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An Implicit Finite Element Method for the Landau-Lifshitz-Gilbert Equation with Exchange and Magnetostriction

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Contents

Abstract

The Landau-Lifshitz-Gilbert Equation describes the dynamics of ferromagnetism, where strong nonlinearity and non-convexity are hard to tackle. Based on the work of S.Bartels and A.Prohl "*Convergence of an implicit finite element method for the Landau-Lifshitz-Gilbert equation*" ([4]), we present in this report a fully implicit finite element scheme with exchange and magnetostriction. We verify unconditional convergence and present numerical examples

1 Introduction

The Landau-Lifshitz-Gilbert equation with exchange and magnetostriction describes the magnetization of non-linear magneto-elastic materials, such as ferromagnetism; let $\alpha \geq 0$ denote the damping factor and $\Omega_T = \Omega \times (0,T)$, with a Lipschitz boundary continuous Γ, which consists in two non-overlapping parts Γ_D and Γ_N. Then the magnetization $\mathbf{m} : \Omega_T \to \mathbb{S}^2$, where $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 | |\mathbf{x}| = 1\}$, and the displacement $\mathbf{u} : \Omega_T \to \mathbb{R}^3$ solve on Ω_T for all $T > 0$:

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = (1+\alpha^2)(\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{h}_\sigma)) \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0 \end{cases} \tag{1}$$

where $\sigma = \sigma(\mathbf{u}, \mathbf{m})$ is a tensor from magneto-dynamic and $\mathbf{h}_\sigma$ is the magnetostriction vector. The system is supplemented by initial conditions

$$\mathbf{m}(0) = \mathbf{m}_0 \in H^1(\Omega, \mathbb{S}^2), \quad \mathbf{u}(0) = \mathbf{u}_0 \in H^1(\Omega), \quad \mathbf{u}_t(0) = \mathbf{v}_0 \in H^1(\Omega)$$

and boundary conditions

$$\frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma_N.$$

The construction of convergent scheme for this system is a non-trivial task due to the non-convex side-contraint $|\mathbf{m}| = 1$ a.e in $(0,T) \times \Omega$, which is difficult to realize in numerical approximation schemes. The authors in [4] have developed an implicit finite element scheme with this property to solve the Landau-Lifshitz-Equation equation with exchange and they show it was better than the standard explicit scheme. In this work, we extended this algorithm with the magnetostriction: we take the lowest order finite element space $\mathbf{V}_h \subset \mathbf{H}^1(\Omega, \mathbb{R}^3)$ subordinate to a triangulation $\mathcal{T}_h$ of Ω and a time-step size $k > 0$. We note $\mathbf{V}_{h,0} = \{\boldsymbol{\varphi} \in \mathbf{V}_h | \boldsymbol{\varphi} = 0 \text{ on } \Gamma_D\}$. Let $\mathbf{m}_h^0 \in \mathbf{V}_h$, $\mathbf{u}_h^0, \mathbf{v}_h^0 \in \mathbf{V}_{h,0}$. Given a time step $k > 0$, $j \geq 0$ and $\mathbf{m}_h^j \in \mathbf{V}_h$, $\mathbf{u}_h^j, \mathrm{d}_t \mathbf{u}_h^j \in \mathbf{V}_{h,0}$, our approximative scheme reads as follows:

Algorithm 1.1 *1. Determine* $\boldsymbol{m}_h^{j+1} \in \boldsymbol{V}_h$ *from*

$$\begin{aligned}(d_t \boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \quad &+ \quad \alpha(\boldsymbol{m}_h^j \times d_t \boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \\ &= \quad (1+\alpha^2)(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h \bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} + P_{\boldsymbol{V}_h} \bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}}, \boldsymbol{\phi}_h)_h \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h.\end{aligned}$$

2. Determine $\boldsymbol{u}_h^{j+1} \in \boldsymbol{V}_{h,0}$ *from*

$$(d_t^2 \boldsymbol{u}_h^{j+1}, \boldsymbol{\varphi}_h) + (\boldsymbol{\sigma}_h^{j+1}, \nabla(\boldsymbol{\varphi}_h)) = 0 \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_{h,0}.$$

3. *set* $j = j + 1$ *and return to* 1.

Here, $(\cdot\,,\cdot)_h$ denote the reduced integration and the terms $\tilde{\Delta}_h$, $\mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}$ are respectively the discrete implementation of the Laplacian and the L^2 projection of the magnetostriction. We use $\mathrm{d}_t\psi^{j+1} = k^{-1}(\psi^{j+1} - \psi^j)$ for $j \geq 1$, $\mathrm{d}_t^2\psi^{j+1} = k^{-2}(\mathrm{d}_t\psi^{j+1} - \mathrm{d}_t\psi^j)$ for $j \geq 2$ and $\bar{\psi}^{j+\frac{1}{2}} = 1/2(\psi^{j+1} + \psi^j)$ for $j \geq 0$ and a sequence ψ^j. The report is organized as follows. We first present the Landau-Lifshitz equation, some of its properties and preliminaries about finite element space (2). Then we prove that the algorithm (1.1) conserves the length of the magnetization $\mathbf{m}$ and verifies a discrete energy inequality:

$$\begin{aligned}\frac{1}{2}\|\nabla\mathbf{m}_h^N\|_{L^2}^2 \quad &+ \quad \frac{\alpha}{1+\alpha^2}\sum_{j=1}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 k \\ &+ \quad \frac{1}{2}\|\mathrm{d}_t\mathbf{u}_h^N\|_{L^2}^2 + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \\ &\leq \quad C(\boldsymbol{\lambda}^e,\boldsymbol{\lambda}^m) + \frac{1}{2}\|\nabla\mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0)\|_{L^2}^2\end{aligned}$$

where $\boldsymbol{\lambda}^e, \boldsymbol{\lambda}^m$ are symmetric tensors which describe the properties of the material. Our main result is theorem **3.20**, which verifies unconditional convergence for the algorithm (1.1). Fixed-point iteration 3.3 is used to solve the non-linearity, whose convergence is established for $k = \mathcal{O}(h^2)$.

We discuss some numerical examples implemented in FreeFem++. An exact solution problem in one dimension (4.1) allows to look at the convergence of the algorithm. The program is also motivated by possible finite time blow-up behaviour of the solution from (17) and the impact of the magnetostriction on the exchange field, (4.2)

Finally, the last problem (4.3) shows the deformation of a ferromagnetic material under the effect of magnetostriction.

2 The Landau-Lifshitz equation

In solid state physics, the Landau-Lifshitz equation, named after Lev Landau (1908−1968) and Evgeny Lifshitz (1915 – 1985), is a partial differential equation describing time evolution of magnetization in solids. Depending on the time variable and generally two or three space variables, the model first appears for the precessional motion of the magnetization in 1935. We introduce here the Landau-Lifshitz equation with some of its properties, the effect of the magnetostriction and the definition of the problem that will be studied throughout this report.

2.1 Maxwell's equations and magnetization

The electromagnetic field on the macroscopic scale is described by the four Maxwell equations:

$$\begin{aligned} \partial_t \mathbf{D} - \nabla \times \mathbf{H} &= -\sigma \mathbf{E} - \mathbf{J}, \\ \partial_t \mathbf{B} + \nabla \times \mathbf{E} &= \mathbf{0}, \\ < \nabla, \mathbf{B} > &= 0, \\ \nabla \cdot \mathbf{D} &= \rho. \end{aligned}$$

The four space and time dependent vector functions $\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}$ are respectively the electric field, electric displacement, magnetic field and magnetic induction. Moreover, ρ is the scalar charge density function, $\mathbf{J}$ is the current density vector and σ is the conductivity. In vacuum or in free space, relations exist beween the electric and magnetic variables:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}$$

where ε_0, μ_0 are respectively the permittivity and magnetic permeability. For linear material, we have the following correspondances:

$$\mathbf{D} = \boldsymbol{\varepsilon} \mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu} \mathbf{H}$$

where $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ are 3×3 dependent tensor. In this work, we are interesting in non-linear materials, such as ferromagnets. In this case, the precedent relation is usually a bit more complicated:

$$\mathbf{D} = \boldsymbol{\varepsilon}_0 \mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu}_0 (\mathbf{H} + \mathbf{m}),$$

where $\mathbf{m}$ is the magnetization of the material, governed by the Landau-Lifshitz equation below.

2.2 The Landau-Lifshitz equation

We search the magnetization $\mathbf{m} : (0,T) \times \Omega \to \mathbb{R}^3$ such that

$$\mathbf{m}_t = \mathbf{m} \times \mathbf{h}_T - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T) \tag{2}$$

with the initial condition $\mathbf{m}(0) = \mathbf{m}_0 \in \mathbf{H}^1(\Omega; S^2)$ and $\partial_{\mathbf{n}} \mathbf{m} = 0$ on $(0,T) \times \partial\Omega$. The term

$$\mathbf{m} \times \mathbf{h}_T$$

is the precession part and

$$\alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T)$$

is the damping part, where $\alpha \geq 0$ denote the damping parameter. The vector $\mathbf{h}_T$ is the total effective field and consist of

$$\mathbf{h}_T = \mathbf{h} + \mathbf{h}_{anis} + \mathbf{h}_{exch} + \mathbf{h}_\sigma,$$

where

1. $\mathbf{h}$ is the magnetic field which comes from Maxwell's equation and is supposing to be known;
2. Magnetic properties depend of the direction. This is given by the anisotropy field $\mathbf{h}_{anis}$ which takes the form

   $$\mathbf{h}_{anis} = K < \mathbf{p}, \mathbf{m} > \mathbf{p},$$

   where $\mathbf{p}$ is a direction of the uniaxial anisotropy and K a constant that depends of that direction;
3. $\mathbf{h}_{exch} = \Delta \mathbf{m}$ is the exchange field which is due to the interactions between the magnetic spins;
4. $\mathbf{h}_\sigma$ is the magnetostriction that will be described later.

A basic model to understand the effect of the effective and damping part is to consider one constant effective field $\mathbf{h}$ and the following Landau-Lifshitz equation:

$$\mathbf{m}_t = \mathbf{m} \times \mathbf{h}. \tag{3}$$

This is the precession part and the solution $\mathbf{m}$ of (3) rotates around $\mathbf{h}$ and the angle between $\mathbf{m}$ and $\mathbf{h}$ remains contant. If we add a damping part, the equation is written

$$\mathbf{m}_t = \mathbf{m} \times \mathbf{h} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}). \tag{4}$$

The magnetization $\mathbf{m}$ of (4) rotates around $\mathbf{h}$ until their mutual angle becomes zero, which means that the magnetization becomes parralel to $\mathbf{h}$, the steady state.

2.3 Conservation of magnitude

In the Landau-Lifshitz Equation, the magnitude of the magnetization $\mathbf{m}$ is conserved. Indeed, a scalar multiplication of (2) and using the relation $< a \times b, a >= 0$ gives

$$< \partial_t \mathbf{m}, \mathbf{m} >= \frac{1}{2} \partial_t |\mathbf{m}|^2 = 0.$$

Consequently, $|\mathbf{m}| = C$. This is an important conservation property of the Landau-Lifshitz Equation. In general, the magnetization will be then usually been in $(0,T) \times \Omega \to \mathbb{S}^2$, where

$$\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 | |\mathbf{x}| = 1\}.$$

2.4 Gilbert form of the Landau-Lifshit equation

The construction of the numerical scheme of this report is based on a reformulation of (2) by T.L.Gilbert. Taking the cross product of (2) with $\mathbf{m}$, we obtain

$$\mathbf{m}_t \times \mathbf{m} = [-\alpha\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T)] \times \mathbf{m} + (\mathbf{m} \times \mathbf{h}_T) \times \mathbf{m},$$

Using that $|\mathbf{m}|^2 = 1$ and the proposition (6.10) we have

$$(\mathbf{m} \times \mathbf{h}_T) \times \mathbf{m} =< \mathbf{m}, \mathbf{m} > \mathbf{h}_T - < \mathbf{m}, \mathbf{h}_T > \mathbf{m} = \mathbf{h}_T - < \mathbf{m}, \mathbf{h}_T > \mathbf{m}.$$

Consequently

$$\mathbf{m}_t = \mathbf{m} \times \mathbf{h}_T + \alpha\mathbf{h}_T - \alpha < \mathbf{m}, \mathbf{h}_T > \mathbf{m},$$

and taking again the cross product with $\mathbf{m}$, the last equation reduces to

$$\mathbf{m}_t \times \mathbf{m} = (\mathbf{m} \times \mathbf{h}_T) \times \mathbf{m} + \alpha\mathbf{h}_T \times \mathbf{m}.$$

We finally have

$$\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T) = -\mathbf{m}_t \times \mathbf{m} - \alpha\mathbf{m} \times \mathbf{h}_T,$$

and inserting this term in the equation (2) yields

$$\begin{aligned} \mathbf{m}_t &= \mathbf{m} \times \mathbf{h}_T - \alpha\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_T) \\ &= \mathbf{m} \times \mathbf{h}_T - \alpha(-\mathbf{m}_t \times \mathbf{m} - \alpha\mathbf{m} \times \mathbf{h}_T) \\ &= \alpha\mathbf{m}_t \times \mathbf{m} + (1 + \alpha^2)\mathbf{m} \times \mathbf{h}_T. \end{aligned}$$

This last assertation gives the following version of equation (2), known as the Landau-Lifshitz-Gilbert equation (written LLG equation) :

$$\mathbf{m}_t + \alpha\mathbf{m} \times \mathbf{m}_t = (1 + \alpha^2)\mathbf{m} \times \mathbf{h}_T. \tag{5}$$

This form will be used throughout the report.

2.5 Magnetostriction

Magnetostriction is a property of ferromagnetic materials that cause them to change their shape or dimension during the process of magnetization. To include magnetostriction effects in the LLG equation, we need some expression of magneto-mechanical energy. We define the stress tensor by

$$\boldsymbol{\sigma} = \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^e(\mathbf{u}, \mathbf{m}), \quad \boldsymbol{\sigma}_{ij} = \sum_{kl} \boldsymbol{\lambda}^e_{ijkl} \boldsymbol{\varepsilon}^e_{kl}, \tag{6}$$

where

$$\boldsymbol{\varepsilon}^e(\mathbf{u}, \mathbf{m}) = \boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}^m(\mathbf{m}).$$

The vector $\mathbf{u}$ is the displacement and the total strain tensor $\boldsymbol{\varepsilon}$ is written

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right).$$

The magnetic part of the total strain is given by

$$\boldsymbol{\varepsilon}^m(\mathbf{m}) = \boldsymbol{\lambda}^m \mathbf{m}\mathbf{m}^T.$$

The tensor $\boldsymbol{\lambda}^e$ and $\boldsymbol{\lambda}^m$, which describes the properties of the materials, are supposed to be symmetric:

$$\boldsymbol{\lambda}_{ijkl} = \boldsymbol{\lambda}_{jikl} = \boldsymbol{\lambda}_{ijlk} = \boldsymbol{\lambda}_{klij}$$

and positive definite

$$\sum_{ijkl} \boldsymbol{\lambda}_{ijkl} \boldsymbol{\xi}_{ij} \boldsymbol{\xi}_{kl} \geq \boldsymbol{\lambda}^* \sum_{ij} \boldsymbol{\xi}_{ij}^2,$$

with bounded elements. We can use all the precedent relation to write the contribution of the magnetostriction $\mathbf{h}_{\boldsymbol{\sigma}}$ to the effective field in the LLG equation. The magnetostrictive field $\mathbf{h}_{\boldsymbol{\sigma}}$ has the form

$$\mathbf{h}_{\boldsymbol{\sigma}} = \boldsymbol{\lambda}^m \boldsymbol{\sigma} \mathbf{m}.$$

which means for each s-component $s = 1, 2, 3$:

$$\mathbf{h}_{\boldsymbol{\sigma},s} = \sum_{klr} \lambda^m_{klrs} \sigma_{kl} m_r.$$

The stress tensor $\boldsymbol{\sigma}$ and displacement vector $\mathbf{u}$ obey the equation of elastodynamics:

$$\varrho \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \quad \text{in} \quad \Omega \times (0, T),$$

where ϱ is the constant mass density independent of the deformation. The initial conditions are:

$$\begin{aligned} \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega, \\ \mathbf{u}_t(0, \mathbf{x}) &= \mathbf{v}_0(\mathbf{x}) \text{ in } \Omega, \end{aligned} \tag{7}$$

and the boundary conditions:

$$\begin{aligned} \mathbf{u} = \mathbf{0} \quad &\text{on} \quad \Gamma_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \mathbf{0} \quad &\text{on} \quad \Gamma_N. \end{aligned} \tag{8}$$

2.6 LLG equation with exchange and magnetostriction

The coupled problem is then written: let $\Omega_T = \Omega \times (0, T)$, with a Lipschitz boundary continuous Γ, which consists in two non-overlapping parts Γ_D and Γ_N. For any $T > 0$, we search for $\mathbf{m} : \Omega_T \to \mathbb{S}^2$ and $\mathbf{u} : \Omega_T \to \mathbb{R}^3$ such that

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = (1 + \alpha^2)(\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{h}_\sigma)) \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0 \end{cases} \tag{9}$$

with the initial conditions:

$$\begin{aligned} \mathbf{m}(0, \mathbf{x}) &= \mathbf{m}_0(\mathbf{x}) \text{ in } \Omega, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega, \\ \mathbf{u}_t(0, \mathbf{x}) &= \mathbf{v}_0(\mathbf{x}) \text{ in } \Omega, \end{aligned} \tag{10}$$

and the boundary conditions:

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \quad &\text{on} \quad \partial\Omega, \\ \mathbf{u} = \mathbf{0} \quad &\text{on} \quad \Gamma_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \mathbf{0} \quad &\text{on} \quad \Gamma_N. \end{aligned} \tag{11}$$

We choose $\mathbf{V} = \mathbf{H}^1(\Omega_T, \mathbb{R}^3)$ and

$$\mathbf{V}_0 = \{\boldsymbol{\varphi} \in \mathbf{V}, \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma_D\}.$$

Using the proposition (6.15), the LLG equation from (17) can be written:

$$\mathbf{m}_t + \alpha\mathbf{m} \times \mathbf{m}_t = (1+\alpha^2)(\nabla \cdot (\mathbf{m} \times \nabla\mathbf{m}) + \mathbf{m} \times \mathbf{h}_\sigma).$$

We multiply then by a function $\boldsymbol{\phi} \in C^\infty(\Omega_T; \mathbb{R}^3)$ and we integrate over Ω_T:

$$\begin{aligned} & \int_{\Omega_T} < \mathbf{m}_t, \boldsymbol{\phi} > d\mathbf{x}dt + \alpha \int_{\Omega_T} < \mathbf{m} \times \mathbf{m}_t, \boldsymbol{\phi} > d\mathbf{x}dt \\ = \; & (1+\alpha^2) \int_{\Omega_T} < \nabla \cdot (\mathbf{m} \times \nabla\mathbf{m}) + \mathbf{m} \times \mathbf{h}_\sigma, \boldsymbol{\phi} > d\mathbf{x}dt. \end{aligned}$$

Using the integration by part for the term on the right, we obtain the weak formulation for the LLG equation:

$$\begin{aligned} & \int_{\Omega_T} < \mathbf{m}_t, \boldsymbol{\phi} > d\mathbf{x}dt + \alpha \int_{\Omega_T} < \mathbf{m} \times \mathbf{m}_t, \boldsymbol{\phi} > d\mathbf{x}dt \\ = \; & -(1+\alpha^2) \int_{\Omega_T} < \mathbf{m} \times \nabla\mathbf{m}, \nabla\boldsymbol{\phi} > d\mathbf{x}dt \\ - \; & (1+\alpha^2) \int_{\Omega_T} < \mathbf{m} \times \mathbf{h}_\sigma, \boldsymbol{\phi} > d\mathbf{x}dt \quad \forall \boldsymbol{\phi} \in C^\infty(\Omega_T; \mathbb{R}^3). \end{aligned}$$

For the mechanical part, we multiply the elastodynamics equation by a vector field in the space

$$C_T^\infty(\Omega_T; \mathbb{R}^3) = \{\boldsymbol{\varphi} \in C^\infty(\Omega_T; \mathbb{R}^3) | \boldsymbol{\varphi}(T) = 0\}.$$

We will see that this condition is necessary to pass the limit for the convergence of the algorithm. Then we integrate over Ω to obtain:

$$(\mathbf{u}_{tt}, \boldsymbol{\varphi}) = (\nabla \cdot \boldsymbol{\sigma}, \boldsymbol{\varphi}).$$

Integrated the left term by part and using the boundary conditions (11) we obtain

$$(\mathbf{u}_{tt}, \boldsymbol{\varphi}) = -(\boldsymbol{\sigma}, \nabla\boldsymbol{\varphi}).$$

We develop then, using the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$:

$$\begin{aligned} < \boldsymbol{\sigma}, \nabla\boldsymbol{\varphi} > &= -\sum_{ij} < \sigma_{ij}, \frac{\partial\varphi_j}{\partial x_i} > \\ &= -\frac{1}{4}\left(\sum_{ij} < \sigma_{ij} + \sigma_{ji}, \frac{\partial\varphi_j}{\partial x_i} > + \sum_{ij} < \sigma_{ji} + \sigma_{ij}, \frac{\partial\varphi_i}{\partial x_j} >\right) \\ &= -\frac{1}{2}\sum_{ij} < \sigma_{ij} + \sigma_{ji}, \varepsilon_{ij}(\boldsymbol{\varphi}) > \\ &= -\frac{1}{2}\sum_{ij}\sum_{kl} < (\lambda^e_{ijkl} + \lambda^e_{jikl})\varepsilon^e_{kl}, \varepsilon_{ij}(\boldsymbol{\varphi}) > \\ &= -\sum_{ij}\sum_{kl} < \lambda^e_{ijkl}(\varepsilon_{kl} - \varepsilon^m_{kl}), \varepsilon_{ij}(\boldsymbol{\varphi}) > \\ &= < \boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}^m - \boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > . \end{aligned}$$

We integrate over Ω_T the precedent relation:

$$\int_{\Omega_T} < \mathbf{u}_{tt}, \boldsymbol{\varphi} > d\mathbf{x}dt \quad + \quad \int_{\Omega_T} < \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > d\mathbf{x}dt$$
$$= \int_{\Omega_T} < \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > d\mathbf{x}dt \quad \forall \boldsymbol{\varphi} \in C_T^\infty(\Omega_T; \mathbb{R}^3).$$

Using the integration by part, we obtain the weak formulation of the elastodynamics part:

$$-\int_{\Omega_T} < \mathbf{u}_t, \boldsymbol{\varphi}_t > d\mathbf{x}dt \quad + \quad \int_{\Omega_T} < \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > d\mathbf{x}dt$$
$$= \int_{\Omega_T} < \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > d\mathbf{x}dt$$
$$+ \int_{\Omega} < \mathbf{u}_t(0), \boldsymbol{\varphi}_t(0) > d\mathbf{x} \quad \forall \boldsymbol{\varphi} \in C_T^\infty(\Omega_T; \mathbb{R}^3).$$

The variational formulation of the problem (17) motivates the following definition:

Definition 2.1 (Weak Solution of the LLG equation with magnetostriction)
Let $\boldsymbol{m}_0 \in \boldsymbol{V}$ *such that* $|\boldsymbol{m}_0| = 1$, $\boldsymbol{u}_0 \in \boldsymbol{V}_0$ *and* $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega_T)$ *. A couple of functions* $(\boldsymbol{m}, \boldsymbol{u})$ *is called a weak solution of the LLG equation with magnetostriction if for all* $T > 0$ *it verifies:*

1. $\boldsymbol{m} \in \boldsymbol{V}, \boldsymbol{u} \in \boldsymbol{V}_0$ *and* $|\boldsymbol{m}| = 1$ *a.e in* Ω_T*;*

2. *for all* $\boldsymbol{\phi} \in C^\infty(\Omega_T; \mathbb{R}^3)$, $\boldsymbol{\varphi} \in C_T^\infty(\Omega_T; \mathbb{R}^3)$ *there holds*

$$\int_{\Omega_T} < \boldsymbol{m}_t, \boldsymbol{\phi} > d\boldsymbol{x}dt + \alpha \int_{\Omega_T} < \boldsymbol{m} \times \boldsymbol{m}_t, \boldsymbol{\phi} > d\boldsymbol{x}dt$$
$$= -(1+\alpha^2) \int_{\Omega_T} < \boldsymbol{m} \times \nabla \boldsymbol{m}, \nabla \boldsymbol{\phi} > d\boldsymbol{x}dt$$
$$- \quad (1+\alpha^2) \int_{\Omega_T} < \boldsymbol{m} \times \boldsymbol{h}_\sigma, \boldsymbol{\phi} > d\boldsymbol{x}dt$$

and

$$-\int_{\Omega_T} < \boldsymbol{u}_t, \boldsymbol{\varphi}_t > d\boldsymbol{x}dt \quad + \quad \int_{\Omega_T} < \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > d\boldsymbol{x}dt$$
$$= \int_{\Omega_T} < \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\boldsymbol{m}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) > d\boldsymbol{x}dt$$
$$+ \int_{\Omega} < \boldsymbol{u}_t(0), \boldsymbol{\varphi}_t(0) > d\boldsymbol{x};$$

3. $\boldsymbol{m}(0) = \boldsymbol{m}_0$ *in the sense of the trace;*

4. *for almost all* $T' \in (0, T)$ *we have the following inegality of energy:*

$$\frac{1}{2}\int_{\Omega} |\nabla \boldsymbol{m}(T')|^2 d\boldsymbol{x} \quad + \quad \frac{1}{2}\int_{\Omega} |\boldsymbol{\varepsilon}(\boldsymbol{u}(T'))|^2 d\boldsymbol{x} + \frac{\alpha}{1+\alpha^2} \int_{\Omega_T} |\boldsymbol{m}_t|^2 d\boldsymbol{x}dt$$
$$+ \quad \frac{1}{2}\int_{\Omega} |\boldsymbol{u}_t(T')|^2 d\boldsymbol{x} \le C(\boldsymbol{m}_0, \boldsymbol{u}_0, \boldsymbol{v}_0).$$

Remark 2.2
The theorem 3.20 below will prove the existence of a couple of weak solution $(\boldsymbol{m},\boldsymbol{u})$ *that solve the LLG equation with exchange and magnetrostriction* (17)*.*

3 An implicit method with magnetostriction

From the difficult task of the conservation of magnitude and the hard non-linearity in the LLG equation, a lot of different numerical methods were developed during years. To preserve the length of the magnetization, projection methods are currently used, with explicit Euler approximation in time ([1], [9], [7]). In [5] is presented an explicit method with energy inequality, good for small two dimensional problems, but which fails to be useful for more complicated simulation due to very hard time-step restriction.

In [4] a norm conservative implicit finite element scheme is presented for the LLG equation with exchange field. The method from this paper is better and more stable than the explicit scheme of [5]. The authors of [4] motive also the use of reduced integration (mass-lumped integral) in the numerical scheme, that allows better stability of the algorithm when the damping factor $\alpha \to 0$. The goal of this chapter is to extend the algorithm developed in [4] to the case of the LLG equation with the exchange and magnetostriction fields.

