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Estimates for the topological degree and related topics

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To Haïm Brezis on his 70th birthday with esteem

Abstract. This is a survey paper on estimates for the topological degree and related topics which range from the characterizations of Sobolev spaces and BV functions to the Jacobian determinant and nonlocal filter problems in Image Processing. These results are obtained jointly with Bourgain and Brezis. Several open questions are mentioned.

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1. Introduction

This is a survey paper on estimates for the topological degree and related topics which range from the characterizations of Sobolev spaces and BV functions to the Jacobian determinant and nonlocal filter problems in Image Processing. These results are obtained in collaboration with Bourgain and Brezis in [18], Bourgain in [21], and Brezis in [32, 33, 34], and by the author in [68, 69, 70, 71, 72, 73, 74]. The first topic is on estimates for the topological degree of maps from a sphere into itself. These estimates, partly joint work with Bourgain and Brezis in [18, 70], are motivated by the work of Bourgain, Brezis, and Mironescu in [17]. These results are discussed in Section 2. The second topic is on the characterizations of Sobolev spaces, partly joint work with Bourgain in [21, 68, 71]. These characterizations deal with the pointwise limit of a family of functionals as a small parameter goes to 0. The corresponding results for functions of bounded variations do not hold completely. This suggests to replace the notion of "pointwise convergence" by the notion of "Γ-convergence," which is more flexible. The Γ-convergence of the family of functionals considered in Section 3 is presented in Section 4; surprisingly the Γ-limit is strictly smaller than the pointwise limit. In Section 5, we deal with properties of Sobolev spaces related to the characterizations discussed in Section 3. More precisely, we discuss variants of the Sobolev inequality, the Poincaré inequality, and the Rellich–Kondrachov compactness criterion in these settings. The next two topics, joint work with Brezis in [32, 33], are on the Jacobian distributional of maps from a sphere into itself and the Jacobian determinant. These are presented in Sections 6 and 7. The last topic, obtained with Brezis in [34], discusses briefly some recent results related to Sobolev spaces and their applications in Image Processing.

2. Estimates for the topological degree

It is known from the work of Bethuel, Brezis, and Hélein in [8] that the number of singular points of solutions to the Ginzburg Landau equations is equal to $|\deg g|$, where g is the given boundary data taking values in the unit circle in two dimensions. Hence good estimates for the degree are of importance. This direction of work was initiated by Bourgain, Brezis, and Mironescu in [17]. Their results are improved later by Bourgain, Brezis, and Nguyen [18] and Nguyen [70]. In this section, we describe these results and highlight the ideas of the proof.

There is a very beautiful and useful formula to compute the degree of a map g from a unit sphere $\mathbb{S}^N$ in $\mathbb{R}^{N+1}$ into itself known as Kronecker's formula:

$$\deg g = \fint_{\mathbb{S}^N} \det(\nabla g)\, d\sigma = \fint_{B_{N+1}} \det(\nabla v)\, dx. \tag{2.1}$$

Here B_{N+1} denotes the unit ball in $\mathbb{R}^{N+1}$ and v is any smooth extension g in B_{N+1}. From the first equality of (2.1), one can easily obtain

$$|\deg g| \le C_N \|\nabla g\|^N_{L^N(\mathbb{S}^N)}. \tag{2.2}$$

Using the second equality of (2.1), one has

$$|\deg g| \le C_N \|\nabla v\|^{N+1}_{L^{N+1}(B_{N+1})},$$

where v is an extension of g in B_{N+1}. Letting v be the harmonic extension of g, i.e., $\Delta v = 0$ in B_{N+1} and $v = g$ on $\mathbb{S}^N$, and using the trace theory:

$$\|\nabla v\|_{L^{N+1}(B_{N+1})} \le C_N |g|^{N+1}_{W^{\frac{N}{N+1}, N+1}(\mathbb{S}^N)},$$

one obtains

$$|\deg g| \le C_N |g|^{N+1}_{W^{\frac{N}{N+1}, N+1}(\mathbb{S}^N)}.$$

Here and in what follows, $W^{s,p}$ $(0 < s < 1,\ p > 1)$ denotes the standard fractional Sobolev space and $|\cdot|$ denotes the corresponding seminorm. This is an improvement of (2.2) since, by interpolation,

$$|g|^{N+1}_{W^{\frac{N}{N+1}, N+1}(\mathbb{S}^N)} \le C_N \|g\|_{L^\infty(\mathbb{S}^N)} \|\nabla g\|^N_{L^N(\mathbb{S}^N)}.$$

Using Gagliardo–Nirenberg's characterization of the fractional seminorm, one derives

$$|\deg g| \le C_N \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{|g(x) - g(y)|^{N+1}}{|x - y|^{2N}}\, dx\, dy. \tag{2.3}$$

The first important improvement of (2.3) is due to Bourgain, Brezis, and Mironescu in [17]. They proved the following theorem.

