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# Fixed-structure LPV Discrete-time Controller Design with Induced $l_2$-Norm and $\mathscr{H}_2$ Performance

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A new method for the design of fixed-structure dynamic output-feedback Linear Parameter Varying (LPV) controllers for discrete-time LPV systems with bounded scheduling parameter variations is presented. Sufficient conditions for the stability, $\mathscr{H}_2$ and induced $l_2$-norm performance of a given LPV system are given through a set of Linear Matrix Inequalities (LMIs) and exploited for design. Extension to the case of uncertain scheduling parameter value is considered as well. Controller parameters appear directly as decision variables in the convex optimisation program, which enables preserving a desired controller structure in addition to the low order. Efficiency of the proposed method is illustrated on a simulation example, with an iterative convex optimisation scheme used for the improvement of the control system performance.

**Keywords:** Linear Parameter-Varying; Fixed-structure Controller Design; Parameter Dependent Lyapunov Function; Induced $l_2$-Norm Performance; $\mathscr{H}_2$ Performance, Scheduling Parameter Uncertainty.

## 1. Introduction

The LPV system modelling and control paradigm arises naturally as a successor of classical gain-scheduling controller design approaches (Leith and Leithead, 2000; Shamma and Athans, 1991). It allows modelling a wide class of nonlinear systems and the use of many tools from the linear systems theory for analysis and control. Consequently, a number of applications has been treated in the LPV framework recently; modelling and control of turbofan engines (W. Gilbert et al., 2010), active braking control (G. Panzani et al., 2012) and semi-active vehicle suspension design (C. Poussot-Vassal et al., 2008), to name just a few.

Over the last 20 years, different continuous-time LPV controller design strategies for LPV systems with state-space description were developed (Apkarian and Adams, 1998; F. Wu et al., 1996; Sato, 2011; Wu, 2001). Some important results for the stability analysis of uncertain and LPV polytopic discrete-time systems are presented in (J. Daafouz and J. Bernussou, 2001; M. C. de Oliveira et al., 1999; R. C. L. F. Oliveira and P. L. D. Peres, 2005). These ideas establish a good starting point for the LPV controller synthesis. A few recent publications cover the case of controller synthesis for discrete-time LPV systems affected by scheduling parameters with limited variations (F. Amato et al., 2005; J De Caigny et al., 2012; R. C. L. F. Oliveira and P. L. D. Peres, 2009), and the observer-based controller design for the LPV system with uncertainty in the measurement of the

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scheduling parameter is considered in (W.P.M. Heemels et al., 2010) .

All enlisted methods result in a controller in either state-feedback or full-order output-feedback form. For online reconstruction of the full-order controller, time-consuming linear algebraic operations need to be employed. Moreover, the order of the controller may be too high since it depends on the augmented plant model order. Some methods for the LPV controller reduction are available (Beck, 2006), but there is no guarantee of preserving stability or performance of the original LPV system with reduced-order controller. On the other side, a state-feedback LPV controller demands state estimation, which is a non-trivial task for LPV systems. Often, the users may have a preference for a certain controller structure. Decentralised (N.R. Sandell et al., 1978) or distributed (R. D'Andrea and G.E. Dullerud, 2003) controller structure may be essential in order to achieve low complexity of the overall control system. However, in both cases of state-feedback or full-order output-feedback controller design methods, controller is restored from the optimisation results by a nonlinear change of variables. This means that the user requested structure in the controller cannot be preserved. As well, in most practical applications, resources available for control are highly limited. This is why a method for the direct design of low-order fixed-structure output-feedback LPV controllers, which are easier and cheaper to implement and with accordingly lower execution times, is highly needed.

Some methods for the fixed-order LPV controller design in the transfer function setting are presented in (S. Formentin et al., 2013; W. Gilbert et al., 2010; Z. Emedi and A. Karimi, 2012). The use of transfer function models is very well aligned with industrial practice and modelling paradigm in the SISO case (Tóth, 2010). However, the extension to the MIMO case can be highly non-trivial comparing to its simplicity in the state-space setting.

The importance of the discrete-time LPV controller design methods comes from the fact that the LPV models produced by identification procedures are usually in discrete-time (Toth et al., 2009; V. Cerone et al., 2012; Verdult, 2002). As well, control is anyway implemented using digital computers in practice. The problem is that preservation of the closed-loop stability under the discretisation of a continuous-time LPV system could require too high sampling frequency (R. Toth et al., 2008).

In (F.D. Adegas and J. Stoustrup, 2011), authors develop a fixed-structure state-space LPV controller design method with guaranteed induced $l_2$-norm performance. Performance analysis conditions from (C.E. de Souza et al., 2006) are convexified around the slack variable matrix, and its value is updated based on its relation with Lyapunov matrix. Appropriate two-step iterative optimisation scheme is used to improve induced $l_2$-norm performance. As this method shares some common assumptions with the proposed method, numerical comparison on the example from (F.D. Adegas and J. Stoustrup, 2011) will be performed.

In this paper a class of discrete-time LPV state-space plants, affine in the scheduling parameter vector, is considered. User imposed controller structure is preserved since controller parameters appear directly as decision variables in the convex optimisation program. The realistic case of limited scheduling parameter variations is treated through the use of Parameter Dependent Lyapunov Functions (PDLF) affine in the scheduling parameter vector. Uncertainty in the scheduling parameter vector, coming from the sensor measurement error, can be considered in design. Upper bound on the $\mathscr{H}_2$ and induced $l_2$-norm performance of a control system is enhanced through the use of iterative convex optimisation procedure.

The paper is organised as follows. First, preliminaries about the LPV system stability and performance are given in Section 2. Stabilising LPV controller design procedure is proposed in Section 3. Extensions of the procedure to $\mathscr{H}_2$ and induced $l_2$-norm performance design are given in Section 4. Numerical comparison with method (F.D. Adegas and J. Stoustrup, 2011) on the simulation example is presented in Section 5. Finally, Section 6 contains concluding remarks.