3.1 Preliminaries

In this section we consider a regular and quasi-uniform triangulation $\mathcal{T}_h$. We define the lowest order finite element space $\mathbf{V}_h \subset \mathbf{H}^1(\Omega;\mathbb{R}^3)$ by:

$$\mathbf{V}_h = \{\boldsymbol{\phi}_h \in C(\bar{\Omega};\mathbb{R}^3) : \boldsymbol{\phi}_{h|K} \in \mathcal{P}_1(K;\mathbb{R}^3) \quad \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}_1(K;\mathbb{R}^3)$ is the space of polynome of degree less or equal to one. We consider moroever in addition

$$\mathbf{V}_{h,0} = \{\boldsymbol{\varphi}_h \in \mathbf{V}_h; \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma_D\}.$$

Definition 3.1 (Interpolation Operator)
For any $l \in L$, we denote by $\boldsymbol{\varphi}_l \in C(\bar{\Omega})$ the basis function which are affine and satisfies $\boldsymbol{\varphi}_l(\boldsymbol{x}_m) = 1$ if $m = l$ and 0 otherwise. Given the set of node of the triangulation $\{\boldsymbol{x}_l : l \in L\}$., we note the local interpolate operator $\mathcal{I}_h : C(\bar{\Omega};\mathbb{R}^3) \to \boldsymbol{V}_h$, such that $\mathcal{I}_h(\boldsymbol{\phi}_h(\boldsymbol{x}_l) = \boldsymbol{\phi}_h(\boldsymbol{x}_l)$. This can be written:

$$\mathcal{I}_h(\boldsymbol{\phi}) = \sum_{l\in L} \boldsymbol{\phi}(\boldsymbol{x}_l)\boldsymbol{\varphi}_l.$$

Definition 3.2 (Discrete inner product of functions)
We define the discrete inner product $(\cdot,\cdot)_h : \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \times \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \to \boldsymbol{V}_h$, by

$$\begin{aligned}(\boldsymbol{\phi},\boldsymbol{\xi})_h &= \int_\Omega \mathcal{I}_h(<\boldsymbol{\phi},\boldsymbol{\xi}>) = \int_\Omega \sum_{j,l\in L} <\boldsymbol{\phi}(\boldsymbol{x}_l),\boldsymbol{\xi}(\boldsymbol{x}_j)> \mathcal{I}_h(\boldsymbol{\varphi}_l\boldsymbol{\varphi}_j)\\ &= \int_\Omega \sum_{j,l\in L} <\boldsymbol{\phi}(\boldsymbol{x}_l),\boldsymbol{\xi}(\boldsymbol{x}_j)> \delta_{jl}\boldsymbol{\varphi}_j\\ &= \sum_{l\in L} B_l <\boldsymbol{\phi}(\boldsymbol{x}_l),\boldsymbol{\xi}(\boldsymbol{x}_l)>,\end{aligned}$$

with $\beta_l = \int_\Omega \boldsymbol{\varphi}_l dx$.

Definition 3.3 (Discrete Laplacian)
The discrete version of the Laplacian $\tilde{\Delta}_h : \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \to \boldsymbol{V}_h$ is given by

$$(\tilde{\Delta}_h\boldsymbol{\phi},\boldsymbol{\varphi}_h)_h = -(\nabla\boldsymbol{\phi},\nabla\boldsymbol{\varphi}_h) \quad \forall\boldsymbol{\varphi}_h \in \boldsymbol{V}_h. \tag{12}$$

Definition 3.4 (L^2 projection on $\mathbf{V}_h$)
We define the L^2-projector $P_{\boldsymbol{V}_h} : L^2(\Omega_T) \to \boldsymbol{V}_h$ by

$$(P_{\boldsymbol{V}_h}\boldsymbol{u}, \boldsymbol{\phi}_h)_h = (\boldsymbol{u}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h.$$

The next standard results are useful to show the convergence of the algorithm:

Proposition 3.5
For all $\boldsymbol{\phi} \in \boldsymbol{V}$, there holds

$$\|(Id - \mathcal{I}_h)\boldsymbol{\phi}\|_{L^2} \leq C \left(\sum_{K \in \mathcal{T}_h} h^2 \|\boldsymbol{\phi}\|^2_{L^2(K)} \right)^{\frac{1}{2}}.$$

Proposition 3.6
For a regular and quasi-uniform triangulation $\mathcal{T}_h$, there exists a constant c_1 such that for all $\boldsymbol{\phi}_h \in V_h$ the following estimation hold:

$$\|\nabla \boldsymbol{\phi}_h\|_{L^2} \leq c_1 h^{-1} \|\boldsymbol{\phi}_h\|_{L^2},$$

where h is the maximal mesh side, $h = \max\{diam(K) : K \in \mathcal{T}_h\}$.

Proposition 3.7 (Equivalence of norms)
For all $\boldsymbol{\phi}_h \in V_h$,

$$\|\boldsymbol{\phi}_h\|^2_{L^2} \leq \|\boldsymbol{\phi}_h\|^2_h \leq 5\|\boldsymbol{\phi}_h\|^2_{L^2}.$$

Proposition 3.8
For all $\boldsymbol{\phi}_h \in V_h$,

$$\|\tilde{\Delta}_h \boldsymbol{\phi}_h\|_h \leq c_1 h^{-1} \|\nabla \boldsymbol{\phi}_h\|_{L^2}.$$

Proof. We choose $\boldsymbol{\varphi}_h = \tilde{\Delta}_h \boldsymbol{\phi}_h$ in (12). Using then the Holder inequality, the proposition (3.6) and (3.7), we obtain:

$$\begin{aligned} \|\tilde{\Delta}_h \boldsymbol{\phi}_h\|^2_h &= -(\nabla \boldsymbol{\phi}_h, \nabla \tilde{\Delta}_h \boldsymbol{\phi}_h) \\ &\leq \|\nabla \boldsymbol{\phi}_h\|_{L^2} \|\nabla \tilde{\Delta}_h \boldsymbol{\phi}_h)\|_{L^2} \\ &\leq c_1 h^{-1} \|\nabla \boldsymbol{\phi}_h\|_{L^2} \|\tilde{\Delta}_h \boldsymbol{\phi}_h\|_h. \end{aligned}$$

□

Proposition 3.9
For all $\boldsymbol{\phi}_h \in V_h$,

$$|\tilde{\Delta}_h \boldsymbol{\phi}_h(\boldsymbol{x}_l)| \leq c_2 h^{-2} \|\boldsymbol{\phi}_h\|_{L^\infty} \quad \forall l \in L.$$

Proof. From the definition of the discrete inner product, we have

$$(\tilde{\Delta}_h \boldsymbol{\phi}_h, \varphi_l \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l))_h = \sum_{k \in L} \beta_l < \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_k), \varphi_l(\mathbf{x}_k) \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l) >= \beta_l |\tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l)|^2.$$

Consequently,

$$\begin{aligned} |\tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l)|^2 &= \beta_l^{-1} < \tilde{\Delta}_h \boldsymbol{\phi}_h, \varphi_l(\mathbf{x}_k) \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l) > \\ &= -\beta_l^{-1} < \nabla \boldsymbol{\phi}_h, \nabla(\varphi_l \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l)) > \\ &= -\beta_l^{-1} \sum_{m \in L \exists K \in \mathcal{T}_h, \mathbf{x}_m, \mathbf{x}_l \in K} < \boldsymbol{\phi}_h(\mathbf{x}_m), \tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l) > (\nabla \varphi_m, \nabla \varphi_l) \\ &\leq \beta_l^{-1} |\tilde{\Delta}_h \boldsymbol{\phi}_h(\mathbf{x}_l)| \sum_{m \in L \exists K \in \mathcal{T}_h, \mathbf{x}_m, \mathbf{x}_l \in K} |\boldsymbol{\phi}_h(\mathbf{x}_m)| (\nabla \varphi_m, \nabla \varphi_l). \end{aligned}$$

On the other hand, $\|\varphi_m\|_{L^2} \le c\beta_l^{\frac{1}{2}}$ for all $m \in L$, which gives:

$$(\nabla\varphi_m, \nabla\varphi_l) \le \|\nabla\varphi_m\|_{L^2}\|\nabla\varphi_l\|_{L^2} \le ch^{-2}\beta_l.$$

Finally, using that the cardinality of the set $\{m \in L : \exists K \in \mathcal{T}_h, \mathbf{x}_m, \mathbf{x}_l \in \mathcal{T}_h\}$ is bounded h independently, we can conclude that

$$|\tilde{\Delta}_h\boldsymbol{\phi}_h(\mathbf{x}_l)|^2 \le Ch^{-2}\|\boldsymbol{\phi}_h\|_\infty|\tilde{\Delta}_h\boldsymbol{\phi}_h(\mathbf{x}_l)|.$$

□

Definition 3.10
For any $T' > 0$, the time is divided $[0, T']$ into N equidistant subintervals $[t_j, t_{j+1}]$ with $t_j = jk$, $i = 0, \dots, N-1$ and with the time step $k = \frac{T'}{N}$. For an approximative solution $\boldsymbol{f}_h$, we denote

$$\boldsymbol{f}_h^{j+1} = \boldsymbol{f}_h(t_{j+1}), \quad d_t\boldsymbol{f}_h^{j+1} = \frac{\boldsymbol{f}_h^{j+1} - \boldsymbol{f}_h^j}{k}, \quad d_t^2\boldsymbol{f}_h^{j+1} = \frac{d_t\boldsymbol{f}_h^{j+1} - d_t\boldsymbol{f}_h^j}{k}.$$

Algorithm 3.11
Let $\boldsymbol{m}_h^0 \in \boldsymbol{V}_h, \boldsymbol{u}_h^0, \boldsymbol{v}_h^0 \in \boldsymbol{V}_{h,0}$. Given a time step $k > 0$, $j \ge 0$, $\boldsymbol{m}_h^j \in \boldsymbol{V}_h, \boldsymbol{u}_h^j, d_t\boldsymbol{u}_h^j \in \boldsymbol{V}_{h,0}$.

1. *Determine $\boldsymbol{m}_h^{j+1} \in \boldsymbol{V}_h$ from*

$$\begin{aligned}(d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \quad &+ \quad \alpha(\boldsymbol{m}_h^j \times d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \\ &= \quad (1+\alpha^2)(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} + P_{\boldsymbol{V}_h}\bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}}, \boldsymbol{\phi}_h)_h \quad \forall\boldsymbol{\phi}_h \in \boldsymbol{V}_h,\end{aligned}$$

with

$$\bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}} = \boldsymbol{\lambda}^m\boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^j) - \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^j))\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}};$$

2. *Determine $\boldsymbol{u}_h^{j+1} \in \boldsymbol{V}_{h,0}$ from*

$$(d_t^2\boldsymbol{u}_h^{j+1}, \boldsymbol{\varphi}_h) + (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) \quad \forall\boldsymbol{\varphi}_h \in \boldsymbol{V}_{h,0}; \tag{13}$$

3. *set $j = j+1$ and return to 1.*

Remark 3.12
The second term in the LLG equation is motivated by the relation

$$\begin{aligned}(\boldsymbol{m}_h^j \times d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi})_h \quad &= \quad \left(\left(\bar{\boldsymbol{m}}^{j+\frac{1}{2}} - \frac{k}{2}d_t\boldsymbol{m}_h^{j+1}\right) \times d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}\right)_h \\ &= \quad (\bar{\boldsymbol{m}}^{j+\frac{1}{2}} \times d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi})_h.\end{aligned}$$

3.2 Convergence of the Algorithm

Proposition 3.13
Suppose that $|\boldsymbol{m}_h^0(\boldsymbol{x}_l)| = 1$ for all $l \in L$. Then the sequences $(\boldsymbol{m}_h^j, \boldsymbol{u}_h^j)_{j\ge0}$ obtained from algorithm (22) satisfies for all $j \ge 0$:

1. $|\boldsymbol{m}_h^{j+1}(\boldsymbol{x}_l)| = 1 \quad \forall l \in L;$

2.
$$\frac{1}{2}d_t\|\nabla \boldsymbol{m}_h^{j+1}\|_{L^2}^2 + \frac{\alpha}{1+\alpha^2}\|d_t\boldsymbol{m}_h^{j+1}\|_h^2 = (d_t\boldsymbol{m}_h^{j+1}, \bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}}).$$

Proof. For the proof of the first assertion, we choose $\boldsymbol{\phi}_h = \varphi_t\bar{\mathbf{m}}_h^{j+\frac{1}{2}}(\mathbf{x}_l) \in \mathbf{V}_h$ in the LLG equation, for all $l \in L$. Using then the relation (3.12) :

$$\begin{aligned} 0 &= (d_t\mathbf{m}_h^{j+1}, \varphi_t\bar{\mathbf{m}}_h^{j+\frac{1}{2}}(\mathbf{x}_l))_h = \\ &= \frac{1}{2k}\left(\mathbf{m}_h^{j+1} - \mathbf{m}_h^j, \varphi_t\mathbf{m}_h^{j+1}(\mathbf{x}_l) + \varphi_t\mathbf{m}_h^j(\mathbf{x}_l)\right)_h \\ &= \frac{\beta_l}{2k} < \mathbf{m}_h^{j+1}(\mathbf{x}_l) - \mathbf{m}_h^j(\mathbf{x}_l), \mathbf{m}_h^{j+1}(\mathbf{x}_l) + \mathbf{m}_h^j(\mathbf{x}_l) > \\ &= \frac{\beta_l}{2k}\left(|\mathbf{m}_h^{j+1}(\mathbf{x}_l)|^2 - |\mathbf{m}_h^j(\mathbf{x}_l)|^2\right), \end{aligned}$$

and thus $|\mathbf{m}_h^{j+1}(\mathbf{x}_l)|^2 = |\mathbf{m}_h^j(\mathbf{x}_l)|^2 = 1$.
In order to verify the second assertion, we first choose

$$\boldsymbol{\phi}_h = -\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} - \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}} \in \mathbf{V}_h$$

in the LLG part of the algorithm (3.11). We obtain with (3.12) and the definition of $-\tilde{\Delta}_h$:

$$\begin{aligned} &(d_t\mathbf{m}_h^{j+1}, -\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} - \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h + \alpha(\mathbf{m}_h^j \times d_t\mathbf{m}_h^{j+1}, -\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} - \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h \\ =\ & (\nabla d_t\mathbf{m}_h^{j+1}, \nabla\bar{\mathbf{m}}_h^{j+\frac{1}{2}}) - (d_t\mathbf{m}_h^{j+1}, \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h \\ +\ & \alpha(\bar{\mathbf{m}}_h^{j+\frac{1}{2}} \times d_t\mathbf{m}_h^{j+1}, -\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} - \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h \\ =\ & \frac{1}{2}\mathrm{d}_t\|\nabla\mathbf{m}_h^{j+1}\|_{L^2}^2 \\ -\ & (d_t\mathbf{m}_h^{j+1}, \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h - \alpha(\bar{\mathbf{m}}_h^{j+\frac{1}{2}} \times d_t\mathbf{m}_h^{j+1}, \tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} + \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h = 0. \end{aligned}$$

On the other hand, taking $\boldsymbol{\phi}_h = \mathrm{d}_t\bar{\mathbf{m}}_h^{j+1}$ in (3.11) yields

$$\begin{aligned} \frac{\alpha}{1+\alpha^2}\|d_t\mathbf{m}_h^{j+1}\|_h^2 &= \frac{\alpha}{1+\alpha^2}(d_t\mathbf{m}_h^{j+1}, d_t\mathbf{m}_h^{j+1})_h \\ &= \alpha(\bar{\mathbf{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} + \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}}), \mathrm{d}_t\bar{\mathbf{m}}_h^{j+1})_h. \end{aligned}$$

Finally, the relation $(a \times b, c) = -(a \times c, b)$ and the definition of the projection operator $\mathrm{P}_{\mathbf{V}_h}$ conclude that

$$\begin{aligned} \frac{1}{2}\mathrm{d}_t\|\nabla\mathbf{m}_h^{j+1}\|_{L^2}^2 + \frac{\alpha}{1+\alpha^2}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 &= (\mathrm{d}_t\mathbf{m}_h^{j+1}, \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})_h \\ &= (\mathrm{d}_t\mathbf{m}_h^{j+1}, \bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}}). \end{aligned}$$

□

Proposition 3.14
The following estimate holds for the solution $\boldsymbol{u}_h$:

$$\begin{aligned}
\frac{1}{2}\| d_t \boldsymbol{u}_h^N \|_{L^2}^2 &+ \frac{1}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^N)\|_{L^2}^2 \\
&+ \frac{1}{2}\sum_{j=1}^{N-1} \left(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h^j)\|_{L^2}^2 + \| d_t \boldsymbol{u}_h^{j+1} - d_t \boldsymbol{u}_h^j \|_{L^2}^2 \right) \\
&\leq C + (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^N), \boldsymbol{\varepsilon}(\boldsymbol{u}_h^N)) - (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^1), \boldsymbol{\varepsilon}(\boldsymbol{u}_h^0)) \\
&- k \sum_{j=0}^{N-1} (\boldsymbol{\lambda}^e d_t \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{u}_h^j)).
\end{aligned}$$

Proof. From the elastodynamics equation:

$$(\mathrm{d}_t^2 \mathbf{u}_h^{j+1}, \boldsymbol{\varphi}_h) + (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)),$$

we choose $\boldsymbol{\phi}_h = \mathbf{u}_h^{j+1} - \mathbf{u}_h^j$ and sum over j:

$$\begin{aligned}
\sum_{j=0}^{N-1} (\mathrm{d}_t^2 \mathbf{u}_h^{j+1}, \mathbf{u}_h^{j+1} - \mathbf{u}_h^j) &+ \sum_{j=0}^{N-1} (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}), \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1} - \mathbf{u}_h^j)) \\
&= \sum_{j=0}^{N-1} (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^j), \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1} - \mathbf{u}_h^j)).
\end{aligned}$$

We develop the first identity with the Abel summation and we obtain :

$$\begin{aligned}
\sum_{j=0}^{N-1} (\mathrm{d}_t^2 \mathbf{u}_h^{j+1}, \mathbf{u}_h^{j+1} - \mathbf{u}_h^j) &= \sum_{j=0}^{N-1} (\mathrm{d}_t \mathbf{u}_h^{j+1} - \mathrm{d}_t \mathbf{u}_h^j, \mathrm{d}_t \mathbf{u}_h^{j+1}) \\
= \frac{1}{2}\|\mathrm{d}_t \mathbf{u}_h^N\|_{L^2}^2 - \frac{1}{2}\|\mathrm{d}_t \mathbf{u}_0\|_{L^2}^2 &+ \frac{1}{2}\sum_{j=1}^{N-1} \|\mathrm{d}_t \mathbf{u}_h^{j+1} - \mathrm{d}_t \mathbf{u}_h^j\|_{L^2}^2
\end{aligned}$$

We do the same for the second sum

$$\sum_{j=0}^{N-1} (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}), \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1} - \mathbf{u}_h^j)).$$

and combining the two precedent identities with the boundeness and positive definition of $\boldsymbol{\lambda}^e$, we finally have

$$\begin{aligned}
&\frac{1}{2}\|\mathrm{d}_t \mathbf{u}_h^N\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_h^N)\|_{L^2}^2 \\
+ &\frac{1}{2}\sum_{j=1}^{N-1} \left(\|\varepsilon(\mathbf{u}_h^{j+1}) - \varepsilon(\mathbf{u}_h^j)\|_{L^2}^2 + \|(\mathrm{d}_t \mathbf{u}_h^{j+1} - \mathrm{d}_t \mathbf{u}_h^j)\|_{L^2}^2 \right) \\
\leq &\underbrace{\frac{1}{2}\|\mathrm{d}_t \mathbf{u}_0\|_{L^2}^2 + \frac{1}{2}\|\boldsymbol{\lambda}^e \varepsilon(\mathbf{u}_0)\|_{L^2}^2}_{=C} + \sum_{j=0}^{N-1} (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}), \boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1} - \mathbf{u}_h^j)).
\end{aligned}$$

Moroever, the right hand side can be written:

$$\begin{aligned}
&\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}-\mathbf{u}_h^j))\\
=\;&(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N),\boldsymbol{\varepsilon}(\mathbf{u}_h^N))-(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1),\boldsymbol{\varepsilon}(\mathbf{u}_h^0))\\
-\;&\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1})-\boldsymbol{\varepsilon}^m(\mathbf{m}_h^j)),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\\
=\;&(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N),\boldsymbol{\varepsilon}(\mathbf{u}_h^N))-(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1),\boldsymbol{\varepsilon}(\mathbf{u}_h^0))\\
-\;&k\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\mathrm{d}_t\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j)),
\end{aligned}$$

which concludes the proof. □

Lemma 3.15
There exists $\eta_i > 0$, $i = 1,\ldots,N$ and $\eta > 0$ sufficiently small such that

$$\begin{aligned}
\frac{1}{2}\|\nabla\boldsymbol{m}_h^N\|_{L^2}^2 \;&+\; \underbrace{\left(\frac{\alpha}{1+\alpha^2}-C\max\eta_i\right)}_{>0}\sum_{j=1}^{N-1}\|d_t\boldsymbol{m}_h^{j+1}\|_h^2k\\
&+\;\frac{1}{2}\|d_t\boldsymbol{u}_h^N\|_{L^2}^2+\underbrace{\left(\frac{1}{2}-\eta\right)}_{>0}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^N)\|_{L^2}^2\\
&\le\; C+\frac{1}{2}\|\nabla\boldsymbol{m}_0\|_{L^2}^2+\frac{1}{2}\|\boldsymbol{v}_0\|_{L^2}^2+\frac{1}{2}\|\varepsilon(\boldsymbol{u}_0)\|_{L^2}^2.
\end{aligned}$$

Proof. We sum the assertions from (3.13) and (3.14) and we integrate over time. With

$$\begin{aligned}
\mathcal{A}\;=\;&\frac{1}{2}\sum_{j=1}^{N-1}\mathrm{d}_t\|\nabla\mathbf{m}_h^{j+1}\|_{L^2}^2k+\frac{\alpha}{1+\alpha^2}\sum_{j=1}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2k\\
+\;&\frac{1}{2}\|\mathrm{d}_t\mathbf{m}_h^N\|_{L^2}^2+\frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{m}_h^N)\|_{L^2}^2\\
+\;&\frac{1}{2}\sum_{j=1}^{N-1}\left(\|\varepsilon(\mathbf{u}_h^{j+1})-\varepsilon(\mathbf{u}_h^j)\|_{L^2}^2+\|\mathrm{d}_t\mathbf{u}_h^{j+1}-\mathrm{d}_t\mathbf{u}_h^j\|_{L^2}^2\right),
\end{aligned}$$

we have :

$$\begin{aligned}
\mathcal{A}\;\le\;&C+\sum_{j=1}^{N-1}(\mathrm{d}_t\mathbf{m}_h^{j+1},\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})k\\
-\;&k\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\mathrm{d}_t\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\\
+\;&(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N),\boldsymbol{\varepsilon}(\mathbf{u}_h^N))-(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1),\boldsymbol{\varepsilon}(\mathbf{u}_h^0)).
\end{aligned}$$

Using then the Cauchy's and Young inequalities, we obtain for any positive η and

η_j, $j = 1, \ldots, N$:

$$\begin{aligned}
\mathcal{A} &\leq C + \frac{1}{2}\sum_{j=1}^{N-1} \eta_N \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_{L^2}^2 k + \frac{1}{2}\sum_{j=1}^{N-1} C_{\eta_N} \|\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}}\|_{L^2}^2 k \\
&+ \frac{1}{2}\sum_{j=0}^{N-1} \eta_j \|(\boldsymbol{\lambda}^e \mathrm{dt}(\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}))\|_{L^2}^2 k + \frac{1}{2}\sum_{j=0}^{N-1} C_{\eta_j} \|\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\|_{L^2}^2 k \\
&+ \frac{1}{2}\eta\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 + \frac{1}{2}C_\eta \|(\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^N)\|_{L^2}^2 + C\|\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^1)\|_{L^2}^2 + C\|\boldsymbol{\varepsilon}(\mathbf{u}_h^0)\|_{L^2}^2.
\end{aligned}$$

Choosing η sufficiently small, using the boundeness of the coefficients $\boldsymbol{\lambda}^e$ and the conservation of the magnitude

$$|\mathbf{m}_h^i| = 1 \quad \forall i = 0, \ldots, N, \tag{14}$$

we have

$$\begin{aligned}
\mathcal{A} - \frac{1}{2}\eta\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 &\leq C + \frac{1}{2}\sum_{j=1}^{N-1} \eta_N \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_{L^2}^2 k + \frac{1}{2}\sum_{j=1}^{N-1} C_{\eta_1} \|\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}}\|_{L^2}^2 k \\
&+ \frac{1}{2}C\sum_{j=0}^{N-1} \eta_j \|\mathrm{d}_t(\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}))\|_{L^2}^2 k + \frac{1}{2}\sum_{j=0}^{N-1} C_{\eta_j} \|\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2 k.
\end{aligned}$$

From the definition of $\boldsymbol{\varepsilon}^m$, the boundeness of the tensor $\boldsymbol{\lambda}^m$ and the proposition (3.13) we have on one hand

$$\|\mathrm{dt}(\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}))\|_{L^2}^2 \leq C\|\mathrm{dt}\mathbf{m}_h^{j+1}\|_{L^2}^2$$

and on the other hand

$$\|\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}}\|_{L^2}^2 = \|\boldsymbol{\lambda}^m \boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\mathbf{u}_h^j) - \boldsymbol{\varepsilon}^m(\mathbf{m}_h^j))\bar{\mathbf{m}}_h^{j+\frac{1}{2}}\|_{L^2}^2 \leq C + C\| \ \boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2.$$

Consequently,

$$\begin{aligned}
\mathcal{A} - \frac{1}{2}\eta\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 &\leq C + C\left(\sum_{j=1}^{N-1} \|\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2 k + \sum_{j=0}^{N-1} \|\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2 k\right) \\
&+ C\frac{1}{2}\underbrace{\max \eta_i}_{i=0,N} \left(\sum_{j=1}^{N-1} \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_{L^2}^2 k + C\sum_{j=0}^{N-1} \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_{L^2}^2 k\right).
\end{aligned}$$

Rassembling the precedent terms, we obtain

$$\begin{aligned}
\mathcal{A} - \frac{1}{2}\eta\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 &\leq C + \|\boldsymbol{\varepsilon}(\mathbf{u}_h^0)\|_{L^2}^2 + 2C\sum_{j=1}^{N-1} \|\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2 k \\
&+ 2C\frac{1}{2}\underbrace{\max \eta_i}_{i=0,N} \sum_{j=1}^{N-1} \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_{L^2}^2 k,
\end{aligned}$$

and then

$$\mathcal{A} - \frac{1}{2}\eta\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 - 2C\frac{1}{2}\underbrace{\max\eta_i}_{i=0,N}\sum_{j=1}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_{L^2}^2 k \quad \leq \quad C + C\sum_{j=0}^{N-1}\|\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2 k.$$

