C_L. Because $\mathcal{C}$ is smooth over W, each residue disk is conformal to the open unit disk in K (see [12, Section I.1]). The ring R_x of rigid functions on a residue disk D_x corresponding to a point $x \in C(k)$ is obtained from the ring $\mathcal{R}_x$ of functions on the formal completion of $\mathcal{C}$ at x by inverting p [22, Lemma 9.7]. Write C^{ord} for the *ordinary locus* of C^{rig}, the union of the residue disks above *ordinary* false elliptic curves.

If $\mathcal{V}$ is a vector bundle on C, we will sometimes write "a rigid-analytic section of $\mathcal{V}$" to mean a section of the associated vector bundle $\mathcal{V}^{\mathrm{rig}}$ on some open subset of C^{rig}; similarly, when we write "locally analytic section", we mean a section of the associated vector bundle $\mathcal{V}^{\mathrm{la}}$ over some open subset of the topological space $C(L)$.

There are three equivalent definitions for a p-adic modular form of weight k over W for the Shimura curve C, analogous to Definitions 3.1–3.3, but working only with ordinary false elliptic curves over p-adically complete W-algebras.

Thus, a *p-adic modular form* for the Shimura curve is a rigid analytic section of the bundle $\underline{\omega}^{\otimes k}$ over C^{ord}. Equivalently, it is a rule F taking in triples (A, t, ω), where A is an *ordinary* false elliptic curve over some p-adically complete W-algebra R, t is level N-structure for A, and $\omega \in \underline{\omega}_{A/R}$, and returning an element of R, subject to compatibility with base change and the rule $F(A, t, r\omega) = r^{-k}F(A, t)$. Equivalently, it is a rule $\tilde{F}$ taking in couples (A, t) and returning a section of $\underline{\omega}_{A/R}^{\otimes k}$, compatible with base change. A *locally analytic modular form* (over some open set in the p-adic topology) is a locally analytic section of the bundle $\underline{\omega}^{\otimes k}$.

3.2 Katz's differential operators arising from the Hodge sequence

The Gauss–Manin connection

$$\nabla : \mathcal{H}^1 \to \mathcal{H}^1 \otimes \Omega_C$$

on the relative de Rham cohomology bundle on the Shimura curve C is compatible with the anti-action of $\mathrm{End}_C(\mathcal{A})$ on $\mathcal{H}^1$ via the rule

$$\nabla \circ \phi = (\phi \otimes 1) \circ \nabla.$$

(See [27, Proposition 2.2].) The Gauss–Manin connection thus naturally restricts to a connection on the bundle $\mathcal{L}_1$ and extends to the symmetric powers $\mathcal{L}_n$ of $\mathcal{L}_1$ by the Leibniz rule

$$\nabla(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_i v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n \otimes \nabla(v_i).$$

(When $n=0$, the connection is just $d : \mathcal{O}_C \to \Omega_C$.) It also extends to the bundle $\mathcal{L}_{n,n}$ via the rule $\nabla(\alpha \otimes \beta) = (\nabla\alpha) \otimes \beta$.

Using the universal principal polarization on $\mathcal{A}$, the Kodaira–Spencer map of deformation theory gives rise to a map

$$\mathrm{KS} : \pi_* \Omega_{\mathcal{A}/C} \otimes \pi_* \Omega_{\mathcal{A}/C} \to \Omega_C.$$

By [27, Theorem 2.5], this map becomes an isomorphism upon restricting to $\underline{\omega} \otimes \underline{\omega}$.

For each j, we get a map $\tilde{\nabla} : \mathcal{L}_j \to \mathcal{L}_{j+2}$ by composing the maps

$$\mathcal{L}_n \xrightarrow{\nabla} \mathcal{L}_n \otimes \Omega_C \xrightarrow{\mathrm{id}\otimes\mathrm{KS}^{-1}} \mathcal{L}_n \otimes \underline{\omega}^{\otimes 2} \longrightarrow \mathcal{L}_n \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_{n+2} \quad ,$$

where the map $\underline{\omega}^{\otimes 2} \to \mathcal{L}_2$ is Sym^2 of the inclusion in the Hodge sequence.

Suppose we have a map $\Psi : \mathcal{H}^1 \to \pi_* \Omega_{\mathcal{A}/C}$ of vector bundles splitting the Hodge sequence. Write $\Psi^n : \mathcal{L}_n \to \underline{\omega}^{\otimes n}$ for the induced map on $\mathcal{L}_n$.

We then get a "differential operator" $\Theta_\Psi : \underline{\omega}^{\otimes n} \to \underline{\omega}^{\otimes n+2}$ by the composition

$$\underline{\omega}^{\otimes n} \to \mathcal{L}_n \xrightarrow{\tilde{\nabla}} \mathcal{L}_{n+2} \xrightarrow{\Psi^n} \underline{\omega}^{\otimes n+2}. \qquad (2)$$

When $n=0$, this is just d, followed the inverse of the Kodaira–Spencer map (for any choice of splitting). In practice, the maps Ψ we use will not be algebraic. Rather, following Katz, we will apply this formalism to attain differential operators on the spaces of smooth and p-adic modular forms.

3.3 The Maass–Shimura operator Θ_∞

A *real-analytic modular form* (of weight k and level N^+) for B is an analytic function $f(z)$ satisfying the usual relation

$$f(\gamma z) = j(\gamma, z)^k f(z)$$

for $\gamma \in \Gamma_{B,N^+}$. Such a function gives rise to a section of the real-analytic-bundle associated with $\underline{\omega}_{\mathbb{C}}^{\otimes k}$ via the rule

$$\omega_f \leftrightarrow (2\pi i)^k f(\tau)\, \mathrm{d}z_1^{\otimes k} \in e\pi_* \Omega_{\mathcal{A}_{\mathcal{H}}}^{\otimes k}.$$

For $\mathcal{V}$ a vector bundle on $C_{\mathbb{C}}$, write $\mathcal{V}_{\mathrm{ra}}$ for the associated real-analytic vector bundle on $C_{\mathbb{C}}$. Hodge theory then gives a splitting $\Psi_\infty : \mathcal{H}^1_{\mathrm{ra}} \to \underline{\omega}_{\mathrm{ra}}$ of real-analytic vector bundles over $C_{\mathbb{C}}$. The differential operator coming from the splitting Ψ_∞ and the recipe in (2) is written Θ_∞ and called a Maass–Shimura operator. It sends real-analytic modular forms to real-analytic modular forms but does not preserve holomorphy. The following is shown in [27, Proposition 2.9], but we give a slightly different argument, as our normalizations differ from Mori's.

Proposition 3.4. The Maass–Shimura operator is given on the space of real-analytic modular forms (viewed as functions on $\mathcal{H}$) by the rule

$$\Theta_\infty(f) = \frac{1}{2\pi i}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{k}{2i\,\mathrm{Im}(\tau)}\right) f. \qquad \square$$

Proof. The real manifold

$$A_\tau = \frac{M_2(\mathbb{R})}{\iota_\infty(\mathcal{O}_B)}$$

is endowed with the (smooth, real-valued) one-forms $\mathrm{d}x_{ij}$ for $i, j = 1, 2$, where the symbols x_{ij} denote the standard coordinates on the universal cover $M_2(\mathbb{R})$.

The identification

$$\phi_\tau : M_2(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^2$$

of real vector spaces given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a\tau + b, c\tau + d)$$

is complex linear for the complex structure on $M_2(\mathbb{R})$ given by right-multiplication by J_τ. Under this identification, the endomorphism given on $M_2(\mathbb{R})$ by left-multiplication by e corresponds to the projection onto the first factor in $\mathbb{C}^2$. In particular, $e\,\mathrm{d}z_1 = \mathrm{d}z_1$ and $e\,\mathrm{d}z_2 = 0$.

By viewing the first projection map $z_1 : \mathbb{C}^2 \to \mathbb{C}$ as a function on $M_2(\mathbb{R})$, one sees that the complex one-form $\mathrm{d}z_1$ on $\mathbb{C}^2$ is given (under the identification ϕ_τ) by the rule

$$\mathrm{d}z_1 = \tau\,\mathrm{d}x_{11} + \mathrm{d}x_{12}, \tag{3}$$

and similarly

$$\mathrm{d}\bar{z}_1 = \bar{\tau}\,\mathrm{d}x_{11} + \mathrm{d}x_{12}. \tag{4}$$

Consequently, one has

$$\begin{aligned} \nabla\,\mathrm{d}z_1 &= \mathrm{d}x_{11} \otimes \mathrm{d}\tau \\ &= \frac{1}{2i\,\mathrm{Im}\,\tau}(\mathrm{d}z_1 - \mathrm{d}\bar{z}_1) \otimes \mathrm{d}\tau. \end{aligned}$$

From this, we compute

$$\begin{aligned} \mathrm{KS}(\mathrm{d}z_1 \otimes \mathrm{d}z_1) &= \langle \mathrm{d}z_1, \nabla \mathrm{d}z_1 \rangle \\ &= \frac{-1}{2i\,\mathrm{Im}\,\tau}\langle \mathrm{d}z_1, \mathrm{d}\bar{z}_1 \rangle\,\mathrm{d}\tau. \end{aligned}$$

Using (3) and (4), the pairing in the above formula simplifies to

$$\langle \mathrm{d}z_1, \mathrm{d}\bar{z}_1 \rangle = 2i\,\mathrm{Im}(\tau)\langle \mathrm{d}x_{11}, \mathrm{d}x_{12} \rangle.$$

To compute $\langle \mathrm{d}x_{11}, \mathrm{d}x_{12} \rangle$, we use the de-Rham-to-Betti comparison isomorphism, viewing $\mathrm{d}x_{11} \wedge \mathrm{d}x_{22}$ as an alternating form on the Lie algebra $M_2(\mathbb{R})$ of A_τ and expressing this form as a multiple of the form

$$(x, y) \mapsto \frac{-1}{N^-} \operatorname{Tr}(exs(ey)^\dagger),$$

which is the Betti realization of the negative of the polarization class $e[P_{A_\tau}]_{\text{Betti}} \in H^2(A_\tau, \mathbb{Z})$ under the usual isomorphism

$$H^2(A_\tau, \mathbb{Z}) = \{\langle\, , \,\rangle \in \operatorname{Alt}^2(M_2(\mathbb{R})) \mid \langle \iota_\infty \mathcal{O}_B, \iota_\infty \mathcal{O}_B \rangle \subset \mathbb{Z}\}.$$

For $M, M' \in M_2(\mathbb{R})$, one has

$$\mathrm{d}x_{11} \wedge \mathrm{d}x_{12}(N, N') = N_{11} N'_{12} - N'_{12} N'_{11}.$$

On the other hand, using the explicit formulas for the matrices $\iota_\infty(e)$ and $\iota_\infty(s)$ one sees that

$$\frac{1}{N^-} \operatorname{Tr}(eMs(eM')^\dagger) = \frac{1}{N^-} \operatorname{Tr} \begin{pmatrix} N^-(M_{11}M'_{12} - M_{12}M'_{11}) & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that the image of $\mathrm{d}x_{11} \wedge \mathrm{d}x_{22}$ under the Betti-to-algebraic-de-Rham comparison isomorphism coincides with the Betti realization of the polarization class.

The comparison isomorphism

$$\phi : H^2_{\text{Betti}}(A_\tau) \to H^2_{\text{dR}}(A_\tau)$$

does not commute with cycle class maps, but rather [35, Theorem I.3], for codimension 1 cycles ξ one has

$$[\xi]_{\text{dR}} = 2\pi i \phi([\xi]_{\text{Betti}}).$$

We deduce that

$$\langle \mathrm{d}x_{11}, \mathrm{d}x_{22} \rangle = \frac{-1}{2\pi \mathrm{i}}$$

and

$$\mathrm{KS}(\mathrm{d}z_1 \otimes \mathrm{d}z_1) = \left(\frac{1}{2\pi \mathrm{i}}\right) \mathrm{d}\tau. \tag{5}$$

Now we may compute the Maass–Shimura operator Θ_∞. Given a modular form $f(\tau)$ of weight k, the associated section of $\underline{\omega}^{\otimes k}$ is

$$(2\pi i)^k f(\tau)\,\mathrm{d}z_1^{\otimes k},$$

and by the Leibniz rule

$$\nabla((2\pi\mathrm{i})^k f(\tau)\,\mathrm{d}z_1^{\otimes k}) = (2\pi\mathrm{i})^k f'(\tau)\,\mathrm{d}z_1^{\otimes k}\otimes \mathrm{d}\tau \\ + (2\pi\mathrm{i})^k \sum_{i=0}^{k-1} f(\tau)\,\mathrm{d}z_1^{\otimes i}\otimes \frac{1}{2\mathrm{i}\,\mathrm{Im}\,\tau}(\mathrm{d}z_1 - \mathrm{d}\bar{z}_1)\otimes \mathrm{d}z_1^{\otimes(k-i)}\otimes \mathrm{d}\tau.$$

Applying the inverse Kodaira–Spencer map and the Hodge splitting (which annihilates $\mathrm{d}\bar{z}_1$), the above expression simplifies to

$$\frac{1}{2\pi i}(2\pi\mathrm{i})^{k+2}\left(\frac{k}{2\mathrm{i}\,\mathrm{Im}\,\tau} f(\tau) + f'(\tau)\right)\mathrm{d}z_1^{\otimes(k+2)},$$

which completes the proof. ■

3.4 The Ramanujan–Atkin–Serre operator θ

There is a Frobenius morphism ϕ on the relative de Rham cohomology bundle $\mathcal{H}^*(\mathcal{A}^{\mathrm{ord}}/C^{\mathrm{ord}})$ over the ordinary locus C^{ord}, semilinear over L, inducing the usual Frobenius morphism ϕ on the fibers of this bundle. Moreover, there is a splitting Ψ_p of the Hodge sequence (of rigid vector bundles over the ordinary locus), where $R^1\pi_*\mathcal{O}_{\mathcal{A}^{\mathrm{ord}}}$ is identified with the sub-bundle of $\mathcal{H}|_{C^{\mathrm{ord}}}$ on which ϕ acts with unit eigenvalue (see [27, Proposition 2.10]).

This splitting Ψ_p and the recipe in (2) give rise to a differential operator θ, taking p-adic modular forms of weight k to p-adic modular forms of weight $k+2$. If one regards the splitting Ψ_p as a map of bundles for the p-adic topology on $C^{\mathrm{ord}}(L)$, the same recipe gives rise to an operator on the space of locally analytic modular forms over the ordinary locus, also written as θ.

3.5 Coincidence of the operators at CM points

The p-adic and real-analytic differential operators can be related by the following fundamental theorem.