## 2. Preliminaries

### 2.1 *LPV plant and controller*

The class of LPV discrete-time systems considered in this paper can be represented by the following model:

$$\begin{aligned}\mathbf{x}_g(k+1) &= A_g(\boldsymbol{\theta})\mathbf{x}_g(k) + B_u(\boldsymbol{\theta})\mathbf{u}(k) + B_w(\boldsymbol{\theta})\mathbf{w}(k)\\ \mathbf{z}(k) &= C_z(\boldsymbol{\theta})\mathbf{x}_g(k) + D_{zu}(\boldsymbol{\theta})\mathbf{u}(k) + D_{zw}(\boldsymbol{\theta})\mathbf{w}(k)\\ \mathbf{y}(k) &= C_y\mathbf{x}(k) + D_{yw}\mathbf{w}(k).\end{aligned} \tag{1}$$

Here $\mathbf{x}_g(k) \in \mathbb{R}^n$ represents the state vector, $\mathbf{u}(k) \in \mathbb{R}^{n_u}$ is the control input vector, $\mathbf{z}(k) \in \mathbb{R}^{n_z}$ is the vector of controlled outputs and $\mathbf{y}(k) \in \mathbb{R}^{n_y}$ is the vector of measured outputs. The time-varying scheduling parameter vector $\boldsymbol{\theta} = [\theta_1(k), \dots, \theta_{n_\theta}(k)]^T$ is assumed to belong to a hyper-rectangle $\Theta \in \mathbb{R}^{n_\theta}$, or equivalently

$$\theta_i(k) \in [-\overline{\theta}_i, \overline{\theta}_i],\ i = 1, \dots, n_\theta. \tag{2}$$

where, without loss of generality, symmetric bounds around $\theta_i = 0$ are assumed. Scheduling parameters $\theta_i$ are assumed to be independent.

Strict properness of the plant model is a non-restricting assumption, since in discrete-time systems there is always a delay of at least one sampling period. For a technical reason matrices $C_y$ and $D_{yw}$ are assumed to be independent of the scheduling parameter vector. However, similar results could be obtained for the case of $C_y$ and $D_{yw}$ depending on $\theta$, and $B_u$ and $D_{zu}$ being constant.

Affine dependence on the scheduling parameter vector is assumed for all $\theta$-dependent matrices. This can be represented, for example for $A_g$, as

$$A_g(\boldsymbol{\theta}(k)) = A_{g_0} + \sum_{i=1}^{n_\theta} \theta_i(k) A_{g_i}. \tag{3}$$

The following fixed-order LPV dynamic output feedback controller structure is considered:

$$\begin{aligned}\mathbf{x}_c(k+1) &= A_c(\boldsymbol{\theta})\mathbf{x}_c(k) + B_c(\boldsymbol{\theta})\mathbf{y}(k)\\ \mathbf{u}(k) &= C_c\mathbf{x}_c(k) + D_c\mathbf{y}(k),\end{aligned} \tag{4}$$

where $\mathbf{x}_c(k) \in \mathbb{R}^{n_c}$ represents the controller state vector. The choice of controller order $n_c$ is fully left to user.

Matrices $A_c(\boldsymbol{\theta})$ and $B_c(\boldsymbol{\theta})$ are supposed to have an affine dependency on scheduling parameter vector. This implies that the closed-loop matrices are as well affine in the scheduling parameters. Closed-loop system equations can be written as

$$\begin{aligned}\mathbf{x}(k+1) &= A_{cl}(\boldsymbol{\theta})\mathbf{x}(k) + B_{cl}(\boldsymbol{\theta})\mathbf{w}(k)\\ \mathbf{y}(k) &= C_{cl}(\boldsymbol{\theta})\mathbf{x}(k) + D_{cl}(\boldsymbol{\theta})\mathbf{w}(k),\end{aligned} \tag{5}$$

where $\mathbf{x}(k) = [\mathbf{x}_g(k)\ \mathbf{x}_c(k)]^T$ and

$$\begin{aligned} A_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} A_g(\boldsymbol{\theta}) + B_u(\boldsymbol{\theta})D_cC_y & B_u(\boldsymbol{\theta})C_c \\ B_c(\boldsymbol{\theta})C_y & A_c(\boldsymbol{\theta}) \end{bmatrix} \\ B_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} B_w(\boldsymbol{\theta}) + B_u(\boldsymbol{\theta})D_cD_{yw} \\ B_c(\boldsymbol{\theta})D_{yw} \end{bmatrix} \\ C_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} C_z(\boldsymbol{\theta}) + D_{zu}(\boldsymbol{\theta})D_cC_y & D_{zu}(\boldsymbol{\theta})C_c \end{bmatrix} \\ D_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} D_{zw}(\boldsymbol{\theta}) + D_{zu}(\boldsymbol{\theta})D_cD_{yw} \end{bmatrix} \end{aligned} \tag{6}$$

**Remark 1:** *The closed-loop matrices in (6) are affine in $\boldsymbol{\theta}$ as some plant matrices are limited to be $\boldsymbol{\theta}$-independent. If this was not the case, the problem could be treated using the homogenous polynomials relaxations (e.g. as in (J De Caigny et al., 2012). However, for the simplicity of presentation we continue with this assumption.*

### 2.2 *Discrete-time LPV system stability conditions*

Assessing the stability of an LPV system through the use of a Lyapunov function quadratic in the state is well treated in the literature (e.g. (M. C. de Oliveira et al., 1999)). In the discrete-time case, keeping the Lyapunov matrix $P$ constant over $\Theta$ is too restrictive even if the scheduling parameters can change from one extremal value to another over the course of one sampling period (J. Daafouz and J. Bernussou, 2001). Usually in practical applications the maximum possible variation of a scheduling parameter is bounded as in

$$\theta_i^+ - \theta_i \in [-\overline{\delta}_i, \overline{\delta}_i],\ 0 < \overline{\delta}_i < 2\overline{\theta}_i,\ i = 1, \ldots, n_\theta, \tag{7}$$

where $\boldsymbol{\theta}^+ = \boldsymbol{\theta}(k+1)$. To exploit the bounds on scheduling parameter variation, Lyapunov matrix affine in the scheduling parameter vector is considered:

$$P(\boldsymbol{\theta}) = P_0 + \sum_{i=1}^{n_\theta} \theta_i P_i > 0,\ \forall \boldsymbol{\theta} \in \Theta. \tag{8}$$

Using (8) well-known stability condition for a discrete-time LPV system can be written as

$$P(\boldsymbol{\theta}) - A_{cl}^T(\boldsymbol{\theta})P(\boldsymbol{\theta}^+)A_{cl}(\boldsymbol{\theta}) > 0. \tag{9}$$

This condition has to be satisfied for all admissible values of $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$. The limits on scheduling parameters (2) and their variations (7) imply that $(\theta_i, \theta_i^+)$ belongs to a set presented by filling on Fig. 1. The set of vertices of hexagon $A_iB_iD_iE_iF_iH_i$ will be denoted by $\Omega_{v_i}$. This means that the pair $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$ always belongs to the polytope $\Omega$ whose vertex set $\Omega_v$ is given by $\Omega_v = \Omega_{v_1} \times \Omega_{v_2} \times \cdots \times \Omega_{v_{n_\theta}}$. The logic behind Fig. 1 is rather intuitive. For example, point $H_i$ comes from the fact that if $\theta_i = -\overline{\theta}_i$, then $\theta_i^+ \leq -\overline{\theta}_i + \overline{\delta}_i$, since $\overline{\delta}_i$ is maximum possible increase of $\theta_i$ over one sample. Points $B_i$, $D_i$ and $F_i$ can be obtained in a similar manner.