Finally, from the fact that

$$\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \leq C + \sum_{j=0}^{N-1}\|\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\|_{L^2}^2 k,$$

the discrete Gronwall inequality give

$$\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \leq C + \exp Nk\sum_{j=0}^{N-1} Nk \leq C.$$

To conclude, if we choose $\eta > 0$, $\max\eta_i > 0$ sufficiently small such that

$$C\max\eta_i < \frac{\alpha}{1+\alpha^2}, \quad \eta < \frac{1}{2},$$

then we have

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{N-1}\mathrm{d}_t\|\nabla\mathbf{m}_h^{j+1}\|_{L^2}^2 k \quad &+ \quad \left(\frac{\alpha}{1+\alpha^2} - C\max\eta_i\right)\sum_{j=1}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 k \\ &+ \quad \frac{1}{2}\|\mathrm{d}_t\mathbf{u}_h^N\|_{L^2}^2 + \left(\frac{1}{2} - \eta\right)\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \leq C.\end{aligned}$$

Moroever, we remark that we can develop the first integral:

$$\begin{aligned}\sum_{j=1}^{N-1}\mathrm{d}_t\|\nabla\mathbf{M}_h^+\|_{L^2}^2 k \quad &= \quad \sum_{j=0}^{N-1}\frac{\|\nabla\mathbf{m}_h^{j+1}\|_{L^2}^2 - \|\nabla\mathbf{m}_h^j\|_{L^2}^2}{k}k \\ &= \quad \|\nabla\mathbf{m}_h^N\|_{L^2}^2 - \|\nabla\mathbf{m}_h^0\|_{L^2}^2,\end{aligned}$$

to finally obtain the relation:

$$\begin{aligned}\frac{1}{2}\|\nabla\mathbf{m}_h^N\|_{L^2}^2 \quad &+ \quad \left(\frac{\alpha}{1+\alpha^2} - C\max\eta_i\right)\sum_{j=1}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 k \\ &+ \quad \frac{1}{2}\|\mathrm{d}_t\mathbf{u}_h^N\|_{L^2}^2 + \left(\frac{1}{2} - \eta\right)\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \\ &\leq \quad C + \frac{1}{2}\|\nabla\mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0)\|_{L^2}^2.\end{aligned}$$

□

Definition 3.16
For $\boldsymbol{x} \in \Omega$, $t \in [t_j, t_{j+1})$ we define

$$\begin{aligned}
&\boldsymbol{M}(t,\boldsymbol{x}) := \frac{t-t_j}{k}\boldsymbol{m}_h^{j+1}(\boldsymbol{x}) + \frac{t_{j+1}-t}{k}\boldsymbol{m}_h^j(\boldsymbol{x});\\
&\boldsymbol{M}^-(t,\boldsymbol{x}) := \boldsymbol{m}_h^j(\boldsymbol{x}), \quad \boldsymbol{M}^+(t,\boldsymbol{x}) := \boldsymbol{m}_h^{j+1}(\boldsymbol{x}), \quad \bar{\boldsymbol{M}}(t,\boldsymbol{x}) := \bar{\boldsymbol{m}}_h^{j+\frac{1}{2}}(\boldsymbol{x});\\
&\boldsymbol{U}(t,\boldsymbol{x}) := \frac{t-t_j}{k}\boldsymbol{u}_h^{j+1}(\boldsymbol{x}) + \frac{t_{j+1}-t}{k}\boldsymbol{u}_h^j(\boldsymbol{x});\\
&\boldsymbol{V}(t,\boldsymbol{x}) := \frac{t-t_j}{k}d_t\boldsymbol{u}_h^{j+1}(\boldsymbol{x}) + \frac{t_{j+1}-t}{k}d_t\boldsymbol{u}_h^j(\boldsymbol{x});\\
&\boldsymbol{U}^-(t,\boldsymbol{x}) := \boldsymbol{u}_h^j(\boldsymbol{x}), \quad \boldsymbol{U}^+(t,\boldsymbol{x}) := \boldsymbol{u}_h^{j+1}(\boldsymbol{x}), \quad \bar{\boldsymbol{U}}(t,\boldsymbol{x}) := \bar{\boldsymbol{u}}_h^{j+\frac{1}{2}}(\boldsymbol{x});\\
&\boldsymbol{V}^-(t,\boldsymbol{x}) := d_t\boldsymbol{u}_h^j(\boldsymbol{x}), \quad \boldsymbol{V}^+(t,\boldsymbol{x}) := d_t\boldsymbol{u}_h^{j+1}(\boldsymbol{x}), \quad \bar{\boldsymbol{V}}(t,\boldsymbol{x}) := [d_t\bar{\boldsymbol{u}}_h]^{j+\frac{1}{2}}(\boldsymbol{x})
\end{aligned}$$

and

$$\bar{\boldsymbol{H}}_\sigma(t,\boldsymbol{x}) = \boldsymbol{\lambda}^m\boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\boldsymbol{U}^-) - \boldsymbol{\varepsilon}^m(\boldsymbol{M}^-))\bar{\boldsymbol{M}}.$$

All these functions have the property to be very close each other when k becomes small. This is illustrated by the following proposition:

Proposition 3.17
For all $t \in [t_j, t_{j+1})$ we have when $k \to 0$,

$$\int_0^T \|\boldsymbol{M} - \boldsymbol{M}^-\|_{L^2} \to 0, \quad \int_0^T \|\boldsymbol{U} - \boldsymbol{U}^-\|_{L^2} \to 0, \quad \int_0^T \|\boldsymbol{V} - \boldsymbol{V}^-\|_{L^2} \to 0.$$

Similar results are available for $\boldsymbol{M}^+$, $\boldsymbol{U}^+$, $\boldsymbol{V}^+$.

Proof. By the lemma (6.19) and (6.18), there exists a constant $C > 0$ such that

$$\|\mathbf{M}_t\|_{L^2} \le C, \quad \|\mathbf{U}_t\|_{L^2} \le C, \quad \sum_{j=0}^N \|\mathbf{V}^+ - \mathbf{V}^-\|_{L^2} \le C.$$

Consequently, using the definition of $\mathbf{M}$ and $\mathbf{M}^-$ yields

$$\begin{aligned}
\int_0^T \|\mathbf{M} - \mathbf{M}^-\|_{L^2}^2 &\le \int_0^T k^2 \left\|\frac{\mathbf{M}^+ - \mathbf{M}^-}{k}\right\|_{L^2}^2\\
&\le \sum_{j=0}^N k^3\|\mathbf{M}_t\|_{L^2}^2 \le Ck^3 \to 0.
\end{aligned}$$

On the other hand, we have for $\mathbf{V}$:

$$\begin{aligned}
\int_0^T \|\mathbf{V} - \mathbf{V}^-\|_{L^2}^2 &\le \int_0^T k^2 \left\|\frac{\mathbf{V}^+ - \mathbf{V}^-}{k}\right\|_{L^2}^2\\
&\le \sum_{j=0}^N k\|\mathbf{V}^+ - \mathbf{V}^-\|_{L^2} \le Ck \to 0.
\end{aligned}$$

The proof is similar for $\mathbf{U}$. □

The next tool is an important result from Sobolev Spaces theory:

Proposition 3.18
Let $\boldsymbol{M}_\nu$ *be a sequence of vector functions. If there exists a constant* $C > 0$ *such that*

$$\|\boldsymbol{M}_\nu\|_{\boldsymbol{H}^1(\Omega_T;\mathbb{R}^3)} \leq C,$$

then there exists a subsequence $\boldsymbol{M}_{\nu_i}$ *and a limit* $\boldsymbol{m} \in \boldsymbol{H}^1(\Omega_T;\mathbb{R}^3)$ *such that*

$$\boldsymbol{M}_{\nu_i} \rightharpoonup \boldsymbol{m} \text{ in } \boldsymbol{H}^1(\Omega_T;\mathbb{R}^3).$$

Proposition 3.19
Let $\boldsymbol{u}_h, \boldsymbol{v}_h$ *be two sequences in* $\boldsymbol{V}_h$*. Let* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$*. Suppose that*

1. $\lim_{h\to 0} |(\boldsymbol{u}_h, \boldsymbol{v}_h)_h - (\boldsymbol{u}_h, \boldsymbol{v}_h)| = 0$;
2. $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ *in* $L^2(\Omega_T)$;
3. $\boldsymbol{v}_h \to \boldsymbol{v}$ *in* $L^2(\Omega_T)$.

Then

$$\lim_{k,h\to 0} \int_0^T (\boldsymbol{u}_h, \boldsymbol{v}_h)_h dt = \int_0^T (\boldsymbol{u}, \boldsymbol{v}) dt.$$

Proof. A triangle inequality gives,

$$\left| \int_0^T (\mathbf{u}_h, \mathbf{v}_h)_h - (\mathbf{u}, \mathbf{v}) dt \right| \leq \int_0^T |(\mathbf{u}_h, \mathbf{v}_h)_h - (\mathbf{u}_h, \mathbf{v}_h)| dt + \int_0^T |(\mathbf{u}_h, \mathbf{v}_h) - (\mathbf{u}, \mathbf{v})| dt$$

By the dominated convergence theorem

$$\int_0^T |(\mathbf{u}_h, \mathbf{v}_h)_h - (\mathbf{u}_h, \mathbf{v}_h)| dt \to 0 \qquad \text{when } k, h \to 0.$$

For the second integral, we use the relation

$$ab - cd = \frac{1}{2}\left((a-c)(b+d) + (a+c)(b-d)\right)$$

to obtain:

$$\begin{aligned} \int_0^T |(\mathbf{u}_h, \mathbf{v}_h) - (\mathbf{u}, \mathbf{v})| dt &\leq \int_0^T |(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h + \mathbf{v})| dt + \int_0^T |(\mathbf{u}_h + \mathbf{u}, \mathbf{v}_h - \mathbf{v})| dt \\ &\leq \underbrace{\int_0^T |(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h + \mathbf{v})| dt}_{\to 0, \text{ because } \mathbf{u}_h \rightharpoonup \mathbf{u} \text{ in } L^2} + \|\mathbf{u}_h + \mathbf{u}\|_{L^2} \underbrace{\|\mathbf{v}_h - \mathbf{v})\|_{L^2}}_{\to 0}. \end{aligned}$$

□

Theorem 3.20
Suppose that $|\boldsymbol{m}_h^0(\boldsymbol{x}_l)| = 1$ *for all* $l \in L$*, and let* $(\boldsymbol{m}_h^j, \boldsymbol{u}_h^j)_{j\geq 0}$ *solve algorithm* (3.11)*. Suppose that* $\boldsymbol{m}_h^0 \to \boldsymbol{m}_0$ *in* $\boldsymbol{H}^1(\Omega, \mathbb{R}^3)$ *and* $\boldsymbol{u}_h^0 \to \boldsymbol{u}_0$ *in* $\boldsymbol{V}_0$ *for* $h \to 0$*. For* $k, h \to 0$ *there exists* $\boldsymbol{m} \in \boldsymbol{H}^1(\Omega_T;\mathbb{R}^3)$*,* $\boldsymbol{u} \in \boldsymbol{V}_0$ *such that* $\boldsymbol{M}$ *subconverges to* $\boldsymbol{m}$ *in* $\boldsymbol{H}^1(\Omega_t, \mathbb{R}^3)$ *and* $\boldsymbol{U}$ *subconverges to* $\boldsymbol{u}$ *in* $\boldsymbol{V}_0$*. Moreover,* $\boldsymbol{m}$ *and* $\boldsymbol{u}$ *are weak solutions of the LLG equation with magnetostriction.*

Proof. From the lemma (6.19), we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla \mathbf{m}_h^N\|_{L^2}^2 &+ \sum_{j=1}^{N-1} \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_h^2 k \\ &+ \frac{1}{2}\|\mathrm{d}_t \mathbf{u}_h^N\|_{L^2}^2 + \|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \\ &\leq C(1 + \frac{1}{2}\|\nabla \mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0))\|_{L^2}^2.\end{aligned}$$

Consequently, using the definition (3.16) this inequality can be written for all $T' > 0$

$$\begin{aligned}\frac{1}{2}\|\nabla \mathbf{M}^+(T')\|_{L^2}^2 &+ \int_0^{T'} \|\mathbf{M}_t\|_h^2 \\ &+ \frac{1}{2}\|\mathbf{U}_t(T')\|_{L^2}^2 + \|\boldsymbol{\varepsilon}(\mathbf{U}^+(T')\|_{L^2}^2 \\ &\leq C + \frac{1}{2}\|\nabla \mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0\|_{L^2}^2.\end{aligned}$$

This bound the proposition (3.18) and (3.17) and the Korn's inequality yields existence of a couple

$$(\mathbf{m}, \mathbf{u}) \in [L^\infty(0, T, \mathbf{H}^1(\Omega, \mathbb{S}^2)) \cap \mathbf{H}^1(\Omega_T; \mathbb{R}^3)] \times \mathbf{V}_0,$$

which is the weak limit (as $k, h \to 0$) of a subsequence $\{(\mathbf{M}, \mathbf{U})\}_{k,h}$ such that

$$\begin{aligned}&(\mathbf{M}; \mathbf{U}) \rightharpoonup (\mathbf{m}; \mathbf{u}) \text{ in } (\mathbf{H}^1(\Omega_T; \mathbb{R}^3); \mathbf{V}_0);\\ &(\nabla \mathbf{M}, \nabla \mathbf{M}^+, \nabla \mathbf{M}^-, \nabla \bar{\mathbf{M}}; \nabla \mathbf{U}, \nabla \mathbf{U}^+, \nabla \mathbf{U}^-, \nabla \bar{\mathbf{U}}) \rightharpoonup (\nabla \mathbf{m}, \nabla \mathbf{u}) \text{ in } L^2(\Omega_T, \mathbb{R}^3).\end{aligned}$$

Moreover, by the theorem of Rellich Kondrachov, $\mathbf{H}^1(\Omega_T; \mathbb{R}^3) \subset L^2(\Omega_T, \mathbb{R}^3)$ with compact embedding. It means that there exists subsequence such that

$$\begin{aligned}(\mathbf{M}, \mathbf{M}^+, \mathbf{M}^-, \bar{\mathbf{M}}) &\to \mathbf{m} \text{ in } L^2(\Omega_T, \mathbb{R}^3),\\ (\mathbf{U}, \mathbf{U}^+, \mathbf{U}^-, \bar{\mathbf{U}}) &\to \mathbf{u} \text{ in } L_0^2(\Omega_T, \mathbb{R}^3).\end{aligned}$$

On the other hand, the proposition (3.14) gives the estimate

$$\frac{1}{2}\|\mathrm{d}_t \mathbf{u}_h^N\|_{L^2}^2 \leq C$$

and then

$$\frac{1}{2}\|\mathbf{V}^+(T')\|_{L^2}^2 \leq C.$$

This bound yields the existence of a limit $\mathbf{u}_t$ in $L^2(\Omega_T; \mathbb{R}^3)$ and a subsequence $\mathbf{V}$ such that

$$\mathbf{V} \rightharpoonup \mathbf{u}_t \text{ in } L^2(\Omega_T, \mathbb{R}^3).$$

We now verify that the limit $(\mathbf{m}, \mathbf{u})$ is a weak solution of the LLG equation with magnetostriction; we verify that it satisfies the four points of the definition (2.1):

1. From the proposition (3.13) $|\mathbf{M}^-(t,\mathbf{x}_l)| = 1$ for every $l \in L$ and almost all $t \in (0,T)$. Consequently, using standard finite element and Cauchy-Schwarz inequalities, there holds for every $K \in \mathcal{T}_h$

$$\begin{aligned} \||\mathbf{M}^-|^2 - 1\|_{L^2(K)} &\leq Ch\|\nabla(|\mathbf{M}^-|^2 - 1)\|_{L^2(K)} = Ch\|2(\nabla|\mathbf{M}^-)\mathbf{M}^-\|_{L^2(K)} \\ &\leq 2Ch\|\nabla|\mathbf{M}^-\|_{L^2}(K), \end{aligned}$$

which means that $|\mathbf{M}^-| \to 1$ in $L^2(K)$, and consequently $|\mathbf{m}| = 1$ a.e in Ω_T.

2. We recall the algorithm (3.11)

   (a) Determine $\mathbf{m}_h^{j+1} \in \mathbf{V}_h$ from

$$\begin{aligned} (\mathrm{d}_t\mathbf{m}_h^{j+1}, \boldsymbol{\phi}_h)_h &+ \alpha(\mathbf{m}_h^j \times \mathrm{d}_t\mathbf{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \\ &= (1+\alpha^2)(\bar{\mathbf{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} + \mathrm{P}_{\mathbf{V}_h}\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}}, \boldsymbol{\phi}_h)_h \quad \forall\boldsymbol{\phi}_h \in \mathbf{V}_h \end{aligned}$$

   with

$$\bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}} = \boldsymbol{\lambda}^m\boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\mathbf{u}_h^j) - \boldsymbol{\varepsilon}^m(\mathbf{m}_h^j))\bar{\mathbf{m}}_h^{j+\frac{1}{2}};$$

   (b) Determine $\mathbf{u}_h^{j+1} \in \mathbf{V}_{h,0}$ from

$$(\mathrm{d}_t^2\mathbf{u}_h^{j+1}, \boldsymbol{\varphi}_h) + (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) \quad \forall\boldsymbol{\varphi}_h \in \mathbf{V}_{h,0};$$

   (c) set $j = j+1$ and return to 1.

   For the LLG equation, using the definition of the subsequences, integrate on $(0,T)$ and taking $\boldsymbol{\phi}_h = \mathcal{I}_h\phi(t,\cdot)$ for $\phi \in C^\infty(\Omega_T;\mathbb{R}^3)$ goes to

$$\begin{aligned} \int_0^T (\mathbf{M}_t, \boldsymbol{\phi}_h)_h + \alpha\int_0^T (\mathbf{M}^- \times \mathbf{M}_t, \boldsymbol{\phi}_h)_h &= (1+\alpha^2)\int_0^T (\bar{\mathbf{M}} \times \tilde{\Delta}_h\bar{\mathbf{M}}, \boldsymbol{\phi}_h)_h \\ &+ (1+\alpha^2)\int_0^T (\bar{\mathbf{M}} \times P_{\mathbf{V}_h}\bar{\mathbf{H}}, \boldsymbol{\phi}_h)_h. \end{aligned}$$

   We need to show that when $h, k \to 0$, we have

$$\begin{aligned} \int_0^T (\mathbf{m}_t, \boldsymbol{\phi})dt + \alpha\int_0^T (\mathbf{m} \times \mathbf{m}_t, \boldsymbol{\phi}) &= -(1+\alpha^2)\int_0^T (\mathbf{m} \times \nabla\mathbf{m}, \boldsymbol{\phi}) \\ &+ (1+\alpha^2)\int_0^T (\mathbf{m} \times \mathbf{h}_\sigma, \boldsymbol{\phi}). \end{aligned}$$

   The effect of the reduced integral is controlled by the relation

$$|(\boldsymbol{\xi}_h, \boldsymbol{\nu}_h)_h - (\boldsymbol{\xi}_h, \boldsymbol{\nu}_h)| \leq Ch\|\boldsymbol{\xi}_h\|_{L^2}\|\nabla\boldsymbol{\nu}_h\|_{L^2}.$$

   Consequently, for almost all $t \in (0,T)$ we have

$$|(\boldsymbol{M}_t, \boldsymbol{\phi}_h)_h - (\boldsymbol{M}_t, \boldsymbol{\phi}_h)| \leq Ch\|\boldsymbol{M}_t\|_{L^2}\|\nabla\boldsymbol{\phi}_h\|_{L^2}.$$

   Moroever, since $\mathbf{M} \rightharpoonup \mathbf{m}$ in $H^1(\Omega_T)$, we have that $\mathbf{M}_t \rightharpoonup \mathbf{m}_t$ in $L^2(\Omega_T)$. On the other hand

$$\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_{L^2(\Omega_T)} = \|(I - \mathcal{I}_h)\phi(t,\cdot)\|_{L^2} \leq Ch\|\phi(t,\cdot)\|_{L^2},$$

which means that $\boldsymbol{\phi}_h \to \boldsymbol{\phi}$ in $L^2(\Omega_t, \mathbb{R}^3)$. We use the proposition (3.19) to conclude that

$$\lim_{k,h\to 0} \int_0^T (\mathbf{M}_t, \boldsymbol{\phi}_h)_h dt = \int_0^T (\mathbf{m}_t, \boldsymbol{\phi}) dt.$$

For the second integral, we have by standard estimate

$$\begin{aligned} & |(\mathbf{M}_t, \mathbf{M}^- \times \boldsymbol{\phi}_h)_h - (\mathbf{M}_t, \mathbf{M}^- \times \boldsymbol{\phi}_h)| \\ \leq \quad & Ch \|\mathbf{M}_t\|_{L^2} \|\nabla(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h)\|_{L^2}. \end{aligned}$$

Using then that $\mathbf{M}^- \times \boldsymbol{\phi}_h \to \mathbf{m} \times \boldsymbol{\phi}$ in $L^2(\Omega_T, \mathbb{R}^3)$, this yields again with the precedent argument

$$\lim_{k,h\to 0} \int_0^T (\bar{\mathbf{M}} \times \mathbf{M}_t, \boldsymbol{\phi}_h)_h dt = \int_0^T (\mathbf{m} \times \mathbf{m}_t, \boldsymbol{\phi}) dt.$$

For the third term of the equation, we write:

$$\begin{aligned} (\bar{\mathbf{M}} \times \tilde{\Delta}_h \bar{\mathbf{M}}, \boldsymbol{\phi}_h)_h &= -(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \tilde{\Delta}_h \bar{\mathbf{M}})_h = -(Id - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \tilde{\Delta}_h \bar{\mathbf{M}})_h) \\ &- (\mathcal{I}_h - Id)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \tilde{\Delta}_h \bar{\mathbf{M}})_h - (\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \tilde{\Delta}_h \bar{\mathbf{M}})_h \\ &= -(Id - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \tilde{\Delta}_h \bar{\mathbf{M}})_h \\ &+ (\nabla((\mathcal{I}_h - Id)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h)), \nabla \bar{\mathbf{M}}) + (\nabla(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h), \nabla \bar{\mathbf{M}}) \\ &= I + II + III. \end{aligned}$$

We use for I the bound from the proposition (3.8)

$$\begin{aligned} |I| &= |(Id - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \tilde{\Delta}_h \bar{\mathbf{M}})_h| \\ &\leq \|(Id - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h\|_{L^2} \|\tilde{\Delta}_h \bar{\mathbf{M}})\|_{L^2} \\ &\leq Ch^{-1} \|(Id - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h)\|_{L^2} \|\nabla \bar{\mathbf{M}}\|_{L^2}. \end{aligned}$$

On the other hand, the proposition (3.9) gives

$$\begin{aligned} \|(Id - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h)\|_{L^2} &\leq Ch^2 \sum_{K \in \mathcal{T}_h} \|D^2(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h)\|_{L^2(K)} \\ &\leq Ch^2 \|\nabla \bar{\mathbf{M}}\|_{L^2} \|\nabla \boldsymbol{\phi}_h)\|_{L^\infty}. \end{aligned}$$

We can then easily conclude that

$$\lim_{h,k\to 0} I = 0.$$

The same argument allows to show that

$$|II| \leq Ch \|\nabla \bar{\mathbf{M}}\|_{L^2}^2 \|\nabla \boldsymbol{\phi}_h)\|_{L^\infty}.$$

For III, the proposition (6.13) gives

$$III = (\nabla(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h), \nabla \bar{\mathbf{M}}) = (\bar{\mathbf{M}} \times \nabla \boldsymbol{\phi}_h, \nabla \bar{\mathbf{M}}).$$

Using one more time the proposition (3.19) we obtain

$$\lim_{k,h\to 0} \int_0^T (\bar{\mathbf{M}} \times \tilde{\Delta}_h \bar{\mathbf{M}}, \boldsymbol{\phi}_h)_h = \int_0^T (\mathbf{m} \times \nabla \boldsymbol{\phi}, \nabla \mathbf{m}) dt = -\int_0^T (\mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\phi}) dt.$$

For the last term, we have by definition of $P_{\mathbf{V}_h}$,

$$\begin{aligned}(\bar{\mathbf{M}} \times P_{\mathbf{V}_h}\bar{\mathbf{H}}, \boldsymbol{\phi}_h)_h &= -(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, P_{\mathbf{V}_h}\bar{\mathbf{H}})_h \\ &= -(\mathcal{I}_h(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h), P_{\mathbf{V}_h}\bar{\mathbf{H}})_h = -(\mathcal{I}_h(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h), \bar{\mathbf{H}}_\sigma) \\ &= (I - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \bar{\mathbf{H}}_\sigma) - (\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \bar{\mathbf{H}}_\sigma).\end{aligned}$$

Linearity of the limit shows that $\boldsymbol{\varepsilon}(\mathbf{U}^-) \rightharpoonup \boldsymbol{\varepsilon}(\mathbf{u})$ in $L^2(\Omega_T, \mathbb{R}^3)$ and then

$$\begin{aligned}\bar{\mathbf{H}}(t, \mathbf{x}) &= \boldsymbol{\lambda}^m \boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\mathbf{U}^-) - \boldsymbol{\varepsilon}^m(\mathbf{M}^-))\bar{\mathbf{M}} \\ &\rightharpoonup \boldsymbol{\lambda}^m \boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}^m(\mathbf{m}))\mathbf{m} = \mathbf{h}_\sigma.\end{aligned}$$

Finally, using

$$\|(I - \mathcal{I}_h)(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \bar{\mathbf{H}}_\sigma)\|_{L^2} \leq Ch \sum_{K \in \mathcal{T}_h} \|\nabla(\bar{\mathbf{M}} \times \boldsymbol{\phi}_h)\|_{L^2(K)} \|\bar{\mathbf{H}}_\sigma\|_{L^2(K)},$$

we obtain

$$\begin{aligned}\lim_{k,h\to 0} \int_0^T (\bar{\mathbf{M}} \times P_{\mathbf{V}_h}\bar{\mathbf{H}}, \boldsymbol{\phi}_h)_h dt &= -\lim_{k,h\to 0} \int_0^T (\bar{\mathbf{M}} \times \boldsymbol{\phi}_h, \bar{\mathbf{H}}_\sigma) \\ &= -\int_0^T (\bar{\mathbf{m}} \times \boldsymbol{\phi}, \mathbf{h}_\sigma) \\ &= \int_0^T (\mathbf{m} \times \mathbf{h}_\sigma, \boldsymbol{\phi}).\end{aligned}$$

For the elastodynamics equation, we choose $\boldsymbol{\varphi}_h = \mathcal{I}_h \varphi(t, \cdot)$ for $\varphi \in C_T^\infty(\Omega_T; \mathbb{R}^3)$. Using the definition of the subsequences and integrate on $(0, T)$ goes to

$$\int_0^T (\partial_t \mathbf{V}, \boldsymbol{\varphi}_h) + \int_0^T (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{U}^+), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = \int_0^T (\boldsymbol{\lambda}^m \boldsymbol{\varepsilon}^m(\mathbf{M}^+), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)). \tag{15}$$

We need to prove that (15) converge when $k, h \to 0$ to

$$\begin{aligned}-\int_0^T (\mathbf{u}_t, \boldsymbol{\varphi}_t) &+ \int_0^T (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) \\ &= \int_0^T (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^e(\mathbf{m}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) \\ &+ (\mathbf{u}_t(0), \boldsymbol{\varphi}_t(0)).\end{aligned}$$

We integrate by part the first integral of (15), and using $\boldsymbol{\varphi}_h(T) = 0$ gives:

$$\begin{aligned}\int_0^T (\partial_t \mathbf{V}, \boldsymbol{\varphi}_h) &= (\mathbf{V}, \boldsymbol{\varphi}_h)\Big|_0^T - \int_0^T (\mathbf{V}, \partial_t \boldsymbol{\varphi}_h) \\ &= -(\mathbf{V}(0), \boldsymbol{\varphi}_h(0)) \\ &- \int_0^T (\mathbf{V}, \partial_t \boldsymbol{\varphi}_h)\end{aligned}$$

We have also seen that there exists a subsequence $\mathbf{V}$ such that

$$\mathbf{V} \rightharpoonup \mathbf{u}_t \text{ in } L_0^2(\Omega_T, \mathbb{R}^3).$$

By hypothesis, $\mathbf{u}_h^0 \to \mathbf{u}_0$ in $\mathbf{V}_0$ and then

$$\lim_{k,h\to 0} (\mathbf{V}(0), \boldsymbol{\varphi}_h(0)) = (\mathbf{u}_t(0), \boldsymbol{\varphi}(0))$$

The proposition (3.19) also concludes that

$$\lim_{k,h\to 0}\int_0^T (\mathbf{V},\partial_t\boldsymbol{\varphi}_h) = \int_0^T (\mathbf{u}_t,\partial_t\boldsymbol{\varphi}).$$

From $\nabla\mathbf{U}^+ \rightharpoonup \nabla\mathbf{u}$ in $L^2(\Omega_T,\mathbb{R}^3)$ we have by linearity of the limit

$$\varepsilon(\mathbf{U}^+) \rightharpoonup \varepsilon(\mathbf{u}) \text{ in } L^2(\Omega_T,\mathbb{R}^3).$$

On the other hand,

$$\begin{aligned}\|\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)-\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^2} &\leq Ch^{-1}\|\boldsymbol{\varphi}_h-\boldsymbol{\varphi}\|_{L^2} = Ch^{-1}\|(\mathcal{I}_h - I)\boldsymbol{\varphi}\|_{L^2}\\ &\leq Ch^{-1}h^2\sum_{K\in\mathcal{T}_h}\|D^2(\varphi)\|_{L^2(K)}.\end{aligned}$$

and consequently $\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)\to\boldsymbol{\varepsilon}(\boldsymbol{\varphi})$ in $L^2(\Omega_T,\mathbb{R}^3)$. We apply again the proposition (3.19) to obtain

$$\lim_{k,h\to 0}\int_0^T (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{U}^+),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = \int_0^T (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\boldsymbol{\varphi})).$$

The fact that

$$\mathbf{M}^- \to \mathbf{m} \text{ in } L^2(\Omega_T,\mathbb{R}^3)$$

allows to show that

$$\boldsymbol{\varepsilon}^m(\mathbf{M}^-)\to\boldsymbol{\varepsilon}^m(\mathbf{m}) \text{ in } L^2(\Omega_T,\mathbb{R}^3).$$

Following the same argument:

$$\lim_{k,h\to 0}\int_0^T (\boldsymbol{\lambda}^m\boldsymbol{\varepsilon}^m(\mathbf{M}^-),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = \int_0^T (\boldsymbol{\lambda}^m\boldsymbol{\varepsilon}^m(\mathbf{m}),\boldsymbol{\varepsilon}(\boldsymbol{\varphi})).$$

Altogether, we obtain that the couple $(\mathbf{m},\mathbf{u})$ verifies the weak LLG equation with magnetostriction.