Theorem 3.5. If g is a modular form on C and $P \in C(M)$ is a CM point for some number field M, then for any choice ω of translation-invariant differential on A, one has

$$(\theta g)(P, \omega) = (\Theta_\infty g)(P, \omega),$$

where both numbers belong to M. (We are using the chosen embeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$ in two ways: first, to get inclusions $C(M) \subset C(\mathbb{C}_p)$ and $C(M) \subset C(\mathbb{C})$, and second, to make sense of the equality.) □

Proof. This is shown in many places; see, for example, [23, Theorems 2.4.5 and 2.6.7] or [27, Proposition 2.11] for the Shimura curve case. The crux is that, for A' a false elliptic curve with complex multiplication, the splitting of $H^1_{\mathrm{dR}}(A'/\mathbb{C})$ coming from Hodge theory and the splitting of $H^1_{\mathrm{dR}}(A'/\mathbb{C}_p)$ coming from Frobenius both come from the splitting of $H^1_{\mathrm{dR}}(A'/M)$ into the subspace where K acts via the identity and the subspace where K acts via conjugation, as discussed in Section 2.8. ■

3.6 Hecke operators and p-adic Hecke operators

Throughout this document, we follow the convention that Hecke operators act on the *right* on the space of modular forms, while differential operators act on the *left*. This convention is unfortunate, as the differential operators and Hecke operators do not commute. The commutation relation is given by Proposition 3.6.

For a prime ℓ, a false elliptic curve A over a field k of characteristic prime to ℓ has $\ell + 1$ cyclic sub-$\mathcal{O}$-modules annihilated by ℓ. Write $C_0, \ldots, C_\ell$ for these subgroups and $\phi_i : A \to A/C_i$ for the false isogenies associated to C_i. If t is a $V_1(N^+)$ level structure on A, and $\ell \nmid N^+$, then $t_i = \phi_i \circ t$ is a $V_1(N^+)$ level structure on A/C_i. If ω is a one-form on A, then there is a unique one-form ω_i on A/C_i with $\phi_i^* \omega_i = \omega$.

The Hecke operator T_ℓ on the space of modular forms of weight k is defined by the averaging rule

$$F|_{T_\ell}(A, t, \omega) = \frac{1}{\ell} \sum_{i=0}^{\ell} F(A/C_i, t_i, \omega_i).$$

Note that the Hecke operators preserve the weight and level of a modular form and also act on the larger space of p-adic modular forms.

Now suppose that A is a false elliptic curve with *ordinary* reduction over a p-adic field L. Then there is a unique p-torsion cyclic sub-$\mathcal{O}$-module C of A which reduces mod p to the kernel of the Frobenius morphism, called the *canonical subgroup* (this is

Theorem 11.1 of [22], although in the case of ordinary reduction one may construct it more simply, following the discussion above the statement of that theorem). Order the p-torsion cyclic sub-$\mathcal{O}$-modules in such a fashion that C_0 is the canonical subgroup. If F is a p-adic modular form, we get another p-adic modular form $F|_V$ by the rule

$$F|_V(A, t, \omega) = F\left(A/C_0, \frac{1}{p}t_0, p\omega_0\right)$$

and a p-adic modular form $F|_U$ by the rule

$$F|_U(A, t, \omega) = \frac{1}{p}\sum_{i=1}^{p} F(A/C_i, t_i, \omega_i).$$

Writing $[p]$ for the operator on the space of modular forms given by

$$F|_{[p]}(A, t, \omega) = F\left(A, pt, \frac{1}{p}\omega\right)$$

one has the relations

$$T_p = U + \frac{1}{p}[p]V \tag{6}$$

and

$$VU = \mathrm{Id}. \tag{7}$$

In particular, the operators UV and $VU - UV$ are idempotent.

Proposition 3.6. For any prime ℓ with $(\ell, N) = 1$, including the case $\ell = p$, one has

$$(\theta f)|_{T_\ell} = \ell\theta(f|_{T_\ell}).$$ □

Proof. In the modular curve case, this is an easy consequence of the formula for T_ℓ on q-expansions (see [32, paragraph 2.1]). We give a proof for Shimura curves in Serre–Tate coordinates in Section 4.6. This proof has the advantage of working for locally analytic modular forms which are rigid on residue disks (for which there is no q-expansion principle, even in the classical case). ■

4 Serre–Tate Coordinates

This section works under a notation scheme that conflicts with the one introduced in Section 2. In this section only, we work exclusively over the ring W of Witt vectors on $k = \bar{\mathbb{F}}_p$, writing L for its field of fractions. In this section, roman A always refers to abelian varieties over k and cursive $\mathcal{A}$ always to abelian schemes over other W-algebras (in particular, $\mathcal{A}$ no longer denotes the universal false elliptic curve and A no longer denotes the fixed false elliptic curve). Also, X refers to C_L and $\mathcal{X}$ refers to its integral model $\mathcal{C}_W$.

Fix a residue disk $D \subseteq X(L)$, the space of points reducing to a fixed ordinary false elliptic curve A/k with level structure t. The ring of rigid analytic functions on D is obtained from the ring of functions on the formal completion of $\mathcal{X}$ at the point corresponding to A on the special fiber by inverting p. There is a canonical formal uniformizer for this ring, coming from Serre–Tate theory, which we will use to give explicit formulas for the operators of the preceding section. Before explaining this, we review the basics of Serre–Tate theory. For a detailed exposition, see [24].

Fix a g-dimensional *ordinary* abelian variety A over k. Write A^t for the dual abelian variety. If R is an Artin local ring with residue field k, then a *deformation* of A to R is an abelian scheme $\mathcal{A}$ over R together with an identification $\mathcal{A} \times k \xrightarrow{\sim} A$.

Write T_pA and T_pA^t for the "physical" Tate modules of A and A^t, that is,

$$T_pA = \varprojlim A[p^n](k).$$

They are free $\mathbb{Z}_p$-modules of rank g (by ordinarity).

Whenever we refer to the Weil pairing on A, we mean the scheme-theoretic Weil pairing (The scheme-theoretic Weil pairing is due to Oda. The standard reference is [25]; the pairing there is the inverse of Oda's pairing and the pairing in [24]. The normalization of the Weil pairing does not affect any of the formulas in this paper.), normalized as in [24, Section 5] (the classical Weil pairing is trivial in characteristic p). It is a non-degenerate alternating pairing of k group schemes

$$e_{p^n} : A[p^n] \times A^t[p^n] \to \mu_{p^n}$$

restricting to a perfect pairing

$$\hat{A}[p^n] \times A^t[p^n](k) \to \mu_{p^n},$$

compatible with the maps $p\colon A[p^n] \to A[p^{n-1}]$. (Here, $\hat{A}$ is the formal completion of A at the origin.)

Let $\mathcal{A}$ be a deformation of A to R. The formal group $\hat{\mathcal{A}}$ represents the functor

$$\{\text{Artin local } R-\text{algebras with residue field } k\} \to \{\text{Groups}\}$$

given by

$$\hat{\mathcal{A}}(B) = \ker(\mathcal{A}(B) \to \mathcal{A}(k)).$$

Then there is a pairing

$$q_{\mathcal{A}} : T_p A \times T_p A^t \to \hat{\mathbb{G}}_m(R)$$

given by the following rule: given $P_n \in A[p^n](k)$ and $Q_n \in A^t[p^n](k)$, pick a lift $\tilde{P}_n$ of P_n to $\mathcal{A}(R)$. Consider $p^n \tilde{P}_n \in \hat{\mathcal{A}}(R) \subseteq \mathcal{A}(R)$. If n is large enough, so that $(1+\mathfrak{m}_R)^{p^n} = 1$, then $\hat{\mathcal{A}}(R)$ is killed by p^n. Thus, it makes sense to compute the Weil pairing $e_{p^n}(p^n \tilde{P}_n, Q_n^t)$, which is an element of $\mu_{p^n}(R)$, which for n large coincides with $\hat{\mathbb{G}}_m(R)$. These elements are compatible for large n, which gives the desired map $q_{\mathcal{A}}$.

The Serre–Tate theorem asserts that this construction gives a bijection

$$\{\text{Isomorphism classes of deformations of } A \text{ to } R\} = \mathrm{Hom}_{\mathbb{Z}_p}(T_p A \otimes T_p A^t, \hat{\mathbb{G}}_m(R))\}.$$

In particular, the left-hand side, which is *a priori* only a set, gains the structure of a $\mathbb{Z}_p$-module. Furthermore, this correspondence is functorial in R. More precisely, writing $\mathcal{M}$ for the functor from the category of Artin local rings to the category of sets given by

$$\mathcal{M}(R) = \{\text{Isomorphism classes of deformations of } A \text{ to } R\},$$

we have

$$\mathcal{M} = \mathrm{Hom}_{\mathbb{Z}_p}(T_p A \otimes T_p A^t, \hat{\mathbb{G}}_m).$$

Because these equivalences are compatible with inverse limits, we may replace the category of Artin local rings with the category of complete local rings in all of the preceding discussion (although the recipe for computing the pairing $q_{\mathcal{A}}$ only makes sense over Artin local rings). The following proposition [24, Theorem 2.1.4] gives us a helpful shortcut in calculating Serre–Tate coordinates.

Proposition 4.1. Given two ordinary abelian varieties A and A' over k with fixed deformations $\mathcal{A}$ and $\mathcal{A}'$ (over a fixed Artin local ring with residue field k), a map $f : A \to A'$ lifts to a map from $\mathcal{A}$ to $\mathcal{A}'$ if and only if for all $P \in T_p A$ and $Q^t \in T_p A'^t$ one has

$$q_{\mathcal{A}}(P, f^t(Q^t)) = q_{\mathcal{A}'}(f(P), Q^t). \qquad \square$$

Now, the functor $\mathcal{M}$ parameterizing all deformations of our fixed ordinary abelian variety A is a formal scheme $\mathrm{Spf}\mathcal{R}$ equipped with a universal formal abelian scheme $\pi : \hat{\mathcal{A}} \to \mathrm{Spf}\mathcal{R}$.

By the above, there is a bilinear pairing

$$q_{\hat{\mathcal{A}}} : T_pA \otimes T_pA^t \to 1 + \mathfrak{m}_{\mathcal{R}},$$

and by universality, given any deformation $\mathcal{A}$ of A over any complete W-algebra R with residue field k, we have a map $\mathcal{R} \to R$ making the following triangle commute:

$$\begin{tikzcd} & T_pA \otimes T_pA^t \arrow[dl, "q_{\hat{\mathcal{A}}}"'] \arrow[dr, "q_{\mathcal{A}}"] & \\ 1 + \mathfrak{m}_{\mathcal{R}} \arrow[rr] & & 1 + \mathfrak{m}_R \end{tikzcd}$$

The ring $\mathcal{R}$ is the completion of the W-algebra generated by the functions $q(P, Q^t) - 1$, as P and Q^t range over T_pA and T_pA^t, respectively, subject to the relations generated by the bilinearity of the pairing q. In particular, suppose that we pick bases $\{P_1, \ldots, P_g\}$ and $\{Q_1^t, \ldots, Q_g^t\}$ of T_pA and T_pA^t. Then, we have g^2 elements $q_{ij} = q(\hat{\mathcal{A}}, P_i, Q_j^t) \in \mathcal{R}$, and, writing

$$T_{ij} = q_{ij} - 1,$$

we get a ring isomorphism

$$\mathcal{R} = W[[T_{ij}]].$$

4.1 Katz's computation of ∇

Serre–Tate coordinates give us a canonical way to compute the Gauss–Manin connection on the formal relative de Rham cohomology bundle $\hat{\mathcal{H}} = \mathbb{R}^1\pi_*(\Omega^{\cdot}_{\mathcal{A}/\mathcal{R}})$ on residue disks over ordinary points. In this formal setting, there are line bundles $\pi_*\Omega_{\hat{\mathcal{A}}/\mathcal{M}}$ and $R^1\pi_*\mathcal{O}_{\hat{\mathcal{A}}} = \mathrm{Lie}(\hat{\mathcal{A}}^t/\mathcal{R})$, sitting in the usual Hodge exact sequence

$$0 \to \pi_*\Omega_{\hat{\mathcal{A}}/\mathcal{M}} \to \hat{\mathcal{H}} \to \mathrm{Lie}(\hat{\mathcal{A}}^t/\mathcal{R}) \to 0.$$

There is likewise a Gauss–Manin connection $\nabla : \hat{\mathcal{H}} \to \hat{\mathcal{H}} \otimes \Omega_{\mathrm{Spf}\mathcal{R}}$ and an $\mathcal{R}$-semilinear Frobenius endomorphism of $\hat{\mathcal{H}}$. We will abuse notation and not distinguish these line bundles from $\mathcal{R}$-modules, starting in the next statement.

Lemma 4.2 (Katz)**.** There are canonical isomorphisms

$$T_pA^t \otimes \mathcal{R} \xrightarrow{\sim} \pi_*\Omega_{\hat{\mathcal{A}}/\mathcal{M}} \tag{$\star$}$$

and

$$\mathrm{Hom}(T_pA, \mathbb{Z}_p) \otimes \mathcal{R} \xrightarrow{\sim} \mathrm{Lie}(\hat{\mathcal{A}}^t/\mathcal{R}) \tag{$\star\star$}$$

such that:

(1) the $\mathcal{R}$-semilinear Frobenius endomorphism Φ of $\hat{\mathcal{H}}$ acts via multiplication by p on T_pA^t under the identification ($\star$);

(2) the sub-$\mathbb{Z}_p$-module of $\hat{\mathcal{H}}$ on which Φ acts via the identity maps isomorphically to $\mathrm{Hom}(T_pA, \mathbb{Z}_p)$ under the map $\hat{\mathcal{H}} \to \mathrm{Lie}(\hat{\mathcal{A}}^t/\mathcal{R})$ and the identification ($\star\star$). □

Proof. We just recall the construction of the isomorphisms here. For the computation of the Frobenius action, see [24, Lemma 4.2.1]. The Weil pairing gives an isomorphism

$$T_pA^t = \mathrm{Hom}_{\mathbb{Z}_p}(\hat{\mathcal{A}}, \hat{\mathbb{G}}_m),$$

so given $P \in T_pA^t$, one gets a differential on $\hat{\mathcal{A}}$ by pulling back $\frac{\mathrm{d}T}{T}$.

Dually (and swapping the roles of A and A^t), the Weil pairing gives rise to an isomorphism

$$\hat{\mathcal{A}}^t = \mathrm{Hom}(T_pA, \hat{\mathbb{G}}_m).$$

Applying the functor Lie to both sides gives the second result, since for any R one has, writing $R[\epsilon]$ for the ring of dual numbers over R:

$$\begin{aligned} \ker(\mathrm{Hom}(T_pA, \hat{\mathbb{G}}_m(R[\epsilon])) \to \mathrm{Hom}(T_pA, \hat{\mathbb{G}}_m(R))) &= \mathrm{Hom}(T_pA, 1+\epsilon R) \\ &\approx \mathrm{Hom}(T_pA, R). \end{aligned}$$ ■

For $Q^t \in T_pA^t$, write $\hat{\omega}_{Q^t}$ for the differential form coming from the lemma. If $\phi \in \mathrm{Hom}(T_pA, \mathbb{Z}_p)$, write $\hat{\eta}_\phi$ for the image in $\hat{\mathcal{H}}$ of the vector field attached to ϕ by the lemma under the Frobenius splitting of the Hodge sequence. Fix a basis $P_1, \ldots, P_g$ of T_pA, and write $P_i^\vee$ for the dual basis of $\mathrm{Hom}(T_pA, \mathbb{Z}_p)$. Then, the Gauss–Manin connection on $\hat{\mathcal{H}}$ is computed as follows:

Theorem 4.3 (Katz). One has

$$\nabla \hat{\eta}_{P_i^\vee} = 0$$

and, for any $Q^t \in T_p A^t$, one has

$$\nabla \hat{\omega}_{Q^t} = \sum_i \hat{\eta}_{P_i^\vee} \otimes d \log q(P_i, Q^t). \qquad \square$$

Proof. This is "version quat." of the Main Theorem of [24], as is stated in Section 4.1 of that paper. ■

The following observation is Lemma 3.5.1 of [24].