**Remark 2:** *There are two limit cases that are covered by this setup. First is the fixed scheduling parameter case, which is defined by $\overline{\delta}_i = 0$. In this case the hexagon $A_iB_iD_iE_iF_iH_i$ collapses into a line $A_iE_i$. In the case of maximum possible variations, defined by $\overline{\delta}_i = 2\overline{\theta}_i$, the filled hexagon degenerates into a square $A_iC_iE_iG_i$. However, the primary focus here is on the non-degenerate case, taking its importance into account.*

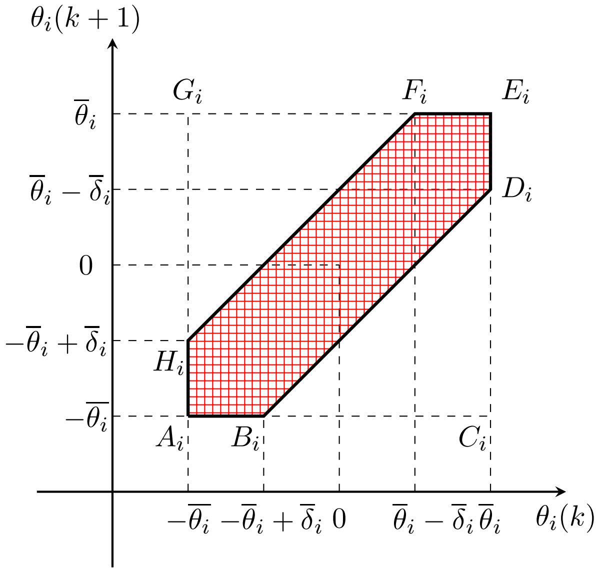


Figure 1. Admissible $(\theta_i, \theta_i^+)$ space (filled).

**Remark 3:** *The case of non-symmetric variation bounds could be treated straightforwardly. Symmetric bounds are assumed for the simplicity of presentation.*

Equivalent representation of (9) in the literature (see e.g. (J. Daafouz and J. Bernussou, 2001)) is

$$\begin{bmatrix} P(\boldsymbol{\theta}) & A_{cl}^T(\boldsymbol{\theta})P(\boldsymbol{\theta}^+) \\ P(\boldsymbol{\theta}^+)A_{cl}(\boldsymbol{\theta}) & P(\boldsymbol{\theta}^+) \end{bmatrix} > 0. \tag{10}$$

As controller variables appearing in $A_{cl}$ multiply unknown Lyapunov matrix $P$ in (10), the controller synthesis problem becomes a Bilinear Matrix Inequality (BMI) optimisation program. As it is a non-convex optimisation problem, obtaining even (good) local solution is far from trivial. Another issue is that multiplication of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^+$ produces the infiinite number of constraints. This can be substituted by a finite number of constraints by application of some relaxation technique (Scherer, 2006). The idea applied in this publication is to substitute the given infinite set of non-convex constraints on design variables by a finite number of linear matrix inequalities in the controller and Lyapunov function parameters.

## 3. Stabilising Fixed-structure Discrete-time LPV Controller Synthesis

Over the last 15 years, stability of uncertain and LPV systems is treated using different "slack matrix variable" approaches (J. Daafouz and J. Bernussou, 2001; M. C. de Oliveira et al., 1999; R. C. L. F. Oliveira and P. L. D. Peres, 2005). Similar conditions are developed in (M. S. Sadabadi and A. Karimi, 2013) and applied to robust fixed-order controller design for uncertain polytopic systems. These results will be extended to LPV systems.

First, the well-known KYP lemma needs to be recalled. KYP lemma for the discrete-time systems states that the biproper transfer function $H(z) = C(zI - A)^{-1}B + D$ is Strictly Positive Real (SPR) if and only if $\exists P = P^T > 0$ such that

$$\begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{bmatrix} < 0. \tag{11}$$

The following lemma based on the theory from (M. S. Sadabadi and A. Karimi, 2013) represents a basis for this LPV fixed-structure controller synthesis approach.

**Lemma 1:** *A Strictly Positive Real (SPR) transfer function $H(z)$ and $H^{-1}(z)$ satisfy discrete-time Kalman-Yakubovic-Popov (KYP) lemma with a common Lyapunov matrix $P$.*

**Lemma 2:** *Matrix inequalities*

$$\begin{bmatrix} P - M^T P M & M^T P - M^T + T^T A_{cl}^T T^{-T} \\ PM - M + T^{-1} A_{cl} T & 2I - P \end{bmatrix} > 0 \tag{12}$$

*and*

$$\begin{bmatrix} P_T - A_{cl}^T P_T A_{cl} & A_{cl}^T P_T - A_{cl}^T X + M_T^T \\ P_T A_{cl} - X A_{cl} + M_T & 2X - P \end{bmatrix} > 0, \tag{13}$$

*where*

$$P_T = T^{-T} P T^{-1},\ M_T = T^{-T} M T^{-1},\ X = T^{-T} T^{-1},$$

*are equivalent.*

*Proof.* This lemma is a consequence of Lemma 1. Inequality (12) represents the KYP lemma inequality for

$$H(z) = \left[\begin{array}{c|c} M & I \\ \hline M - T^{-1} A_{cl} T & I \end{array}\right] \tag{14}$$

Inequality (13) represents the KYP lemma inequality for

$$H^{-1}(z) = \left[\begin{array}{c|c} T^{-1} A_{cl} T & I \\ \hline T^{-1} A_{cl} T - M & I \end{array}\right] \tag{15}$$

which is pre- and postmultiplied by block-diagonal matrix $diag(T^{-T}, T^{-T})$ and its transpose. □

Alternatively, the equivalence of (12) and (13) can be proven using the matrix

$$L = \begin{bmatrix} T^{-1} & 0 \\ MT^{-1} - T^{-1} A_{cl} & T^{-1} \end{bmatrix}. \tag{16}$$

Namely, (13) is obtained as (12) pre- and post-multiplied by $L^T$ and $L$. Since pre- and post-multiplication of matrix by the invertible matrix and its transpose do not change its positive definiteness, the matrix inequalities (12) and (13) are equivalent.

**Remark 4:** *It can be noticed that Schur stability of both matrices $A$ and $M$ is implied through the positive definiteness of the upper left blocks of given matrix inequalities.*

### 3.1 *Fixed-structure LPV Controller Design Conditions*

Using Lemma 2, a sufficient condition for the fixed-structure LPV controller synthesis is proposed.