3. By hypothesis, $\mathbf{m}_h^0\to\mathbf{m}_0$ in $\mathbf{H}^1(\Omega_T;\mathbb{R}^3)$. By the continuity of the trace operator, we have $\mathbf{m}(0)=\mathbf{m}_0$ in the sense of traces.

4. To prove the energy inequality, remember that the norm in $\mathbf{H}^1(\Omega_T;\mathbb{R}^3)$ is lower semi-continuous, which means that for all sequence such that $\mathbf{a}_\nu\rightharpoonup\mathbf{a}$ in $\mathbf{H}^1(\Omega_T;\mathbb{R}^3)$, we have

$$\lim_{\nu\to\infty}\|\mathbf{a}_\nu\|_{\mathbf{H}^1}\geq\|\mathbf{a}\|_{\mathbf{H}^1}.$$

Consequently, we have

$$\begin{aligned}\lim_{k,h\to 0} -\frac{1}{2}\|\nabla\mathbf{M}^+(T')\|_{L^2}^2 &- \frac{\alpha}{1+\alpha^2}\int_0^{T'}\|\mathbf{M}_t\|_h^2\\ &- \frac{1}{2}\|\mathbf{U}_t(T')\|_{L^2}^2 + \|\boldsymbol{\varepsilon}(\mathbf{U}^+(T')\|_{L^2}^2\\ &+ C + \|\nabla\mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0\|_{L^2}^2 \geq\\ -\frac{1}{2}\|\nabla\mathbf{m}(T')\|_{L^2}^2 &- \frac{\alpha}{1+\alpha^2}\int_0^{T'}\|\mathbf{m}_t\|_h^2\\ &- \frac{1}{2}\|\mathbf{u}_t(T')\|_{L^2}^2 + \left(\frac{1}{2}-\eta\right)\|\boldsymbol{\varepsilon}(\mathbf{u}(T')\|_{L^2}^2\\ &+ C(1+\|\nabla\mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0\|_{L^2}^2)\geq 0,\end{aligned}$$

which concludes the proof.

□

3.3 Solving the non-linear system

We employ a fixed-point iteration to solve the non linear system in the step 1 of the algorithm (3.11). Given $\mathbf{m}_h^j \in \mathbf{V}_h$ and $\mathbf{u}_h^j \in \mathbf{V}_{h,0}$ for $j \geq 0$, we compute $\mathbf{m}_h^{j+1}$ by intermediate $\mathbf{m}_h^{j+1,l+1}$ for some $l \in \mathbb{N}$. We choose a condition such that after a certain number of iteration,

$$\mathbf{m}_h^{j+1,l+1} \approx \mathbf{m}_h^{j+1}$$

sufficiently precisely. To simplify the algorithm, instead of computing $\mathbf{m}_h^{j+1,l+1}$ we compute

$$\mathbf{w}^{j+1,l+1} = \frac{\mathbf{m}_h^{j+1,l+1} + \mathbf{m}_h^j}{2} \approx \bar{\mathbf{m}}_h^{j+\frac{1}{2}}.$$

It means that

$$\mathbf{m}_h^{j+1,l+1} = 2\mathbf{w}^{j+1,l+1} - \mathbf{m}_h^j.$$

We write also

$$\mathbf{h}_{\sigma_h}^l = \boldsymbol{\lambda}^m \boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\mathbf{u}_h^j) - \boldsymbol{\varepsilon}^m(\mathbf{m}_h^j))\mathbf{w}^l$$

Algorithm 3.21
We set $\boldsymbol{m}_h^{j+1,0} := \boldsymbol{m}_h^j$ *and* $l = 0$. *Given* $\boldsymbol{u}_h^j \in \boldsymbol{V}$

1. *compute* ($\boldsymbol{w}_h^{l+1} \in \boldsymbol{V}_h$ *such that for all* $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ *there holds*

$$\begin{aligned} & \frac{2}{k}(\boldsymbol{w}_h^{l+1}, \boldsymbol{\phi}_h)_h + \frac{2\alpha}{k}(\boldsymbol{m}_h^j \times \boldsymbol{w}_h^{l+1}, \boldsymbol{\phi}_h)_h \\ - & (1+\alpha^2)(\boldsymbol{w}_h^{l+1} \times (\tilde{\Delta}_h \boldsymbol{w}_h^l + P_{\boldsymbol{V}_h} \boldsymbol{h}_{\sigma_h}^l), \boldsymbol{\phi}_h)_h \\ = & \frac{2}{k}(\boldsymbol{m}_h^j, \boldsymbol{\phi}_h)_h \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h; \end{aligned}$$

2. *If* $\|\tilde{\Delta}_h(\boldsymbol{w}_h^{l+1} - \boldsymbol{w}_h^l) + P_{\boldsymbol{V}_h}(\boldsymbol{h}_\sigma^l - \boldsymbol{h}_{\sigma_h}^{l-1})\|_h \leq \varepsilon$, *then stop and set* $\boldsymbol{m}_h^{j+1} := 2\boldsymbol{w}_h^{l+1} - \boldsymbol{m}_h^j$;

3. *Set* $l := l + 1$ *and go to* 1.

The following proposition shows that the iteration converge, provided that $\varepsilon = 0$ *and* $k = \mathcal{O}(h^2)$.

Proposition 3.22
Suppose there exists a constant C >0 *such that* $|\boldsymbol{m}_h^j(\boldsymbol{x}_m)| \leq C, |\nabla \boldsymbol{u}_h^j(\boldsymbol{x}_l)| \leq C$ *for all* $m \in L$. *Then for all* $l \geq 0$, *there exists a unique solution* $\boldsymbol{w}_h^{j+1,l+1}$ *to* (3.21). *For all* $l \geq 1$ *there holds*

$$\|\boldsymbol{w}_h^{l+1} - \boldsymbol{w}_h^l\|_h \leq \Theta \|\boldsymbol{w}_h^l - \boldsymbol{w}_h^{l-1}\|_h,$$

provided that $\Theta = k(1+\alpha^2)C(c_1^2\sqrt{5}h^{-2} + C_\sigma) < 1$. *Moreover, for all* $l \geq 0$ *and all* $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$, *we have*

$$\begin{aligned} |(d_t \boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h & + \alpha(\boldsymbol{m}_h^j \times d_t \boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \\ & - (1+\alpha^2)(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h \bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} + P_{\boldsymbol{V}_h} \boldsymbol{h}_{\sigma_h}^{j+\frac{1}{2}}, \boldsymbol{\phi}_h)_h| \\ & \leq (1+\alpha^2)(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times \boldsymbol{R}^j, \boldsymbol{\phi}_h)_h, \end{aligned}$$

with $\|\boldsymbol{R}^j\| \le \varepsilon$. *By the Banach fixed-point theorem, the above contraction property implies for* $|\boldsymbol{m}_h^j(\boldsymbol{x}_m)| = 1$ *and for all* $m \in L$ *the existence of a unique* $\boldsymbol{m}_h^{j+1,*} \in \boldsymbol{V}_h$ *which solves algorithm* (3.11) *and satisfies lemma* (3.13).

Proof. We choose $\boldsymbol{\phi}_h = \mathbf{w}_h^{l+1} \in \mathbf{V}_h$. We obtain for the left hand-side:

$$\frac{1}{k}\|\mathbf{w}_h^{l+1}\|_h^2 \ge 0,$$

which means that the bilinear form formed by the left hand side is positive definite. Thus, the algorithm admits a unique solution. We control now the L^∞ norm : choose $m \in L$ such that

$$\|\mathbf{w}_h^{l+1}\|_{L^\infty} = |\mathbf{w}_h^{l+1}(\mathbf{x}_m)|.$$

We take then $\boldsymbol{\phi}_h = \varphi_m \mathbf{w}_h^{l+1}(\mathbf{x}_m)$ in (3.21) and it gives

$$\frac{1}{k}|\mathbf{w}_h^{l+1}(\mathbf{x}_m)|^2 = \frac{2}{k}(\mathbf{m}_h^j, \varphi_m \mathbf{w}_h^{l+1}(\mathbf{x}_m))_h \le \frac{2}{k}|\mathbf{m}_h^j||\varphi_m \mathbf{w}_h^{l+1}(\mathbf{x}_m)| \le \frac{C}{k}|\mathbf{w}_h^{l+1}(\mathbf{x}_m)|$$

Consequently,

$$\|\mathbf{w}_h^{l+1}\|_{L^\infty} \le C.$$

Substraction of two subsequent equations in the fixed-point iteration yields

$$\begin{aligned}
& \frac{2}{k}(\mathbf{w}_h^{l+1} - \mathbf{w}_h^l, \boldsymbol{\phi}_h)_h + \frac{2\alpha}{k}(\mathbf{m}_h^j \times (\mathbf{w}_h^{l+1} - \mathbf{w}_h^l), \boldsymbol{\phi}_h)_h \\
- \;& (1+\alpha^2)((\mathbf{w}_h^{l+1} \times (\tilde{\Delta}_h \mathbf{w}_h^l + P_{\mathbf{V}_h}\mathbf{h}_\sigma^l, \boldsymbol{\phi}_h)_h \\
+ \;& (1+\alpha^2)((\mathbf{w}_h^l \times (\tilde{\Delta}_h \mathbf{w}_h^{l-1} + P_{\mathbf{V}_h}\mathbf{h}_{\sigma_h}^{l-1}, \boldsymbol{\phi}_h)_h \\
= \;& \frac{2}{k}(\mathbf{w}_h^{l+1} - \mathbf{w}_h^l, \boldsymbol{\phi}_h)_h + \frac{2\alpha}{k}(\mathbf{m}_h^j \times (\mathbf{w}_h^{l+1} - \mathbf{w}_h^l), \boldsymbol{\phi}_h)_h \\
- \;& (1+\alpha^2)((\mathbf{w}_h^{l+1} \times (\tilde{\Delta}_h \mathbf{w}_h^l + P_{\mathbf{V}_h}\mathbf{h}_\sigma^l, \boldsymbol{\phi}_h)_h \\
+ \;& (1+\alpha^2)(\mathbf{w}_h^l \times (\tilde{\Delta}_h \mathbf{w}_h^l + P_{\mathbf{V}_h}\mathbf{h}_{\sigma_h}^l, \boldsymbol{\phi}_h)_h \\
- \;& (1+\alpha^2)(\mathbf{w}_h^l \times (\tilde{\Delta}_h \mathbf{w}_h^l + P_{\mathbf{V}_h}\mathbf{h}_{\sigma_h}^l, \boldsymbol{\phi}_h)_h \\
+ \;& (1+\alpha^2)((\mathbf{w}_h^l \times (\tilde{\Delta}_h \mathbf{w}_h^{l-1} + P_{\mathbf{V}_h}\mathbf{h}_{\sigma_h}^{l-1}, \boldsymbol{\phi}_h)_h \\
= \;& \frac{2}{k}(\mathbf{w}_h^{l+1} - \mathbf{w}_h^l, \boldsymbol{\phi}_h)_h + \frac{2\alpha}{k}(\mathbf{m}_h^j \times (\mathbf{w}_h^{l+1} - \mathbf{w}_h^l), \boldsymbol{\phi}_h)_h \\
- \;& (1+\alpha^2)((\mathbf{w}_h^{l+1} - \mathbf{w}_h^l) \times (\tilde{\Delta}_h \mathbf{w}_h^l + P_{\mathbf{V}_h}\mathbf{h}_\sigma^l, \boldsymbol{\phi}_h)_h \\
- \;& (1+\alpha^2)(\mathbf{w}_h^l \times (\tilde{\Delta}_h (\mathbf{w}_h^l - \mathbf{w}_h^{l-1}) + P_{\mathbf{V}_h}\mathbf{h}_\sigma^l - P_{\mathbf{V}_h}\mathbf{h}_{\sigma_h}^{l-1}, \boldsymbol{\phi}_h)_h \\
= \;& 0.
\end{aligned}$$

Choosing $\boldsymbol{\phi}_h = \mathbf{w}_h^{l+1} - \mathbf{w}_h^l$ in the precedent identity gives:

$$\begin{aligned}
& \|\mathbf{w}_h^{l+1} - \mathbf{w}_h^l\|_h \\
\le \;& k(1+\alpha^2)C(\|\tilde{\Delta}_h(\mathbf{w}_h^l - \mathbf{w}_h^{l-1})\|_h + \|P_{\mathbf{V}_h}(\mathbf{h}_{\sigma_h}^l - \mathbf{h}_{\sigma_h}^{l-1})\|_h) \\
\le \;& k(1+\alpha^2)C(c_1^2\sqrt{5}h^{-2}\|\mathbf{w}_h^l - \mathbf{w}_h^{l-1}\|_h + \|\mathbf{h}_{\sigma_h}^l - \mathbf{h}_{\sigma_h}^{l-1}\|_h) \\
\le \;& k(1+\alpha^2)C(c_1^2\sqrt{5}h^{-2} + C_\sigma)\|\mathbf{w}_h^l - \mathbf{w}_h^{l-1}\|_h,
\end{aligned}$$

when we use that

$$\|\mathbf{h}_{\sigma_h}^l - \mathbf{h}_{\sigma_h}^{l-1}\|_h \le |\boldsymbol{\lambda}^m \boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\mathbf{u}_h^j) - \boldsymbol{\varepsilon}^m(\mathbf{m}_h^j))|\|\mathbf{w}^l - \mathbf{w}^{l-1}\|_h \le C_\sigma\|\mathbf{w}^l - \mathbf{w}^{l-1}\|_h.$$

Consequently if we choose

$$k < \frac{1}{(1+\alpha^2)C(c_1^2\sqrt{5}h^{-2}+C_\sigma)},$$

there exists a unique solution of the algorithm (3.21) by the Banach fixed-point theorem. To connect the algorithms (3.11) and (3.21), we use the fact that for all l

$$\begin{aligned} & \frac{2}{k}(\mathbf{w}_h^{l+1},\boldsymbol{\phi}_h)_h + \frac{2\alpha}{k}(\mathbf{m}_h^j \times \mathbf{w}_h^{l+1},\boldsymbol{\phi}_h)_h \\ - \quad & (1+\alpha^2)(\mathbf{w}_h^{l+1} \times (\tilde{\Delta}_h \mathbf{w}_h^l + P_{\mathbf{V}_h}\mathbf{h}_\sigma^l,\boldsymbol{\phi}_h)_h \\ = \quad & \frac{2}{k}(\mathbf{m}_h^j,\boldsymbol{\phi}_h)_h. \end{aligned}$$

Finally,

$$\begin{aligned} (\mathrm{d}_t\mathbf{m}_h^{j+1},\boldsymbol{\phi}_h)_h \quad + \quad & \alpha(\mathbf{m}_h^j \times \mathrm{d}_t\mathbf{m}_h^{j+1},\boldsymbol{\phi}_h)_h \\ - \quad & (1+\alpha^2)(\bar{\mathbf{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h\bar{\mathbf{m}}_h^{j+\frac{1}{2}} + \mathrm{P}_{\mathbf{V}_h}\mathbf{h}_{\sigma h}^{j+\frac{1}{2}},\boldsymbol{\phi}_h)_h \\ = \quad & (1+\alpha^2)(\bar{\mathbf{m}}_h^{j+\frac{1}{2}} \times \mathbf{R}^j,\boldsymbol{\phi}_h)_h, \end{aligned}$$

where

$$\mathbf{R}^j = \tilde{\Delta}_h(\mathbf{w}_h^{l+1} - \mathbf{w}_h^l) + P_{\mathbf{V}_h}(\mathbf{h}_\sigma^l - \mathbf{h}_{\sigma_h}^{l-1}).$$

Then, if $\|\mathbf{R}^j\|_{L^2} < \varepsilon$ for ε sufficiently small, we have also the convergence of the algorithm (3.11), which achieves the proof. □

At each iteration l of the algorithm (3.21), we need to solve a linear system to find $\mathbf{w}_h^{l+1}$. Since the matrix of the system is sparse, we can choose a direct method such UMFPACK (from LU decomposition). On more complicated problem in three dimensions or when h has small values, we can use iterative methods like GMRES. For the elastodynamics equation, we generally took direct solvers.

Remark 3.23
When we consider the LLG equation only with magnetostriction, the time step restriction becomes:

$$k < \frac{1}{(1+\alpha^2)C(C_\sigma)}.$$

In this case the restriction is less important provided the tensors $\boldsymbol{\lambda}$ *are reasonably small. It allows an important gain of time in the computation.*

Remark 3.24
We can easily extend the algorithm (3.11)*, the proof of the convergence and the non-linear system when we consider the magnetic field and the anisotropy. Indeed, the term*

$$(\boldsymbol{w}_h^{l+1} \times (\boldsymbol{h} + C < \boldsymbol{p}, \boldsymbol{w}_h^l > \boldsymbol{p}, \boldsymbol{\phi}_h)_h$$

present no difficulties in the precedent algorithm and the convergence is obvious. At worst, we should maybe choose k *a bit smaller, depending of the value of* $\boldsymbol{p}$ *and* C*.*

4 Numerical Experiments

4.1 LLG with exact solution

We test the accuracy of our implicit algorithm on a one dimensional example with exchange, magnetostriction and magnetic field. We choose the exact solutions

$$\mathbf{m}(x,t) = \begin{pmatrix} \sin(3t) \\ \cos(x)\cos(3t) \\ \sin(x)\cos(3t) \end{pmatrix}$$

and

$$u(x,t) = \cos(x)\cos(3t).$$

The others parameters used were $\boldsymbol{\lambda}^e = \lambda^e_{3333} = \lambda^m_{3333} = 1, \alpha = 1, \eta = 1, \Omega = (0,1), T = 1$ and

$$\mathbf{h}(t) = (\sin(0.9t), \cos(0.9t), 0).$$

Consequently, the magnetostriction takes the form

$$\mathbf{h}_\sigma(x,t) = \begin{pmatrix} 0 \\ 0 \\ u_x m_3 - m_3^3 \end{pmatrix}.$$

Such that the above functions are the exact solutions, we need to add appropriate right-hand sides, and the problem becomes in this case:

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = (1+\alpha^2)(\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{h}_\sigma + \mathbf{h})) \\ -(1+\alpha^2)(\mathbf{m} \times (\mathbf{f}_\Delta + \mathbf{f}_{\mathbf{h}_\sigma} + \mathbf{h})) + \mathbf{f}_{\mathbf{m}_t} + \alpha(\mathbf{m} \times \mathbf{f}_{\mathbf{m}_t}) \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}_\mathbf{u} \end{cases} \tag{16}$$

with Dirichlet boundary conditions and the right hand-sides:

$$\mathbf{f}_\Delta = \begin{pmatrix} 0 \\ -\cos(x)\cos(3t) \\ -\sin(x)\cos(3t) \end{pmatrix} \quad \mathbf{f}_{\mathbf{m}_t} = \begin{pmatrix} 3\cos(3t) \\ -3\cos(x)\sin(3t) \\ -3\sin(x)\sin(3t) \end{pmatrix},$$

$$\mathbf{f}_{\mathbf{h}_\sigma} = \begin{pmatrix} 0 \\ 0 \\ -\sin(x)^2\cos(3t)^2 - \sin(x)^3\cos(3t)^3 \end{pmatrix},$$

$$\mathbf{f}_\mathbf{u} = -8\cos(x)\cos(3t) + 2\sin(x)\cos(x)\cos(3t)^2.$$

Remark 4.1
With these right hand-sides, the algorithm will no longer conserve the norm of the magnetization because of the term

$$(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times \boldsymbol{f}_{\boldsymbol{m}_t}).$$

Nevertheless, for k sufficiently small, the magnitude stays very close to one.

In tables $1-3$ below, we present the following errors for differents problem:

$$\mathrm{Err}(\mathbf{m}) = \max_j \|\mathbf{m}_h^j - \mathbf{m}\|_{L^2} + \max_j \|\nabla \mathbf{m}_h^j - \nabla \mathbf{m}\|_{L^2},$$

and

$$\mathrm{Err}(\mathbf{u}) = \max_j \|\mathbf{u}_h^j - \mathbf{u}\|_{L^2} + \max_j \|\nabla \mathbf{u}_h^j - \nabla \mathbf{u}\|_{L^2}.$$

In table 1, we can see the $\mathbf{H}^1$ error with different value of k and N. The algorithm (3.11) is expected to have at least a convergence of $\mathcal{O}(h) + \mathcal{O}(k)$, but this rate is not observable in that example. Indeed, for a space discretization N fixed, the functions $\mathrm{Err}(\mathbf{m})$ and $\mathrm{Err}(\mathbf{u})$ do not decrease when we diminish the time step k. For example, if $N = 5$, the error will stay around $\frac{C}{5}$, because of the relation

$$\|\mathbf{m} - \mathbf{m}_h^j\|_{H^1} \leq C(k + h).$$

This restriction comes from the small value of k we need to choose to solve the non-linear system ($k = \mathcal{O}(h^2)$ from the proposition (3.22)). Thus, the convergence rate in that example can only depend of the space discretization:

$$\|\mathbf{m} - \mathbf{m}_h^j\|_{H^1} \leq C(\mathcal{O}(h^2) + h) \leq Ch.$$

which is then visible in table 1.

In table 2, we solve only the linear wave equation so we do not have restriction on k because the system is linear. Consequently, for a sufficiently small h, the function $\mathrm{Err}(\mathbf{u})$ diminish when we decrease k.

The functions $\mathrm{Err}(\mathbf{m})$ and $\mathrm{Err}(\mathbf{u})$ are shown in table 3 for the case when we solve the LLG equation only with the magnetostriction field. In this case, the time step restriction does not depend of h, consequently we can observe a linear decreasing of these functions for sufficiently small k. Indeed, taking for example $k = 0.001$ and $h = 1/10$ we have $\mathrm{Err}(\mathbf{m})$=0.031 and $\mathrm{Err}(\mathbf{u})$=0.025. Choosing then $k = 0.0001$ and $h = 1/100$, the values of $\mathrm{Err}(\mathbf{m})$ and $\mathrm{Err}(\mathbf{u})$ are divided by ten.

These computations have been done on my personal computer, resulting sometimes in a large CPU time. For $N = 100$ and $k = 0.0001$, the time of computation is almost 60'000[s]. For more complicated problem, development of parallel solver and computation on a cluster is then highly recommended.

k	0.001		0.0001		0.00001	
N	**u**	**m**	**u**	**m**	**u**	**m**
5	0.053	0.062	0.052	0.061	0.053	0.061
10	0.026	0.031	0.025	0.029	0.026	0.030
20	N.A.N	N.A.N	0.012	0.015	0.012	0.015
50	N.A.N	N.A.N	N.A.N	N.A.N	0.0059	0.008

Table 1: Error for the LLG equation with exchange and magnetostriction

N \ k	0.001	0.0001	0.00001
5	0.17	0.06	0.056
10	0.16	0.034	0.026
20	0.16	0.022	0.013
40	0.16	0.019	0.0066
50	0.16	0.018	0.0057
80	0.16	0.017	0.0034
100	0.16	0.017	0.0031

Table 2: Error for the Wave equation

k	0.1		0.01		0.001		0.0001	
N	**m**	**u**	**m**	**u**	**m**	**u**	**m**	**u**
5	1.124	0.27	0.11	0.023	0.062	0.052	0.061	0.052
10	1.59	0.30	0.16	0.039	0.031	0.025	0.029	0.025
20	2.21	0.31	0.22	0.039	0.022	0.013	0.015	0.012
40	3.05	0.31	0.30	0.039	0.030	0.0072	0.0075	0.0062
50	3.39	0.31	0.34	0.039	0.034	0.0059	0.0060	0.0050
80	4.23	0.31	0.42	0.040	0.042	0.0044	0.0042	00031
100	4.73	0.31	0.47	0.040	0.047	0.0042	0.0032	0.0025

Table 3: Error for the LLG equation with magnetostriction

4.2 A Blow Up Example

4.2.1 Introduction

In this section we discuss numerical experiments which motivate finite-time blow-up. This example has been developed in [4] and [5] for respectively the previous implicit method without magnetostriction and an explicit scheme. We extend the experiment with the magnetostriction and study the impact of the tensor $\boldsymbol{\lambda}$ and the parameter α on the blow-up time.