Lemma 4.4. Given a map $f : A \to B$ of ordinary abelian varieties in characteristic p that deforms to a map of universal formal abelian varieties $\mathbf{f} : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$, and $P^t \in T_p B^t$, one has

$$\mathbf{f}^* \hat{\omega}_{P^t} = \hat{\omega}_{f^*(P^t)}. \qquad \square$$

4.2 Serre–Tate coordinates for Shimura curves

Now assume that A is a false elliptic curve over k. In this case, we have a subfunctor $\mathcal{M}^{\text{false}}$ of $\mathcal{M} = \mathrm{Spf}\mathcal{R}$ taking an Artin local ring R with residue field k to the set of "false deformations" of A to R, where a false deformation is a deformation $\mathcal{A}$ of A to R together with an embedding $\mathcal{O}_B \to \mathrm{End}_{\mathcal{R}}(\mathcal{A})$ deforming the given embedding $\mathcal{O}_B \to \mathrm{End}_k(A)$ (deformations of the extra endomorphisms, if they exist, are unique—see [24, Theorem 2.4]).

Proposition 4.5. The subfunctor $\mathcal{M}^{\text{false}}$ of $\mathcal{M}$ is a formal subgroup-scheme. The ring of formal functions $\mathcal{R}^{\text{false}}$ on $\mathcal{M}^{\text{false}}$ is the quotient of $\mathcal{R}$ by the closed ideal generated by the relations

$$q(bP, Q^t) = q(P, b^\dagger Q^t)$$

for $b \in \mathcal{B}$. $\square$

Proof. Recall that, by definition of a false elliptic curve, under the embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}(\mathcal{A})$, the Rosati involution restricts to $\dagger$. The relations $q(bP, Q^t) = q(P, b^\dagger Q^t)$ then follow as an endomorphism is adjoint, with respect to the Weil pairing, to its image under the Rosati involution. To see that these are the only relations (which is the remaining content of the proposition), see [27, Proposition 3.3]. ■

Restricting the Hodge sequence of vector bundles on $\mathcal{M}$ to $\mathcal{M}^{\text{false}}$ recovers the Hodge sequence for the universal false elliptic curve $\mathcal{A}_{\mathcal{M}^{\text{false}}}/\mathcal{M}^{\text{false}}$, and for a class $\eta \in$

$\mathcal{H}^1(\mathcal{M}/\mathcal{R})$, one has

$$(\nabla\eta)|_{\mathcal{M}^{\text{false}}} = \nabla(\eta|_{\mathcal{M}^{\text{false}}})$$

by the functoriality of the construction of the Gauss–Manin connection.

Lemma 4.6. The Katz isomorphisms of Lemma 4.2 are †-equivariant for the action of $\mathrm{End}(\mathcal{A})$; that is, one has

$$\hat{\omega}_{bQ^t}|_{\mathcal{M}^{\text{false}}} = [b^\dagger]^*\hat{\omega}_{Q^t}|_{\mathcal{M}^{\text{false}}}$$

for $b \in \mathrm{End}(\mathcal{A})$, and similarly, for $\phi \in \mathrm{Hom}(T_pA, \mathbb{Z}_p)$, one has

$$\hat{\eta}_{b^*\phi} = (b^\dagger)^*\hat{\eta}_\phi. \qquad \square$$

Proof. By definition,

$$\hat{\omega}_{bQ^t} = \psi_b^*\frac{\mathrm{d}T}{T},$$

where ψ_b is the Weil-pairing map

$$(bQ_t, \_\_).$$

Writing ψ for the map $(Q_t, \_\_)$, the map ψ_b decomposes as $\psi_b = \psi \circ b^\dagger$, and the first result follows.

The second argument is similar. The construction of the isomorphism says to view $b\phi$ as a map to $1+\epsilon R$ given by Weil-pairing against some $\xi \in \mathrm{Lie}(\hat{A})$. But now ϕ will be Weil-pairing against $b^\dagger\xi$. ■

Pick a basis $\{P_1, P_2\}$ for T_pA such that $eP_1 = P_1$ and $eP_2 = 0$; denote by P_1^t, P_2^t the images of P_1 and P_2 in T_pA^t under the canonical principal polarization. These choices give rise to sections of the formal bundles $\hat{\pi}_*\mathcal{A}/\mathcal{X}$ and $\mathrm{Lie}(\hat{\mathcal{A}}^t/\mathcal{X})$ via the Katz isomorphisms of Lemma 4.2, which we denote by $\hat{\omega}_{P_i^t}$ and $\hat{\eta}_{P_i^\vee}$. To compute the Gauss–Manin connection in the situation that we desire, we will compute it on these sections over the formal four-fold $\mathcal{M}$, restrict to $\mathcal{M}^{\text{false}}$, then apply e.

Because the Katz isomorphisms in Lemma 4.2 are equivariant for the action of $\mathrm{End}(\mathcal{A})$, one has

$$e^*\begin{pmatrix} \hat{\omega}_{P_1^t}|_{\mathcal{M}^{\text{false}}} \\ \hat{\omega}_{P_2^t}|_{\mathcal{M}^{\text{false}}} \\ \hat{\eta}_{P_1^\vee}|_{\mathcal{M}^{\text{false}}} \\ \hat{\eta}_{P_2^\vee}|_{\mathcal{M}^{\text{false}}} \end{pmatrix} = \begin{pmatrix} \hat{\omega}_{P_1^t}|_{\mathcal{M}^{\text{false}}} \\ 0 \\ \hat{\eta}_{P_1^\vee}|_{\mathcal{M}^{\text{false}}} \\ 0 \end{pmatrix}.$$

For the remainder of this document, abbreviate $e^*\hat{\omega}_{P_1}$ as $\hat{\omega}$ and $e^*\hat{\eta}_{P_1}$ as $\hat{\eta}$. The choice of basis of T_pA gives us functions $q(P_i, P_j^t)$ for $i, j = 1, 2$; abbreviate the particular function $q(P_1, P_1^t)$ as just q.

Theorem 4.7. One has

$$\nabla\hat{\omega} = \hat{\eta} \otimes d\log q,$$

$$\nabla\hat{\eta} = 0. \qquad \square$$

Proof. This follows from Theorem 4.3 and the rule $\nabla \circ e = (1 \otimes e) \circ \nabla$. ■

As a consequence, we see that the horizontal sections for ∇ are spanned by $\hat{\eta}$ and $\hat{\omega} - \log q\hat{\eta}$. As another consequence, the theorem gives

$$\mathrm{KS}(\hat{\omega}^{\otimes 2}) = d\log q. \tag{8}$$

4.3 The operator θ in coordinates

Recall that θ is defined by composing the maps

$$\underline{\omega}^k \to \mathcal{L}_k \xrightarrow{\nabla} \mathcal{L}_k \otimes \Omega \xrightarrow{\Psi_p^r} \hat{\underline{\omega}}^k \otimes \Omega \xrightarrow{\mathrm{KS}^{-1}} \underline{\omega}^{k+2}.$$

To compute the effect of this map on a section ω of the bundle $\mathcal{L}_k$ over the ordinary locus C^{ord}, we compute separately in each residue disk. Thus, along a fixed residue disk D, write

$$\omega = F(T)\hat{\omega}^{\otimes k},$$

where $T = q - 1$ is the canonical uniformizer of $\mathcal{R}$ coming from Serre–Tate theory and F is a power series in T.

Then it follows from Katz's computation of ∇ and the Leibniz rule that

$$\nabla\omega = \sum_{i=0}^{k-1} F(T)\hat{\omega}^{\otimes i} \otimes \hat{\eta} \otimes \hat{\omega}^{\otimes k-1-i} \otimes d\log q + F'(T)\hat{\omega}^{\otimes k}.$$

By Lemma 4.2, the splitting Ψ_p sends $\hat{\eta}$ to 0, so

$$\theta\omega = F'(T)\hat{\omega}^{\otimes k} \otimes \mathrm{KS}^{-1}(\mathrm{d}T).$$

Now

$$\mathrm{d}T = q\,d\log q = (1+T)\,\mathrm{d}\log q,$$

so

$$\theta f = (T+1)F'(T)\hat{\omega}^{\otimes k+2}$$

by formula (8). Thus,

$$\theta f = (T+1)\frac{\mathrm{d}}{\mathrm{d}T}F(T)\hat{\omega}^{\otimes k+2}.$$

This formula is also derived in the proof of [27, Theorem 3.6].

4.4 Hecke operators in coordinates

We handle T_ℓ for $\ell \neq p$ first. Suppose $\psi : A \to A/C$ is an isogeny of degree prime to p. Then, the subgroup C and the map ψ deform uniquely to a subgroup scheme $\mathcal{C}$ and a map $\psi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ for any lift $\mathcal{A}$ of A to $L = \mathrm{Frac}(W)$. Recall the fixed generator P_1 of eT_pA. Because ψ and ψ^t both induce isomorphisms of p-adic Tate modules, $Q_1^t := (\psi^t)^{-1}(P_1^t)$ is a generator of $eT_p^t(A/C)$. Write $\hat{\omega}_1 := e\hat{\omega}_{P_1^t}$ for the canonical formal one-form on the disk $\tilde{D} \subset X(L)$ whose points correspond to characteristic 0 abelian surfaces reducing to A/C with level structure $\psi(t)$. By Lemma 4.4, it satisfies

$$\psi^*\hat{\omega}_1 = \hat{\omega}.$$

Given this choice of Q_1^t, there is a corresponding basis element of $T_p(A/C)$ via the canonical principal polarization of A/C, which is $Q_1 = \frac{1}{\deg\psi}\psi^t(Q_1^t)$. Thus, there is a Serre–Tate coordinate $\tilde{q} = q(\_, Q_1, Q_1^t)$ on the disk $\tilde{D}$.

Lemma 4.8. The function f_ψ on D given by $\mathcal{A} \mapsto \tilde{q}(\mathcal{A}/\mathcal{C})$ can be computed as

$$f_\psi = q^{\frac{1}{\deg\psi}}. \qquad \square$$

Proof. It suffices to check that the two functions agree for any deformation $\mathcal{A}$ of A to an Artin local ring R with residue field k. It follows from Proposition 4.1 that

$$\begin{aligned} f_\psi(\mathcal{A}) &= q(\mathcal{A}/\mathcal{C}, Q_1, Q_1^t) \\ &= q\left(\mathcal{A}/\mathcal{C}, \frac{1}{\deg\psi}\psi_t(P_1), Q_1^t\right) \end{aligned}$$

$$= q\left(\mathcal{A}, \frac{1}{\deg\psi} P_1, P_1^t\right)$$

$$= q(\mathcal{A})^{\frac{1}{\deg\psi}}. \qquad \blacksquare$$

Note that the final expression does not depend on any choices, as the Serre–Tate parameter q is a principal unit and $\deg\psi$ is prime to p.

Corollary 4.9. For each cyclic degree ℓ isogeny $A \to A/C_i$ of A, write $\hat{\omega}_i$ for the canonical one-form on the disk R_i of points reducing to A/C_i, as normalized above. Suppose f is a modular form such that for each i, the Serre–Tate expansion on the disk D_i is

$$f = F_i(T_i)\hat{\omega}_i^{\otimes k}.$$

Then on the disk D, $f \mid T_\ell$ is given by

$$\sum_{i=1}^{\ell+1} F_i((1+T)^{1/\ell} - 1)\hat{\omega}^{\otimes k}. \qquad \square$$

We move on to the operators U and V. Write ϕ for the mod p Frobenius. For D an ordinary residue disk corresponding to a false elliptic curve A with level structure t, write D^ϕ for the disk corresponding to A^{frob} with level structure $\frac{1}{p}t^\phi$. Note that, because of the extra factor of $\frac{1}{p}$ on the level structure, D^ϕ is *not* the image of D under the map $X^{\mathrm{ord}} \to X^{\mathrm{ord}}$ under the canonical p-isogeny $\mathcal{A} \to \mathcal{A}/\mathcal{C}_0$.

We are going to pick Serre–Tate coordinates on these disks in a compatible way. One cannot ensure, as in the prime-to-p case, that the canonical formal one-form pulls back to the canonical formal one-form. However, using ordinarity, we do at least have that $Q_1 := \phi P_1$ is a basis for eT_pA^ϕ. Write $\{Q_1^t, Q_2^t\}$ for the corresponding basis for $T_pA^{\phi,t}$, using the principal polarization on A. Then one has

$$\phi^t(Q_i^t) = pP_i^t.$$

Write q for the function on D^ϕ corresponding to this basis. There is a function f on $D = D_0$ given by $(\mathcal{A}, t) \mapsto q_1(\mathcal{A}/C_{\mathcal{A}}, \frac{1}{p}t^\phi)$, where $\mathcal{C}_{\mathcal{A}}$ is the canonical subgroup of $\mathcal{A}$.

Lemma 4.10. One has

$$f = q^p$$

as functions on D. $\square$

Proof. Again it suffices to check that the two functions agree for any deformation $\mathcal{A}$ of A to an Artin local ring R with residue field k. One has

$$\begin{aligned} f(\mathcal{A}) &= q(\mathcal{A}/C, \phi P_1, Q_1^t) \\ &= q(\mathcal{A}, P_1, \phi^t Q_1^t) \\ &= q(\mathcal{A}, P_1, pP_1^t) \\ &= q(\mathcal{A})^p. \end{aligned}$$ ■

Let $\hat{\omega}_0$ be the canonical formal relative one-form for D_0 attached to the Tate-module generators $\{\phi P_1, \phi P_2\}$. Write $\Phi : D \to D_0$ for the "quotient by the canonical subgroup" map.

Lemma 4.11. One has

$$\Phi^* \hat{\omega}_0 = p\hat{\omega}.$$ □

Proof. This follows from Lemma 4.4. ■

Corollary 4.12. If $f = \sum F(T)\hat{\omega}_0^{\otimes k}$ is a modular form on D^ϕ expressed in Serre–Tate coordinates, then the corresponding modular form $f|_V$ is given in Serre–Tate coordinates on D by

$$f|_V = F((1+T)^p - 1)\hat{\omega}^{\otimes k}.$$ □

Finally, we compute U. Write $D_\star$ for the image of D under the map $X^{\mathrm{ord}} \to X^{\mathrm{ord}}$ under any of the (not-canonical) p-isogenies $\mathcal{A} \to \mathcal{A}/\mathcal{C}_i$. Then, the map $D \to D_\star \to (D_\star)^\phi$ is the identity map (because of the factor of $1/p$ in the level structure). For each subgroup $\mathcal{C}$ other than the canonical subgroup of the universal false elliptic curve over D, there is a function g_C given by $(\mathcal{A}, t) \mapsto q_{D_\star}(\mathcal{A}/C, \mathrm{Image} t)$, where $C = \mathcal{C}_\mathcal{A}$.

Lemma 4.13. One has

$$g_C^p = q.$$ □

Proof. This follows from Lemma 4.10, because $(\mathcal{A}/C)^t \to \mathcal{A}^t$ is the canonical isogeny for $(\mathcal{A}/C)^t$. ■

Fix a primitive pth root of unity $\zeta \in \mathbb{C}_p$.