**Theorem 1:** *Assume that are given a discrete-time LPV plant affine in scheduling parameter vector $\boldsymbol{\theta}$, bounds on the scheduling parameter vector and its variation as in Preliminaries. Furthermore, assume an LPV controller structure (4). Given matrices $M$ and $T$, there exists an LPV* *controller stabilising the given LPV plant for all admissible scheduling parameter trajectories if*

$$\begin{bmatrix} P(\boldsymbol{\theta}) - M^T P(\boldsymbol{\theta}^+) M & (*) \\ P(\boldsymbol{\theta}^+) M - M + T^{-1} A_{cl}(\boldsymbol{\theta}) T & 2I - P(\boldsymbol{\theta}^+) \end{bmatrix} > 0, \tag{17}$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v,$$

*with* $(*)$ *representing the terms completing the symmetric matrix.*

*Proof.* First it can be observed that the left-hand side of (17) is affine in pair $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$. This means that its validity for $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$ can be proven using an appropriate convex combination of vertex inequalities.

Next, it has to be proven that validity of (17) implies stability condition for the closed-loop system $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$. Similarly to the alternative proof of Lemma 2, the following matrix can be considered:

$$L(\boldsymbol{\theta}) = \begin{bmatrix} T^{-1} & 0 \\ MT^{-1} - T^{-1} A_{cl}(\boldsymbol{\theta}) & T^{-1} \end{bmatrix}. \tag{18}$$

Pre- and post-multiplication of (17) by $L^T(\boldsymbol{\theta})$ and $L(\boldsymbol{\theta})$ imply positive-definiteness of

$$\begin{bmatrix} P_T(\boldsymbol{\theta}) - A_{cl}^T(\boldsymbol{\theta}) P_T(\boldsymbol{\theta}^+) A_{cl}(\boldsymbol{\theta}) & (*) \\ P_T(\boldsymbol{\theta}^+) A_{cl}(\boldsymbol{\theta}) - X A_{cl}(\boldsymbol{\theta}) + M_T & 2X - P_T(\boldsymbol{\theta}^+) \end{bmatrix} \tag{19}$$

for $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$, with the same shorthands as in Lemma 2. The top left block of 19 represents the stability condition (9) for the closed-loop LPV system. Since its positivity for $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$ is guaranteed by the Schur complement lemma, stability of the closed-loop system is guaranteed for all allowable scheduling parameter trajectories. □

**Remark 5:** *The total number of constraints in the non-degenerate case corresponds to the cardinality of the set* $\Omega_v$*, which equals* $6^{n_\theta}$*. Considering that in realistic applications there are rarely more than 3 scheduling parameters (F. Wu et al., 1996), this number of LMIs should be numerically tractable in acceptable execution time.*

### 3.2 *Fixed-structure LPV Controller Synthesis Algorithm*

In the continuous-time LPV controller design method presented in (Z. Emedi and A. Karimi, 2013), the idea for choosing matrix $M$ is based on the design of initial controllers for all vertices of $\Theta$, and solving the inverse of the synthesis problem. Similar idea can be applied here to find appropriate values for $M$ and $T$.

**Remark 6:** *It is important to emphasise that the fixed-structure controller design is not a trivial task even for an LTI plant, being a non-convex optimisation problem as well. A few approaches are available in the form of Matlab® toolbox for* $\mathscr{H}_\infty$ *and* $\mathscr{H}_2$ *controller design, for example* `hinfstruct` *(Apkarian and Noll, 2006),* `HIFOO` *(Burke et al., 2006) and* `FDRC` *(M. Sadeghpour et al., 2012). Since we need an LTI controller just to initialise the algorithm (not necessarily an optimal one, in any sense), one of these or similar methods should suffice.*

Suppose that initial controllers $K_i, i = 1, \ldots, 2^{n_\theta}$ correspond to the vertices of hyper-rectangle $\Theta$. This means that for each LTI system obtained by fixing $\boldsymbol{\theta}_v \in \Theta_v$ one of the above-mentioned fixed-structure LTI controller design methods is used to design appropriate stabilising LTI controller $K_i$. The next step is the choice of matrices $M$ and $T$. Based on $K_i, i = 1, \ldots, 2^{n_\theta}$, closed-loop matrices $A_{cl}(\boldsymbol{\theta}_v)$ can be calculated. By introducing $A_{cl}(\boldsymbol{\theta}_v)$ into (19), feasible $X$, $M_T$ and $P_T$ can be obtained. Then, from matrix $X$ the similarity transform matrix $T$ can be reconstructed by Cholesky factorisation, and from $M_T$ and $T$ rises $M = T^T M_T T$. Now the controller design phase can be performed using $M$ and $T$ in (17).

If this method fails in the first phase, an alternative set of constraints can be used. The idea is to replace (17) and (19) by

$$\begin{bmatrix} \sigma^2 P(\boldsymbol{\theta}) - M^T P(\boldsymbol{\theta}^+) M & (*) \\ P(\boldsymbol{\theta}^+) M - M + T^{-1} A_{cl}(\boldsymbol{\theta}) T & 2I - P(\boldsymbol{\theta}^+) \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} \sigma^2 P_T(\boldsymbol{\theta}) - A_{cl}^T(\boldsymbol{\theta}) P_T(\boldsymbol{\theta}^+) A_{cl}(\boldsymbol{\theta}) & (*) \\ P_T(\boldsymbol{\theta}^+) A_{cl}(\boldsymbol{\theta}) - X A_{cl}(\boldsymbol{\theta}) + M_T & 2X - P_T(\boldsymbol{\theta}^+) \end{bmatrix} > 0. \tag{21}$$

Now iterating between (21) and (20) is performed, until minimal $\sigma$ is obtained. This corresponds to the exponential decay minimisation, and $\sigma \leq 1$ guarantees stability of the closed-loop system.

This algorithm can be summarised in the 4 following steps:

***step 1*:** choose small $\epsilon > 0$; design the initial controllers $K_i^0, i = 1, \ldots, 2^{n_\theta}$ for $\boldsymbol{\theta}_v \in \Theta_v$; set $j = 0$
***step 2*:** for $\forall \boldsymbol{\theta}_v \in \Theta_v$ calculate $A_{cl}(\boldsymbol{\theta}_v)$ using $K^{j-1}$; set (21) for $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v$ using $A_{cl}(\boldsymbol{\theta})$ and find feasible $X$, $M_T$ and $P_T(\theta)$ while minimising $\sigma$ by bisection; reconstruct $T$ from $X = T^{-T} T^{-1}$ and subsequently $M = T^T M_T T$;
***step 3*:** set (20) for $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v$ using $M$ and $T$ obtained in **step 2** and search for feasible $K^{j+1}$, $P_T(\boldsymbol{\theta})$ and minimal (by bisection) $\sigma^j$;
***step 4*:** if $\sigma^j - \sigma^{j-1} > \epsilon$ set $j = j + 1$ and jump to the **step 2**; otherwise stop.

Equivalence of (20) and (21) ensures that at worst case in **step 3** we will obtain exactly the same controller and $\sigma^j$ as those applied in **step 2**. Therefore stability indicator (and exponential decay parameter) $\sigma^j$ is monotonically non-increasing in this synthesis procedure.