4.2.2 Initial Data

In [5], the initial data is chosen to be an harmonic map that solves

$$-\Delta \mathbf{m} = |\nabla \mathbf{m}|^2 \mathbf{m}.$$

This function is the static solution of the LLG equation and is crucial to the formation of singularity in the case of the LLG equation with exchange field. The initial condition is given by: $\Omega = (-0.5, 0.5)^2$ and $\mathbf{m}_0 : \Omega \to \mathbb{S}^2$ defined by

$$\mathbf{m}_0(\mathbf{x}) = \begin{cases} (0,0,-1) & \text{for } |\mathbf{x}| \geq 0.5 \\ \frac{(2\mathbf{x}A, A^2 - |\mathbf{x}|^2}{(A^2 + |\mathbf{x}|^2} & \text{for } |\mathbf{x}| \leq 0.5, \end{cases}$$

where $A := \frac{(1-2|\mathbf{x}|)^4}{s}$.
We solve the following LLG problem:

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = \gamma(1+\alpha^2)(\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{h}_\sigma)), \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0. \end{cases} \tag{17}$$

where γ is a given constant. For the magnetostriction, we took $\boldsymbol{\lambda}^e$ and $\boldsymbol{\lambda}^m$ 2×2 tensors such that

$$\boldsymbol{\lambda}^e = \lambda^e_{1111} = \lambda^e_{2222} = C_1, \quad \boldsymbol{\lambda}^m = \lambda^m_{1111} = \lambda^m_{2222} = C_2.$$

for given constant C_1, C_2. The initial condition for the elastodynamics equation are $\mathbf{u}_0 = \mathbf{0}, \mathbf{v}_0 = \mathbf{0}$ and homogeneous Dirichlet boundary conditions. The triangulation $\mathcal{T}_l$ used in the numerical simulation are defined through a positive integer l and consists of 2^{2l+1} halved square with edge length $h = 2^{-l}$. The restriction on k by the non-linear algorithm (3.12) motives to choose

$$k = \frac{h^2}{10 + \alpha^2}.$$

We set $\varepsilon = 2^{-16}$ in (3.12) and the non-linear system terminates after at most 13 iterations. As discrete intial data we employ the nodal interpolation

$$\mathbf{m}^0 = \mathcal{I}_{\mathcal{T}_l} \mathbf{m}_0.$$

4.2.3 LLG with exchange

In this example we solve the LLG equation with only the exchange field $\mathbf{h}_T = \Delta\mathbf{m}$:

$$\mathbf{m}_t + \alpha\mathbf{m} \times \mathbf{m}_t = (1+\alpha^2)(\mathbf{m} \times (\Delta\mathbf{m}). \tag{18}$$

In figure 1 and 2 we display snapshots of the numerical approximation provided by algorithm (3.11) without magnetostriction and with $\alpha = 1$, $s = 1$ and $l = 4$. The plots display the first two components at various time. A zoom towards the central nodes is seen in figure 2. We observe that at time $t \approx 0.0529$ the vector at the origin points in another direction than all surrounding vectors, resulting in a large $\mathbf{W}^{1,\infty}$ norm. This reveals than in this experiment, regularity of the exact solution cannot be expected. Figure 3 shows similar snapshots for $\alpha = 1/64$, $s = 1$ and $l = 4$. In this case the numerical solution is even less regular than in the previous experiment and at time $t = 0.3$ the solution is still unsteady.

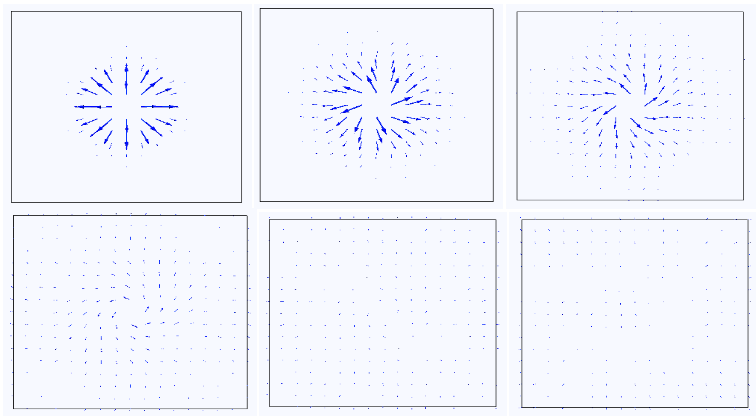

Figure 1: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with exchange (18), with $s = 1$, $\alpha = 1$, $l = 4$ at time $t = 0, 0.0119, 0.0297, 0.0529, 0.0588, 0.06$.

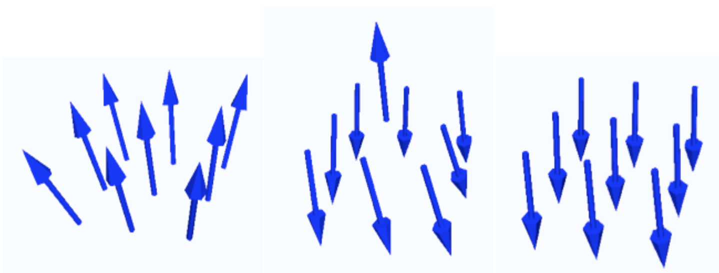

Figure 2: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with exchange (18), with $s = 1$, $\alpha = 1$, $l = 4$, at time $t = 0, 0.0.0529, 0.06$.

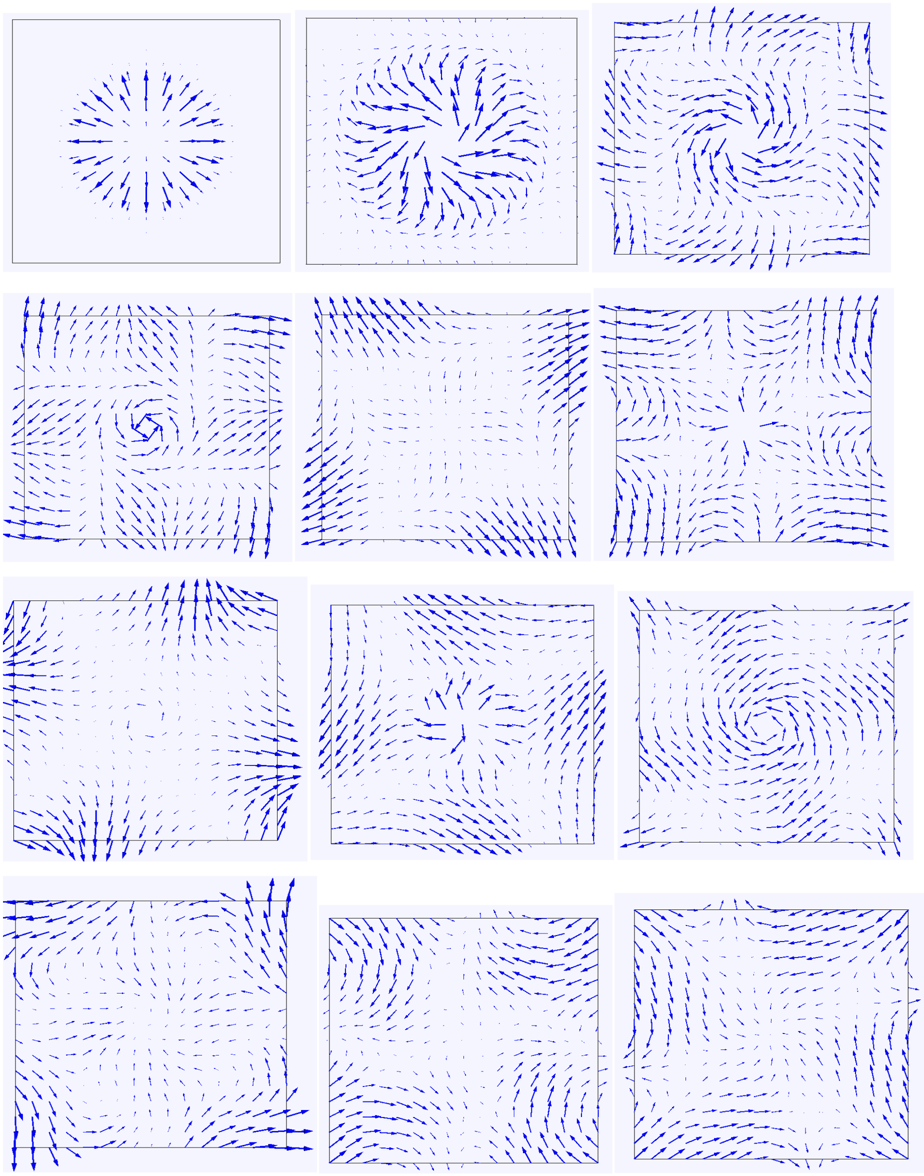

Figure 3: Snapshots of the Numerical Approximation of the magnetization $\mathbf{m}$ for the problem (18), with $s = 1$, $\alpha = 1/64$, $l = 4$, at time $t = 0, 0.0102, 0.0297, 0.0492, 0.0687, 0.1078, 0.1371, 0.1664, 0.2054, 0.2347, 0.2738, 0.2999$.

For $\mathbf{h}_T = \Delta\mathbf{m}$ and without displacement, the lemma (6.19) is reduced to the following relation:

$$\frac{1}{2}\int_\Omega |\nabla\mathbf{m}(t,\cdot)|^2 + \frac{\alpha}{\gamma(1+\alpha^2)}\int_0^t \|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 = \frac{1}{2}\int_\Omega |\nabla\mathbf{m}_0|^2.$$

Consequently, we expect that the energy

$$E(\mathbf{m},\mathbf{u},t) = \frac{1}{2}\int_\Omega |\nabla\mathbf{m}(t,\cdot)|^2.$$

is a decreasing function of time because

$$p(t) = \frac{\alpha}{\gamma(1+\alpha^2)}\int_0^t \|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2$$

is a strictly non-decreasing function. We define also the $\mathbf{W}^{1,\infty}$ semi-norm:

$$|\mathbf{m}(t)|_{1,\infty} = \|\nabla\mathbf{m}(t)\|_{L^\infty}$$

In figure 4 we show the energy and the $\mathbf{W}^{1,\infty}$ semi-norm for different values of h. We observe that the maximum value of $\|\nabla\mathbf{m}\|_{L^\infty}$ is $2\sqrt{2}h^{-1}$ and when h becomes small, the blow-up time (when the $\|\nabla\mathbf{m}\|_{L^\infty}$ is maximal) seems to approach $t = 0.03$.

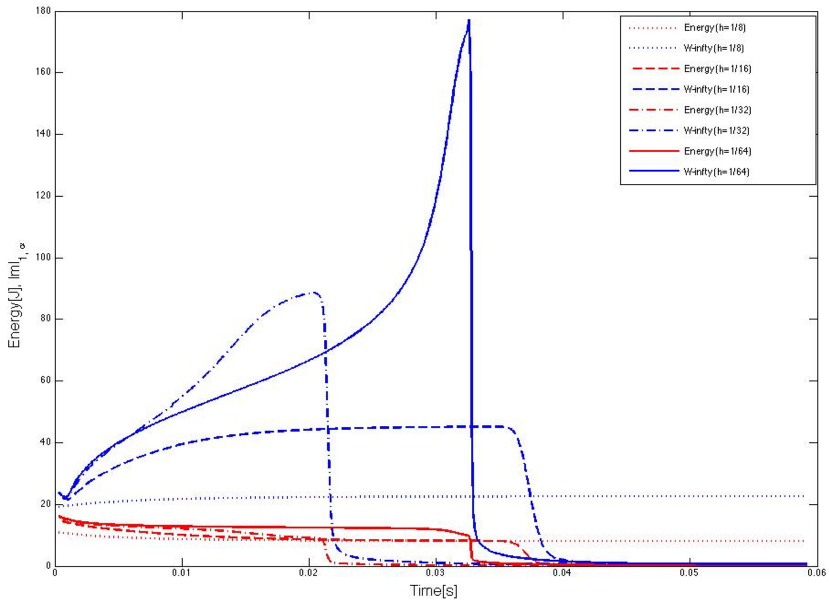

Figure 4: Energy and $\mathbf{W}^{1,\infty}$ semi-norm for the magnetization $\mathbf{m}$ from the problem (18), with $\alpha = 1$, $s = 4$ and for different values of h.

Behaviour of the blow-up time when α has small value is displayed in figure 5. The blow-up time seems to converge at $t = 0.085$ when $\alpha \to 0$. For $\alpha \geq 1/256$, the $\mathbf{W}^{1,\infty}$ semi-norm oscillates and the energy decreases very slowly, meaning possible unstability. Finally, figure 6 shows that the Blow-Up time decreases for larger value of s.

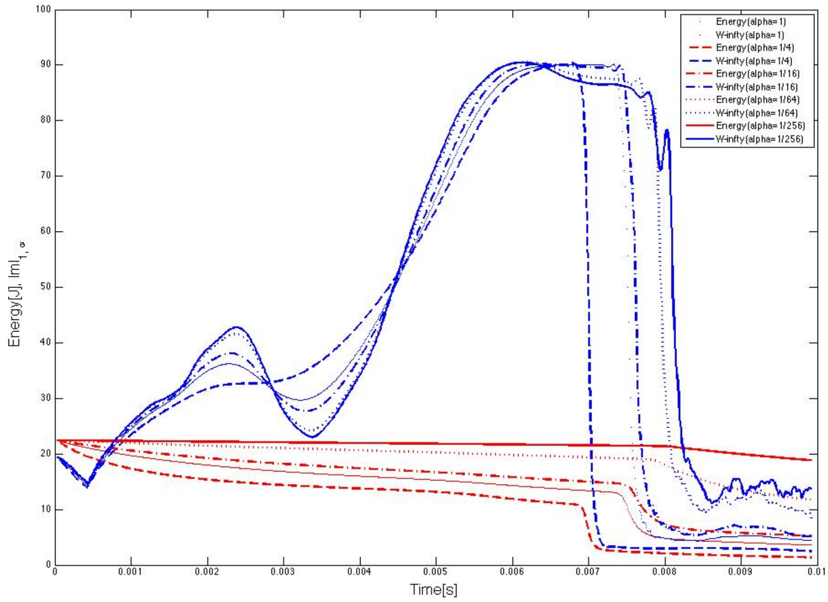

Figure 5: Energy and $\mathbf{W}^{1,\infty}$ semi-norm for the magnetization $\mathbf{m}$ from the problem (18), with $h = 1/32$, $s = 4$ and for different values of α.

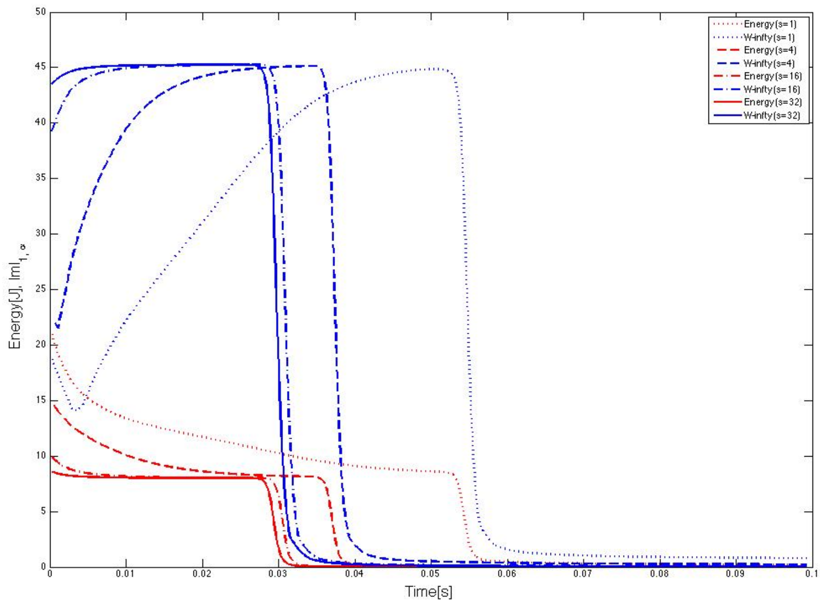

Figure 6: Energy and $\mathbf{W}^{1,\infty}$ semi-norm for the magnetization $\mathbf{m}$ from the problem (18), with $h = 1/32$, $\alpha = 1$ and for different values of s.

4.2.4 LLG with magnetostriction

We solve now the LLG equation with magnetostriction:

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = \gamma(1+\alpha^2)(\mathbf{m} \times (\mathbf{h}_\sigma)), \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0. \end{cases} \tag{19}$$

where $\gamma = 4$ and $s = 1$. For the magnetostriction, we took $\boldsymbol{\lambda}^e$ and $\boldsymbol{\lambda}^m$ 2×2 tensors such that

$$\boldsymbol{\lambda}^e = \lambda^e_{1111} = \lambda^e_{2222} = 5, \quad \boldsymbol{\lambda}^m = \lambda^m_{1111} = \lambda^m_{2222} = 5.$$

The initial conditions for the elastodynamics equation are $\mathbf{u}_0 = \mathbf{0}, \mathbf{v}_0 = \mathbf{0}$ and homogeneous Dirichlet boundary conditions. Snapshots of the magnetization $\mathbf{m}$ with $\alpha = 1$ are shown in figure . After $t \approx 0.5[s]$, we observe that the vectors near the center converge to $(0, 0, 1)$ and the ones near the boundary to $(0, 0, -1)$.

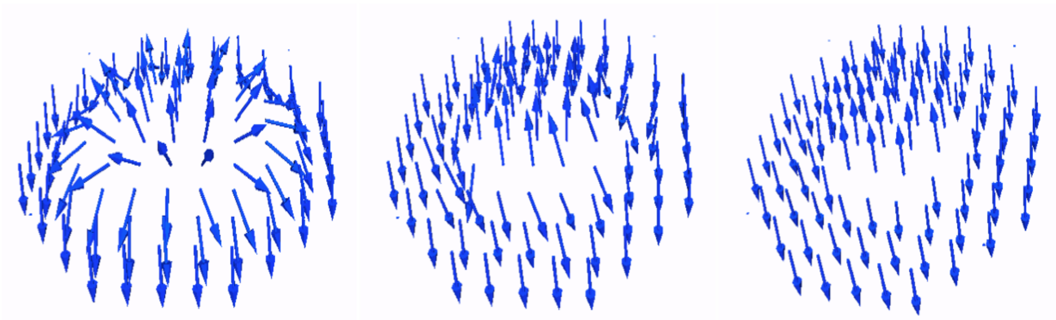

Figure 7: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with magnetostriction (19), with $\alpha = 1$, $h = 1/16$, for time $t = 0, 0.2, 1$.

The displacement, shown in figure 8, spreads rapidly on the diagonal of the square, with stronger component in u_1. It progressively vanishes when the system becomes steady.

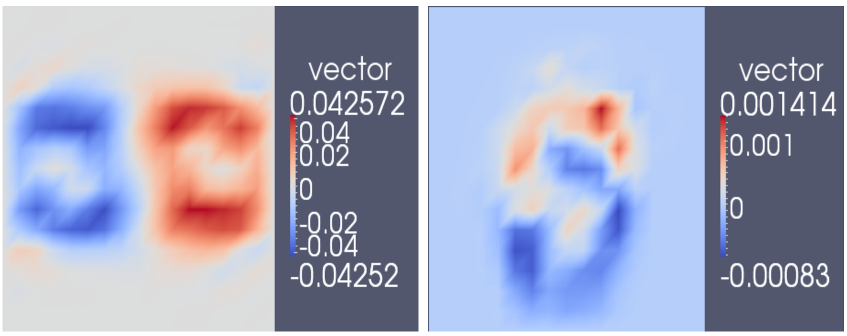

Figure 8: Isoclines of the numerical approximation of the displacement $u_1(t, \cdot)$, $u_2(t, \cdot)$ for the elastodynamic equation from (19), with $\alpha = 1$, $h = 1/16$ and at time $t = 0.25$.

In figure 9 and 10 we show the evolution of the magnetization for $\alpha = 1/16$. The vector turns around their nodes, diagonaly symmetric, and the steady state is only reached after $t = 5[s]$. The spread of the displacement (figure 11) is stronger due to the small value of α, and the vectors on the diagonal start to turn after $t = 0.2[s]$ until the steady state is reached.

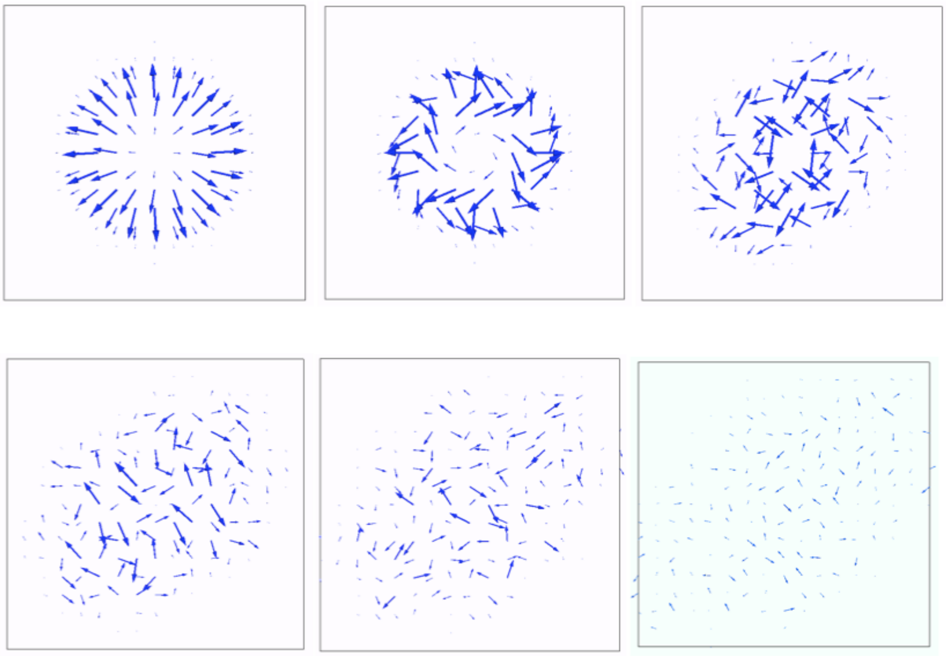

Figure 9: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with magnetostriction (19) with $\alpha = 1/16$, $h = 1/16$, for time $t = 0, 0.015, 0.1, 0.4, 1, 3$.

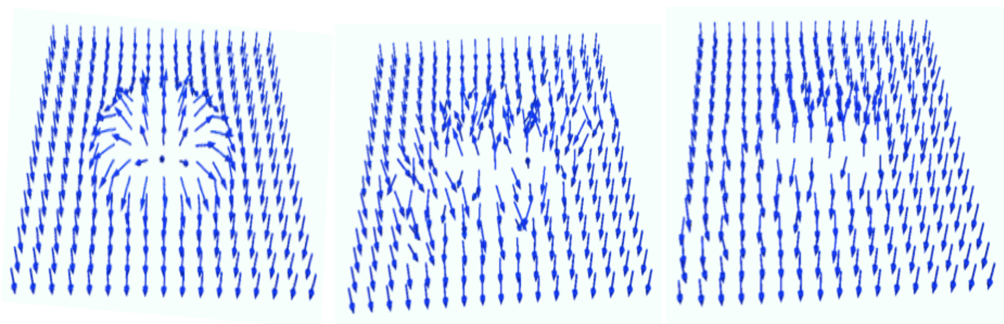

Figure 10: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with magnetostriction (19) with $\alpha = 1/16$, $h = 1/16$, for time $t = 0, 1, 3$.

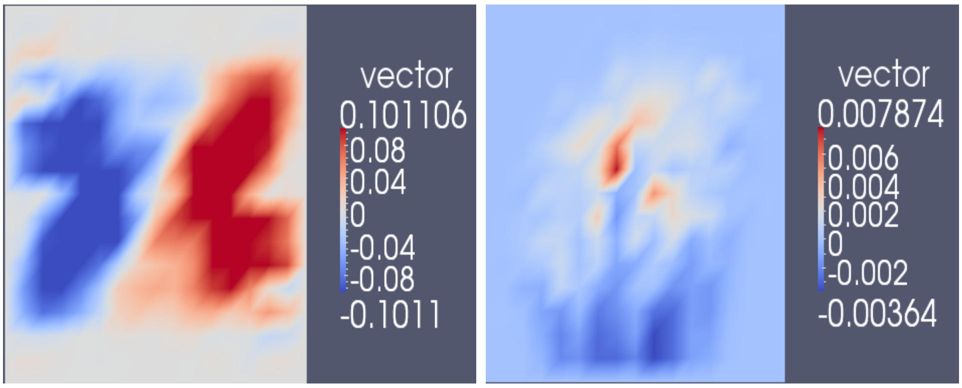

Figure 11: Isoclines for the numerical approximation of the displacement $u_1(t,\cdot)$, $u_2(t,\cdot)$ for the elastodynamic equation from (19), with $\alpha = 1/16$, $h = 1/16$ and at time $t = 0.25$.

4.2.5 LLG with exchange and magnetostriction

We solve the LLG equation with exchange and magnetostriction:

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = \gamma(1+\alpha^2)(\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{h}_\sigma)), \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0. \end{cases} \tag{20}$$

where $\gamma = 4$ and $s = 1$. We took $\boldsymbol{\lambda}^e$ and $\boldsymbol{\lambda}^m$ 2×2 diagonal tensors such that

$$\lambda^e_{ij} = 40\delta_{ij}, \quad \lambda^m_{ij} = 10\delta_{ij}.$$

These tensors have been chosen with high value in order to have an impact on the blow-up time. On figure 12 we display snapshots of the numerical magnetization for $\alpha = 1$. The vectors turn around their nodes with a diagonal symmetry. In figure 13 , we observe that at time $t \approx 0.074$, that the vector at the origin point in another direction than all surrounding vectors. It means that the blow-up happened again with the magnetostriction. Nevertheless, the first component of the displacement is strong (figure 14) and consequently the vectors move after the blow-up time, and so the solution fails to become stationary for $t \leq 0.1$.