Lemma 4.14. There is an ordering on the subgroups such that on C_i one has

$$g_{C_i} = \zeta^i q. \qquad \square$$

Proof. It follows from the preceding lemma that there is some root of unity making the formula true for any given C_i; the content of the lemma is that each distinct root of unity appears exactly once. To determine which root of unity shows up after killing C_i, we may evaluate at the CM point $q = 1$. This calculation is done by Brakočević in the GL_2 case, using Shimura's reciprocity law for GL_2 (see the proof of [8, Lemma 7.2]).

To reduce to the GL_2 case, note that, by the argument on p. 919 of [31], there is a false isogeny (defined over a number field in which p is unramified)

$$\lambda : \mathcal{A} \to \mathcal{E}_1 \times \mathcal{E}_2$$

of $\mathcal{A}$ with a product of elliptic curves, with degree prime to p. The result then follows from Proposition 4.1. ■

The pth roots of q in the ring of $\mathbb{C}_p$-valued functions on D are given by Taylor-expanding $\zeta^i(1+T)^{1/p}$. Write $\Phi_i : D \to D_\star$ for the map killing the ith (noncanonical) subgroup.

Lemma 4.15. One has $\Phi^*\hat{\omega}_i = \hat{\omega}$. □

Proof. This follows from Lemma 4.11, since the degree of Φ_i is p. ■

Proposition 4.16. If $f = \sum F(T)\hat{\omega}^{\otimes k}$ is a rigid-analytic modular form on D_{-1} expressed in Serre–Tate coordinates, then the corresponding modular form $f|_U$ is given in Serre–Tate coordinates on D by

$$f|_U(T) = \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+T)^{1/p} - 1)\hat{\omega}^{\otimes k}. \qquad \square$$

Proof. Just a restatement of the preceding two lemmas. Note that it makes sense to evaluate F at $(\zeta^i(1+T)^{1/p} - 1)$, as the constant coefficient of $\zeta^i(1+T)^{1/p} - 1$ is $\zeta^i - 1$, which lives in the maximal ideal of $W[\zeta]$. ■

The importance of the above formulas is that they give a formula for the composition UV of Hecke operators (the composition VU is the identity).

Proposition 4.17. Suppose that f has Serre–Tate expansion

$$f = F(T)\hat{\omega}^{\otimes k}$$

on the disk D. Then $f|_{UV}$ has Serre–Tate expansion

$$f|_{UV}(T) = \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+T)-1)\hat{\omega}^{\otimes k}.$$

Moreover, if $F(T) \in W[[T]]$, then $\frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+T)-1) \in W[[T]]$. □

Proof. Write $f = F_\phi(T)\hat{\omega}^{\otimes k}$ in Serre–Tate coordinates on the disk D^ϕ.

We compute

$$\begin{aligned}(F(T)\hat{\omega}^{\otimes k})|_{UV} &= (F_\phi((1+T)^p - 1)\hat{\omega}_0^{\otimes k})|_U \\ &= \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+(1+T)^p-1)^{1/p}-1)\hat{\omega}^{\otimes k} \\ &= F(\zeta^i(1+T)-1)\hat{\omega}^{\otimes k}.\end{aligned}$$

The integrality claim for this expression is well known (see e.g. [11, p. 16]). To prove it, note that

$$\sum_{i=0}^{p-1} F(\zeta^i(1+T)-1)$$

has coefficients in the maximal ideal $\mathfrak{p}$ of $W[\zeta]$, because it reduces to 0 mod $(1-\zeta)$. Thus, the coefficients lie in $\mathfrak{p} \cap W = (p)$. ■

4.5 Continuity properties of the operators

Write Θ for the operator $(1+T)\frac{\mathrm{d}}{\mathrm{d}T}$ on the ring $W[[T]]$, so what we have seen so far is that

$$\theta(F(T)\hat{\omega}^{\otimes k}) = (\Theta F)(T)\hat{\omega}^{\otimes k+2}.$$

In this section, we investigate elementary continuity properties of the operator Θ, and then use them to deduce similar properties for θ on the space of p-adic modular forms.

Proposition 4.18. The Θ operator satisfies the continuity condition

$$\Theta^i F \equiv \Theta^j F \mod p^n$$

for any F and any $i, j \geq n$ such that $i \equiv j \mod (p-1)p^{n-1}$. □

Proof. First suppose that F is a polynomial. Setting $x = 1 + T$, on the ring $W[x] = W[T] \subseteq W[[T]]$, we have $\Theta = x\frac{\mathrm{d}}{\mathrm{d}x}$, so $\theta^i \sum a_n x^n = \sum n^i a_n x^n$ and the result follows (using Fermat's little theorem for the terms with $(p, n) = 1$ and the condition $i, j \geq n$ for the others).

To establish the result for a general power series $F = \sum b_n T^n$, we may fix n and prove that the congruence holds for the coefficients of T^n in $\Theta^i F$ and $\Theta^j F$. Note that the coefficient of T^n in ΘF depends only on the coefficients of T^n and T^{n+1} in F. Thus, the coefficient of T^n in $\Theta^i F$ depends only on the numbers $b_n, b_{n+1}, \ldots, b_{n+i}$, and similarly for $\Theta^j F$. It follows that there exists a polynomial truncation G of F such that the coefficients of T^n in $\Theta^i G$ and $\Theta^j G$ are the same as those for F. Since the congruence holds for polynomials, the result follows. ■

Corollary 4.19. Suppose that f is a p-integral modular form, that is, that f is a modular form over some subring of W. Then, for any ordinary pair (A, ω), one has

$$\theta^i f(A, \omega) \equiv \theta^j f(A, \omega) \mod p^n$$

whenever $i \equiv j \mod (p-1)p^{n-1}$. □

Write $f^\flat = f|_{VU-UV}$, and similarly for $F \in W[[T]]$, we write $F^\flat = F|_{\mathbf{VU}-\mathbf{UV}}$, where $\mathbf{UV}$ is the formal operator on power series of Theorem 4.17 and $\mathbf{VU}$ is the identity operator.

Proposition 4.20. One has

$$F^\flat = \lim_{i\to\infty} \Theta^{p^i(p-1)} F.$$ □

Proof. The limit on the right-hand side makes sense because of Proposition 4.18. Writing $\Theta^{(p-1)p^\infty}$ for the operator $\lim_{i\to\infty} \Theta^{p^i(p-1)}$, we see that $\Theta^{(p-1)p^\infty}$ is a continuous W-linear operator on $W[[T]]$. As this is also the case for the operator $\flat$, it suffices to check the putative equality on the polynomials $F_m = (1+T)^m$, since the linear span of these polynomials is dense. One has $\Theta^i F_m = m^i F_m$, and so

$$\Theta^{(p-1)p^\infty} F_m = \begin{cases} 0, & p \mid m, \\ F_m, & (p,m) = 1. \end{cases}$$

On the other hand, using Proposition 4.17, we compute

$$\begin{aligned} F_m^\flat(T) &= F_m(T) - \frac{1}{p}\sum_{i=0}^{p-1} F_m(\zeta^i(1+T)-1) \\ &= F_m(T) - \frac{1}{p}\sum_{i=0}^{p-1} \zeta^{mi}(1+T)^m. \end{aligned}$$

If p is prime to m, then the sum is zero, since ζ^{mi} ranges over a complete set of pth roots of unity. If p divides m, the sum is $F_m(T)$. In either case, $F_m^\flat = \Theta^{(p-1)p^\infty} F_m$ as desired. ■

We return to the operators θ, U, and V on the space of p-adic modular forms. If f is a p-adic modular form, and (A', t', ω') is a triple consisting of a false elliptic curve over L with ordinary reduction, level structure, and a translation-invariant one-form, then the limit

$$\lim_{i\to\infty} \theta^{p^i(p-1)} f(A', t', \omega')$$

exists and equals $f^\flat(A', t')$, since this statement can be checked on residue disks. In particular, if j is a negative integer, it makes sense to write

$$\theta^j f(A', t', \omega')$$

to mean

$$\lim_{i\to\infty} \theta^{j+p^i(p-1)} f.$$

A priori, $\theta^j f$ is a locally analytic modular form, rigid when restricted to a fixed residue disk. Note that, in spite of the notation, one has $\theta^k\theta^{-k} f = f^\flat$, not f.

4.6 Proof of Proposition 3.6

We conclude by proving the formula

$$(\theta f)|_{T_\ell} = \ell\theta(f|_{T_\ell})$$

as promised. In each case, the result follows from the explicit formulas for the Hecke operators (using the chain rule). For $\ell \neq p$, this is a simple calculation. For T_p it will follow from related formulas for U and V, using the formula

$$T_p = U + \frac{1}{p}[p]V.$$

Letting ψ denote the automorphism of X mapping (A, t) to (A, pt), it follows directly from the modularity of f that

$$f|_{[p]} = p^k f \circ \psi,$$

and thus,

$$\theta(f|_{[p]}) = p^{-2}(\theta f)|_{[p]}$$

(because θ boosts the weight of f by 2). The chain rule argument gives that

$$\theta(f|_V) = p(\theta f)|_V$$

and

$$p\theta(f|_U) = (\theta f)|_U,$$

which is the desired result.

5 Cohomology of Shimura Curves and Coleman's Theory

5.1 Deligne's twisted cohomology groups

Let $X/\mathbb{C}$ be a variety, and suppose that $\mathbb{V}$ is a local system on $X(\mathbb{C})$, that is, a sheaf locally (for the complex topology) isomorphic to the constant sheaf $\mathbb{C}^g$. Deligne [16] then showed how to recover the cohomology groups $H^i(X(\mathbb{C}), \mathbb{V})$ algebraically, generalizing the case $\mathbb{V} = \mathbb{C}$ of algebraic de Rham cohomology. Recall that the vector bundle $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$ is algebraic, as is the connection $\mathcal{V} \to \mathcal{V} \otimes \Omega$ for which $\mathbb{V}$ is the sheaf of horizontal sections. Then $H^i(X(\mathbb{C}), \mathbb{V})$ coincides with the hypercohomology of the complex

$$0 \xrightarrow{\nabla} \mathcal{V} \otimes \Omega \xrightarrow{\nabla} \mathcal{V} \otimes \wedge^2 \Omega \xrightarrow{\nabla} \cdots. \tag{9}$$

Write $H^i_{\mathrm{dR}}(X, \mathcal{V}, \nabla)$ for the ith hypercohomology group of this complex. Of course these algebraic definitions all make sense over an arbitrary base field k (they are not useful unless the characteristic of k is zero).

Given two vector bundles $\mathcal{V}, \mathcal{V}'$ with flat connections ∇, ∇', the natural map of the complexes (9) for $\mathcal{V}, \mathcal{V}'$, and $\mathcal{V} \otimes \mathcal{V}'$ gives rise to maps

$$H^i(X, \mathcal{V}, \nabla) \otimes H^j(X, \mathcal{V}', \nabla') \to H^{i+j}(X, \mathcal{V}, \nabla \otimes \nabla').$$

We will apply this observation in particular to the case where $(\mathcal{V}, \nabla)$ is self-dual, so that the target is the ordinary algebraic de Rham cohomology of X.

5.2 Coleman's rigid analytic theory

Let $X/\mathbb{C}_p$ be a curve with good reduction and $\mathcal{V}$ a vector bundle on X with flat connection. Write X^{rig} for the associated rigid analytic space and $\mathcal{V}^{\mathrm{rig}}$ for the analytification of $\mathcal{V}$.

For any point $P \in X(\mathbb{C}_p)$, write D_P for the residue disk containing P, which is isomorphic as a rigid space to the open unit disk. Fixing one such isomorphism taking P to 0 allows us to speak of the affinoid subdomain of D_P given by "the closed disk of radius r" for any $r < 1$, denoted $D_{P,r}$. Fixing a finite number of points $P_1, \ldots, P_n$, consider the affinoid space

$$\mathcal{X}^0 = X^{\mathrm{rig}} \setminus D_{P_1} \setminus \cdots \setminus D_{P_n}$$

and its "basic wide open neighborhoods" (for various choices of radii r_i with $0 < r_i < 1$)

$$\mathcal{W}_{r_1,\ldots,r_n} = X^{\mathrm{rig}} \setminus D_{P_1,r_1} \setminus \cdots \setminus D_{P_n,r_n}.$$

For ω a one-form on an annulus, Coleman has defined a notion of "residue" that coincides with the algebraic notion of residue when ω comes from an algebraic one-form. Using this definition, it makes sense to speak of residues of vector-bundle-valued one-forms on annuli, provided that the vector bundle comes with a flat connection that trivializes on sufficiently small disks. Coleman's residue is only well-defined up to a sign (depending on the "orientation" of the annulus), but he shows in Corollary 3.7a of [13] that one can compatibly orient all the annuli $D_{P_i} \setminus D_{P_i,r_i}$ by choosing a uniformizer of the deleted point as a uniformizer in the ring of rigid functions on the annulus (instead of choosing the reciprocal of a uniformizer). Here, "compatibility" implies that the residue of a meromorphic one-form on a Zariski open will agree with the residue of the same form thought of as a rigid one-form on an annulus, rather than with its negative.

Coleman has shown that the algebraic de Rham cohomology of the affine curve $X \setminus \{P_1, \ldots, P_n\}$ can be computed analytically as the "honest" de Rham cohomology of any wide open neighborhood of $\mathcal{X}^0$:

Theorem 5.1 (Coleman)**.** For any wide open neighborhood $\mathcal{W}$, the natural map

$$H^1_{\mathrm{dR}}(X \setminus \{P_1, \ldots, P_n\}, \mathcal{V}, \nabla) \to \frac{\mathcal{V}_{\mathcal{W}} \otimes \Omega_{\mathcal{W}}}{\nabla \mathcal{V}_{\mathcal{W}}}$$

is an isomorphism. □

Write $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla)$ for

$$\frac{\mathcal{V}_{\mathcal{W}} \otimes \Omega_{\mathcal{W}}}{\nabla \mathcal{V}_{\mathcal{W}}}.$$

The following consequence of this theorem is immediate:

Corollary 5.2. Any inclusion of wide opens $\mathcal{X}^0 \subset \mathcal{W} \subset \mathcal{W}'$ induces an isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{W}', \mathcal{V}, \nabla) \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla).$$ □

One also has the following:

Corollary 5.3. The image of the natural map

$$H^1_{\mathrm{dR}}(X, \mathcal{V}, \nabla) \to \frac{\mathcal{V}_{\mathcal{W}} \otimes \Omega_{\mathcal{W}}}{\nabla \mathcal{V}_{\mathcal{W}}}$$

is the space of classes of rigid 1-forms on $\mathcal{W}$ with residue zero at each of the points P_i. □

This follows from the usual description of the algebraic de Rham cohomology of the affine $X \setminus \{P_1, \ldots, P_n\}$ (with $\mathcal{V}$ coefficients) and the compatibility of the algebraic and rigid residue maps.