The final value of $\sigma$ depends on the choice of initial controllers. In the case that the final value is not satisfactory, or that its value is above 1, another set of initial controllers should be used. Similar re-initialisation is proposed in both `hinfstruct` or `HIFOO` for LTI controller design.

### 3.3 *Treatment of Scheduling Parameter Uncertainty*

In reality, the exact value of the scheduling parameter $\boldsymbol{\theta}$ is never available. Even if the scheduling parameter is directly measured (i.e. not estimated), what will be available in the real-time is the value affected by the measurement error of measurement device. Assume that the maximum absolute error of the measurement device for the $i^{\text{th}}$ component of the scheduling parameter vector is $\overline{\Delta}_{e_i} > 0$. If we denote by $\hat{\boldsymbol{\theta}}$ measured value of the scheduling parameter and considering $\boldsymbol{\theta}$ as an exact value, this means that $\hat{\theta}_i - \theta_i \in [-\overline{\Delta}_{e_i}, \overline{\Delta}_{e_i}]$, $i = 1, \ldots, n_\theta$.

Current values of the controller matrices are calculated online based on the available value of the scheduling parameter, so what will be used to control the given system is a controller $(A_c(\hat{\theta}), B_c(\hat{\theta}), C_c(\hat{\theta}), D_c(\hat{\theta}))$. This means that the closed-loop system matrices are affected by both

$\theta$ and $\hat{\theta}$ in the affine fashion as following:

$$\begin{aligned} A_{cl}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) &= \begin{bmatrix} A_g(\boldsymbol{\theta}) + B_u(\boldsymbol{\theta})D_cC_y & B_u(\boldsymbol{\theta})C_c \\ B_c(\hat{\boldsymbol{\theta}})C_y & A_c(\hat{\boldsymbol{\theta}}) \end{bmatrix} \\ B_{cl}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) &= \begin{bmatrix} B_w(\boldsymbol{\theta}) + B_u(\boldsymbol{\theta})D_cD_{yw} \\ B_c(\hat{\boldsymbol{\theta}})D_{yw} \end{bmatrix} \\ C_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} C_z(\boldsymbol{\theta}) + D_{zu}(\boldsymbol{\theta})D_cC_y & D_{zu}(\boldsymbol{\theta})C_c \end{bmatrix} \\ D_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} D_{zw}(\boldsymbol{\theta}) + D_{zu}(\boldsymbol{\theta})D_cD_{yw} \end{bmatrix}. \end{aligned} \tag{22}$$

Assume again Lyapunov function quadratic in the state $V(k) = \boldsymbol{x}(k)^T P(\boldsymbol{\theta}(k))\boldsymbol{x}(k)$. Taking into account dynamics affected by the uncertainty as in (22), Lyapunov function difference over one sampling period is

$$\begin{aligned} V(k+1) - V(k) &= \boldsymbol{x}^T(k+1)P(\boldsymbol{\theta}(k+1))\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P(\boldsymbol{\theta}(k))\boldsymbol{x}(k) = \\ &= \boldsymbol{x}^T(k)A_{cl}^T(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k))P(\boldsymbol{\theta}(k+1))A_{cl}^T(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k))\boldsymbol{x}(k) - \boldsymbol{x}(k)^T P(\boldsymbol{\theta}(k))\boldsymbol{x}(k) = \\ &= \boldsymbol{x}^T(k)[A_{cl}^T(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k))P(\boldsymbol{\theta}(k+1))A_{cl}(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k)) - P(\boldsymbol{\theta}(k))]\boldsymbol{x}(k). \end{aligned} \tag{23}$$

Consequently, the following condition has to be satisfied to guarantee the closed-loop stability:

$$P(\boldsymbol{\theta}(k)) - A_{cl}^T(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k))P(\boldsymbol{\theta}(k+1))A_{cl}(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k)) > 0,\ \forall(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k),\boldsymbol{\theta}(k+1)). \tag{24}$$

Assume that for each index $i$ the vertex set of an allowable space of $(\theta_i,\theta_i^+,\hat{\theta}_i)$ is denoted by $\Omega_{v_i}^u$, illustrated on Fig. 2. This means that the triplet $(\boldsymbol{\theta},\boldsymbol{\theta}^+,\hat{\boldsymbol{\theta}})$ always belongs to the polytope $\Omega^u$ whose vertex set $\Omega_v^u$ is given by $\Omega_v^u = \Omega_{v_1}^u \times \Omega_{v_2}^u \times \cdots \times \Omega_{v_{n_\theta}}^u$. But, as $A_{cl}(\boldsymbol{\theta}(k),\hat{\boldsymbol{\theta}}(k))$ is affine in the couple $(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})$, we can replace (20) and (21) by

$$\begin{bmatrix} \sigma^2 P(\boldsymbol{\theta}) - M^T P(\boldsymbol{\theta}^+)M & (*) \\ P(\boldsymbol{\theta}^+)M - M + T^{-1}A_{cl}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) & 2I - P(\boldsymbol{\theta}^+) \end{bmatrix} > 0, \tag{25}$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall(\boldsymbol{\theta},\boldsymbol{\theta}^+,\hat{\boldsymbol{\theta}}) \in \Omega_v^u,$$

and

$$\begin{bmatrix} \sigma^2 P_T(\boldsymbol{\theta}) - A_{cl}^T(\boldsymbol{\theta},\hat{\boldsymbol{\theta}})P_T(\boldsymbol{\theta}^+)A_{cl}^T(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) & (*) \\ P_T(\boldsymbol{\theta}^+)A_{cl}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) - XA_{cl}(\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) + M_T & 2X - P_T(\boldsymbol{\theta}^+) \end{bmatrix} > 0, \tag{26}$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall(\boldsymbol{\theta},\boldsymbol{\theta}^+,\hat{\boldsymbol{\theta}}) \in \Omega_v^u.$$

So, in the presence of non-negligible uncertainty in the scheduling parameter vector, stabilising DT LPV controller can be designed using similar algorithm as in Subsection 3.2, with $\Omega_v^u$ replacing $\Omega_v$, $(\boldsymbol{\theta},\boldsymbol{\theta}^+,\hat{\boldsymbol{\theta}})$ replacing $(\boldsymbol{\theta},\boldsymbol{\theta}^+)$ and (25) and (26) replacing (20) and (21), respectively.