Theorem 2.1 (Bourgain, Brezis, and Mironescu [17]**).**

$$|\deg g| \le C_{p,N} \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{|g(x) - g(y)|^p}{|x - y|^{2N}} \, dx \, dy \tag{2.4}$$

for all $g \in C^0(\mathbb{S}^N, \mathbb{S}^N)$ *and all* $p > 1$.

Estimate (2.4) gets better and better when p becomes larger and larger. This follows from the fact that

$$|g(x) - g(y)|^q \le 2^{q-p} |g(x) - g(y)|^p$$

for $q \ge p$ since $|g(x) - g(y)| \le 2$ for $x, y \in \mathbb{S}^N$.

Their proof of (2.4) is based on an original use of Kronecker's formula and the machinery of trace theory in fractional Sobolev spaces. Instead of choosing the harmonic extension v of g, their extension, based on v, is

$$u := \begin{cases} v/|v| & \text{if } |v| > 1/2, \\ 2v & \text{if } |v| < 1/2. \end{cases} \tag{2.5}$$

For every $x \in \mathbb{S}^N$, let $\rho(x)$ be the length of the largest radial interval coming from $x \in \mathbb{S}^N$ on which $|v| > 1/2$. It is clear that $\det(\nabla u) = 0$ in $\{x \in B_{N+1}; |v| > 1/2\}$ which is a subset of $\bigcup_{x \in \mathbb{S}^N} [0, (1 - \rho(x))x]$. Applying Kronecker's formula for u and using the fact that $|\nabla v(x)| \le C/(1 - |x|)$, after straightforward computations, they obtained the following inequality:

$$|\deg g| \le C_N \int_{\substack{\mathbb{S}^N \\ \rho(x) < 1}} \frac{1}{|\rho(x)|^N} \, dx. \tag{2.6}$$

Inequality (2.6) is the crucial point in their proof and for the later development on this topic. The rest of their proof is based on the machinery of trace theory in fractional spaces $W^{s,p}$ which is not discussed here. We next present a different approach due to Bourgain, Brezis, and Nguyen in [18] and Nguyen in [70]. The new approach is more elementary and provides better estimates for the degree.

It is well known that $\deg g \neq 0$ implies that g is surjective. In view of this fact, Bourgain, Brezis, and Mironescu in [17] asked the question whether or not the inequality

$$|\deg g| \le C_{\delta,N} \int_{\mathbb{S}^N} \int_{\substack{\mathbb{S}^N \\ |g(x) - g(y)| > \delta}} \frac{1}{|x - y|^{2N}} \, dx \, dy \tag{2.7}$$

holds for δ small enough. It is clear that (2.7) implies (2.4). In their unpublished note [14], they gave a positive answer to this question for $N = 1$ and for small δ, $\delta = 1/10$ is allowed. Their technique is quite involved. It is not clear if their method can be applied for higher dimensions.

In a joint work with Bourgain and Brezis in [18], we presented a positive answer to this question in arbitrary dimensions. More precisely, we proved the following result.

Theorem 2.2. *Let $N \geq 1$. For every $0 < \delta < \sqrt{2}$, there exists a positive constant $C_{N,\delta}$ such that*

$$|\deg g| \leq C_{\delta,N} \underset{|g(x)-g(y)|>\delta}{\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}} \frac{1}{|x-y|^{2N}}\,dx\,dy \quad \forall\, g \in C^0\big(\mathbb{S}^N, \mathbb{S}^N\big).$$