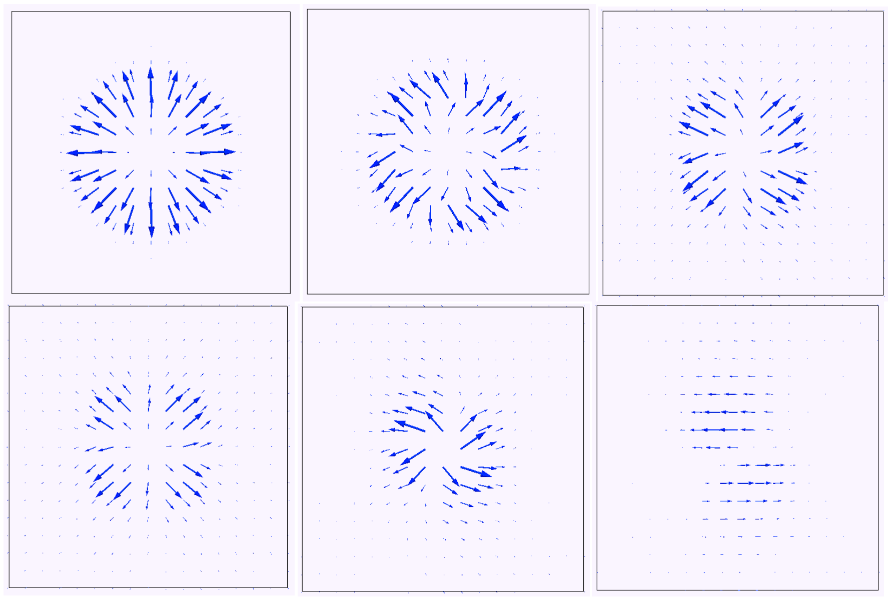

Figure 12: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with exchange and magnetostriction (20), with $\alpha = 1$, $h = 1/16$, for time $t = 0, 0.00015, 0.01, 0.06, 0.0725, 0.1$.

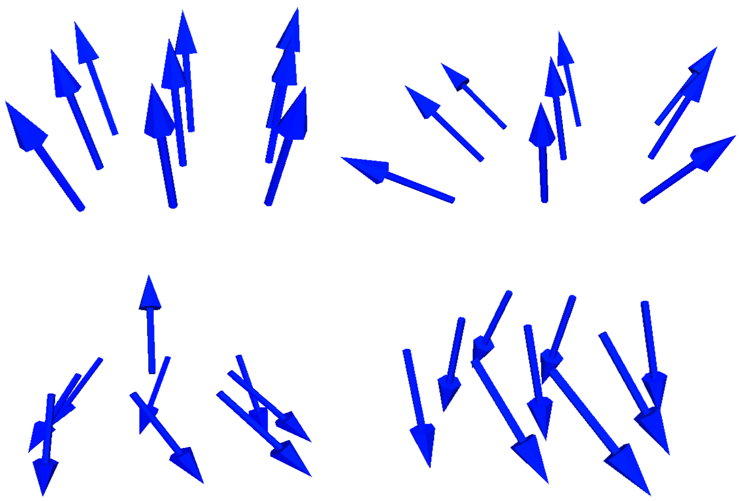

Figure 13: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with exchange and magnetostriction (20), with $\alpha = 1$, $h = 1/16$, for time $t = 0, 0.065, 0.07, 0.08$.

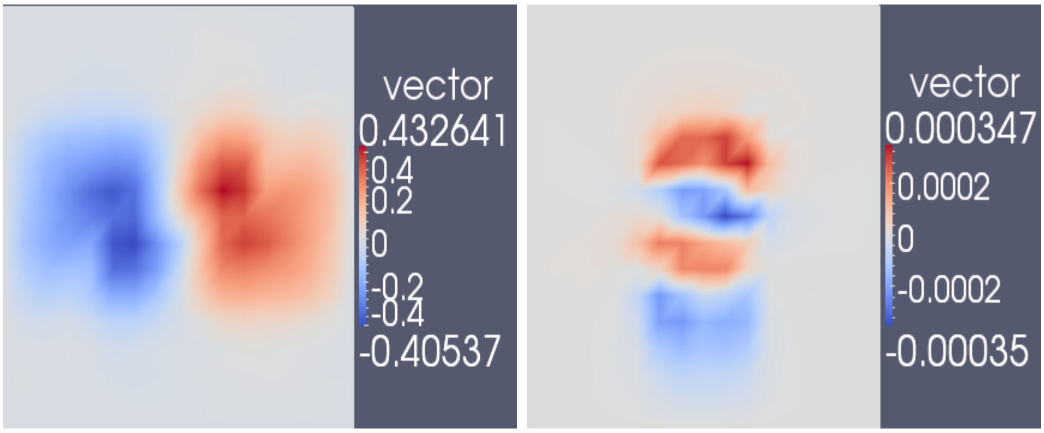

Figure 14: Isoclines of the numerical approximation of the displacement $u_1(t,\cdot)$, $u_2(t,\cdot)$ for the elastodynamic equation from (20), with $h = 1/16$, $\alpha = 1$ and at time $t = 0.1$.

In figure 15 we show snapshots for the magnetization with $\alpha = 1/4$. The solution is less regular than for $\alpha = 1$ and the blow-up time is smaller (figure 16). The displacement has the same order and consequently the steady state is still not reached for $t \leq 0.1$.

We define the following energies:

$$\begin{aligned} E_T(\mathbf{m},\mathbf{u},t) &= \frac{1}{2}\|\nabla \mathbf{m}_h(t)\|_{L^2}^2 + \frac{\alpha}{\gamma(1+\alpha^2)}\int_0^t \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_h^2 \\ &+ \frac{1}{2}\|\mathrm{d}_t \mathbf{u}_h(t)\|_{L^2}^2 + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}_h(t))\|_{L^2}^2; \\ E(\mathbf{m},\mathbf{u},t) &= \frac{1}{2}\|\nabla \mathbf{m}_h(t)\|_{L^2}^2 + \frac{1}{2}\|\mathrm{d}_t \mathbf{u}_h(t)\|_{L^2}^2 + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{u}_h(t))\|_{L^2}^2; \\ &= E_{\mathbf{m}}(\mathbf{u},t) + E_{u_t}(\mathbf{u},t) + E_{u_x}(\mathbf{u},t). \end{aligned}$$

By the lemma (6.19),

$$\begin{aligned} E_T(\mathbf{m},\mathbf{u},t) &= E(\mathbf{m},\mathbf{u},t) + \frac{\alpha}{\gamma(1+\alpha^2)}\int_0^t \|\mathrm{d}_t \mathbf{m}_h^{j+1}\|_h^2 \\ &\leq C(\boldsymbol{\lambda}_e,\boldsymbol{\lambda}_m) + \frac{1}{2}\|\nabla \mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0)\|_{L^2}^2 \end{aligned}$$

We cannot expect that the function $E(\mathbf{m},\mathbf{u},t)_\sigma$ decreases, because the constant $C(\boldsymbol{\lambda}_e,\boldsymbol{\lambda}_m)$ only depend of the tensors $\boldsymbol{\lambda}$, which have high value in our experiment. We show these different energies for $\boldsymbol{\lambda}_e = 40$, $\boldsymbol{\lambda}_m = 10$ in figure 17 (after rescaling). We note that the total energy increases quickly and then stabilizes around a constant of order $10^2 \leq C(\boldsymbol{\lambda}_e,\boldsymbol{\lambda}_m)$.

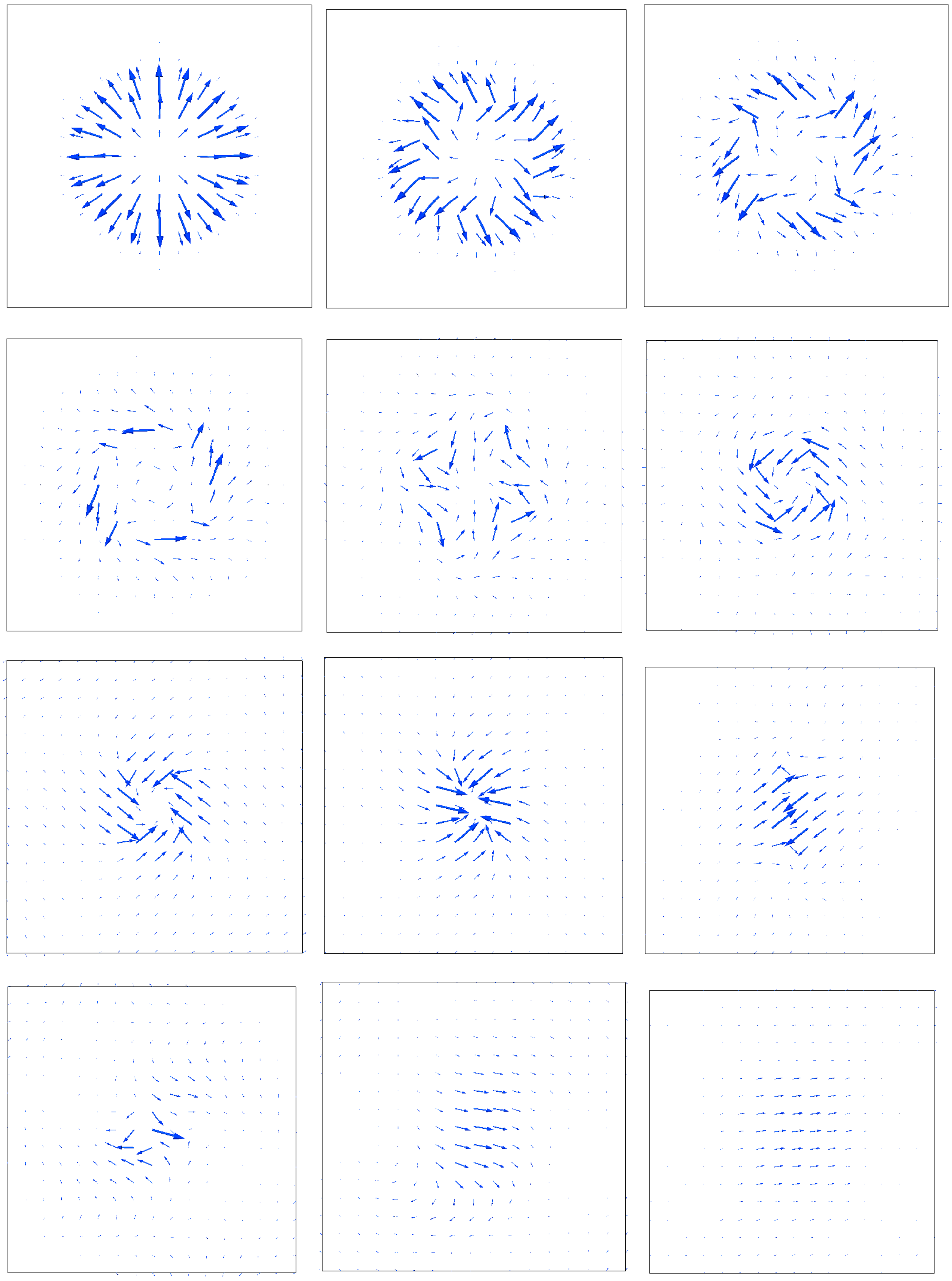

Figure 15: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with exchange and magnetostriction (20), with $\alpha = 1/4$, $h = 1/16$, for time $t = 0, 0.0002, 0.0005, 0.001, 0.002, 0.003, 0.01, 0.024, 0.041, 0.05, 0.06$.

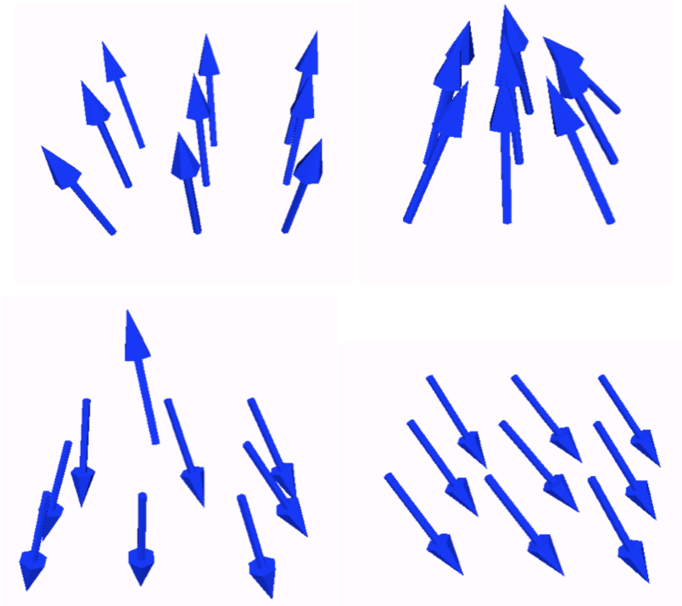

Figure 16: Snapshots of the numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with exchange and magnetostriction (20), with $\alpha = 1/4$, $h = 1/16$, for time $t = 0, 0.02874, 0.04869, 0.06564$.

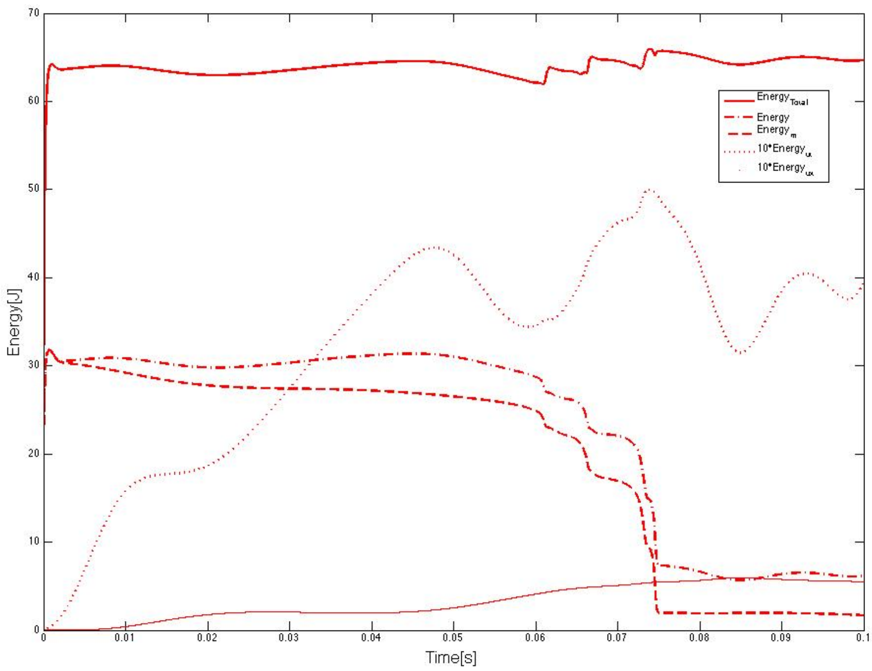

Figure 17: The different energies for the magnetization $\mathbf{m}$ and the displacement $\mathbf{u}$ from the problem (20), with $h = 1/16$, $\boldsymbol{\lambda}_e = 40$, $\boldsymbol{\lambda}_m = 10$ and $\alpha = 1$.

Impact of the tensors $\boldsymbol{\lambda}$ on the Blow-Up time : On figure 18 and 19 we plot the energy and the $\mathbf{W}^{1,\infty}$ semi-norm for different values of $\boldsymbol{\lambda}^m$ respectively $\boldsymbol{\lambda}^e$. The magnetostriction have an impact on the exchange field only for $\boldsymbol{\lambda}^m \boldsymbol{\lambda}^e \geq 300$. For higher values, the Blow-Up time increases quickly and its existence is therefore not guarantee.

Impact of α on the Blow-Up time :

Finally, we show on figure 20 the energy and the $\mathbf{W}^{1,\infty}$ semi-norm for different values of α. The more we decrease α, the more the Blow-Up time decreases. The oscillations of the energy and $\mathbf{W}^{1,\infty}$ semi-norm for $\alpha \geq 1/20$ mean that the algorithm start to become unstable for too small value of α.

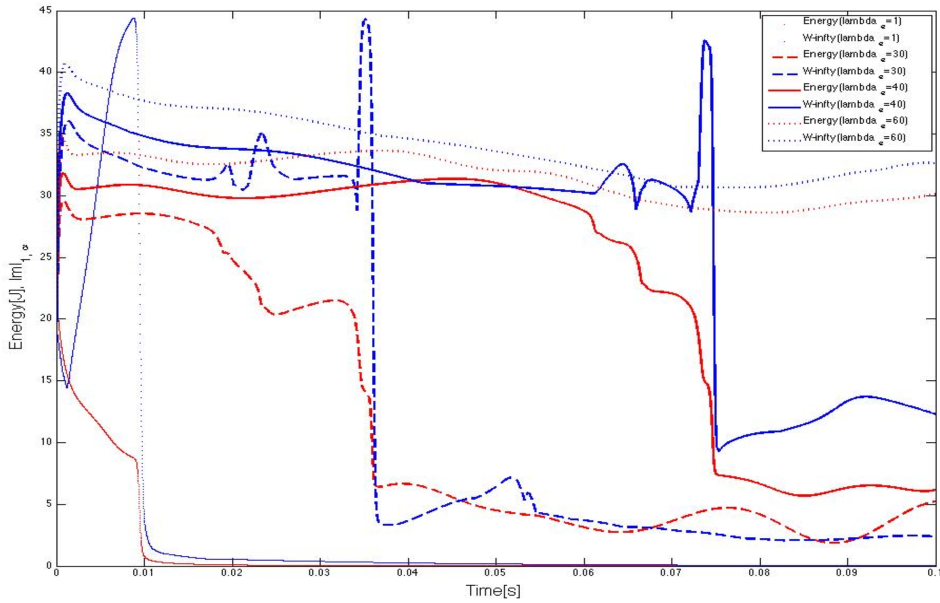

Figure 18: Energy and $\mathbf{W}^{1,\infty}$ semi-norm for the magnetization $\mathbf{m}$ and the displacement $\mathbf{u}$ from the problem (20), with $h = 1/16$, $\boldsymbol{\lambda}_m = 10$, $\alpha = 1$ and for different values of $\boldsymbol{\lambda}_e$.

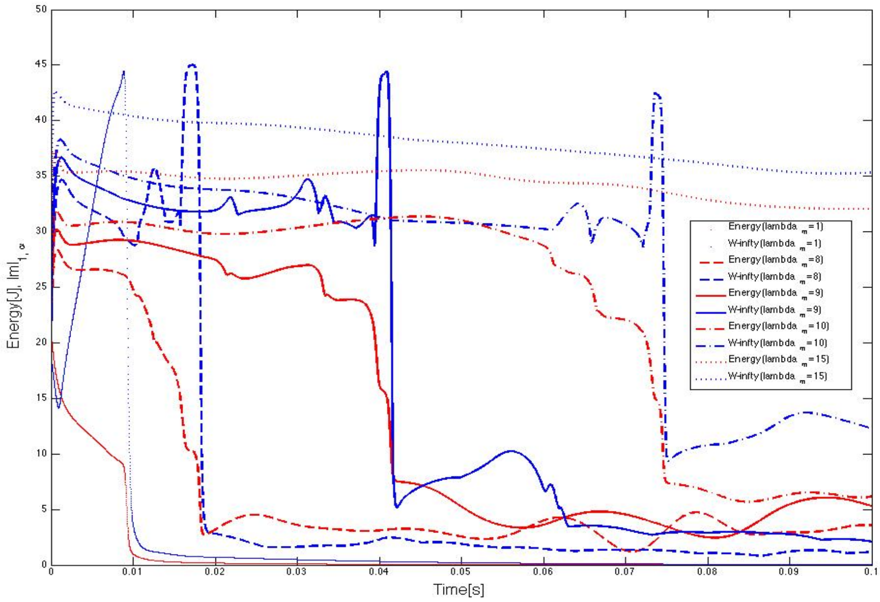

Figure 19: Energy and $\mathbf{W}^{1,\infty}$ semi-norm for the magnetization $\mathbf{m}$ and the displacement $\mathbf{u}$ from the problem (20), with $h = 1/16$, $\boldsymbol{\lambda}_e = 40$, $\alpha = 1$ and for different values of $\boldsymbol{\lambda}_m$.

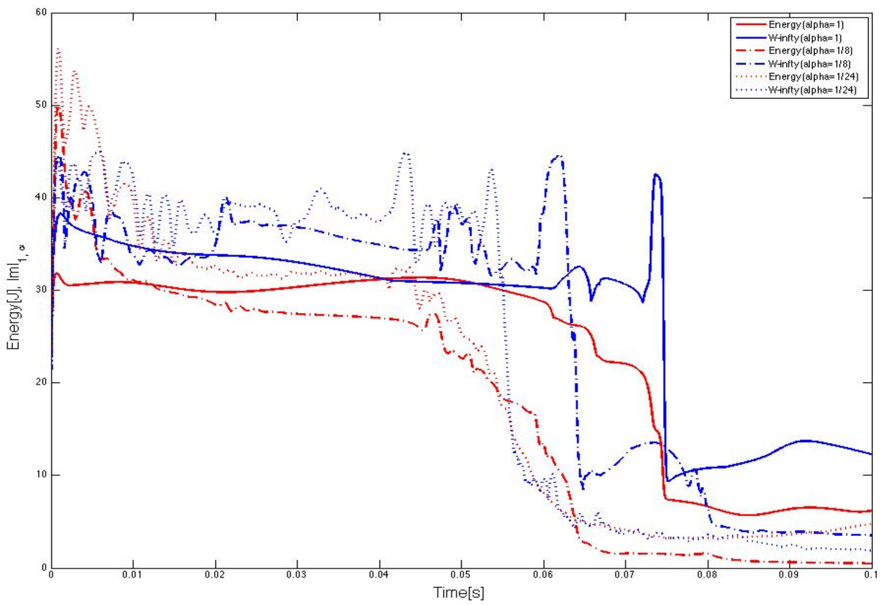

Figure 20: Energy and $\mathbf{W}^{1,\infty}$ semi-norm for the magnetization $\mathbf{m}$ and the displacement $\mathbf{u}$ from the problem (20), with $h = 1/16$, $\boldsymbol{\lambda}_m = 10$, $\boldsymbol{\lambda}_e = 40$ and different values of α.

4.3 A problem with magnetrostriction

We take $\Omega = (0,1)^2$ and $\mathbf{m}_0 : \Omega \to \sqrt{2}\mathbb{S}^2$ be defined by

$$\mathbf{m}_0(\mathbf{x}) = \begin{pmatrix} \exp^{-200|\mathbf{x}-\mathbf{x}_0|^2} \\ \sqrt{2-(\exp^{-200|\mathbf{x}-\mathbf{x}_0|^2})^2} \\ 0 \end{pmatrix}$$

with $\mathbf{x}_0 = (0.5, 0.5)$. We solve the LLG equation with magnetostriction:

$$\begin{cases} \mathbf{m}_t + \alpha \mathbf{m} \times \mathbf{m}_t = |\mathbf{m}|\gamma(1+\alpha^2)\mathbf{m} \times \mathbf{h}_\sigma, \\ \mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = 0. \end{cases} \tag{21}$$

with $\gamma = 4$ and $|\mathbf{m}| = \sqrt{2}$. For the magnetostriction, we took $\boldsymbol{\lambda}^e$ and $\boldsymbol{\lambda}^m$ 2×2 tensors such that

$$\boldsymbol{\lambda}^e = \lambda^e_{1111} = \lambda^e_{2222} = 1, \quad \boldsymbol{\lambda}^m = \lambda^m_{1111} = \lambda^m_{2222} = 1.$$

For the elastodynamics equation we take homogeneous Dirichlet condition with $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{v}_0 = \mathbf{0}$. The triangulation $\mathcal{T}_l$ used in the numerical simulation are defined through a positive integer l and consists of 2^{6l+1} halved square with edge length $h = 2^{-6l}$. The center of the triangulation has been refined. The others parameters used were

$$\alpha = 1, \eta = 4, \quad \varepsilon = 10^{-16}, \quad k = 10^{-2} \text{ (weakly dependence with h)} .$$

The initial condition of the magnetization is radially symmetric. When we compute the LLG equation with the magnetostriction, the displacement evoluate from around the center and the symmetry is still conserved. The magnetization, the displacement and the mesh can be seen on figure 21 and 22 at time $t = 1$ and on figure 23 for $t = 0$ and $t = 0.5$.

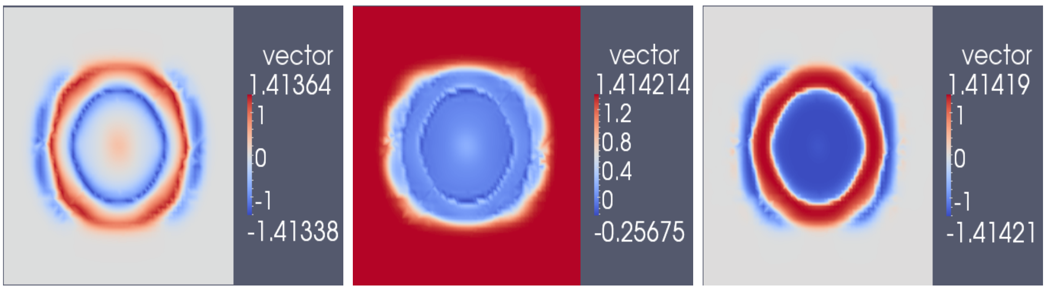

Figure 21: Isoclines of the numerical approximation of the magnetization $m_1(t,\cdot)$, $m_2(t,\cdot)$ and $m_3(t,\cdot)$ for the LLG equation with magnetostriction (21), with $h = 1/30$, $\alpha = 1$ and at time $t = 1$.

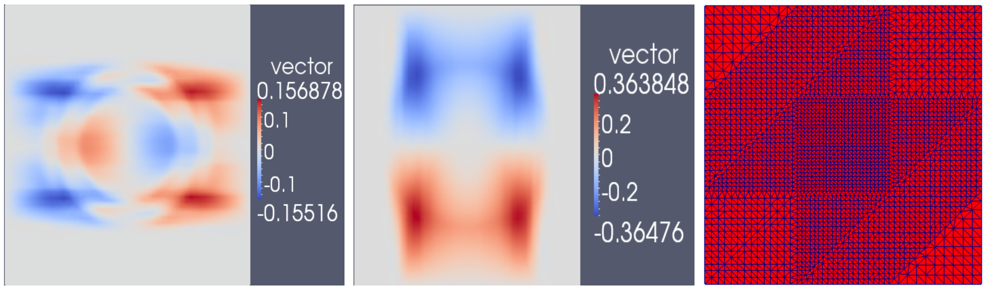

Figure 22: Isocline of the numerical approximation of the displacement $u_1(t,\cdot)$, $u_2(t,\cdot)$ for the LLG equation with magnetostriction (21), and the mesh $\mathcal{T}_h$, with $h = 1/30$, $\alpha = 1$ and at time $t = 1$.