If $\mathcal{V}$ is a vector bundle with flat connection, then a *primitive* for a $\mathcal{V}$-valued one-form ω (over some open set) is a section F_ω of $\mathcal{V}$ with $\nabla F_\omega = \omega$. Of course, a primitive is only unique up to horizontal sections of $\mathcal{V}$. In the p-adic setting, Coleman has shown a canonical way to write down a primitive for sections of $\mathcal{V}$ in the event that $\mathcal{V}$ is equipped with some extra structure coming from the Frobenius map on the reduction of X. For more details on the following, the reader should consult Section 10 of [14]. (In that section, Coleman uses the phrase "overconvergent F-crystal" to mean what this document and others call an "overconvergent Frobenius isocrystal". Moreover, Coleman does not limit his theory to the good-reduction case, which requires him to distinguish between the "flab"-sheaf of locally analytic sections of $\mathcal{V}$ and a certain "flog" sub-sheaf.)

The reduction X^0_p of $\mathcal{X}^0$ is a smooth affine curve that admits the p-power absolute Frobenius map to itself. If the set $\{P_1, \ldots, P_n\}$ is Frobenius stable (which one may always assume by adding more points to it), then, using the good reduction hypotheses, Coleman shows that this Frobenius map lifts to a semilinear map $\phi : \mathcal{X}^0 \to \mathcal{X}^0$. Fix a choice of such a ϕ once and for all.

Definition 5.4. A Frobenius neighborhood of $\mathcal{X}^0$ in $\mathcal{W}$ is a pair $(\mathcal{W}', \phi)$, where $\mathcal{W}'$ is a basic wide open neighborhood with $\mathcal{X}^0 \subset \mathcal{W}' \subset \mathcal{W}$ and $\phi : \mathcal{W}' \to \mathcal{W}$ restricts to ϕ on $\mathcal{X}^0$. □

Definition 5.5. An *overconvergent Frobenius isocrystal* on the affinoid $\mathcal{X}^0$ is a pair $(\mathcal{V}, \mathrm{Fr})$ where $\mathcal{V}$ is a vector bundle with flat connection ∇ on $\mathcal{W}$ and Fr is a ∇-horizontal morphism

$$\mathrm{Fr} : \phi^*\mathcal{V}|_{\mathcal{W}'} \to \mathcal{V}|_{\mathcal{W}'}$$

on some Frobenius neighborhood $\mathcal{W}'$ of $\mathcal{X}^0$ in $\mathcal{W}$. □

Given an overconvergent Frobenius isocrystal $\mathcal{V}$, there is an endomorphism Φ of the space

$$H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla)$$

given by the composition

$$H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla) \to H^1_{\mathrm{dR}}(\mathcal{W}', \phi^*\mathcal{V}, \nabla) \to H^1_{\mathrm{dR}}(\mathcal{W}', \mathcal{V}, \nabla) = H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla).$$

Definition 5.6. A polynomial $P(T) \in F[T]$ is a *Coleman polynomial* for a class $[\omega] \in H^1(\mathcal{W}, \mathcal{V}, \nabla)$ if the following hold:

(1) $P(\Phi)([\omega]) = 0$.

(2) $P(\Phi)$ induces an automorphism of the space of locally analytic sections of $\mathcal{V}$ that are horizontal for ∇.

(3) $P(1) \neq 0$. □

Theorem 5.7 (Coleman). Suppose that ω is a $\mathcal{V}$-valued one-form on $\mathcal{W}$ such that the cohomology class $[\omega] \in H^1(\mathcal{W}, \mathcal{V}, \nabla)$ admits a Coleman polynomial $P(T)$. Then there is a unique locally analytic primitive F_ω for ω such that $P(\Phi)F_\omega$ is a rigid section of $\mathcal{V}$ on some Frobenius neighborhood of $\mathcal{X}^0$ in $\mathcal{W}$. Moreover, F_ω is rigid on any fixed residue disk of $\mathcal{X}^0$. □

The function F_ω is called the *Coleman primitive* for ω. It turns out that it depends on none of the choices involved in stating Theorem 5.7—that is, it does not depend on

$\mathcal{W}'$, the extension of ϕ to $\mathcal{W}'$, nor on the Coleman polynomial (provided that a polynomial exists).

Of course, we will apply this theory in the setting of $\mathcal{L}_{2r}$ on the ordinary locus of C (and its trivial enlargement $\mathcal{L}_{2r,2r}$). In this case, the overconvergent Frobenius isocrystal structure comes from an extension of the canonical morphism $\mathcal{A}_{\mathrm{ord}} \to \mathcal{A}_{\mathrm{ord}}$ to a wide open neighborhood $\mathcal{W}$ of the ordinary locus, as is constructed in [22, Chapter 11]. For ease of notation, we do not symbolically denote restriction of bundles to $\mathcal{W}$ or the ordinary locus in the following proof.

Proposition 5.8. There exists a Coleman polynomial P for ω_f (where ω_f is viewed as a section of $\mathcal{L}_{2r} \otimes \Omega$ via the Kodaira–Spencer isomorphism). □

Proof. The Hecke polynomial $P(X) = X^2 - a_p X + p^{k-1}\epsilon_f(p)$, when evaluated at Frobenius, annihilates the class of ω_f, essentially by design. We can see this concretely as follows: the operator Φ on $\mathcal{L}_{2r}$, when restricted to $\underline{\omega}^{\otimes 2r}$, satisfies $\Phi = \frac{1}{p}[p]V$, as follows from Lemmas 4.10 and 4.11, the action of $[p]$ undoing the extra factor of $\frac{1}{p}$ in the level structure in the function f computed in Lemma 4.10. (This is an analog of the same result on GL_2, cf. [9, Lemme 4.3.2] for a proof using q-expansions, bearing in mind our convention that Hecke operators act on the right.) We deduce that $P(\Phi)$ annihilates ω_f from the classical compatibilities of the actions of T_p and $[p]$ on f with those on ω_f and the formal factorization

$$X^2 - T_p X + \frac{1}{p}[p] = \left(X - \frac{1}{p}[p]V\right)(X - U)$$

(in the ring $\mathbb{T}_p[X]$, where $\mathbb{T}_p$ is the noncommutative algebra generated over the Hecke algebra by formal variables U and V, subject to the relations (6) and (7)).

As remarked after 4.7, on any residue disk, the space of horizontal sections for ∇ on $\mathcal{H}^1$ is spanned by $(\hat{\eta})$ and $(\hat{\omega} - \log(q) \otimes \hat{\eta})$, on which Frobenius acts by 1 and p, respectively. Thus, $P(\Phi)$ is diagonal with respect to the basis of horizontal sections of $\mathcal{L}_{2r}$ given by symmetric powers of these, with eigenvalues $P(p^i)$, so to check the second and third conditions defining a Coleman polynomial, we need only check that p^i cannot be a root of P for $i \geq 0$. As f is cuspidal, this is a consequence of the Weil conjectures. ■

The following lemma will be useful in the proof of Proposition 7.1.

Lemma 5.9. Given an overconvergent Frobenius isocrystal $(\mathcal{V}, \mathrm{Fr})$ on an affinoid $\mathcal{X}^0$, a pairing on $\mathcal{V}$ that is compatible with the connection, $[\omega] \in H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla)$ a cohomology class on a wide open neighborhood of $\mathcal{X}^0$ admitting a Coleman primitive F_ω, and $[\eta^{\mathrm{frob}}] \in H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla)$ a cohomology class that is fixed by Frobenius, one has

$$\sum_{i=1}^{n} \mathrm{res}_{P_i} \langle F_\omega, \eta^{\mathrm{frob}} \rangle = 0. \qquad \square$$

Proof. This is Lemma 3.20 of [3]. ■

6 Construction of the Cycle

6.1 Projectors on Kuga–Sato varieties

Recall that C is the Shimura curve over F, f is a modular form of weight $k = 2r + 2$ on C, $\mathcal{A}_r$ is the r-fold fiber product of the universal false elliptic curve over C with itself, A is a fixed "CM false elliptic curve", and $W_r = \mathcal{A}_r \times A^r$. In this section, we construct a homologically trivial cycle on W_r, and then begin our discussion of the p-adic Abel–Jacobi map, as applied to this cycle. For the first two subsections, we assume $r > 0$. The case $r = 0$ (so $W_r = C$) is treated separately in Section 6.4.

As in [3], our cycle will be the graph of a morphism of false elliptic curves, modified by an idempotent in the ring of correspondences on a Kuga–Sato variety. All rings of correspondences in this section are taken with *rational* coefficients.

Recall the bundle $\mathcal{L}_{2r} = \mathrm{Sym}^{2r} e\mathcal{H}^1$. The following is Theorem 5.8.iii of [5]:

Theorem 6.1 (Besser)**.** There is a projector P in the ring

$$\mathrm{Corr}_C(\mathcal{A}_r)$$

of algebraic correspondences on $\mathcal{A}_r$ fibered over C, with the property

$$P\mathcal{H}^*(\mathcal{A}_r/C) = P\mathcal{H}^{2r}(\mathcal{A}_r/C) = \mathcal{L}_{2r}. \qquad \square$$

Lemma 6.2. For any $r > 0$,

$$H^0(C, \mathcal{L}_{2r}, \nabla) = 0. \qquad \square$$

Proof. This can be computed after base changing to $\mathbb{C}$, and thus (thanks to GAGA for differential operators as in [16]), it suffices to show it for the local system

$$\mathrm{Sym}^{2r} eR^1 \pi_* \mathbb{Q}$$

on the Riemann surface $C(\mathbb{C})$. This local system corresponds to the representation $\mathrm{Sym}^{2r} e(\mathbb{C}^4)$ of $\pi_1(C_{\mathbb{C}}) = \Gamma$ for which there are no fixed points. ■

Corollary 6.3. The projector P satisfies

$$P H^*_{\mathrm{dR}}(\mathcal{A}_r) = H^1(C, \mathcal{L}_{2r}, \nabla). \qquad \square$$

Proof. We first show

$$PH^{2r+1}_{\mathrm{dR}}(\mathcal{A}_r) = \bigoplus_{p+q=2r+1} H^p(C, P\mathcal{H}^q, \nabla) = H^1(C, \mathcal{L}_{2r}, \nabla).$$

The first equality, known (without the P) as Lieberman's trick, is true for any abelian scheme $X \to S$. Lieberman's observation is that the Leray spectral sequence

$$E_2^{p,q} = H^p(S, \mathcal{H}^q(X/S), \nabla) \Rightarrow H^{p+q}_{\mathrm{dR}}(X)$$

degenerates at page 2, as the multiplication maps $[m]: X \to X$ must, on the one hand, commute with the edge maps but, on the other, induce multiplication by m^q on $E_2^{p,q}$. This identifies $H^p(S, \mathcal{H}^q(X/S), \nabla)$ with the m^q eigenspace of $[m]$ on $H^{p+q}_{\mathrm{dR}}(X)$. The second equality follows from Proposition 6.1.

To see that

$$P H^*_{\mathrm{dR}}(\mathcal{A}_r) \subset H^{2r+1}_{\mathrm{dR}}(\mathcal{A}_r),$$

observe that P annihilates $H^p_{\mathrm{dR}}(C, \mathcal{H}^q, \nabla)$ unless $p = 0, 1$ and $q = 2r$. As $r > 0$, the latter bundle has no global sections by Lemma 6.2. ■

As a summand of $H^{2r+1}_{\mathrm{dR}}(\mathcal{A}_r)$, the cohomology group $H^1_{\mathrm{dR}}(C, \mathcal{L}_{2r}, \nabla)$ inherits its Hodge filtration (which coincides with the filtration defined directly from the hypercohomology spectral sequence). On the other hand, Kodaira–Spencer gives a map from the space $H^0(C, \underline{\omega}^{\otimes 2r+2})$ of modular forms to $H^0(C, \underline{\omega}^{\otimes 2r} \otimes \Omega_C)$, which includes (again from the hypercohomology spectral sequence) into $H^1_{\mathrm{dR}}(C, \mathcal{L}_{2r}, \nabla)$. It follows from our computation of the Kodaira–Spencer map over $\mathbb{C}$ (formula (5)) that the section ω_f of $\underline{\omega}^{2r+2}$ corresponds to the holomorphic section $(2\pi i)^{2r+1}\, \mathrm{d}z_1^{2r} d\tau$ of $\underline{\omega}^{\otimes k} \otimes \Omega$. In particular, working over any characteristic zero field, we have a map $H^0(C, \underline{\omega}^{\otimes 2r+2}) \to \mathrm{Fil}^{2r+1} H^1_{\mathrm{dR}}(C, \mathcal{L}_{2r}, \nabla)$ (as the filtration can be computed after a base change).

Proposition 6.4. There is a projector $\epsilon_A \in \mathrm{Corr}^{2r}(A^r, A^r)$ such that

$$\epsilon_A H^*_{\mathrm{dR}}(A^r) = \mathrm{Sym}^{2r} e H^1_{\mathrm{dR}}(A). \qquad \square$$

Proof. Take the image of Besser's projector P under the "evaluation at τ" map

$$\mathrm{Corr}_C(W_r) \to \mathrm{Corr}_F(A^r). \qquad \blacksquare$$

Consider the variety W_r together with the projector $\epsilon = P\epsilon_A$ in $\mathrm{Corr}^r_C(W_r, W_r)$. Recall the local system $\mathcal{L}_{2r,2r}$ on C which is $\mathcal{L}_{2r} \otimes \mathrm{Sym}^{2r} e H^1_{\mathrm{dR}}(A)$; the fiber of $\mathcal{L}_{2r,2r}$ at a point P of $C(F)$ corresponding to a false elliptic curve A' is $\mathrm{Sym}^{2r} e H^1(A') \otimes \mathrm{Sym}^{2r} e H^1_{\mathrm{dR}}(A)$.

Proposition 6.5. One has

$$\epsilon H^*_{\mathrm{dR}}(W_r) = H^1_{\mathrm{dR}}(C, \mathcal{L}_{2r}, \nabla) \otimes \mathrm{Sym}^{2r} e H^1_{\mathrm{dR}}(A) \subseteq H^{2r+1}_{\mathrm{dR}}(\mathcal{A}^r) \otimes H^{2r}_{\mathrm{dR}}(A^r) \subseteq H^{4r+1}_{\mathrm{dR}}(W_r). \qquad \square$$

Proof. This follows immediately from the Künneth formula. $\blacksquare$

6.2 The generalized Heegner cycle and the p-adic Abel–Jacobi map

Recall the fixed level $V_1(N^+)$-structure t on A. For any false isogeny $\phi : A \to A'$ whose kernel intersects the image of t trivially, there is a point P_ϕ on C corresponding to the pair $(A', \phi \circ t)$, and an embedding of the graph Γ_ϕ into the fiber $A \times A'$ of W_1 above P_ϕ. Write Υ_ϕ for the rth power of Γ_ϕ. The cycles studied in this paper are given by

$$\Delta_\phi = \epsilon \Upsilon_\phi.$$

Note that Υ_ϕ has codimension $2r+1$ in W_r. It follows that the cycle class map takes Υ_ϕ to $H^{4r+2}_{\mathrm{dR}}(W_r)$, and so Δ_ϕ is cohomologically trivial by Proposition 6.5. It follows from [33, Theorem 3.2] that each P_ϕ is defined over F.

6.3 The p-adic Abel–Jacobi map

Write F_p for the completion of F at the place above p induced by the chosen embedding $\bar{\mathbb{Q}} \to \mathbb{C}_p$. Recall that for a variety X, $\mathrm{CH}^i_0(X)$ denotes the group of homologically trivial cycles of codimension i, modulo rational equivalence.

