OMP with Unknown Filters for Multipath Channel Estimation

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***Abstract*—We study a modification of the orthogonal matching pursuit (OMP) for estimating sparse multipath channels. The reflectors that generate the multipath components are not ideal; rather, they act as filters, so that the returned pulses are reshaped and widened. To deal with this, we introduce unknown filters into the OMP, and then search for the best filtered set of atoms, together with the optimal set of filters. Our algorithm extends naturally to unknown pulses lying in any known subspace. We show how this observation allows us to reconstruct sums of Gaussians with unknown widths.**

I. Introduction

The OMP aims to decompose a signal into a linear combination of a few atoms. One assumes that the data vector $\boldsymbol{x}$ to decompose is a sparse sum of the dictionary atoms—columns of $\boldsymbol{\Phi}$—corrupted by additive errors,

$$\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{d} + \boldsymbol{e}. \tag{1}$$

We can use OMP to estimate the structure of a sparse multipath channel: a probing pulse is emitted and the received signal is a convolution of the channel response and the probing pulse. The channel response is assumed to be a collection of Dirac delta functions, so that the resulting waveform is a sum of shifted and scaled pulses. Examples of applications are in wireless communications [1], [2] and in room acoustics [3].

A limiting factor is that (1) inaccurately models the error. Reflections are not only delays, because reflectors (*e.g.* walls) are never perfect; they are frequency selective, acting as linear filters with short impulse responses. Successive reflections become longer, and the error $\boldsymbol{e}$ is structured. We propose to modify the standard OMP to account for this filtering: at every iteration, we select the atom that is best correlated with the residual response, while allowing it to be filtered by any unknown finite impulse response (FIR) filter of fixed length. We demonstrate numerically that this outperforms OMP in channel estimation, simply because it respects physics.

II. OMP with Unknown Filters

Denote by $\phi(n)$ the emitted pulse of length N, and by $x(n)$ the measured signal. Our signal model is

$$x(n) = \sum_{k=1}^{K} \alpha_k [\phi * h_k](n - n_k) = \left[\phi * \sum_{k=1}^{K} \alpha_k h_k(\,\cdot - n_k) \right](n),$$

where h_k denotes the filter corresponding to the kth multipath component, and n_k is the delay of the kth multipath component. We assume all h_k to be relatively short FIR filters, of length at most M. The sequence $h(n) = \sum_{k=1}^{K} \alpha_k h_k(n - n_k)$ is the sought impulse response.

Let $\boldsymbol{r}_n^{(i)}$ denote the nth segment of $x^{(i)}$ (in ith iteration) of length $L \stackrel{\text{def}}{=} N + M - 1$, that is, $\boldsymbol{r}_n^{(i)} \stackrel{\text{def}}{=} [x(n), \ldots, x(n + L - 1)]^T$. In standard OMP, we choose the next atom according to $m_i = \arg\max |\langle \boldsymbol{r}_n^{(i)},\ \boldsymbol{\phi} \rangle|$. We propose to change this rule so as to jointly find the best atom (best delay) and the best filter, making sure that the filtered atom is normalized,

$$m_i = \underset{n,\, g,\, \phi * g \neq 0}{\arg\max} \left| \left\langle \boldsymbol{r}_n^{(i)}, \boldsymbol{\Phi}_M \boldsymbol{g} / \|\boldsymbol{\Phi}_M \boldsymbol{g}\| \right\rangle \right|. \tag{2}$$

Here, $\boldsymbol{\Phi}_M$ is the Toeplitz (convolution) matrix corresponding to $\phi(n)$ and $\boldsymbol{g}$ is the length-M unknown filter.

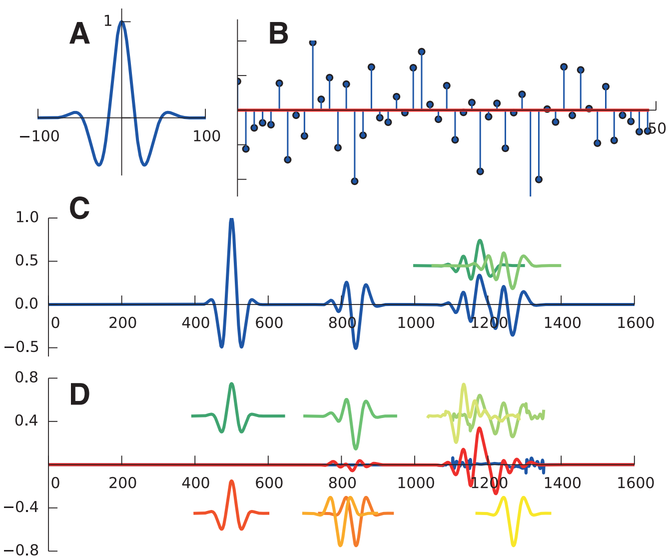

Fig. 1. *An illustration of the proposed algorithm, and a comparison with the standard OMP*. The emitted pulse (A) is convolved repeatedly with a random FIR filter (B), so that every successive pulse is longer than the previous. The last two delays are overlapping (C). The residuals (D) after 4 iterations for the OMP (red), and for the proposed algorithm (blue) suggest that the proposed algorithm performs better than the OMP. Detected atoms are shown above in green shades for the proposed algorithm (filtered), and below in orange shades for the ordinary OMP. Clearly, the OMP fails to detect the delays correctly. Length of the unknown filter was fixed to 50 (thus the last pulse—after three "reflections"—was filtered by a filter of length ≈ 150).