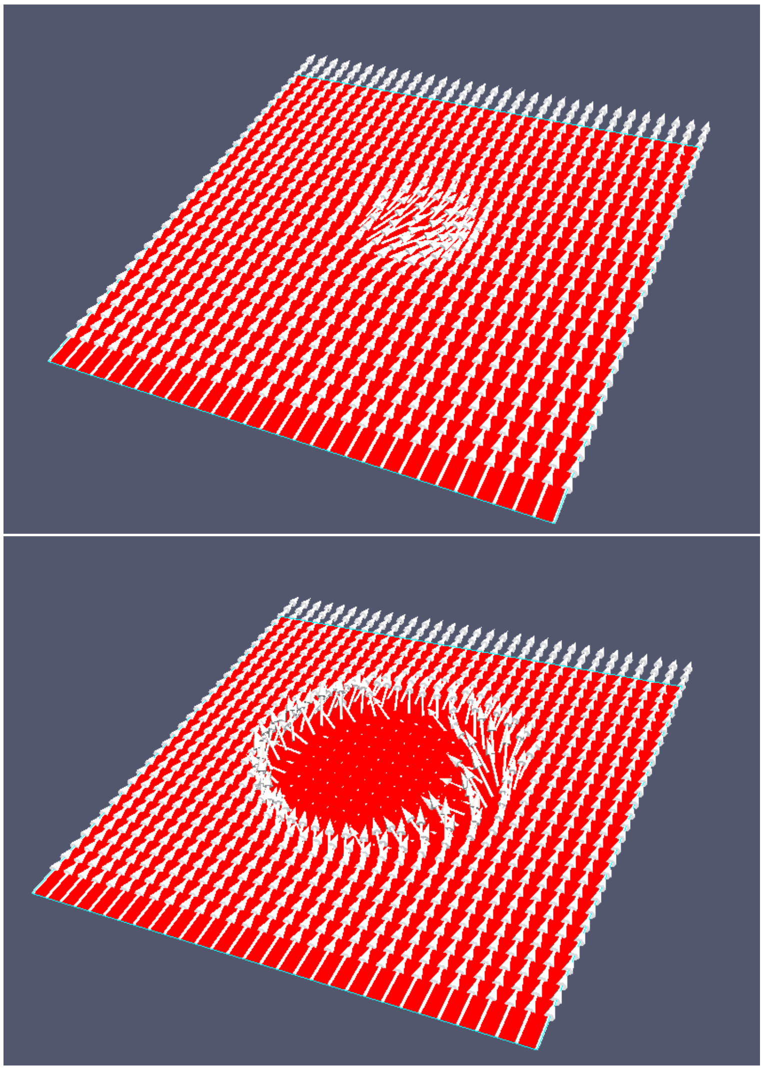

Figure 23: Numerical approximation of the magnetization $\mathbf{m}$ for the LLG equation with magnetostriction (21), with $h = 1/30$, $\alpha = 1$ and at time $t = 0, 0.5$.

5 Conclusion

We have presented an implicit convergent finite element method to solve the Landau-Lifshitz-Gilbert equation with exchange and magnetostriction (17). The convergence of the fixed-point iteration is established for $k = \mathcal{O}(h^2)$ and the order of convergence mesured is $\mathcal{O}(k) + \mathcal{O}(h)$.

Numerical experiments motivating blow-up have been discussed. The magnetostriction field has an impact on the blow-up time only if the values for the tensors $\boldsymbol{\lambda}$ are chosen enough high. In this case the blow-up time increases quickly and can possibly disappears. These high values for $\boldsymbol{\lambda}$ imply also that the energy can no longer decreases uniformly, due to the remaining term in the energy inequality:

$$\int_0^T (\mathrm{d}_t \mathbf{m}_h^{j+1}, \boldsymbol{\lambda}^m \boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}) \bar{\mathbf{m}}_h^{j+\frac{1}{2}}).$$

The algorithm suffers sometimes from long computations time for small values of h and k, which can motivates to use for example a newton algorithm instead of the fixed-point iteration to solve the non-linear system.

In order to improve the efficiency of the method, we could also solve the following coupled system:

Algorithm 5.1
Let $\boldsymbol{m}_h^0 \in \boldsymbol{V}_h$, $\boldsymbol{u}_h^0, \boldsymbol{v}_h^0 \in \boldsymbol{V}_{h,0}$. *Given a time step* $k > 0$, $j \geq 0$ *and* $\boldsymbol{m}_h^j \in \boldsymbol{V}_h$, $\boldsymbol{u}_h^j, d_t \boldsymbol{u}_h^j \in \boldsymbol{V}_{h,0}$, *determine* $(\boldsymbol{u}_h^{j+1}, \boldsymbol{m}_h^{j+1}) \in (\boldsymbol{V}_h; \boldsymbol{V}_{h,0})$ *from*

$$\left\{ \begin{array}{lll} (d_t^2 \boldsymbol{u}_h^{j+1}, \boldsymbol{\varphi}_h) & + & (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = (\boldsymbol{\lambda}^e \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h \\ (d_t \boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h & + & \alpha(\boldsymbol{m}_h^j \times d_t \boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h \\ & = & (1+\alpha^2)(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h \bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} + P_{\boldsymbol{V}_h} \bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}}, \boldsymbol{\phi}_h)_h \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_{h,0}. \end{array} \right. \quad ll \tag{22}$$

with

$$\bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}} = \boldsymbol{\lambda}^m \boldsymbol{\lambda}^e (\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1}) - \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^{j+1})) \bar{\boldsymbol{m}}_h^{j+\frac{1}{2}}$$

This algorithm is more complicated than the one we use, because the coupled term is mixed with the non-linearity.

Finally, we can extend our algorithm with the Maxwell system of [3] and the anisotropy field to obtain a complete implicit algorithm using reduced integration for the LLG equation.

6 Appendix

6.1 Notations and Preliminaries

6.1.1 Functional Analysis

If not stated otherwise, we take $\Omega \subset \mathbb{R}^N$, $(N = 1, 2, 3)$ to be a domain with Lipschitz continuous boundary Γ. For two vector functions $\mathbf{u}, \mathbf{v}$ we define the inner product as

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega < \mathbf{u}, \mathbf{v} > .$$

Definition 6.1 ($\mathbf{L}^p$ Space)
We denote by $\boldsymbol{L}^p(\Omega)$ the space of $p-$integrable vector function $\boldsymbol{u}$ defined on Ω, with the norm

$$\|\boldsymbol{u}\|_{L^p(\Omega)} = \left(\int_\Omega |\boldsymbol{u}(\boldsymbol{x})|^p\right)^{\frac{1}{p}} .$$

Here $|\cdot|$ denote the standard euclidian norm.

Proposition 6.2 (Holder's Inequality)
Let $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, then

$$\|\boldsymbol{u}\boldsymbol{v}\| \leq \|\boldsymbol{u}\|_{L^p}\|\boldsymbol{v}\|_{L^q},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. When $p = q = 2$, we obtain the usual Cauchy-Schwarz inequality.

Proposition 6.3 (Korn's Inequality)
Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, then there exists constant $C_1 > 0$, $C_2 > 0$ such that

$$C_1\|\boldsymbol{u}\|_{H^1} \leq \|\nabla\boldsymbol{u}\|_{L^2} \leq C_2\|\boldsymbol{u}\|_{H^1}.$$

Proposition 6.4 (Young Inequality)
For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, we have for all arbitrary $\eta > 0$

$$(\boldsymbol{u}, \boldsymbol{v}) \leq \eta\frac{1}{2}\|\boldsymbol{u}\|^2_{L^2} + C_\eta\frac{1}{2}\|\boldsymbol{v}\|^2_{L^2}.$$

Lemma 6.5 (Gronwall Discrete Version)
Let a_i, b_i be sequences of non-negative real numbers and let $k \geq 0$. If for $N \in \mathbb{N}$ holds

$$a_N \leq b_N + \sum_{i=1}^{N-1} a_i k,$$

then

$$a_N \leq b_N + \exp Nk \sum_{i=1}^{N-1} b_i k.$$

Lemma 6.6 (Abel's summation)
Let $a_i \in \mathbb{R}$ for $i = 0, \ldots, N$. Then

$$\sum_{i=1}^{N}(a_i - a_{i-1}, a_i) = \frac{1}{2}\|a_j\|^2 - \frac{1}{2}\|a_0\|^2 + \frac{1}{2}\sum_{i=1}^{N}\|a_i - a_{i-1}\|^2.$$

Definition 6.7 (Sobolev Space)
We denote by $\boldsymbol{W}^{k,p}(\Omega)$ *the space of vector functions with weak* $k-th$ *derivatives from* $\boldsymbol{L}^p$*, with the norm*

$$\|\boldsymbol{f}\|_{k,p} = \sum_{|\alpha|\leq k} \|D^{\alpha}\boldsymbol{u}\|_{L^p},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ *is a multi-index and* $D^{\alpha}\boldsymbol{u} = \frac{\partial^{|\alpha|}\boldsymbol{u}}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}}$*. When* $p = 2$*, we note* $\boldsymbol{W}^{k,2}(\Omega) = \boldsymbol{H}^k(\Omega)$*.*

Theorem 6.8 (Sobolev Embedding)
For the space $\boldsymbol{H}^k(\Omega)$*, the following embeddings hold:*

1. *if* $0 \leq 2k < N$*, then* $\boldsymbol{H}^k(\Omega) \subset \boldsymbol{L}^{p*}(\Omega), \quad p* = \frac{2N}{N-2k}$;
2. *if* $2k = N$*, then* $\boldsymbol{H}^k(\Omega) \subset \boldsymbol{L}^q(\Omega), \quad q \in [2, \infty)$;
3. *if* $2(k-m) > N$*, then* $\boldsymbol{H}^k(\Omega) \subset \boldsymbol{C}^m(\bar{\Omega})$.

6.1.2 Cross Product

The particularity of the LLG equation is the term that contains a cross product between the magnetization and the effective field. Here, $\mathbf{V}$ will denote a enough regular Sobolev space of vector field in $\mathbb{R}^3$.

Definition 6.9 (Cross Product)
The cross-product of two vector fields $\mathbf{u}$*,* $\boldsymbol{v} \in \boldsymbol{V}$ *is defined by*

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

The cross-product of a vector field $\mathbf{u}$ *and the Jacobian matrix* $\nabla\boldsymbol{v}$ *is given by*

$$\boldsymbol{u} \times \nabla\boldsymbol{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} \nabla v_1 \\ \nabla v_2 \\ \nabla v_3 \end{pmatrix}.$$

It can be written

$$\begin{aligned} \boldsymbol{u} \times \nabla\boldsymbol{v} &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} \nabla v_1 \\ \nabla v_2 \\ \nabla v_3 \end{pmatrix} = \begin{pmatrix} u_2\nabla v_3 - u_3\nabla v_2 \\ u_3\nabla v_1 - u_1\nabla v_3 \\ u_1\nabla v_2 - u_2\nabla v_1 \end{pmatrix} \\ &= \begin{pmatrix} u_2\partial_{x_1}y_3 - u_3\partial_{x_1}v_2, & \ldots & , u_2\partial_{x_3}y_3 - u_3\partial_{x_3}v_2 \\ & \ldots & \\ u_1\partial_{x_1}v_2 - u_2\partial_{x_1}v_1, & \ldots & , u_1\partial_{x_3}v_2 - u_2\partial_{x_3}v_1 \end{pmatrix}. \end{aligned}$$

Proposition 6.10
For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$*, we have*

1. $\boldsymbol{u} \times \boldsymbol{u} = 0$;

2. $\boldsymbol{u}\times\boldsymbol{v}=-\boldsymbol{v}\times\boldsymbol{u}$;

3. $<\boldsymbol{u}\times\boldsymbol{v},\boldsymbol{u}>=0$;

4. $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})=<\boldsymbol{u},\boldsymbol{w}>\boldsymbol{v}-<\boldsymbol{u},\boldsymbol{v}>\boldsymbol{w}$;

5. $<\boldsymbol{u}\times\boldsymbol{v},\boldsymbol{w}>=<\boldsymbol{w}\times\boldsymbol{u},\boldsymbol{v}>=<\boldsymbol{v}\times\boldsymbol{w},\boldsymbol{u}>$;

6. $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})+\boldsymbol{v}\times(\boldsymbol{w}\times\boldsymbol{u})+\boldsymbol{w}\times(\boldsymbol{u}\times\boldsymbol{v})=0$.

Proof. Standard computations. □

Remark 6.11
The last property from the precedent proposition is called the Jacoby identity. In fact, the cross product can be seen as a simple Lie product.

Definition 6.12 (Norm L^p of Jacobian Matrix)
For a vector field $\boldsymbol{u}\in\boldsymbol{V}$, the L^2 norm of the Jacobian Matrix $\nabla\boldsymbol{u}$ is defined with respect to the scalar product of two matrix A,B:

$$<A,B>=Tr(A^TB).$$

Consequently,

$$\begin{aligned}\|\nabla\boldsymbol{u}\|_{L^2} &= \int_\Omega <\nabla\boldsymbol{u},\nabla\boldsymbol{u}>dx=\int_\Omega Tr(\nabla\boldsymbol{u}^T\nabla\boldsymbol{u})dx\\ &= \sum_{ij}\int_\Omega\left(\frac{\partial u_i}{\partial u_j}\right)^2dx=\sum_{ij}\left\|\frac{\partial u_i}{\partial u_j}\right\|^2_{L^2}.\end{aligned}$$

Proposition 6.13
For all $\boldsymbol{u},\boldsymbol{w}\in\boldsymbol{V}$, we have

$$<\nabla\boldsymbol{u},\nabla(\boldsymbol{u}\times\boldsymbol{v})>=<\nabla\boldsymbol{u},\boldsymbol{u}\times\nabla\boldsymbol{v}>.$$

Proof. It sufficies to see that $\nabla(\mathbf{u}\times\mathbf{v})=\nabla\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\nabla\mathbf{v}$. Then,

$$<\nabla\mathbf{u},\nabla(\mathbf{u}\times\mathbf{v})>=<\nabla\mathbf{u},\nabla\mathbf{u}\times\mathbf{v}>+<\nabla\mathbf{u},\mathbf{u}\times\nabla\mathbf{v}>.$$

On the other hand, the matrix $\nabla\mathbf{u}$ is orthogonal to the matrix $\nabla\mathbf{u}\times\mathbf{v}$. Consequently,

$$<\nabla\mathbf{u},\nabla\mathbf{u}\times\mathbf{v}>=0,$$

which achieves the proof. □

Definition 6.14 (Divergence of a matrix)
Let $A\in\boldsymbol{V}\times\boldsymbol{V}$ be a matrix of function. The divergence of A is defined by

$$\nabla\cdot A=\begin{pmatrix}\nabla\cdot(a_{11},\dots,a_{1n})\\ \cdots\\ \nabla\cdot(a_{n1},\dots,a_{nn})\end{pmatrix}.$$

Proposition 6.15
For all $\boldsymbol{m}$ we have

$$\nabla \cdot (\boldsymbol{m} \times \nabla \boldsymbol{m}) = \boldsymbol{m} \times \Delta \boldsymbol{m}.$$

Proof. We show the assertion for the first component. We have

$$\begin{aligned}(\nabla(\mathbf{m} \times \nabla \mathbf{m}))_1 &= \nabla \cdot (m2\nabla m3 - m3\nabla m2)\\ &= m2\Delta m3 - m3\Delta m2 + \nabla m2 \cdot \nabla m3 - \nabla m3 \cdot \nabla m2\\ &= m2\Delta m3 - m3\Delta m2 = (\mathbf{m} \times \Delta \mathbf{m})_1.\end{aligned}$$

The same computation for the others components finishes the proof. □

6.2 Alternative Energy Inequality

The following proof is more elegant than the one presented, but the non-linear system associated can be more complicated.

Algorithm 6.16
Let $\boldsymbol{m}_h^0 \in \boldsymbol{V}_h$ and $\boldsymbol{u}_h^0, \boldsymbol{v}_h^0 \in \boldsymbol{V}_{h,0}$. Given a time step $k > 0$, $j \geq 0$ and $\boldsymbol{m}_h^j \in \boldsymbol{V}_h$, $\boldsymbol{u}_h^j, d_t\boldsymbol{u}_h^j \in \boldsymbol{V}_{h,0}$.

1. *Determine $\boldsymbol{m}_h^{j+1} \in \boldsymbol{V}_h$ from*

$$\begin{aligned}(d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h &+ \alpha(\boldsymbol{m}_h^j \times d_t\boldsymbol{m}_h^{j+1}, \boldsymbol{\phi}_h)_h\\ &= (1+\alpha^2)(\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} \times (\tilde{\Delta}_h \bar{\boldsymbol{m}}_h^{j+\frac{1}{2}} + P_{\boldsymbol{V}_h}\bar{\boldsymbol{h}}_{\sigma_h}^{j+\frac{1}{2}}), \boldsymbol{\phi}_h)_h \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{V}_h.\end{aligned}$$

with

$$\bar{\boldsymbol{h}}_{\sigma_h}^{j+1} = \boldsymbol{\lambda}^m\boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}(\boldsymbol{u}_h^j) - \boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^{j+1}))\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}};$$

2. *Determine $\boldsymbol{u}_h^{j+1} \in \boldsymbol{V}_{h,0}$ from*

$$(d_t^2\boldsymbol{u}_h^{j+1}, \boldsymbol{\varphi}_h) + (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) = (\boldsymbol{\lambda}^e\bar{\boldsymbol{m}}_h^{j+\frac{1}{2}}, \boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)) \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h; \qquad (23)$$

3. *set $j = j+1$ and return to 1.*

The only difference with the algorithm (3.11) is $\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1})$ instead of $\boldsymbol{\varepsilon}^m(\mathbf{m}_h^j)$ in the LLG equation. Consequently, supposing that $|\mathbf{m}_h^0(\mathbf{x}_l)| = 1$ for all $l \in L$, the sequences $(\mathbf{m}_h^j, \mathbf{u}_h^j)_{j\geq 0}$ obtained from algorithm (6.16) satisfies the property of the proposition (3.13) for all $j \geq 0$:

1. $|\mathbf{m}_h^{j+1}(\mathbf{x}_l)| = 1 \quad \forall l \in L$,
2. $$\frac{1}{2}\mathrm{d}_t\|\nabla \mathbf{m}_h^{j+1}\|_{L^2}^2 + \frac{\alpha}{1+\alpha^2}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 = (\mathrm{d}_t\mathbf{m}_h^{j+1}, \bar{\mathbf{h}}_{\sigma_h}^{j+\frac{1}{2}})$$

The following proof is more elegant than the lemma (6.19). We need nevertheless the following supposition on the magneto-dynamical tensors:

Definition 6.17

$$\begin{aligned}&\sum_{lqrkst} \lambda_{qrkl}^m\lambda_{qrst}^e < d_t m_{h,l}^{j+1}\bar{m}_{h,k}^{j+\frac{1}{2}}, \varepsilon_{st}(u_h^{j+1}) >\\ =\ &\sum_{lqrkst} \lambda_{qrkl}^m\lambda_{qrst}^e < d_t m_{h,k}^{j+1}\bar{m}_{h,l}^{j+\frac{1}{2}}, \varepsilon_{qr}(u_h^{j+1}) > .\end{aligned}$$

Proposition 6.18
The following estimate holds for the solution $\boldsymbol{u}_h$*:*

$$\begin{aligned} \frac{1}{2}\|d_t\boldsymbol{u}_h^N\|_{L^2}^2 &+ \frac{1}{2}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^N)\|_{L^2}^2 \\ &+ \frac{1}{2}\sum_{j=1}^{N-1}\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^{j+1})-\boldsymbol{\varepsilon}(\boldsymbol{u}_h^j)\|_{L^2}^2+\|d_t\boldsymbol{u}_h^{j+1}-d_t\boldsymbol{u}_h^j\|_{L^2}^2\Big) \\ &\leq C+(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^N),\boldsymbol{\varepsilon}(\boldsymbol{u}_h^N))-(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\boldsymbol{m}_h^1),\boldsymbol{\varepsilon}(\boldsymbol{u}_h^0)) \\ &- k\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\lambda}^m d_t\boldsymbol{m}_h^{j+1}\bar{\boldsymbol{m}}_h^{j+\frac{1}{2},T},\boldsymbol{\varepsilon}(\boldsymbol{u}_h^j)). \end{aligned}$$

Proof. From the elastodynamics equation:

$$(\mathrm{d}_t^2\mathbf{u}_h^{j+1},\boldsymbol{\varphi}_h)+(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h))=(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\boldsymbol{\varphi}_h)),$$

we choose $\boldsymbol{\phi}_h=\mathbf{u}_h^{j+1}-\mathbf{u}_h^j$ and sum over j to have:

$$\begin{aligned} \sum_{j=0}^{N-1}(\mathrm{d}_t^2\mathbf{u}_h^{j+1},\mathbf{u}_h^{j+1}-\mathbf{u}_h^j) &+ \sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}-\mathbf{u}_h^j)) \\ &= \sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^j),\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}-\mathbf{u}_h^j)). \end{aligned}$$

We use the Abel summation on the first sum

$$\begin{aligned} \sum_{j=0}^{N-1}(\mathrm{d}_t^2\mathbf{u}_h^{j+1},\mathbf{u}_h^{j+1}-\mathbf{u}_h^j) &= \sum_{j=0}^{N-1}(\mathrm{d}_t\mathbf{u}_h^{j+1}-\mathrm{d}_t\mathbf{u}_h^j,\mathrm{d}_t\mathbf{u}_h^{j+1}) \\ = \frac{1}{2}\|\mathrm{d}_t\mathbf{u}_h^N\|_{L^2}^2-\frac{1}{2}\|\mathrm{d}_t\mathbf{u}_0\|_{L^2}^2 &+ \frac{1}{2}\sum_{j=1}^{N-1}\|\mathrm{d}_t\mathbf{u}_h^{j+1}-\mathrm{d}_t\mathbf{u}_h^j\|_{L^2}^2 \end{aligned}$$

and we do the same for the second one:

$$\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}-\mathbf{u}_h^j)).$$

Adding the last two identities, we obtain

$$\begin{aligned} &\frac{1}{2}\|\mathrm{d}_t\mathbf{u}_h^N\|_{L^2}^2+\frac{1}{2}\|\varepsilon(\mathbf{u}_h^N)\|_{L^2}^2 \\ +\ &\frac{1}{2}\sum_{j=1}^{N-1}\Big(\|\varepsilon(\mathbf{u}_h^{j+1})-\varepsilon(\mathbf{u}_h^j)\|_{L^2}^2+\|(\mathrm{d}_t\mathbf{u}_h^{j+1}-\mathrm{d}_t\mathbf{u}_h^j)\|_{L^2}^2\Big) \\ \leq\ &\underbrace{\frac{1}{2}\|\mathrm{d}_t\mathbf{u}_0\|_{L^2}^2+\frac{1}{2}\|\boldsymbol{\lambda}^e\varepsilon(\mathbf{u}_0)\|_{L^2}^2}_{=C}+\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}-\mathbf{u}_h^j)). \end{aligned}$$

The last sum can be written

$$\begin{aligned}
&\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^{j+1}-\mathbf{u}_h^j))\\
=\ &(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N),\boldsymbol{\varepsilon}(\mathbf{u}_h^N))-(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1),\boldsymbol{\varepsilon}(\mathbf{u}_h^0))\\
-\ &\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e(\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1})-\boldsymbol{\varepsilon}^m(\mathbf{m}_h^j)),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\\
=\ &(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N),\boldsymbol{\varepsilon}(\mathbf{u}_h^N))-(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1),\boldsymbol{\varepsilon}(\mathbf{u}_h^0))\\
-\ &k\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\mathrm{d}_t\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j)).
\end{aligned}$$

We now show that

$$k\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\mathrm{d}_t\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))=\sum_{j=0}^{N-1}\boldsymbol{\lambda}^m\boldsymbol{\lambda}^e(\mathrm{d}_t(\mathbf{m}_h^{j+1})\bar{\mathbf{m}}_h^{j+\frac{1}{2},T},\boldsymbol{\varepsilon}(\mathbf{u}_h^j)).$$

Using the symmetry of $\boldsymbol{\varepsilon}(\mathbf{u}_h^j)$ and $\boldsymbol{\lambda}^e\boldsymbol{\lambda}^m$ we have:

$$\begin{aligned}
&k\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\mathrm{d}_t\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\\
=\ &\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\lambda}^m(\mathbf{m}_h^{j+1}\mathbf{m}_h^{j+1,T}-\mathbf{m}_h^j\mathbf{m}_h^{j,T}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\\
=\ &\sum_{j=0}^{N-1}(\boldsymbol{\lambda}^e\boldsymbol{\lambda}^m(\mathbf{m}_h^{j+1}\mathbf{m}_h^{j+1,T}-\mathbf{m}_h^j\mathbf{m}_h^{j+1,T}+\mathbf{m}_h^{j+1}\mathbf{m}_h^{j,T}-\mathbf{m}_h^j\mathbf{m}_h^{j,T}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j))\\
=\ &2k\sum_{j=0}^{N-1}\left(\boldsymbol{\lambda}^e\boldsymbol{\lambda}^m\left(\frac{\mathbf{m}_h^{j+1}-\mathbf{m}_h^j}{2k}\right)\mathbf{m}_h^{j+1,T}+\left(\frac{\mathbf{m}_h^{j+1}-\mathbf{m}_h^j}{2k}\right)\mathbf{m}_h^{j,T}),\boldsymbol{\varepsilon}(\mathbf{u}_h^j)\right)\\
=\ &\sum_{j=0}^{N-1}\boldsymbol{\lambda}^m\boldsymbol{\lambda}^e(\mathrm{d}_t(\mathbf{m}_h^{j+1})\bar{\mathbf{m}}_h^{j+\frac{1}{2},T},\boldsymbol{\varepsilon}(\mathbf{u}_h^j)),
\end{aligned}$$

where in the last step we used the assertion (6.17). □

Lemma 6.19
There exists $\eta_1 > 0$ and $\eta_2 > 0$ sufficiently small such that

$$\begin{aligned}
\frac{1}{2}\|\nabla\boldsymbol{m}_h^N\|_{L^2}^2\ &+\ \underbrace{\left(\frac{2\alpha}{1+\alpha^2}-\eta_1\right)}_{>0}\sum_{j=0}^{N-1}\|d_t\boldsymbol{m}_h^{j+1}\|_h^2k\\
&+\ \frac{1}{2}\|d_t\boldsymbol{u}_h^N\|_{L^2}^2+\underbrace{\left(\frac{1}{2}-\frac{1}{2}\eta_2\right)}_{>0}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h^N)\|_{L^2}^2\\
&\leq\ C+\frac{1}{2}\|\nabla\boldsymbol{m}_0\|_{L^2}^2+\frac{1}{2}\|\boldsymbol{v}_0\|_{L^2}^2+\frac{1}{2}\|\varepsilon(\boldsymbol{u}_0)\|_{L^2}^2.
\end{aligned}$$

Proof. We sum the assertions from (3.13) and (6.18) and we integrate over time. With

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\sum_{j=0}^{N-1} \mathrm{d}_t\|\nabla \mathbf{m}_h^{j+1}\|_{L^2}^2 k + \frac{\alpha}{1+\alpha^2}\sum_{j=0}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 k \\ &+ \frac{1}{2}\|\mathrm{d}_t\mathbf{m}_h^N\|_{L^2}^2 + \frac{1}{2}\|\boldsymbol{\varepsilon}(\mathbf{m}_h^N)\|_{L^2}^2 \\ &+ \frac{1}{2}\sum_{j=1}^{N-1}\Big(\|\varepsilon(\mathbf{u}_h^{j+1}) - \varepsilon(\mathbf{u}_h^j)\|_{L^2}^2 + \|\mathrm{d}_t\mathbf{u}_h^{j+1} - \mathrm{d}_t\mathbf{u}_h^j\|_{L^2}^2\Big) \\ &- \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 - \frac{1}{2}\|\boldsymbol{\lambda}^e\varepsilon(\mathbf{u}_0)\|_{L^2}^2,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{A} &\leq C - \sum_{j=0}^{N-1}(\mathrm{d}_t\mathbf{m}_h^{j+1}, \boldsymbol{\lambda}^m\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1})\bar{\mathbf{m}}_h^{j+\frac{1}{2}})k \\ &+ (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N), \boldsymbol{\varepsilon}(\mathbf{u}_h^N)) - (\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1), \boldsymbol{\varepsilon}(\mathbf{u}_h^0)).\end{aligned}$$

Using then the Cauchy's and Young inequalities, we obtain for any positive η_1 and η_2:

$$\begin{aligned}\mathcal{A} &\leq C + \frac{1}{2}\sum_{j=0}^{N-1}\eta_1\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_{L^2}^2 k + \frac{1}{2}\sum_{j=0}^{N-1} C_{\eta_1}\|\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1})\bar{\mathbf{m}}_h^{j+\frac{1}{2}}\|_{L^2}^2 k \\ &+ \frac{1}{2}\eta_2\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 + \frac{1}{2}C_{\eta_2}\|(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N)\|_{L^2}^2 + C\|\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1)\|_{L^2}^2 + C\|\boldsymbol{\varepsilon}(\mathbf{u}_h^0)\|_{L^2}^2.\end{aligned}$$

Choosing η_1, η_2 sufficiently small such that

$$\eta_1 < \frac{2\alpha}{1+\alpha^2}, \quad \eta_2 < 1,$$

and using the boundeness of the coefficients of $\boldsymbol{\lambda}^e$ and that

$$|\mathbf{m}_h^i| = 1 \quad \forall i = 0, \ldots, N, \tag{24}$$

there exists a constant $C > 0$ such that

$$\|\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^{j+1})\bar{\mathbf{m}}_h^{j+\frac{1}{2}}\|_{L^2}^2 \leq C, \quad \|\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^1)\|_{L^2}^2 \leq C, \quad \|(\boldsymbol{\lambda}^e\boldsymbol{\varepsilon}^m(\mathbf{m}_h^N)\|_{L^2}^2 \leq C.$$

Consequently,

$$\mathcal{A} \leq C.$$

which is written

$$\begin{aligned}\frac{1}{2}\|\nabla\mathbf{m}_h^N\|_{L^2}^2 &+ \left(\frac{2\alpha}{1+\alpha^2} - \eta_1\right)\sum_{j=0}^{N-1}\|\mathrm{d}_t\mathbf{m}_h^{j+1}\|_h^2 k \\ &+ \frac{1}{2}\|\mathrm{d}_t\mathbf{u}_h^N\|_{L^2}^2 + \left(\frac{1}{2} - \frac{1}{2}\eta_2\right)\|\boldsymbol{\varepsilon}(\mathbf{u}_h^N)\|_{L^2}^2 \\ &\leq C + \frac{1}{2}\|\nabla\mathbf{m}_0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_0\|_{L^2}^2 + \frac{1}{2}\|\varepsilon(\mathbf{u}_0)\|_{L^2}^2\end{aligned}$$

and completes the proof. □

6.3 Implementation with FreeFem++

6.3.1 FreeFem++

FreeFem++ is a software focused in solving partial differential equation using the finite element method. Written in C++, FreeFem++ has been developed since more than twenty years at the University Pierre et Marie-Curie in Paris. The software is free and can be downloaded on the homepage : http://www.freefem.org

The manual contains a few examples of linear and flow partial differential equations. Nevertheless, some specific functionalities for the LLG equation were hard to find or not in the manual. We present below some tools very useful for the implementation of the LLG equation with FreeFem++.