There is a sequence for étale cohomology with supports: for a closed immersion $Z \hookrightarrow X$ of schemes with complement U, then for any sheaf $\mathcal{F}$ one has

$$\cdots \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}_Z(X, \mathcal{F}) \to \cdots .$$

If Z and X are smooth over an algebraically closed field and $\mathcal{F}$ is a locally constant ℓ^n-torsion sheaf, then there are also (functorial) Gysin maps computing the relative cohomology groups in terms of the ordinary cohomology groups of Z; writing c for the codimension of Z in X, the Gysin map identifies

$$H^{j-2c}(Z, \mathcal{F}(-c)) = H^j_Z(X, \mathcal{F}).$$

We apply these facts in the following context: write W_P for the fiber of W_r above P (which has codimension 1) and $W^\flat$ for its complement. By construction, the cycle Δ_ϕ is supported on W_P. Choose $i = 4r+1$ and $\mathcal{F} = \mathbb{Z}_p(2r+1)$. After base changing to the algebraic closure and applying ϵ to the Gysin sequence, we get an exact sequence of Galois modules

$$0 \to \epsilon H^{4r+1}(\bar{W}_r, \mathbb{Q}_p(2r+1)) \to \epsilon H^{4r+1}(\bar{W}^\flat, \mathbb{Q}_p(2r+1)) \to \epsilon H^{4r}(\bar{W}_P, \mathbb{Q}_p(2r)) \to 0 \tag{10}$$

using Proposition 6.5 for exactness at the left and right.

There is a map $\mathbb{Q}_p \to \epsilon H^{4r}(\bar{W}_P, \mathbb{Q}_p(2r+1)$ sending 1 to the class of Δ_ϕ. Define

$$\xi \in \mathrm{Ext}^1_{\text{Galois modules}}(\mathbb{Q}_p, \epsilon H^{4r+1}(\bar{W}_r, \mathbb{Q}_p(2r+1))) = H^1(F_p, \epsilon H^{4r+1}(\bar{W}_r, \mathbb{Q}_p(2r+1))) \tag{11}$$

by pushing out the sequence (10) along this map.

Write V for the Galois representation $\epsilon H^{4r+1}(\bar{W}_r, \mathbb{Q}_p(2r+1))$. It follows from work of Nekovář [29] that the class ξ lies in the subgroup $H^1_f(F_p, \epsilon H^{4r+1}(\bar{W}_r, \mathbb{Q}_p(2r+1)))$ defined in [7], that is, that the corresponding extension of Galois modules is crystalline. The subgroup H^1_f is the image of the Bloch–Kato exponential map, which is the connecting map in the long exact sequence in cohomology coming from the short exact sequence of Galois modules

$$0 \to W \to B_{\mathrm{cris}}^{\phi=1} \otimes V \oplus \mathrm{Fil}^0 B_{\mathrm{dR}} \otimes V \to B_{\mathrm{dR}} \otimes V \to 0.$$

Because $D_{\mathrm{cris}}(V)^{\phi=1} = 0$, the inverse of the exponential map induces a well-defined "logarithm" map

$$\log_{\mathrm{BK}} : H^1_f(F_p, V) \to \frac{D_{\mathrm{dR}}(V)}{\mathrm{Fil}^0}.$$

The element $\log_{\mathrm{BK}}(\xi)$ lives in

$$\frac{D_{\mathrm{dR}}(\epsilon H^{4r+1}(\bar{W}_r, \mathbb{Q}_\ell(2r+1)))}{\mathrm{Fil}^0} = \frac{\epsilon H^{4r+1}_{\mathrm{dR}}(W_{r/F_p})}{\mathrm{Fil}^{(2r+1)}}.$$

By Poincare duality, the last space is identified with $\mathrm{Fil}^{2r+1}\epsilon H^{4r+1}_{\mathrm{dR}}(W_{r/F_p})^\vee$, which maps to $(M_{2r+2}(F_p) \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A_{/F_p}))^\vee$ (this follows from the Künneth formula and our earlier remarks on the Hodge filtration).

Definition 6.6. The p-adic Abel–Jacobi map

$$\mathrm{AJ}_p : \epsilon \mathrm{CH}^{2r+1}_0(W_{r/F_p}) \to (M_{2r+2}(F_p) \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A_{/F_p}))^\vee$$

sends a cycle Z to the image of the Bloch–Kato logarithm of the extension class ξ_Z as in (10) under the composition of the maps

$$\frac{\epsilon H^{4r+1}_{\mathrm{dR}}(W_{r/F_p})}{\mathrm{Fil}^{(2r+1)}} = \mathrm{Fil}^{2r+1}\epsilon H^{4r+1}_{\mathrm{dR}}(W_{r/F_p})^\vee \to (M_{2r+2}(F_p) \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A_{/F_p}))^\vee. \qquad \square$$

6.4 The case of weight two

In the case that $r = 0$, the variety W_r is just C, the projectors defined above are all trivial, and a homologically trivial cycle of codimension $2r+1$ is a degree zero divisor. Write $\mathbb{T}_{\mathbb{Q}}$ for the Hecke algebra of level N^+. Then, there is a projector $\epsilon_f \in \mathrm{Corr}(C) \otimes \mathbb{Q}$ which lies in the image of the map $\mathbb{T}_{\mathbb{Q}} \to \mathrm{Corr}(C) \otimes \mathbb{Q}$ and satisfies

$$\epsilon_f H^*_{\mathrm{dR}}(C/F) = F\omega_f \subset H^1_{\mathrm{dR}}(C/F).$$

Let Δ_0 be an arbitrary divisor on C. Then, $\epsilon_f \Delta_0$ is automatically homologically trivial. The projector ϵ_f also gives an extension class attached to f using the Gysin sequence above, so it makes sense to apply the p-adic Abel–Jacobi map to $\epsilon_f \Delta_0$.

In the weight two case, the p-adic Abel–Jacobi map can be identified with a formal group logarithm as follows: writing J for the Jacobian of C, there is, for each differential form $\omega \in \Omega_J$, a unique group homomorphism $\log_\omega : J(F_p) \to F_p$ with $d\log_\omega = \omega$. If we pick an F_p-rational point of C to get a (classical) Abel–Jacobi map $C \to J$, we then get a map $C(F_p) \to F_p$, which coincides with AJ_p. (This map depends on our choice of rational point, but the induced map on $\mathrm{Pic}^0(C)$ does not.)

7 Computation of the p-adic Abel–Jacobi Map

7.1 The p-adic Abel–Jacobi map and Coleman integration

We will work with sums Δ of generalized Heegner cycles fibered above points $P_1, \ldots, P_m$; we assume this set to be Frobenius stable. Write also $Q_1, \ldots, Q_n$ for a choice of point on each supersingular residue disk. Write $S_P = \{P_1, \ldots, P_m\}$, $S_Q = \{Q_1, \ldots, Q_m\}$ and $S = S_P \cup S_Q$.

We will apply the formalism of Section 5 to the affinoid $\mathcal{X}^0 = C \setminus \bigcup_{P \in S} D(P, 1)$ and some choice of wide open neighborhood $\mathcal{W}$.

Proposition 7.1. If Δ is a sum of generalized Heegner cycles fibered above points in S_P, where the point P_i corresponds to the false elliptic curve A_i with level structure t_i, the generalized Heegner cycle Δ_i above P_i is given by the false isogeny $\phi_i : A \to A_i$, and $\alpha \in \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A)$ is arbitrary, then

$$AJ_p(\Delta)(\omega_f \wedge \alpha) = \sum_{P_i \in S_P} \langle F_f(P_i) \wedge \alpha, \mathrm{Cl}_{P_i}(\Delta_i) \rangle.$$

(Here, $\mathrm{Cl}_{P_i}(\Delta_i)$ denotes the image of Δ_i under the cycle class map attached to the fiber A_i^r of C above P_i, not the global cycle class map, which annihilates Δ by construction.) □

Proof. This argument mimics the proof of [3, Proposition 3.18]—in fact, it is strictly simpler, as that paper must deal with issues related to cusps of modular curves. To compute $AJ_p(\Delta)(\omega_f \wedge \alpha)$, we need to compute

$$\langle \log_{BK}(\xi_\Delta), \omega_f \wedge \alpha \rangle.$$

Here, ξ_Δ denotes the extension class (11) of Galois modules.

Applying D_{dR} gives the extension class D_Δ, which sits in the exact sequence of filtered Frobenius modules

$$0 \to H^1(C, \mathcal{L}_{2r,2r}, \nabla)(2r+1) \to D_\Delta \to F \to 0,$$

thought of as a class in $H^1_f(F_p, V)$. Explicitly, D_Δ is the set of pairs (η, β), where $\beta \in F$ and η is a cohomology class in

$$H^1(C \setminus S_P, \mathcal{L}_{2r,2r}, \nabla)(2r+1)$$

whose residue at each P_i is $\beta \mathrm{Cl}_{P_i}(\Delta)$.

To write down $\log_{BK}(\xi_\Delta)$, we must find a class $\eta_{\mathrm{hol}} \in \mathrm{Fil}^0 D_\Delta$ and a class $\eta_{\mathrm{frob}} \in (D_\Delta)^{\phi^{\deg(F_{\mathfrak{p}}/\mathbb{Q}_p)}=1}$, both mapping to 1 in F, and then take their difference, which is well-defined up to $\mathrm{Fil}^{2r+1} H^1(C, \mathcal{L}_{2r,2r}, \nabla)$. We will think of η_{hol} and η_{frob} as classes in $H^1(C \backslash S_P, \mathcal{L}_{r,r}, \nabla)$, both required to have residue $\mathrm{Cl}_{P_i}(\Delta_i)$ at P_i.

One has

$$\mathrm{Fil}^0 H^1(C - S_P, \mathcal{L}_{2r,2r}, \nabla)(2r+1) = H^0(C \setminus S_P, \underline{\omega}^r) \otimes \mathrm{Sym}^{2r} e H^1_{\mathrm{dR}}(A).$$

In particular, η_{hol} is represented by an $\mathcal{L}_{r,r}$-valued one-form that is holomorphic away from S_P and has a simple pole at each $P_i \in S_P$ with residue Δ_i (or is holomorphic at P_i if $\Delta_i = 0$). Possibly enlarging S_P, we may assume the centers of the deleted disks include all the poles of η_{frob}.

To compute

$$\langle \log_{BK}(\xi_\Delta), \omega_f \wedge \alpha \rangle,$$

we need to pick primitives for $\omega \wedge \alpha$ in each disk, multiply by $\eta_{\mathrm{hol}} - \eta_{\mathrm{frob}}$, and sum the residues over the points in S. Now Lemma 5.9 tells us that if we pick the global Coleman primitive, then the contribution to the sum from S_Q cancels. Hence the sum simplifies to

$$\sum_{P_i \in S_P} \mathrm{res}_{P_i}(\eta, F_f \wedge \alpha) = \langle F_f(P_i) \wedge \alpha, \mathrm{cl}_{P_i}(\Delta) \rangle$$

(we are using the fact that $F_f \wedge \alpha$ is the Coleman primitive for $\omega_f \wedge \alpha$). ■

The next proposition, which is proved as in [3] and only needed in the higher weight case, shows that we can move this result of the previous proposition from the various $P_i \in S$ to the point P_A corresponding to the fixed false elliptic curve A.

Proposition 7.2 (BDP 3.21). If Δ_ϕ is supported over a single point $P'_{A'}$, then we have

$$AJ_p(\Delta_\phi)(\omega_f \wedge \alpha) = \langle \phi^* F_f(P_{A'}), \alpha \rangle_A.$$

where the pairing occurs on $e\mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(A)$. □

7.2 Computing Coleman primitives for p-adic modular forms

In this section, we will use the following conventions, which are slightly different from those of [3]. A lowercase letter is a p-adic or locally analytic modular form, and the corresponding capital letter is its Serre–Tate expansion, a power series in T (on some fixed ordinary residue disk). As in Section 4, write θ for the operator on the space of locally analytic modular forms and $\Theta = (1+T)\frac{\mathrm{d}}{\mathrm{d}T}$ for the corresponding operator on power series; that is, if $g = G(T)\hat{\omega}^k$ in a fixed residue disk of C, then

$$\theta g = (\Theta G)\hat{\omega}^{k+2}.$$

Because the main theorem of this section may be of use in other situations, we state it for arbitrary p-adic modular form f of weight $\rho+2$ (for the fixed f from Section 2, of course, $\rho = 2r$ is even—note that our proof of the existence of a Coleman polynomial does not assume even weight). Consider f as a section ω_f of $\underline{\omega}^\rho \otimes \Omega \subseteq \mathrm{Sym}^\rho e\mathcal{H}^1 \otimes \Omega$ using the Kodaira-Spencer map, and write g for the Coleman primitive of ω_f. In particular, g is a section of $\mathrm{Sym}^\rho e\mathcal{H}^1$ satisfying $\nabla g = \omega_f$.

In terms of the Serre–Tate basis for $\mathrm{Sym}^\rho e\mathcal{H}^1$ given by $\hat{\omega}^{\rho-i}\hat{\eta}^i$ for $i = 0, \ldots, \rho$, we may write

$$g = \sum_{i=0}^{\rho} G_i(T)\hat{\omega}^{\rho-i}\hat{\eta}^i. \tag{12}$$

(Here, we are using the fact that g is rigid on residue disks.)

The formal power series $G_i(T)$ are actually the T-expansions of locally analytic modular forms of weight $2\rho - i$. To see this, recall that the $\mathcal{O}_X$-linear cup product pairing on $\mathcal{H}^1$ extends to a pairing on $\mathrm{Sym}^\rho e\mathcal{H}^1$ by the rule

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_\rho, \beta_1 \otimes \cdots \otimes \beta_\rho \rangle = \frac{1}{\rho!} \sum_{\sigma \in S_\rho} \prod_i \langle \alpha_i, \beta_{\sigma i} \rangle. \tag{13}$$

Following [BDP], we define a locally analytic modular form $\tilde{g}_i$ by the rule

$$\tilde{g}_i(A, t) = \langle g(A, t), \omega^i \eta^{\rho-i} \rangle \omega^{2\rho-i},$$

where $\omega \in \underline{\omega}(D)$ and $\eta \in \mathcal{H}^1(D)$ are chosen with $\langle \omega, \eta \rangle = 1$. (Replacing ω by $\lambda\omega$ has the effect of replacing η by $\lambda^{-1}\eta$, so the form does not depend on any choices.)

Combining (12) and (13) shows that the Serre–Tate expansion of $\tilde{g}_i$ is given by

$$\tilde{g}_i = \frac{(-1)^i}{\binom{\rho}{i}} G_i(T)\hat{\omega}^{2\rho-i}. \tag{14}$$

Note that $\tilde{g}_i$ is a locally analytic modular form on all of $\mathcal{X}$, not just on the ordinary locus (where its T-expansions make sense and where formula (14) holds).

These components can be computed by inverting the differential operator θ:

Theorem 7.3. One has

$$\tilde{g}_i^\flat = i!\theta^{-1-i} f^\flat. \qquad \square$$

Proof. The theorem is equivalent to the statement that

$$\theta^{1+i}\tilde{g}_i = i!\, f$$

(the flat operator arises upon inverting θ, which has trivial kernel on the space of forms satisfying $f = f^\flat$). It suffices to show that $\theta\tilde{g}_0 = f$ and $\theta\tilde{g}_i = i\tilde{g}_{i-1}$ for $i > 0$.