Lemma 1. *The optimal delay (index of the optimal atom) is*

$$m_i = \underset{n}{\arg\max}\ \langle \boldsymbol{r}_n^{(i)}, \mathcal{P}_{\boldsymbol{\Phi}_M} \boldsymbol{r}_n^{(i)} \rangle, \tag{3}$$

and the corresponding optimal filter is $\boldsymbol{g}^{(i)} = \boldsymbol{\Phi}_M^{\dagger} \boldsymbol{r}_{m_i} / \|\mathcal{P}_{\boldsymbol{\Phi}_M} \boldsymbol{r}_{m_i}\|$, *where* $\mathcal{P}_{\boldsymbol{\Phi}_M}$ *denotes the orthogonal projection onto the range of* $\boldsymbol{\Phi}_M$.

In the algorithm, we first initialize the residual to $\boldsymbol{r}^{(0)} = \boldsymbol{x}$. In ith iteration, we select the best new filtered atom by the rule (3). We compute the optimal filter $\boldsymbol{g}^{(i)}$ and construct the current approximation basis matrix $\boldsymbol{\Psi}$ from the atoms detected thus far, with appropriate filtering. The jth column of $\boldsymbol{\Psi}^{(i)}$ corresponds to $[\phi * g^{(j)}](n - n_j)$. In the established notation we can write

$\mathbf{\Psi}^{(i)} = [\boldsymbol{\psi}_1, \ \cdots, \ \boldsymbol{\psi}_i]$, where

$$\boldsymbol{\psi}_j \stackrel{\text{def}}{=} [\mathbf{0}_{1\times n_j}, \ [\mathbf{\Phi}_M \boldsymbol{g}^{(j)}]^T, \ \mathbf{0}_{1\times(T-L-n_j)}]^T \tag{4}$$

The approximation in the ith iteration (*i.e.*, the current denoised version) is computed as the projection of the signal onto the current basis,

$$\boldsymbol{x}^{(i)} = \mathcal{P}_{\mathbf{\Psi}^{(i)}} \boldsymbol{x}, \quad \boldsymbol{r}^{(i)} = \boldsymbol{x} - \boldsymbol{x}^{(i)}. \tag{5}$$

The described procedure is outlined in Algorithm 1. Fig. 1 illustrates how the proposed algorithm obviates the shortcomings of the ordinary OMP in channel estimation.

Algorithm 1 OMP with Unknown Filters

```
function OMP-UF(x, Φ_M, K)
    r^(0) ← x, Ψ^(0) ← [ ], i ← 0                         ▷ Initialize
    repeat
        i ← i + 1
        m_i ← arg max_n ⟨ r_n^(i), P_{Φ_M} r_n^(i) ⟩        ▷ Best tap
        g_i ← Φ_M^† r_{m_i} / ||P_{Φ_M} r_{m_i}||           ▷ Optimal filter
        Ψ^(i) ← [Ψ^(i−1) | ψ^(i)]                          ▷ Equation (4)
        r^(i) ← x − P_{Ψ^(i)} x                             ▷ New residual
    until i = K
    return {m_i}, {Φ_M g_i}, {g_i}
end function
```

III. From Shift-Invariant to a General Subspace

Note that the pulse shape $\phi(n)$ enters the algorithm only through the projection operator $\mathcal{P}_{\mathbf{\Phi}_M}$. There is nothing special about the Toeplitz structure of $\mathbf{\Phi}_M$; instead of the subspace of the shifts of $\phi(n)$, we could look at *any* subspace.

Some interesting parametric signal classes can be approximated as living in a low-dimensional subspace. A good example are Gaussian pulses of varying widths. Consider the following set of signals:

$$\mathcal{F} = \left\{ x \mapsto f_\sigma(x) = \mathrm{e}^{-\frac{1}{2}(x/\sigma)^2} \ : \ \sigma \in [\sigma_1, \sigma_2] \right\}. \tag{6}$$

For a wide range of widths σ, $\mathcal{F}$ can be approximated by a low-dimensional subspace. A simple way one could find such a subspace is by PCA-like procedure. Suppose we discretize x into a grid of sampling instants $\boldsymbol{x} = [x_1, \ldots, x_P]$, and similarly discretize the range of widths into $\boldsymbol{\sigma} = [\sigma_1, \ldots, \sigma_Q]$. Define $\boldsymbol{f}_\sigma \stackrel{\text{def}}{=} [f_\sigma(x_1), \ldots, f_\sigma(x_P)]^T$, and let $\boldsymbol{F} = [\boldsymbol{f}_{\sigma_1}, \ldots, \boldsymbol{f}_{\sigma_Q}]$. It turns out that we can obtain a good approximation of $\mathcal{F}$ in the subspace spanned the dominant eigenvectors of $\boldsymbol{F}\boldsymbol{F}^T$, as illustrated in Fig. 2.