6.3.2 Reduced Integration

The reduced integration of two functions f, g is given by the command

```
int2d(Th,qft=qf1pTlump)(f*g);
```

By the definition of the reduced integration, the matrix associated to

$$(\mathbf{f},\mathbf{g})_h$$

is given by

$$R = \begin{pmatrix} \int_\Omega \varphi_1 & & & \\ & \int_\Omega \varphi_2 & & \\ & & \ddots & \\ & & & \int_\Omega \varphi_l \end{pmatrix},$$

which will be useful for the implementation of the Laplacian and the magnetostriction.

6.3.3 Discrete Laplacian

For the discrete laplacian, we have by definition:

$$\begin{aligned} \tilde{\Delta}_h \mathbf{m}(\mathbf{x}_j) &= \frac{(\tilde{\Delta}_h \mathbf{m}, \varphi_j)_h}{\int_\Omega \varphi_j} \\ &= -\frac{(\nabla \mathbf{m}, \nabla \varphi_j)}{\int_\Omega \varphi_j} = -\frac{A\mathbf{m}}{R_{jj}}, \end{aligned}$$

where $\mathbf{m} = (\mathbf{m}_0, \ldots, \mathbf{m}_l)$ and $A_{ij} = (\nabla\varphi_j, \nabla\varphi_i)$. Consequently, the discrete Laplacian is given by :

$$\tilde{\Delta}_h \mathbf{m} = -DA \cdot \mathbf{m},$$

where $D_{ii} = \frac{1}{R_{ii}}$ is a diagonal matrix.
In freefem, the implementation of the Laplacian is done using an auxiliary bilinear

form to compute the matrix A_{ij}:

```
//Computation of the Laplacian
//Definition of a bilinear form to compute the matrix A
varf lap(m1,n1)=int2d(Th)(grad(m1)'*grad(n1));
matrix A=lap(Vh,Vh);
//Definition of a bilinear form to compute the matrix R
varf d(m1,n1)=int2d(Th,qft=qf1pTlump)(m1*n1);
matrix R=d(Vh,Vh);
//Compute Dii at each nodes of the triangulation
real [int,int] R(Th.nv,Th.nv);
for (int i;i< Th.nv;i++)
Di(i,i)=1/(R(i,i));
endl;
//Define the sparse matrix D
matrix D=Di;
//Definition of the sparse Laplacian Matrix
matrix D5=D*A;
// Definition of the Laplacian
Vh Lap1,Lap2,Lap3;

//Computation of the Laplacian for m1,m2,m3 known.
Lap1[]=-D5*m1[];
Lap2[]=-D5*m2[];
Lap3[]=-D5*m3[];
```

6.3.4 L^2 -projector of the magnetostriction

By the definition of $P_{\mathbf{V}_h}$, we have, for $l = 1, 2, 3$

$$P_{\mathbf{V}_h} h_{\sigma,l}(\mathbf{x}_j) = \frac{(P_{\mathbf{V}_h} h_{\sigma,l}, \varphi_j)_h}{\int_\Omega \varphi_j} = \frac{(h_{\sigma,l}, \varphi_j)}{R_{jj}}.$$

In a similar way, we have

$$P_{\mathbf{V}_h} h_{\sigma,l}(\mathbf{x}) = D(h_{\sigma,l}, \boldsymbol{\varphi}).$$

```
//defintion of h: known functions form the value of
//the tensors lambda and precedent computation of m and u
func h1,h2,h3;
//first component of h with basis elements
varf Poisson1(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
+ int2d(Th)( h1*v );
//Compute the vector of the integration
real[int] int1=Poisson1(0,Vh);

//second component of h with basis elements
varf Poisson2(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
+ int2d(Th)( h2*v );
//Compute the vector of the integration
real[int] int2=Poisson2(0,Vh);

//third component of h with basis elements
varf Poisson3(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
+ int2d(Th)( h3*v );
//Compute the vector of the integration
real[int] int3=Poisson3(0,Vh);

//Definition of a bilinear form to compute the matrix R
varf d(m1,n1)=int2d(Th,qft=qf1pTlump)(m1*n1);
matrix R=d(Vh,Vh);
//Compute Dii at each nodes of the triangulation
real [int,int] R(Th.nv,Th.nv);
for (int i;i< Th.nv;i++)
Di(i,i)=1/(R(i,i));
endl;
//Define the sparse matrix D
matrix D=Di;
//Definition of Pvh h_{sigma}
Vh magne1,magne2,magne3;
magne1=D*int1;
magne2=D*int2;
magne3=D*int3;
```

6.3.5 FreeFem and Paraview

The two dimension visualization window from FreeFem is not adapted for the LLG equation. Post-processing of the results is then made with Paraview, an open source programm for scientific visualization. At each time step, we save the solutions in format "vtk" with the command "savevtk". We create a folder that contains all the vtk file of the simulation, such that we can play a movie in Paraview.

First we create the folder:

```
//Date of the computation
string Date = "20120612-1_";
//Create the Directory
string dir = Date+"LLG_with_Magnetostriction/";
string mkdir = "mkdir -p "+dir;
exec(mkdir);
string dirVTK = dir+"VTK/";
string mkdirVTK = "mkdir -p "+dirVTK;
exec(mkdirVTK);
```

We solve the problem and save at each time step the solution in a vtk file:

```
while (time<Tfinal)
{
...
solve magnetrostriction;
...
real w;
w=time/dt;
savevtk(dirVTK+"solm_"+w+".vtk",Th,[m1,m2,m3],order=ff,dataname="vm") ;
savevtk(dirVTK+"solu_"+w+".vtk",Th,[u1,u2,u3],order=ff,dataname="vu") ;
}
```

The command

```
int[int] ff=[1];
```

has been fixed to 1 in order to export the solution at every vertices of the triangles of the mesh. If we put 0, then the solution is seen at the center of the triangles by an average of the three vertices. In the case of the LLG equation this is an important detail because it has an impact on the conservation of the magnitude.

6.4 Code

Here is the complete FreeFem++ code for the example 4.2.

```
//Create a folder for Paraview
// Date and name of the simulation
string Date = "22June2012";
string dir = Date+"alpha1L3/";
string mkdir = "mkdir -p "+dir;
exec(mkdir);
string dirVTK = dir+"VTK/";
string mkdirVTK = "mkdir -p "+dirVTK;
exec(mkdirVTK);

//Necessary for the function "savevtk"
load "iovtk"
real dt(0.00001);
real Tfinal(0.5);
```

```
real N(Tfinal/dt);
real time(0);
real alpha(1);
real lambdae(60);
real lambdam(10);
real s(1);
real gamma(4);
real eta(1);
//Create the mesh
real x0=-0.5,x1=0.5;
real y0=-0.5,y1=0.5;
int n=16,m=16;
mesh Th=square(n,m,[x0+(x1-x0)*x,y0+(y1-y0)*y]);

//Finite Element Spac
fespace Vh(Th,P1);

//Initial condition for the LLG equation in the Blow Up problem
func AA=((1-2*sqrt(x^2+y^2))^4)/s;
func B=AA*AA+(x^2+y^2);
func f1=(2*x*AA)/B;
func f2=(2*y*AA)/B;
func f3=(AA*AA-(x^2+y^2))/B;

real [int] Mo1(Th.nv);
real [int] Mo2(Th.nv);
real [int] Mo3(Th.nv);
for (int i; i<Th.nv; i++)
if (Th(i).x*Th(i).x + Th(i).y*Th(i).y <0.25) {Mo1(i)=f1(Th(i).x,Th(i).y);
Mo2(i)=f2(Th(i).x,Th(i).y);
Mo3(i)=f3(Th(i).x,Th(i).y);

}
else {Mo1(i)=0;
Mo2(i)=0;
Mo3(i)=-1;
}
endl;
Vh m1old,m2old,m3old;
m1old[]=Mo1;
m2old[]=Mo2;
m3old[]=Mo3;

Vh m1,m2,m3,n1,n2,n3,dtm1,dtm2,dtm3;
Vh m1N = m1old;
Vh m2N = m2old;
Vh m3N = m3old;

Vh p1=0;
```

```
Vh p2=0;
Vh p3=1;

//Initial Condition for the Magnetostriction Equation
Vh u1,u2,u3,v1,v2,v3;

Vh u1old=0;
Vh u2old=0;
Vh u3old=0;

Vh du1=0;
Vh du2=0;
Vh du3=0;

//Create vector to export the results (Energy and Norm Infty) on Matlab
int[int] ffordervel=[1];
real [int,int] NO(N+100,1);
real [int,int] E(N+100,1);
real [int,int] ET(N+100,1);
real [int,int] Em(N+100,1);
real [int,int] Eut(N+100,1);
real [int,int] Eux(N+100,1);
real [int,int] Esigma(N+100,1);
real [int,int] Ninfty(N+100,1);
real [int,int] NB(N+100,1);
real[int] dtm(N+100);
real[int] sum(N+100);
sum(0)=0.0001;
//Macro for the gradient, the cross product, the anisotropy
macro grad(m1) [dx(m1),dy(m1)] // EOM
macro cross(m1,m2,m3,n1,n2,n3) [m2*n3-m3*n2,
m3*n1-m1*n3,m1*n2-m2*n1] // EOM
macro crossp(n1,n2,n3,m1,m2,m3,p1,p2,p3) [n2*p3*(p1*m1+p2*m2+p3*m3)-
n3*p2*(p1*m1+p2*m2+p3*m3),n3*p1*(p1*m1+p2*m2+p3*m3)-
n1*p3*(p1*m1+p2*m2+p3*m3),n1*p2*(p1*m1+p2*m2+p3*m3)-
n2*p1*(p1*m1+p2*m2+p3*m3)] // EOM

//Computation of the Laplacian
Vh Lap1, Lap2, Lap3;
Vh magne1, magne2, magne3;

//Definition of a bilinear form to compute the matrix A
varf lap(m1,n1)=int2d(Th)(grad(m1)'*grad(n1));
matrix A=lap(Vh,Vh);
//Definition of a bilinear form to compute the matrix D
varf d(m1,n1)=int2d(Th,qft=qf1pTlump)(m1*n1);
matrix Di=d(Vh,Vh);
//Compute Dii at each nodes of the triangulation
real [int,int] Diag(Th.nv,Th.nv);
```

```
for (int i;i< Th.nv;i++)
Diag(i,i)=1/(Di(i,i));
endl;
//Create the diagonal matrix D
matrix D=Diag;
//Definition of the Laplacian Matrix
matrix L=D*A;
// Definition of the Laplacian
Vh Lap1old,Lap2old,Lap3old;

Lap1old[]=L*m1old[];
Lap2old[]=L*m2old[];
Lap3old[]=L*m3old[];

Vh Lap1N=Lap1old;
Vh Lap2N=Lap2old;
Vh Lap3N=Lap3old;

while (time<Tfinal)
{
Vh magne1old,magne2old,magne3old;

Vh du=dx(u1old);
Vh du2=dy(u2old);
varf Poisson1(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
+ int2d(Th)( lambdam*lambdae*
(-lambdam*m1old*m1old*m1old+du*m1old)*v );
real[int] u=Poisson1(0,Vh);

varf Poisson2(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
+ int2d(Th)( lambdam*lambdae*
(-lambdam*m2old*m2old*m2old+du2*m2old)*v );
real[int] uu=Poisson2(0,Vh);
magne1old[]=D*u;
magne2old[]=D*uu;
magne3old=0;
//Definition of the Magnetostriction
Vh magne1N=magne1old;
Vh magne2N=magne2old;
Vh magne3N=magne3old;

//Fixed Point Algorithm for the LLG equation
problem FixedPoint([m1,m2,m3],[n1,n2,n3]) =
int2d(Th,qft=qf1pTlump)
( 2*alpha*(cross(m1old,m2old,m3old,m1,m2,m3)'*[n1,n2,n3])/dt )
+ int2d(Th,qft=qf1pTlump) ( 2*(m1*n1 + m2*n2+ m3*n3)/dt )
- int2d(Th,qft=qf1pTlump) ( 2*(m1old*n1 + m2old*n2 + m3old*n3)/dt )
```

```
// EXCHANGE PART
+ int2d(Th,qft=qf1pTlump) (gamma*(1+alpha*alpha)*
(cross(m1,m2,m3,Lap1N,Lap2N,Lap3N)'*[n1,n2,n3]) )
// MAGNETIC PART
//- int2d(Th,qft=qf1pTlump) ( (1+alpha*alpha)*
( cross(m1,m2,m3,sin(2*pi*time),cos(2*pi*time),20)'*[n1,n2,n3]) )
// MAGNETOSTRICTION PART
+ int2d(Th,qft=qf1pTlump) ( gamma*(1+alpha*alpha)*
( cross(m1,m2,m3,magne1N,magne2N,magne3N)'*[n1,n2,n3]) );
// ANISOTROPY PART
//- int2d(Th,qft=qf1pTlump) ( 1000*gamma*
(1+alpha*alpha)*( crossp(m1,m2,m3,m1N,m2N,m3N,0,0,1)'*[n1,n2,n3]) );


//Weak Formulation for the Wave Equation
problem Waveequation ([u1,u2,u3],[v1,v2,v3]) =
int2d(Th) ( eta*([u1,u2,u3]'*[v1,v2,v3])/(dt*dt)  )
- int2d(Th) ( eta*([u1old,u2old,u3old]'*[v1,v2,v3])/(dt*dt)  )
-int2d(Th) ( eta*([du1,du2,du3]'*[v1,v2,v3])/dt  )
+ int2d(Th) ( lambdae*dx(u1)*dx(v1)) + int2d(Th) ( lambdae*dy(u2)*dy(v2))
- int2d(Th) ( lambdae*lambdam*m1old*m1old*dx(v1))
- int2d(Th) ( lambdae*lambdam*m2old*m2old*dy(v2))
+on(1,2,3,4,u1=0,u2=0,u3=0);

time = time + dt;
cout << endl;
cout << " --> Time : " << time << endl;
cout << endl;

real inc=1;
int nb=0;

while (inc > 2e-16)
{
nb = nb + 1;
//Resolution of the Fixed Point Algorithm
FixedPoint;

//Definition of the New Laplacian term
Lap1[]=L*m1[];
Lap2[]=L*m2[];
Lap3[]=L*m3[];

//Check the condition
inc = sqrt( int2d(Th) ( (m1-m1N)*(m1-m1N)
+ (m2-m2N)*(m2-m2N) + (m3-m3N)*(m3-m3N)) );
cout << nb << " : " << inc << endl;

//Definition of the New Magnetostriction term
```

```
varf Poisson2(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
- int2d(Th)( lambdam*lambdae*
(-(lambdam*m1old*m1old*m1)+dx(u1old)*m1)*v );
real[int] q=Poisson2(0,Vh);
varf Poisson3(u,v) =
int2d(Th)( dx(u)*dx(v) + dy(u)*dy(v))
- int2d(Th)( lambdam*lambdae*
(-(lambdam*m2old*m2old*m2)+dy(u2old)*m2)*v );
real[int] qq=Poisson3(0,Vh);
magne1[]=D*q;
magne2[]=D*qq;
magne3[]=0;

m1N=m1;
m2N=m2;
m3N=m3;
Lap1N=Lap1;
Lap2N=Lap2;
Lap3N=Lap3;
magne1N=magne1;
magne2N=magne2;
magne3N=magne3;
}

//Definition of the final value of m
Vh m1f=2*m1-m1old;
Vh m2f=2*m2-m2old;
Vh m3f=2*m3-m3old;

dtm1=(m1f-m1old)/dt;
dtm2=(m2f-m2old)/dt;
dtm3=(m3f-m3old)/dt;

m1old = m1f;
m2old = m2f;
m3old = m3f;
//Resolution of the Wave Equation
Waveequation;

du1=(u1-u1old)/dt;
du2=(u2-u2old)/dt;
du3=(u3-u3old)/dt;
u1old=u1;
u2old=u2;
u3old=u3;

//Definition of the norm, energy and norm infty
real w;
```

```
w=time/dt;
dtm(w)=int2d(Th,qft=qf1pTlump)(dtm1^2+dtm2^2+dtm3^2);
if (sum(w-1)>0)
{sum(w)=sum(w-1)+dtm(w);}
else
{sum(w)=sum(w-2)+dtm(w);}
//Computation of the total energy
func EnergyT=int2d(Th) ( (dx(m1f)^2 + dy(m1f)^2 + dx(m2f)^2
+ dy(m2f)^2 + dx(m3f)^2 + dy(m3f)^2)/2 )
+ dt*(alpha/(gamma*(1+alpha*alpha)))*sum(w)
+int2d(Th) ( (dx(u1)^2+dy(u2)^2)/2 )
+ int2d(Th) ( (du1^2+du2^2)/2 );
//Energy with m and u
func Energy=int2d(Th) ( (dx(m1f)^2 + dy(m1f)^2 + dx(m2f)^2
+ dy(m2f)^2 + dx(m3f)^2 + dy(m3f)^2)/2 )
+int2d(Th) ( (dx(u1)^2+dy(u2)^2)/2 ) + int2d(Th) ( (du1^2+du2^2)/2 );
// Energy with m
func Energym=int2d(Th) ( (dx(m1f)^2 + dy(m1f)^2 + dx(m2f)^2
+ dy(m2f)^2 + dx(m3f)^2 + dy(m3f)^2)/2 );
// Energy for ux and ut
func Energyut=int2d(Th) ( (du1^2+du2^2)/2 ) ;
func Energyux=int2d(Th) ( (dx(u1)^2+dy(u2)^2)/2 );
// Alternative Energy
func Energysigma=int2d(Th) ( (dx(m1f)^2 + dy(m1f)^2
+ dx(m2f)^2 + dy(m2f)^2 + dx(m3f)^2 + dy(m3f)^2)/2 )+
int2d(Th) ( ( lambdae*(dx(u1)-lambdam*m1f*m1f)*
(dx(u1)-lambdam*m1f*m1f))/2 )
+ int2d(Th) ( ( lambdae*(dy(u2)-lambdam*m2f*m2f)*
(dy(u2)-lambdam*m2f*m2f))/2 ) ;
Vh inf= sqrt(dx(m1f)^2 +dy(m1f)^2 + dx(m2f)^2
+ dy(m2f)^2 + dx(m3f)^2 + dy(m3f)^2);
func Norminfty=inf[].max;

//Save the value of the energy, norm and norm infty in vectors
NO(w-1,0)=n;
E(w-1,0)=Energy;
ET(w-1,0)=EnergyT;
Em(w-1,0)=Energym;
Eut(w-1,0)=Energyut;
Eux(w-1,0)=Energyux;
Esigma(w-1,0)=Energysigma;
Ninfty(w-1,0)=Norminfty;
NB(w-1,0)=nb;
//Save the data into Vtk file for Post-Processing in Paraview
savevtk(dirVTK+"solutionu_"+w+".vtk",Th,
[u1,u2,0],order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionm_"+w+".vtk",Th,
[m1f,m2f,m3f],order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionm1_"+w+".vtk",Th,
```

```
m1f,order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionm2_"+w+".vtk",Th,
m2f,order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionm3_"+w+".vtk",Th,
m3f,order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionu1_"+w+".vtk",Th,
u1,order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionu2_"+w+".vtk",Th,
u2,order=ffordervel,dataname="vector") ;
savevtk(dirVTK+"solutionm2d_"+w+".vtk",Th,
[m1f,m2f,0],order=ffordervel,dataname="vector") ;

cout << w << endl;
}

//Create a file to plot the data (Energy and Norm Infty) in Matlab
{
ofstream ff("Ela1L3.txt");
for (int i=0;i<N;i++)
{
ff << E(i,0) <<",";
}
}

{
ofstream ff("Suma1FL3.txt");
for (int i=0;i<N;i++)
{
ff << sum(i) <<",";
}
}

{
ofstream ff("ETa1L3.txt");
for (int i=0;i<N;i++)
{
ff << ET(i,0) <<",";
}
}

{
ofstream ff("Inftya1L3.txt");
for (int i=0;i<N;i++)
{
ff << Ninfty(i,0) <<",";
}
}

{
```

```
ofstream ff("Ema1L3.txt");
for (int i=0;i<N;i++)
{
ff << Em(i,0) <<",";
}
}

{
ofstream ff("Euxa1L3.txt");
for (int i=0;i<N;i++)
{
ff << Eux(i,0) <<",";
}
}

{
ofstream ff("Euta1L3.txt");
for (int i=0;i<N;i++)
{
ff << Eut(i,0) <<",";
}
}

{
ofstream ff("N_Esigmaa1L3.txt");
for (int i=0;i<N;i++)
{
ff << Esigma(i,0) <<",";
}
}
// Give the max number of iteration of the non linear system
cout << "Maxiter= " << NB.max << endl;
```

References

[1] L. Banas *On dynamical Micromagnetism with Magnetostriction.* Thesis, Ghent University, 2005

[2] L. Banas, M. Slodicka *Error Estimate for Landau-Lischitz-Gilbert Equation with magnetostriction.* Article, Elsevier, Applied Numerical Mathematics 56, pp.1019-1039, 2006,

[3] L. Banas, S. Bartels, A. Prohl *A convergent implicit discretization of the Maxwell-Landau-Lischitz-Gilbert Equation.* Paper, Ghent University, 2005

[4] S. Bartels, A.Prohl *Convergence of an implicit finite element method for the Laudau-Lifshitz-Gilbert Equation.* Article, SIAM J. Numer Anal., Vol. 44, No 4, pp. 1405-1419, 2006.

[5] S. Bartels, J.Ko, A.Prohl *Numerical Approximation of the Landau-Lischitz-Gilbert Equation and finite time Blow-Up of Weak Solution.* Paper, Zurich, 2005.

[6] S.C. Brenner, L. R. Scott *The mathematical Theory of finite element method.* Third Edition, Springer New-York, 2008.

[7] W. C. Nam, M.H. Cho, Y.Lee *Micromagnetic Simulation with Landau-Lifshitz-Gilbert Equation.* Quantum Photonic Science Research Center, Hanyang University, Seoul, Korea, 2001.

[8] D. Deparis *Numerical Approximation of Partial Differential Equation.* Course SMA, 2011.

[9] P. Goldenits, G. Hrkac, M. Page, D. Praetorius, D. Suess *Convergent Geometric Integrator for LLG Equation with Magnetostriction.* Workshop, Institute for Analysis and Scientific Computing, Vienna, 2011

[10] F. Hecht *FreeFem++, Third Version.* Laboratoire Jacques-Louis Lions, Universite Pierre et Marie-Curie, Paris, 2012

[11] M Kruzik, A. Prohl *Recent Developments in modeling, analysis and numerics of ferromagnetism.* Article, Zurich, Prague, 2005.

[12] K. Moreland *The Paraview Tutorial, v.3.12.* Sandia National Laboratories, 2012

[13] C. Serpico, I.D. Mayergoyz, G.Bertotti *Numerical Technique for integration of the Landau-Lischitz equation.* Article, Journal of Applied Physics, Volume 89, Number 11, 2001.

[14] A. Soyeur and F. Alouges *On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness.* Nonlinear Anal., 18 (1992), pp. 1071-1084.

[15] R. Temam *Navier-Stokes Equations, Theory and Numerical Analysis.* North Hoalland Publishing COmpany, 1977