Using the Leibniz rule and Katz's computation of ∇ on the basis $\{\hat{\omega}, \hat{\eta}\}$ yields

$$\begin{aligned}\nabla(G_i\hat{\omega}^{\rho-i}\hat{\eta}^i) &= G_i'\hat{\omega}^{\rho-i}\hat{\eta}^i \otimes \mathrm{d}T + G_i\nabla(\hat{\omega}^{\rho-i}\hat{\eta}^i)\\ &= \Theta G_i\hat{\omega}^{\rho-i}\hat{\eta}^i \otimes \mathrm{d}\log q + (\rho - i)G_i\hat{\omega}^{\rho-i-1}\hat{\eta}^{i+1} \otimes \mathrm{d}\log q.\end{aligned}$$

Summing this equality over i and reindexing gives

$$\nabla g = \Theta G_0\hat{\omega}^\rho \otimes \mathrm{d}\log q + \sum_{i=1}^{\rho}(\Theta G_i + (\rho - i + 1)G_{i-1})\hat{\omega}^{\rho-i}\hat{\eta}^i \otimes \mathrm{d}\log q.$$

On the other hand, since g is a primitive,

$$\begin{aligned}\nabla g &= KS(f)\\ &= KS(F\hat{\omega}^{\rho+2})\\ &= F\hat{\omega}^\rho \otimes \mathrm{d}\log q.\end{aligned}$$

It follows that $\Theta G_0 = F$ and that $\Theta G_i = -(\rho - i + 1)\ G_{i-1} = 0$ for $i < \rho$. The result now follows from (14). (An earlier draft of this article remarked as a consequence of this calculation that $\tilde{g}_i^\flat$, *a priori* only locally analytic, must in fact be rigid on the ordinary locus. As pointed out to the author by Yifeng Liu, this is true, but not obvious, and needs a result from rigid analytic geometry, namely, that if the limit of a family of rigid sections of a bundle exists (as a locally analytic section) and if moreover the convergence is uniform, then the limit is a rigid analytic section. We omit this argument, as it is not needed for our results, which only depend on the values of $\tilde{g}_i^\flat$ at CM points.) ■

In the weight two case ($r=0$), the operator $\mathcal{O}_C \to \underline{\omega}^{\otimes 2}$ is just d, followed by the Kodaira–Spencer isomorphism. The content of the above proposition is then that the limit defining θ^{-1} exists.

Consider the particular f fixed in Section 2, so that $\rho = 2r$, and write g_j for the jth component of the Coleman primitive as before. The following proposition, which is proved using Proposition 7.2 as in [3], relates the components $\tilde{g}_i$ to the p-adic Abel–Jacobi map:

Proposition 7.4 (BDP 3.22)**.** Write d for the degree of the false isogeny $\phi : A \to A'$. Then

$$AJ_p(\Delta_\phi)(\omega_f \wedge \omega^j \eta^{2r-j}) = d^j g_j(A', t', \omega'). \qquad \square$$

Lemma 7.5. Suppose that the weight of f is 2. Then for any zero-cycle Δ on C, one has

$$(\theta^{-1} f^\flat)(\epsilon_f \Delta_0) = (\theta^{-1} f^\flat)(\Delta_0). \qquad \square$$

Proof. Because the operators U and V commute with all the operators T_ℓ, the p-adic modular form $f^\flat$ is still an eigenform with the same Hecke eigenvalues as f away from p. It follows from Proposition 3.6, and the definition of θ^{-1} as a limit of iterates of θ, that

$$(\theta^{-1} f^\flat)|_{T_\ell} = \ell^{-1}\theta^{-1}(f^\flat|_{T_\ell}) = a_\ell \ell^{-1}\theta^{-1} f^\flat.$$

Write $T_\ell^* \Delta_0$ for the zero-cycle on C given by the Hecke orbit of Δ_0. For g a modular function, one has $g|_{T_\ell}(P) = \frac{1}{\ell} g(T_\ell^* P)$. But then

$$\begin{aligned}(\theta^{-1} f^\flat)(T_\ell^* \Delta_0) &= \ell(\theta^{-1} f^\flat)|_{T_\ell}(\Delta_0)\\ &= a_\ell(\theta^{-1} f^\flat)(\Delta_0).\end{aligned}$$

Write $\lambda_f : \mathbb{T} \to F$ for the homomorphism attached to the newform f. Then $\lambda_f(T_\ell) = a_\ell$, so the above computation shows that for an arbitrary $T \in \mathbb{T}$ one has

$$(\theta^{-1} f^\flat)(T^* \Delta_0) = \lambda_f(T)(\theta^{-1} f^\flat)(\Delta_0).$$

By design $\lambda_f(\epsilon_f) = 1$, so we are done. ■

8 The p-adic L-function

8.1 Spaces of Hecke characters

The p-adic L-function we study interpolates special values of the Rankin–Selberg L-function $L(f, \chi, s)$ as χ varies over Hecke characters of K. We now describe a few spaces of Hecke characters on K. A Hecke character χ with infinity type (ℓ_1, ℓ_2) is *critical* for f if one of the following conditions holds:

(1) (The type 1 case): $1 \leq \ell_1, \ell_2 \leq k-1$.

(2) (The first type 2 case): $\ell_1 \geq k$ and $\ell_2 \leq 0$.

(3) (The second type 2 case): $\ell_1 \leq 0$ and $\ell_2 \geq k$.

As is explained in [3, Section 4.1], χ is critical for f precisely when the center of the functional equation for $L(f, \chi^{-1}, s)$ is a critical value in the sense of Deligne. Because an even number of primes divide N^-, the sign in the global functional equation for $L(f, \chi^{-1}, s)$ depends only on the archimedean epsilon factor; in the cases that we have called "type 2", this sign is positive, and in the case we have called "type 1", it is negative.

One says that χ is *central critical* if in addition $\ell_1 + \ell_2 = k$ (which is equivalent to the center of the functional equation occurring at $s = 0$) and the central character of χ matches the nebentypus of f (which forces the same L-function to occur on both sides of the functional equation). We will write $\Sigma_{\mathrm{cc}}^{(1)}$ for the set of central critical characters of type 1 and $\Sigma_{\mathrm{cc}}^{(2)}$ for the set of central critical characters in the first type 2 case. Because the values of critical Hecke characters are algebraic, we may view them as p-adic numbers via our fixed embedding. As is explained in the discussion before [3, Remark 5.8], the set $\Sigma_{\mathrm{cc}}^{(2)}$ inherits a p-adic topology as a subspace of the space of functions from the prime-to-p ideles of K to $\mathcal{O}_{\mathbb{C}_p}$.

Write $\hat{\Sigma}_{\mathrm{cc}}^{(2)}$ for the completion of $\Sigma_{\mathrm{cc}}^{(2)}$ with respect to this topology. Write h for the class number of K; for each integer t, there is a Hecke character ψ_t given by the rule

$$\psi_t(\mathfrak{a}) = a^{6t} / \bar{a}^{6t},$$

where $(a) = \mathfrak{a}^h$. Note that the infinity type of ψ_t is $(6h, -6h)$. It follows that $\chi\psi_t$ is central critical of type 2 for χ central critical of any type (and t large) and $\chi\psi_{p^n(p-1)} \to \chi$ as $n \to \infty$. It follows that we may view $\Sigma_{\mathrm{cc}}^{(1)}$ as a subset of $\hat{\Sigma}_{\mathrm{cc}}^{(2)}$.

8.2 The Waldspurger-type result

Using the fixed complex structure J_τ on $M_2(\mathbb{R})$, thought of as a map

$$J_\tau : M_2(\mathbb{R}) \to \mathbb{C}^2,$$

we get a differential form $\omega_{\mathbb{C}} = J_\tau^*(2\pi i\, \mathrm{d}z_1)$ on $M_2(\mathbb{R})$ (holomorphic for this complex structure). Abusing notation, also write $\omega_{\mathbb{C}}$ for the corresponding form on $B \otimes \mathbb{R}$ (which is really $\iota_\infty^* \omega_{\mathbb{C}}$).

There is a bijective correspondence between $\iota_\infty(\mathcal{O}_B)$-stable sublattices of $\mathbb{C}^2$ and pairs (A, ω) of a false elliptic curve over $\mathbb{C}$ and a section of $e\Omega_{A/\mathbb{C}}$. To a pair (A, ω) we attach the lattice

$$\mathcal{O}_B \left\{ \int_\gamma \omega \mid \gamma \in eH_1(A) \right\},$$

and to an $\mathcal{O}_B$-stable lattice Λ we assign the false elliptic curve $\mathbb{C}^2/\Lambda$ together with the form $2\pi i\, \mathrm{d}z_1$.

Again using the complex structure J_τ on $M_2(\mathbb{R})$, we may view a modular form g as a function on pairs (Λ, t), where Λ is an $\mathcal{O}_B$-stable sublattice of $B \otimes \mathbb{R}$ and $t = et$ is an element of exact order N^+ in $\frac{B\otimes\mathbb{R}}{\Lambda}$. Explicitly, this function is given by the rule

$$g(\Lambda, t) = g\left(\frac{B \otimes \mathbb{R}}{\Lambda}, t, \omega_{\mathbb{C}}\right).$$

Scaling the lattice Λ by some $\lambda \in \mathbb{C}$ multiplies each period integral by λ, so the corresponding one-form $\omega_{\mathbb{C}}$ is divided by λ. Hence, for g of nebentypus ϵ_g, we have

$$g(\lambda\Lambda, \lambda t) = \lambda^{-k} \epsilon_g(\lambda) g(\Lambda, t).$$

Write t for the Heegner level N^+ structure on the false elliptic curve A_τ, as described in Section 2.5, and for $\mathfrak{a}$ an ideal of $\mathcal{O}_K$ prime to N, write $t_{\mathfrak{a}}$ for the induced level structure on the false elliptic curve $A_{\mathfrak{a}\star\tau}$.

Lemma 8.1. Let $\mathfrak{a}$ be an ideal prime to N and let χ be a central critical Hecke character of infinity type $(k+j, -j)$. Then, for any t, the expression

$$\chi^{-1}(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-j} \Theta_\infty^j f(\mathfrak{a}_B^{-1}, t_{\mathfrak{a}})$$

only depends on the class of $\mathfrak{a}$ in $\mathrm{Cl}(\mathcal{O}_K)$ (here $\mathfrak{a}_B$ is as defined in Section 2.5). $\square$

Proof. Scaling the pair $(\mathfrak{a}, t)$ by $\lambda \in K$, we obtain

$$\Theta_\infty^j f(\lambda^{-1}\mathfrak{a}_B^{-1}, t_{\lambda\mathfrak{a}}) = \epsilon_f(\lambda)\lambda^{k+2j} f(\mathfrak{a}, t).$$

On the other hand,

$$\chi^{-1}(\lambda\mathfrak{a}) = \epsilon_\chi^{-1}\lambda^{-k-j}\bar{\lambda}^j\chi^{-1}(\mathfrak{a})$$

and

$$\mathbf{N}(\lambda\mathfrak{a})^{-j} = \lambda^{-j}\bar{\lambda}^{-j}(\mathbf{N}\mathfrak{a})^{-j}.$$

The result follows from the assumption $\epsilon_f = \epsilon_\chi$. ∎

The following result follows from [31, Theorem 3.2], together with the local computations of [3, Section 4]:

Theorem 8.2. Let χ be an unramified Hecke character of K of infinity type $(k+j, -j)$ whose central character is the nebentypus of f. Then one has, for some $\alpha(f, f_{\mathrm{GL}_2}) \in K$

$$C(f, \chi)L(f, \chi^{-1}, 0) = \alpha(f, f_{\mathrm{GL}_2}) \left| \sum_{\mathfrak{a}\in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a})\mathbf{N}\mathfrak{a}^{-j} \cdot (\Theta_\infty^j f)(\mathfrak{a}_B^{-1}, t_{\mathfrak{a}}) \right|^2,$$

where

$$C(f, \chi) = \frac{1}{4}\pi^{k+2j-1}\Gamma(j+1)\Gamma(k+j)w_K\sqrt{|d_K|}2^{\#S_f}\prod_{\ell|N^-}\frac{\ell-1}{\ell+1}$$

and S_f is the set of primes which ramify in K that divide N^+ but do not divide the conductor of the Nebentypus of f. □

The element $\alpha(f, f_{\mathrm{GL}_2})$ is the quotient of the Petersson inner products

$$\frac{\langle f_{\mathrm{GL}_2}, f_{\mathrm{GL}_2}\rangle}{\langle f, f\rangle}.$$

Because of our normalization of f, it is an element of K by [20, Theorem 12.3]. In fact, by [31, Theorem 2.4], it is integral at p provided that $p > k+1$ and $p \nmid \prod_{\ell|N}(\ell-1)(\ell)(\ell+1)$.

8.3 CM points and CM triples

This section eliminates the absolute value signs that occur in the statement of Theorem 8.2 by comparing the complex conjugation action on the space of modular forms with an Atkin–Lehner involution. Fix for now a primitive N^+th root of unity $\zeta \in \bar{\mathbb{Q}}$, that is, a trivialization of μ_N^+. (We will make a particular choice later.) Suppose that L is some field containing K, A'/L is a false elliptic curve with normalized CM by $\mathcal{O}_K$, and $P = eP$ is a torsion point of exact order N^+ on A'. Then there is a point of $C(F)$ given by the false elliptic curve A' together with the level structure

$$\mu_N^+ \times \mu_N^+ \approx \mathbb{Z}/N^+ \oplus \mathbb{Z}/N^+ \xrightarrow{\binom{1}{0} \mapsto P} A'[\mathfrak{N}^+].$$

Such a point will be denoted (A, P).

A *CM triple* over L is an isomorphism class of triple (A, P, ω), where $\omega \in e\Omega_{A/L}$ is nonvanishing. Using the above formalism, one can think of a CM triple as a point on the underlying space of the bundle $\underline{\omega}_L$.

There is an action $\star$ of $\mathrm{Cl}(\mathcal{O}_K)$ on the set of CM triples, given by the rule

$$\mathfrak{a} \star (A', P, \omega) = (A'/A'[\mathfrak{a}], P_0, \omega_0),$$

where P pushes forward to P_0 and ω_0 pulls back to ω.

Assuming also that $\sqrt{-N^+} \in L$, there is an Atkin–Lehner involution, denoted by w_N^+, on the underlying space of the bundle $\underline{\omega}_L$ (it is not an automorphism of line bundles, but rather lies over an involution on C, which we also call an Atkin–Lehner involution and also write w_N^+). It is described by the following rule:

$$(A', P, \omega) \mapsto (A'/P, P', \sqrt{-N^+}\omega),$$

where $P' = eP'$ is chosen so that the Weil pairing $(\mathrm{Image}(P), P') = \zeta$.

There is a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ semilinear complex conjugation action on $C_{\mathbb{C}}$ and the underlying space of $\underline{\omega}_{\mathbb{C}}$, given on arbitrary points (not just CM triples) by

$$\overline{(A', t, \omega)} = (A'^{\sigma}, t^{\sigma}, \omega^{\sigma}),$$

where σ denotes base change along the nontrivial map $\mathbb{C} \to \mathbb{C}$. (Note that even if A' and A'^{σ} are isomorphic as abelian surfaces, they will not be isomorphic as false elliptic curves.)