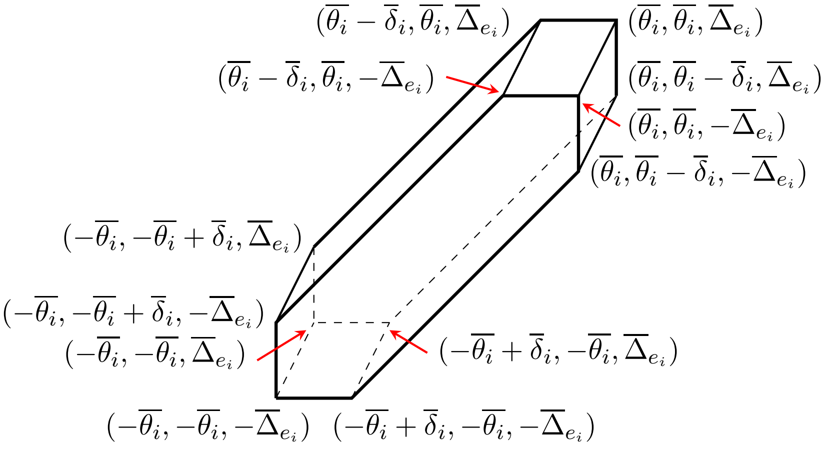


Figure 2. Admissible $(\theta_i, \theta_i^+, \hat{\theta}_i)$ space is a polytope with 12 vertices.

## 4. Induced $l_2$-Norm and $\mathscr{H}_2$ Performance Specifications

While ensuring stability of the controlled system, it is important to optimise some performance indices of the closed-loop system. A widely used performance measure for the LPV control systems is the induced $l_2$-norm, an extension of the $\mathscr{H}_\infty$ norm of LTI systems. In general, it gives a good upper bound on the ratio of "energy" of the performance output and external excitation. The other standard performance measure that will be considered here is $\mathscr{H}_2$ norm. It represents the upper bound on the "energy" of the performance output if the external input is white noise with identity covariance matrix.

### 4.1 *Induced $l_2$-Norm Performance Controller Design*

A formal definition of induced $l_2$-norm performance is given as follows (J De Caigny et al., 2012):

**Definition 1:** *Suppose that the external input $w(k)$ belongs to $l_2$, the set of all discrete-time signals with bounded 2-norm. Then $\gamma$ is an upper bound on the induced $l_2$-norm performance of the LPV system (5) if*

$$\sup_{w \neq 0} \frac{\|z\|_2^2}{\|w\|_2^2} < \gamma \tag{27}$$

*for all allowable scheduling parameter trajectories.*

Induced $l_2$-norm performance of an LTI system can be characterised through the well-known Bounded Real Lemma. Its extension to the LPV system case can be found in the literature (similar to e.g. (C.E. de Souza et al., 2006)):

**Lemma 3:** *$\gamma$ is the upper bound on the induced $l_2$-norm of the LPV system (5) if*

$$P - A_{cl}^T P^+ A_{cl} - \gamma^{-1} C_{cl}^T C_{cl} - (B_{cl}^T P^+ A_{cl} + \gamma^{-1} D_{cl}^T C_{cl})^T \left(I - \gamma^{-1} D_{cl}^T D_{cl} - B_{cl}^T P^+ B_{cl}\right)^{-1} (B_{cl}^T P^+ A_{cl} + \gamma^{-1} D_{cl}^T C_{cl}) > 0 \tag{28}$$

*is satisfied for $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$, where dependence on $\boldsymbol{\theta}$ is omitted, and $P^+ = P(\boldsymbol{\theta}^+)$.*

Our goal is to propose a method for fixed-structure discrete-time LPV controller design, guaranteeing good induced $l_2$-norm performance for a given LPV system. Similarly to the stabilising LPV controller design problem, constraints (28) define a non-convex set in the space of design variables.

The following theorem proposes an inner convex approximation of the non-convex solution set.

**Theorem 2:** *Assume that are given a discrete-time LPV plant affine in scheduling parameter vector* $\boldsymbol{\theta}$ *, bounds on the scheduling parameter vector and its variation as in Preliminaries. Furthermore, suppose that the LPV controller structure is given by (4). Given decoupling matrix* $M$ *and state transformation matrix* $T$*, there exists an LPV controller stabilising the given LPV plant and ensuring the induced* $l_2$*-norm performance to be at most* $\gamma$ *for all admissible scheduling parameter trajectories if*

$$\begin{bmatrix} P(\boldsymbol{\theta}) - M^T P(\boldsymbol{\theta}^+) M & (*) & (*) & (*) \\ P(\boldsymbol{\theta}^+) M - M + T^{-1} A_{cl}(\boldsymbol{\theta}) T & 2I - (\boldsymbol{\theta}^+) & (*) & (*) \\ 0 & B_{cl}^T(\boldsymbol{\theta}) T^{-T} & I & (*) \\ C_{cl}(\boldsymbol{\theta}) T & 0 & D_{cl}(\boldsymbol{\theta}) & \gamma I \end{bmatrix} > 0, \tag{29}$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v.$$

*Proof.* As the expression (29) is affine in the pair $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$, we can conclude that its validity for $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v$ guarantees the validity for $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$ as well. Next, we will prove that validity of (29) for $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$ implies the satisfaction of (28). Consider the non-singular matrix

$$L_{\infty_1}(\boldsymbol{\theta}) = \begin{bmatrix} T^{-T} & T^{-T} M^T - A_{cl}^T(\boldsymbol{\theta}) T^{-T} & 0 & -\gamma^{-1} C_{cl}^T(\boldsymbol{\theta}) \\ 0 & B_{cl}^T(\boldsymbol{\theta}) T^{-T} & -I & \gamma^{-1} D_{cl}^T(\boldsymbol{\theta}) \end{bmatrix}. \tag{30}$$

Pre- and post-multiplication of (29) by $L_{\infty_1}(\boldsymbol{\theta})$ and $L_{\infty_1}^T(\boldsymbol{\theta})$, and then immediate application of Schur complement lemma around the bottom-right block, produces exactly (28) with $P_T = T^{-T} P T^{-1}$ instead of $P$. This guarantees the upper bound $\gamma$ on the induced $l_2$-norm performance for all possible scheduling parameter trajectories. □

To be able to choose $M$ and $T$, we propose a matrix inequality equivalent to (29) in which matrices $M$, $T$ and $P$ are decoupled.

**Lemma 4:** *The matrix inequality*

$$\begin{bmatrix} P_T(\boldsymbol{\theta}) - A_{cl}^T(\boldsymbol{\theta}) P_T(\boldsymbol{\theta}^+) A_{cl}(\boldsymbol{\theta}) & (*) & (*) & (*) \\ P_T(\boldsymbol{\theta}^+) A_{cl}(\boldsymbol{\theta}) - X A_{cl}(\boldsymbol{\theta}) + M_T & 2X - P_T(\boldsymbol{\theta}^+) & (*) & (*) \\ B_{cl}(\boldsymbol{\theta}) M_T - B_{cl}(\boldsymbol{\theta}) X A_{cl}(\boldsymbol{\theta}) & B_{cl}^T(\boldsymbol{\theta}) X & I & (*) \\ C_{cl}(\boldsymbol{\theta}) & 0 & D_{cl}(\boldsymbol{\theta}) & \gamma I \end{bmatrix} > 0 \tag{31}$$

*is equivalent to (29) for* $\forall (\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$.