Imagine now that the received signal is of the form

$$x(n) = \sum_{k=1}^{K} \alpha_k f_{\sigma_k}(nT - t_k), \tag{7}$$

where T is the sampling period, and that we seek to estimate the delays t_k. To do that, we can use the exact same algorithm we proposed for the case of unknown filters, but instead of searching for an unknown filter, we search for unknown basis expansion coefficients (which in turn can be mapped to the width of the pulse).

In Fig. 2 we demonstrate the effectiveness of the proposed algorithm for Gaussian pulses of unknown widths.

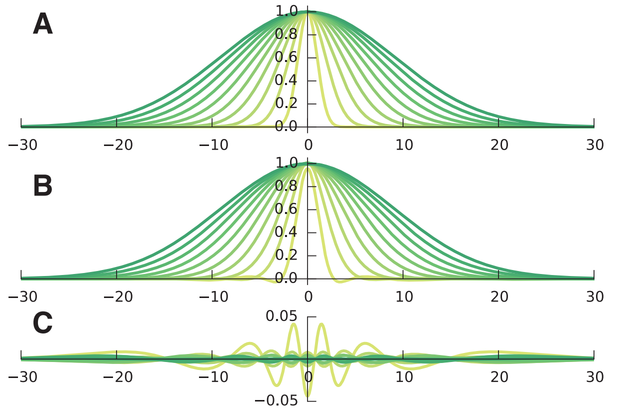

Fig. 2. Approximation of Gaussian pulses in a linear subspace. (A) original (exact) pulses with $\sigma \in [1, 10]$; (B) approximation of the pulses from (A) in a 5-dimensional linear subspace; (C) error signal [(B) - (A)].

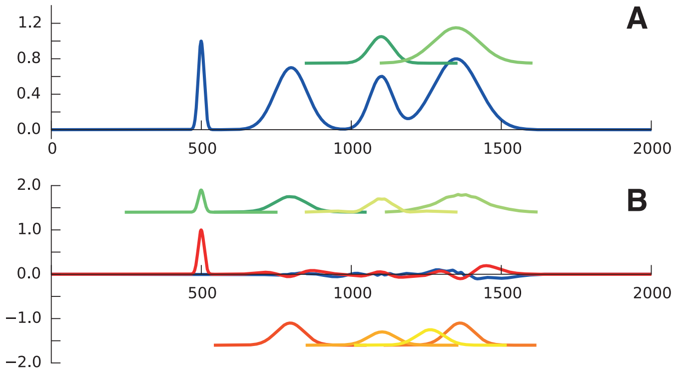

Fig. 3. Illustration of the proposed algorithm applied to Gaussian pulses of unknown widths. (A) the received signal composed of four pulses, with two overlapping pulses shown explicitly in green; (B) residuals for the proposed algorithm (blue) and for the standard OMP (red) with an *average* pulse ($\sigma = 5$). In (B) we also show the detected pulses for the two algorithms (above for the proposed algorithm, and below for the OMP). The OMP completely fails to recover the first (narrow) pulse.

IV. Conclusion

We demonstrated a variation of the OMP for sparse channel estimation, which takes into account the unknown filtering of the multipath components by the non-ideal reflectors. Initial numerical simulations show that the proposed scheme is more effective than the OMP for filtered pulses. Our algorithm generalizes straightforwardly to the case when the pulses live in a known subspace. This is useful for recovering mixtures of unknown pulses from known parametric signal classes. Future work will focus on convergence analysis and comparisons in terms of the performance in channel equalization and acoustic echo cancellation.

References

[1] S. F. Cotter and B. D. Rao, “Sparse Channel Estimation via Matching Pursuit with Application to Equalization,” *IEEE Trans. Commun.*, vol. 50, no. 3, pp. 374–377, 2002.

[2] G. Z. Karabulut and A. Yongacoglu, “Sparse Channel Estimation Using Orthogonal Matching Pursuit Algorithm,” in *Proc. IEEE VTC-Fall*. IEEE, 2004, pp. 3880–3884.

[3] G. Defrance, L. Daudet, and J. D. Polack, “Detecting Arrivals Within Room Impulse Responses Using Matching Pursuit ,” in *Proc. DAFx*, vol. 10, Espoo, Finland, 2008, pp. 307–316.