By [3, Lemma 5.2], the compatibility of the Atkin–Lehner involution with the operation of $\mathrm{Cl}(K)$ on the set of CM triples is given over $\mathbb{C}$ by the rule

$$\mathfrak{a} \star w_N^+(A', P, \omega_{\mathbb{C}}) = w_N^+(\mathfrak{a} \star (A', \mathbf{N}\mathfrak{a}^{-1} P, \omega_{\mathbb{C}})). \tag{15}$$

Since K satisfies the Heegner hypothesis for N^+, there is as before an ideal $\mathfrak{N}^+$ of K with norm N^+. In the course of establishing the following proposition, we will fix a particular ζ, which depends on the fixed CM elliptic curve A; from now on, any reference to an Atkin–Lehner involution is with respect to this ζ.

Proposition 8.3. There exists an ideal $\mathfrak{b}$ of $\mathcal{O}_K$, and a scalar $b_N^+ \in \mathcal{O}_K$, with the property that for any CM triple, one has

$$\overline{(A', P, 2\pi \mathrm{i}\, \mathrm{d}z)} = \mathfrak{b} \star w_N^+ \left(A, P, \frac{b_N^+}{\sqrt{-N^+}} 2\pi \mathrm{i}\, \mathrm{d}z \right). \qquad \square$$

Proof. Because A^σ has false endomorphisms by $\mathcal{O}_K$, there is a false isogeny $A \to A^\sigma$ whose kernel is of the form $A[\mathfrak{f}]$ for some ideal $\mathfrak{f}$ of K. If necessary, multiply by a scalar to ensure $(\mathfrak{f}, \mathfrak{N}^+) = 1$. Pick the ideal $\mathfrak{b}$ to be prime to $\mathfrak{f}\mathfrak{N}^+$ and to satisfy

$$\mathfrak{b}\mathfrak{N}^+\mathfrak{f}^{-1} = (b_N^+)$$

for some scalar b_N^+. Then multiplication by b_N^+, followed by the natural projection, gives an identification

$$\frac{A[N^+]}{A[\mathfrak{N}^+]} \to A^\sigma[\bar{\mathfrak{N}}^+].$$

In particular, one may lift $\bar{P}$ to $P' = eP' \in A[N]$. Set $\zeta = (P, P')$. The result is now plain from formula (15). ■

As in the modular curve case, there is an involution $g \mapsto g_\rho$ on the space of weight k modular forms for $C/\mathbb{C}$ by the rule

$$g_\rho(A, t, \omega) := \overline{g(\overline{A, t, \omega})}.$$

Lemma 8.4. If g is an eigenform with $T_\ell g = a_\ell g$, then g_ρ is an eigenform with $T_\ell g_\rho = \bar{a}_\ell g_\rho$. $\square$

Proof. One has

$$\begin{aligned} g_\rho|_{T_\ell}(A, t, \omega) &= \frac{1}{\ell} \overline{\sum_{i=0}^{\ell} g(A^\sigma/C_i^\sigma, t_i^\sigma, \omega_i^\sigma)} \\ &= \overline{g_{T_\ell}(A^\sigma, t^\sigma, \omega^\sigma)} \end{aligned}$$

$$= \bar{a}_\ell \overline{g(A^\sigma, t^\sigma, \omega^\sigma)}$$

$$= \bar{a}_\ell g_\rho(A, t, \omega). \qquad \blacksquare$$

A similar computation shows that if g has nebentypus ϵ_g, then g_ρ has nebentypus $\bar{\epsilon}_g$. It follows as in the proof of [3, Lemma 5.2] that there is a complex scalar W_g of norm one, depending only on g and our choice of ζ, such that

$$g_\rho(w_N(E, P, \omega)) = W_g g(E, t, \omega).$$

For $\chi \in \Sigma^{(2)}$, set

$$W(f, \chi) = W_f \epsilon_f(\mathbf{N}\mathfrak{b})^{-1} \chi_j(\mathfrak{b})(-N)^{k/2+j} b_N^{-k-2j}.$$

Abbreviate $\chi \mathbf{N}^{-j}$ as χ_j. Then one has:

Proposition 8.5. Under the hypotheses of Theorem 8.2, one has

$$C(f, \chi) L(f, \chi^{-1}, 0) = \alpha(f, f_{\mathrm{GL}_2}) W(f, \chi) \left(\sum_{\mathfrak{a} \in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-j} \cdot (\Theta_\infty^j f)(\mathfrak{a} \star (A, P, \omega_{\mathbb{C}})) \right)^2. \qquad \square$$

Proof. The formula

$$\overline{\chi_j^{-1}(\mathfrak{a}) \Theta_\infty^j f(\mathfrak{a} \star (A, P, \omega_{\mathbb{C}}))} = w_f (-N)^{k/2+j} b_{N^+}^{-k-2j} \chi_j(\mathfrak{b}) \epsilon_f(\mathbf{N}\mathfrak{b})^{-1} \chi_j(\bar{\mathfrak{a}}\mathfrak{b})^{-1} \Theta_\infty^j f(\bar{\mathfrak{a}}\mathfrak{b} \star (A, P, \omega_C))$$

is established as in the proof of Theorem 5.4 of [3], except with Remark (1) in that proof replaced by Proposition 8.3 above. The result follows from summing this formula over $\mathfrak{a}$, using Theorem 8.2. $\blacksquare$

The following lemma expresses the operator $\flat$ on the space of locally analytic p-adic modular forms in terms of the action of $\mathrm{Cl}(K)$ on CM triples.

Lemma 8.6. If g is a locally analytic p-adic modular form of integer weight k satisfying

$$T_p g = b_p g$$

and

$$\langle p \rangle = \epsilon(p) g$$

for some character ϵ, and (A', t, ω') is a CM triple, then

$$g^{\flat}(A', t, \omega') = g(A', t', \omega') - \frac{\epsilon(p) b_p}{p^k} g(\mathfrak{p} \star (A', t', \omega')) + \frac{\epsilon(\mathfrak{p})}{p^{k+1}} g(\mathfrak{p}^2 \star (A', t', \omega')). \qquad \square$$

Proof. This computation is the same as that of [3, Lemma 3.23]: since A' has complex multiplication, the canonical subgroup of A' is $A'[\mathfrak{p}]$. Thus, $g|_V(A', t', \omega') = \mathfrak{p} \star (A', p^{-1}t', p\omega')$ and $g|_{[p]V^2} = \mathfrak{p}^2 \star (A', p^{-1}t', p\omega')$. The result then follows from ■

$$VU - UV = 1 - T_p V + \frac{1}{p}[p]V^2.$$

8.4 The p-adic L-function

Recall the fixed nonvanishing global section ω of the line bundle $e\Omega_{A/H}$ on A, defined over the Hilbert class field of K. Define a period $\Omega \in \mathbb{C}$ by the rule

$$\omega = \Omega \omega_C.$$

Define also a p-adic period $\Omega_p \in \mathbb{C}_p$ by the rule

$$\omega = \Omega_p \hat{\omega},$$

where $\hat{\omega}$ is the formal section picked in Section 4 (which depended upon a choice of basis for $eT_p\tilde{A}$, where $\tilde{A}$ denotes the reduction of A mod $\mathfrak{p}$).

Proposition 8.7. For $\chi \in \Sigma_{\mathrm{cc}}^{(2)}$ of infinity type $(k+j, -j)$, with $j \geq 0$, the quantity

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) := \alpha(f, f_{\mathrm{GL}_2})^{-1} W(f, \chi)^{-1} C(f, \chi, c) L(f, \chi, 0) / \Omega^{(2(k+2j))}$$

belongs to $\bar{\mathbb{Q}}$, and is computed by the formula

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) = \left(\sum_{[\mathfrak{a}] \in \mathrm{Cl}(\mathcal{O})} \chi_j^{-1}(\mathfrak{a}) \cdot \Theta_\infty^j f(\mathfrak{a} \star (A, t, \omega)) \right)^2. \qquad \square$$

Proof. See [3, Theorem 5.5]. ■

The following corollary then follows immediately from the equality of the values of the forms θf and $\Theta_{\mathbb{R}} f$ on CM points.

Corollary 8.8. For $\chi \in \Sigma_{cc}^{(2)}$ of infinity type $(k+j, -j)$, with $j \geq 0$,

$$L_{\mathrm{alg}}(f, \chi^{-1}, 0) = \left(\sum_{[\mathfrak{a}] \in \mathrm{Cl}(\mathcal{O}_K)} \chi_j^{-1}(\mathfrak{a}) \cdot \theta^j f(\mathfrak{a} \star (A, t, \omega)) \right)^2 . \qquad \square$$

Now set

$$L_p(f, \chi) = \Omega_p^{2(k+2j)} (1 - \chi^{-1}(\bar{\mathfrak{p}}) a_p + \chi^{-2}(\bar{\mathfrak{p}}) \epsilon_f(p) p^{k-1})^2 L_{\mathrm{alg}}(f, \chi^{-1}, 0).$$

The following proposition expresses the Euler factor dropped at $\bar{\mathfrak{p}}$ in terms of the operator $\flat$ on the space of p-adic modular forms; the computation is in [3, Theorem 5.9]. (Note that we have replaced the algebraic form ω with the p-adic form $\hat{\omega}$.)

Proposition 8.9. For $\chi \in \Sigma_{cc}^{(2)}$ of infinity type $(k+j, -j)$, with $j \geq 0$, one has

$$L_p(f, \chi) = \left(\sum_{[\mathfrak{a}] \in \mathrm{Cl}(\mathcal{O}_K)} \chi_j^{-1}(\mathfrak{a}) \cdot \theta^j f^\flat(\mathfrak{a} \star (A, t, \hat{\omega})) \right)^2 . \qquad \square$$

8.5 Special values of L_p

This section investigates the properties of L_p outside the range of interpolation.

Proposition 8.10. The function $\chi \mapsto L_p(f, \chi)$ extends to a continuous function on all of $\hat{\Sigma}_{cc}$ (which we will still write as L_p.) $\square$

Proof. If two characters χ_1 and χ_2 are sufficiently close in the topology on $\hat{\Sigma}_{cc}^2$, then their infinity types satisfy the congruence

$$j_1 \equiv j_2 (\mathrm{mod} (p-1) p^{M-1}).$$

(to see this, evaluate on ideles congruent to 1 mod $\mathcal{N}$). It follows from Proposition 4.19 that

$$\theta^{j_1} f(A, t, \hat{\omega}) \equiv \theta^{j_2} f^\flat(A, t, \hat{\omega}) \mod p^M$$

at any ordinary CM point. The result follows from the formula of Proposition 8.9, which computes the value of L_p in terms of values of $f^\flat$ at ordinary CM points; moreover, it follows that the formula of Proposition 8.9 computes the values of L_p for any $\chi \in \hat{\Sigma}_{cc}^{(2)}$. ■

The following theorem is the main result of this document. Write $\phi_{\mathfrak{a}} : A \to A/A[\mathfrak{a}]$ for the natural map and $\Delta_{\mathfrak{a}}$ for the associated generalized Heegner cycle.

Theorem 8.11. Suppose χ is a central critical character with infinity type $(k-1-j, 1+j)$, with $0 \le j \le 2r$. Then

$$\frac{L_p(f,\chi)}{\Omega_p^{2(k-2-2j)}} = (1-\chi^{-1}(\bar{\mathfrak{p}})a_p + \chi^{-2}(\bar{\mathfrak{p}})\epsilon_f(p)p^{k-1})^2 \cdot \left(\frac{1}{j!}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \chi^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a}) \cdot \mathrm{AJ}_p(\Delta_{\mathfrak{a}})(\omega_f \wedge \omega_A^j \eta_A^{k-2-j})\right)^2. \qquad \square$$

Proof. The proof of the preceding proposition establishes the formula

$$L_p(f,\chi) = \left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j} f^{\flat}(\mathfrak{a}\star(A,t,\hat{\omega}))\right)^2.$$

By definition of Ω_p, we have (using that the weight of $\theta^{-1-j}f^{\flat}$ is $k-2-2j$) that

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = \left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j} f^{\flat}(\mathfrak{a}\star(A,t,\omega))\right)^2.$$

Lemma 7.3 shows that the value of θ^{-1-j} acting on $f^{\flat}$ is $\frac{1}{j!}\tilde{g}_j^{\flat}$, where $\tilde{g}_j$ denotes the jth component of the Coleman primitive for f, which gives

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = \left(\frac{1}{j!}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j}\tilde{g}_j^{\flat}(\mathfrak{a}\star(A,t,\omega))\right)^2.$$

By Lemma 8.6 (and a rearrangement of the sum), one can remove the operator $\flat$ on $\tilde{g}_j$ by dropping an Euler factor:

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = (1-\chi^{-1}(\bar{\mathfrak{p}})a_p + \chi^{-2}(\bar{\mathfrak{p}})\epsilon_f(p)p^{k-1})^2 \cdot \left(\frac{1}{j!}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j}\tilde{g}_j(\mathfrak{a}\star(A,t,\omega))\right)^2.$$

Now apply Lemma 7.4 to the Heegner isogeny $\phi_{\mathfrak{a}}$, of degree $(\mathbf{N}\mathfrak{a})^2$, to attain the final result.

(In the case of weight two, we are implicitly using Lemma 7.5 to compute with the cycle $\Delta_\chi = \sum \chi^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a})P_\chi$ rather than the cycle $\epsilon_f \Delta_\chi$.) ■

8.6 Fields of definition

We have stated our results in terms of cycles defined over the ray class field of K mod N^+. One may restate Theorem 8.11 in terms of a cycle defined over a smaller field of definition than F, as is explained (in the GL_2 case) in [4, Section 4.2]. For the purposes of applications, we give a slightly different argument to this effect in the case that $k=2$. In this case, Theorem 8.11 reads:

Proposition 8.12. Suppose that f has weight 2, and χ is a central critical character with infinity type $(1,1)$. Then the cycle $\epsilon_f \Delta_\chi \in \mathrm{Div}(C)(F) \otimes \bar{\mathbb{Q}}$ satisfies

$$L_p(f,\chi) = (1 - \chi^{-1}(\bar{\mathfrak{p}})a_p + \chi^{-2}(\bar{\mathfrak{p}})\epsilon_f(p)p)^2 \cdot \log_{\omega_f}(\epsilon_f \Delta_\chi). \qquad \square$$

Consider the quotient C' of C whose complex points are $\mathcal{H}/\Gamma_{0,N^+}$. Writing $\phi : C \to C'$ for the natural map, one has the rule for p-adic logarithms

$$\log_{\phi^*\omega}(P) = \log_\omega(\phi(P)).$$

for $P \in C'(\mathbb{C}_p)$. Writing Δ'_χ for $\phi(\epsilon_f \Delta_\chi)$, and ω'_f for the differential form on C' attached to f, Shimura's reciprocity law implies that Δ'_χ is defined over K. Thus, one has

Proposition 8.13. The cycle $\Delta'_\chi \in \mathrm{Div}(C')(K) \otimes \bar{\mathbb{Q}}$ satisfies

$$L_p(f,\chi) = (1 - \chi^{-1}(\bar{\mathfrak{p}})a_p + \chi^{-2}(\bar{\mathfrak{p}})\epsilon_f(p)p)^2 \cdot \log_{\omega'_f}(\Delta'_\chi). \qquad \square$$

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