*Proof.* Observe the matrix

$$L_{\infty_2}(\boldsymbol{\theta}) = \begin{bmatrix} T^{-T} & T^{-T} M^T - A_{cl}^T(\boldsymbol{\theta}) T^{-T} & 0 & 0 \\ 0 & T^{-T} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{32}$$

Pre- and post-multiplication of (29) by $L_{\infty_2}(\boldsymbol{\theta})$ and $L_{\infty_2}^T(\boldsymbol{\theta})$ gives exactly (31). Since the matrix $L(\boldsymbol{\theta})$ is non-singular, these two matrix inequalities are equivalent by the same argument of Lemma 2. □

Now similar algorithm to the one in Section 3 can be developed. Here the initialisation can be performed directly using the previously designed stabilising LPV controller. The optimal cost $\gamma_i$ will be monotonically non-increasing for the reason of equivalence of (31) and (29).

### 4.2 $\mathscr{H}_2$ Performance Controller Design

Similarly to Definition 1, we can give a formal definition of $\mathscr{H}_2$ performance (J De Caigny et al., 2012).

**Definition 2:** *Assume that the white noise with the identity covariance matrix acts as the external input $w(k)$. We say that $\eta$ is an upper bound on the $\mathscr{H}_2$ performance of the LPV system (5) if*

$$\limsup_{T\to\infty} \mathbb{E}\left\{\sum_{k=0}^{T} z^T(k)z(k)\right\} < \eta \tag{33}$$

*for all allowable scheduling parameter trajectories.*

The following representation of the $\mathscr{H}_2$ performance guarantee condition can be found in the literature (similarly to e.g. (K.A. Barbosa et al., 2002)):

**Lemma 5:** *$\eta$ is the upper bound on the $\mathscr{H}_2$ performance of the LPV system (5) if there exist $P(\boldsymbol{\theta})$ and $W(\boldsymbol{\theta})$ such that*

$$\begin{bmatrix} P(\boldsymbol{\theta}^+) - A_{cl}(\boldsymbol{\theta})P(\boldsymbol{\theta})A_{cl}^T(\boldsymbol{\theta}) & B_{cl}(\boldsymbol{\theta}) \\ B_{cl}^T(\boldsymbol{\theta}) & I \end{bmatrix} > 0,$$

$$\begin{bmatrix} W(\boldsymbol{\theta}) & C_{cl}(\boldsymbol{\theta})P(\boldsymbol{\theta}) & D_{cl}(\boldsymbol{\theta}) \\ P(\boldsymbol{\theta})C_{cl}^T(\boldsymbol{\theta}) & P(\boldsymbol{\theta}) & 0 \\ D_{cl}^T(\boldsymbol{\theta}) & 0 & I \end{bmatrix} > 0, \tag{34}$$

$$trace(W(\boldsymbol{\theta})) < \eta$$

*is satisfied for $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$.*

**Remark 7:** *To avoid technical problems, we will assume here that $C_z(\theta) = C_z$ and $D_{zu} = 0$. This leads to matrix $C_{cl}$ not depending on $\theta$ nor the optimisation variables. If these assumptions are not met, but $B_w(\theta) = B_w$ and $D_{yw} = 0$, we could write the other form of (34) in which instead of $C_{cl}$ the matrix $B_{cl}$ multiplies $P$.*

We propose the following LPV controller design conditions based on (34).

**Theorem 3:** *Suppose that the discrete-time LPV plant, which is affine in scheduling parameter vector $\boldsymbol{\theta}$, has bounds on the scheduling parameter vector and its variation as defined in Preliminaries. Furthermore, suppose that the LPV controller structure is given by (4). Given decoupling matrix $M$ and state transformation matrix $T$, there exists an LPV controller stabilising given LPV plant and ensuring the $\mathscr{H}_2$-norm to be at most $\eta$ for all admissible scheduling parameter trajectories if there exist such $P(\boldsymbol{\theta})$ and $W(\boldsymbol{\theta})$ that*

$$\begin{bmatrix} P(\boldsymbol{\theta}^+) - MP(\boldsymbol{\theta})M^T & (*) & (*) \\ P(\boldsymbol{\theta})M^T - M^T + T^T A_{cl}(\boldsymbol{\theta})T^{-T} & 2I - P(\boldsymbol{\theta}) & (*) \\ B_{cl}^T(\boldsymbol{\theta})T^{-T} & 0 & I \end{bmatrix} > 0,$$

$$\begin{bmatrix} W(\boldsymbol{\theta}) & C_{cl}TP(\boldsymbol{\theta}) & D_{cl}(\boldsymbol{\theta}) \\ P(\boldsymbol{\theta})T^TC_{cl}^T & P(\boldsymbol{\theta}) & 0 \\ D_{cl}^T(\boldsymbol{\theta}) & 0 & I \end{bmatrix} > 0, \tag{35}$$

$$trace(W(\boldsymbol{\theta})) < \eta,$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v.$$

*Proof.* From the affineity of (35) in the pair $(\boldsymbol{\theta}, \boldsymbol{\theta}^+)$, we can conclude that (35) is valid for $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$. Next, observe the non-singular matrix

$$L_{2_1}(\boldsymbol{\theta}) = \begin{bmatrix} T & TM - A_{cl}(\boldsymbol{\theta})T & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{36}$$

By pre- and post-multiplication of the first inequality in (35) by $L_{2_1}(\boldsymbol{\theta})$ and $L_{2_1}^T(\boldsymbol{\theta})$ we obtain that

$$\begin{bmatrix} P_T(\boldsymbol{\theta}^+) - A_{cl}(\boldsymbol{\theta})P_T(\boldsymbol{\theta})A_{cl}^T(\boldsymbol{\theta}) & B_{cl}(\boldsymbol{\theta}) \\ B_{cl}^T(\boldsymbol{\theta}) & I \end{bmatrix} > 0, \tag{37}$$

with $P_T = TPT^T$, is satisfied for $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$. Similarly we can define the matrix $L_{2_2}(\boldsymbol{\theta}) = diag(I, T, I)$. Pre- and post-multiplication of the second inequality in (35) by $L_{2_2}(\boldsymbol{\theta})$ and $L_{2_2}^T(\boldsymbol{\theta})$ leads to

$$\begin{bmatrix} W(\boldsymbol{\theta}) & C_{cl}P_T(\boldsymbol{\theta}) & D_{cl}(\boldsymbol{\theta}) \\ P_T(\boldsymbol{\theta})C_{cl}^T & P_T(\boldsymbol{\theta}) & 0 \\ D_{cl}^T(\boldsymbol{\theta}) & 0 & I \end{bmatrix} > 0. \tag{38}$$

Finally, the third inequality of (35) with (37) and (38) ensures that (34) is satisfied $\forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega$. □

The following lemma can be used for the initial choice of $M$ and $T$.

