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Reduced basis heterogenenous multiscale methods

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Abstract :

Numerical methods for partial differential equations with multiple scales that combine numerical homogenization methods with reduced order modeling techniques are discussed. These numerical methods can be applied to a variety of problems including multiscale nonlinear elliptic and parabolic problems or Stokes flow in heterogenenous media.

Keywords : homogenization; heterogeneous multiscale method; reduced basis method

1 Introduction

In this paper we discuss the combination of a model reduction algorithm, the reduced basis (RB) method, with a numerical homogenization method such as the heterogenenous multiscale method (HMM). For many applications modeled by partial differential equations (PDEs), numerical homogenization methods or methods based on representative volume elements (RVEs) have proved successful. Yet for problems in high dimensions, nonlinear problems and/or time-dependent problems, the bottleneck of these methods is the large numbers of micro problems or RVE to be solved. Indeed, fully discrete a priori error estimates [1] reveal that the microscopic degrees of freedom must increase proportionally to the macroscopic degrees of freedom (DOF) to guarantee optimal convergence with minimal computational cost. Thus, we face the issue of solving a large number of micro problems with an increasing number of DOF during a macroscopic mesh refinement process. To overcome these difficulties we introduce a reduced order modeling method for the numerical homogenization procedure. The new algorithm called reduced basis finite element heterogeneous multiscale method (RB-FE-HMM) has been introduced in [2, 3] for linear problems, in [4, 5] for nonlinear problems and in [6, 7] for Stokes problems in porous media.

2 Reduced basis numerical homogenization

Let Ω be an open bounded polygonal domain in $\mathbb{R}^d$, V be a Hilbert space and consider the following (multiscale) problem: find $u^\varepsilon \in V$ such that

$$L^\varepsilon(u^\varepsilon, a^\varepsilon) = f \quad \text{in } \Omega, \tag{1}$$

with appropriate boundary conditions. Here L^ε denotes a differential operator, a^ε highly oscillatory data and f a given right-hand side. We assume that u^ε converges (weakly, up to a subsequence) in V to u as $\varepsilon \to 0$, where the function u (a homogenized solution) solves a homogenized problem of the form $L(u, a) = f$ in Ω. The FE-HMM relies on (at least) two solvers and on a data recovery process. A macroscopic solver L_{HMM} for the effective problem L is defined on a macroscopic finite element space $V_H(\Omega)$ based on piecewise polynomials on each element K of the macroscopic triangulation $\mathcal{T}_H$ of Ω *with a priori unknown effective data* $\{a^h(x_{K_j}\}_{j=1}^J, x_{K_j} \in K$ for each $K \in \mathcal{T}_H$. A microscopic solver involving the operator L^ε constrained by the macroscopic state is defined on a microscopic finite element space $V_h(K_{\delta_j})$ (with meshsize h resolving the fine scales) on domains $K_{\delta_j} = x_{K_j} + \delta(-1/2, 1/2)^d$, $j = 1, \ldots, N$, of size $\delta \simeq \varepsilon$ centered at quadrature points $x_{K_j} \in K$. The effective data $a^h(x_{K_j})$ are recovered at these quadrature points by a suitable average of microscopic solutions on K_{δ_j}.

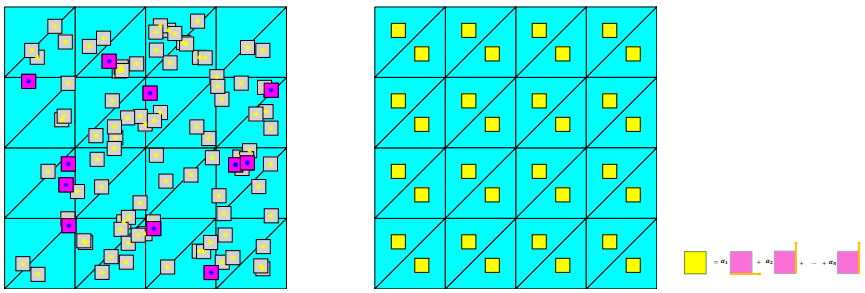

Figure 1: Sketch of offline/online stage of the reduced basis method

Reduced basis for micro solvers. To avoid the computation of the effective data $a^h(x_{K_j})$ at each quadrature point x_{K_j} for each macro element K and to avoid recomputing these data when refining the macro mesh, we adopt an offline online strategy described here for the simple case of linear problems.

- Offline stage: we search for the most representative locations of sampling domains and build in a nested way the space of reduced basis micro solutions $\mathcal{S}_n = span\{\psi_{\tau_k}^{\xi_k}(\cdot),\ k = 1, .., n\}$, where $\psi_{\tau_k}^{\xi_k}(\cdot)$ is an accurate solution of a micro problem in $K_{\tau_k} = \tau_k + \delta(-1/2, 1/2)^d,\ j = 1, \ldots, N$ with force field biven by ξ_k. For a succesfull application of the RB method, the dimension n of the space $\mathcal{S}_n$ must be small. The optimal location $\tau_k \in \Sigma_\Omega$ (Σ_Ω is a uniformly distributed sampling set of Ω) and force field $\xi_k \in \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is selected at each step by a Greedy algorithm controlled by a posteriori error estimators (this is illutrated by the pink sampling domains in the first picture in Figure 2).
- Online stage: we compute a solution of the macro solver by computing the data $a^n(x_{K_j})$ at the correct quadrature points of the macro mesh but in the reduced basis space $\mathcal{S}_n$, which amount in solving only linear systems of small size n (this is illustrated by the second and third pictures in Figure 2).

Similar procedures with possibly more parameters in the offline stage can be used for nonlinear multiscale problems and multiscale Stokes problems.

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