**Lemma 6:** *The system of matrix inequalities*

$$\begin{bmatrix} P_T(\boldsymbol{\theta}^+) - A_{cl}(\boldsymbol{\theta})P_T(\boldsymbol{\theta})A_{cl}^T(\boldsymbol{\theta}) & (*) & (*) \\ P_T(\boldsymbol{\theta})A_{cl}^T(\boldsymbol{\theta}) - XA_{cl}^T(\boldsymbol{\theta}) + M_T^T & 2X - P_T(\boldsymbol{\theta}) & (*) \\ B_{cl}^T(\boldsymbol{\theta}) & 0 & I \end{bmatrix} > 0,$$

$$\begin{bmatrix} W(\boldsymbol{\theta}) & C_{cl}P_T(\boldsymbol{\theta}) & D_{cl}(\boldsymbol{\theta}) \\ P_T(\boldsymbol{\theta})C_{cl}^T & P_T(\boldsymbol{\theta}) & 0 \\ D_{cl}^T(\boldsymbol{\theta}) & 0 & I \end{bmatrix} > 0, \tag{39}$$

$$trace(W(\boldsymbol{\theta})) < \eta,$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall(\boldsymbol{\theta}, \boldsymbol{\theta}^+) \in \Omega_v,$$

*with* $P_T = TPT^T$, $M_T = TMT^T$ *and* $X = TT^T$, *is equivalent to (35).*

*Proof.* By the pre-multiplication of the first inequality in (35) by

$$L_{2_3}(\boldsymbol{\theta}) = \begin{bmatrix} T & TM - A_{cl}(\boldsymbol{\theta})T & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{bmatrix} \tag{40}$$

and post-multiplication by $L_{2_3}^T(\boldsymbol{\theta})$ exactly the first inequality in (39) is obtained. As already mentioned, the second inequality of (39) can be obtained from the second inequality of (35) using $L_{2_2}(\boldsymbol{\theta})$. Now, as both $L_{2_2}(\boldsymbol{\theta})$ and $L_{2_3}(\boldsymbol{\theta})$ are square and invertible, equivalence of (39) and (35) is ensured. □

An algorithm similar to the one in Section 3 and ensuring the monotonically non-increasing behavior of $\eta$ can be used for the iterative controller improvement.

## 5. Simulation results

To illustrate the potential of the proposed method and compare it with the method developed in (F.D. Adegas and J. Stoustrup, 2011), simulation example from (F.D. Adegas and J. Stoustrup, 2011) is used. Plant matrices are given as following:

$$A(\theta) = \begin{bmatrix} 0.7370 & 0.0777 & 0.0810 & 0.0732 \\ 0.2272 & 0.9030 & 0.0282 & 0.1804 \\ -0.0490 & 0.0092 & 0.7111 & -0.2322 \\ -0.1726 & -0.0931 & 0.1442 & 0.7744 \end{bmatrix} + \theta \begin{bmatrix} 0.0819 & 0.0086 & 0.0090 & 0.0081 \\ 0.0252 & 0.1003 & 0.0031 & 0.0200 \\ -0.0055 & 0.0010 & 0.0790 & -0.0258 \\ -0.0192 & -0.0103 & 0.0160 & 0.0860 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{zw} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Bounds on the scheduling parameter are given as $\theta \in [-1, 1]$. Analysis of the system for fixed values of the scheduling parameter shows that the number of unstable poles changes over the interval, as all the poles lie inside the unit circle for $\theta = -1$, but one pole is outside of it for $\theta = 1$.

In (F.D. Adegas and J. Stoustrup, 2011), variation of the scheduling parameter is assumed to belong to the interval $[-0.01, 0.01]$. Here, we assume larger bounds as $\delta \in [-1, 1]$, which means that the scheduling parameter can move over the half of its bounding interval over one sampling period. Control goal defined in (F.D. Adegas and J. Stoustrup, 2011) is to design a fourth order decentralised controller. It is shown that the goal can be achieved after 46 iterations and that the final $4^{th}$ order decentralised controller is obtained with optimal $\gamma$ equal to 4.78.

In this paper, much simpler decentralised static output-feedback controller is designed instead of the $4^{th}$ order decentralised controller. Initial decentralised static output-feedback controllers $K_1^0$

and $K_2^0$ for $\theta = -1$ and $\theta = 1$ are designed using `hinfstruct`. Obtained controllers are

$$K_1^0 = \begin{bmatrix} 0.0101 & 0 \\ 0 & 0.03838 \end{bmatrix},\ K_2^0 = \begin{bmatrix} 1.26 & 0 \\ 0 & 0.4108 \end{bmatrix},$$

with corresponding $H_\infty$ performances of 0.0977 and 1.8392. Note that these values correspond to the square root of the induced-$l_2$ norm performance indicator $\gamma$ used in (F.D. Adegas and J. Stoustrup, 2011) and here, so the comparable value from (F.D. Adegas and J. Stoustrup, 2011) is $\sqrt{4.78} = 2.1863$. Starting from the presented initial controllers, in only two iterations presented algorithm converges to a decentralised static output-feedback LPV controller

$$K(\theta) = \begin{bmatrix} 0.7056 & 0 \\ 0 & 0.3549 \end{bmatrix} + \theta \begin{bmatrix} 0.5549 & 0 \\ 0 & 0.0559 \end{bmatrix}.$$

The controller is designed using SDPT3 (Toh et al., 1999) as a convex optimisation solver, and obtained performance indicator is $\sqrt{\gamma} = 1.8449$.

This means that a better level of performance is reached with simpler controller than the one obtained in (F.D. Adegas and J. Stoustrup, 2011), as well for larger possible variations of the scheduling parameter. Also, obtained level of performance is just marginally worse than the one obtained with the LTI decentralised static output-feedback for the second vertex (1.8392). To further illustrate obtained level of performance and usefulness of fixed-structure controller design, for 51 values of $\theta$ from $[-1, 1]$ optimal full-order output-feedback LTI controllers are designed using `hinfsyn` of Matlab®. The worst-case $H_\infty$ norm obtained for these controllers is 1.6214. Given relatively low loss of performance for the gain of much simpler controller structure (full-order output-feedback vs. decentralised static output-feedback), it may be concluded that given method provides a good alternative control solution.

## 6. Conclusion

In this paper a method for designing fixed-structure dynamic output-feedback Linear Parameter Varying (LPV) controllers for discrete-time LPV systems with bounded scheduling parameter variations is presented. Proposed controller design scheme can iteratively improve induced $l_2$-norm performance of the controlled system. Provided simulation comparison illustrates that good performance can be achieved in a relatively low number of iterations, even for an LPV controller with very limited order and structure.

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