The idea of the proof is as follows. We first choose an extension in the spirit of (2.5). However, instead of choosing v as the harmonic extension, we take v as the average extension; i.e.,

$$v(X) = \fint_{B(x,r)} g(s)\,ds,$$

where $x = X/|X|$ and $r = 2(1-|X|)$ and we define u as follows:

$$u = \begin{cases} v/|v| & \text{if } |v| > \alpha, \\ v/\alpha & \text{if } |v| < \alpha \end{cases} \tag{2.8}$$

for some $\alpha > 0$ small. Here $B(x,r)$ denotes the ball centered at x of radius r in $\mathbb{S}^N$. The requirement for the smallness of α will be clear in a moment. This choice of v was used in the definition of the degree for VMO (vanishing mean oscillation) maps in [35] and is very suitable for our setting, which is close to VMO one. Similar to (2.6), we obtain the key estimate

$$|\deg g| \leq C_N \underset{\rho(x)<1}{\int_{\mathbb{S}^N}} \frac{1}{|\rho(x)|^N}\,dx, \tag{2.9}$$

where $\rho(x)$ is the length of the largest radial interval coming from $x \in \mathbb{S}^N$ on which $|v| > \alpha$. Hence it suffices to prove that, for $x \in \mathbb{S}^N$ with $\rho(x) < 1$,

$$\frac{1}{|\rho(x)|^N} \leq C_{\delta,N} \underset{|g(y)-g(x)|>\delta}{\int_{\mathbb{S}^N}} \frac{1}{|x-y|^{2N}}\,dy. \tag{2.10}$$

For $x \in \mathbb{S}^N$ such that $\rho(x) < 1$, we have

$$\left| \fint_{B\left(x,2\rho(x)\right)} g(y)\,dy \right| = \alpha. \tag{2.11}$$

Since α is small (the smallness of α depends on δ) and $|g(x)| = 1$, it follows that[1]

$$\Big| \big\{ y \in B\big(x, \rho(x)\big); |g(y) - g(x)| > \delta \big\} \Big| \geq C_{\delta,N}\,\rho(x)^N, \tag{2.12}$$

which implies (2.10).

In view of the fact that $\deg g \neq 0$ implies that g is surjective, it is natural to ask whether (2.7) holds for every $0 < \delta < 2$. Surprisingly, (2.7)

[1] Here and in what follows $|A|$ denotes the measure of a set A.

does *not* hold for every $0 < \delta < 2$. When $\delta > \sqrt{2}$, inequality (2.12) should be replaced by

$$\Big|\big\{(\xi,\eta) \in B\big(x, 2\rho(x)\big) \times B(x, 2\rho(x)); |g(\xi) - g(\eta)| > \delta\big\}\Big| \geq C_\delta r^{2N} \quad (2.13)$$

as long as (2.11) holds. In fact, (2.13) holds only for $\delta \leq \ell_N$ defined by

$$\ell_N = \sqrt{2 + \frac{2}{N+1}}, \quad (2.14)$$

which is the side length of an $(N+2)$ regular simplex inscribed in $\mathbb{S}^N$. This suggests that ℓ_N is an upper bound for (2.7). In fact, this is true by considering a sequence of continuous maps g_n whose ranges concentrate more and more on the vertices of such a regular simplex. Moreover, we can show that ℓ_N is optimal. To this end, we establish (2.13) for $\delta < \ell_N$. We are also able to show that the constant C_δ in (2.13) is independent of δ for $\delta < \ell_N$. The main ingredient of the proof is the following geometric lemma which is a discrete version of (2.13) and is interesting in itself [70, Lemma 3].

Lemma 2.3. *Let $N \geq 1$ and $A_i \in \mathbb{S}^N$ $(1 \leq i \leq N+2)$. Assume that*

$$\operatorname{dist}\big(0, \operatorname{conv}\{A_i;\, 1 \leq i \leq N+2\}\big) \leq \frac{1}{N+1}.$$

Then

$$|A_i - A_j| \geq \ell_N \quad \textit{for some } i, j.$$

Here $\operatorname{conv}(\cdot)$ denotes the convex hull of a subset in $\mathbb{R}^{N+1}$. We hence reach the following result [70, Theorem 1].

Theorem 2.4. *Let $N \geq 1$. Then ℓ_N is optimal for (2.7); i.e.,*

$$|\deg g| \leq C_N \underset{|g(x)-g(y)|\geq \ell_N}{\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}} \frac{1}{|x-y|^{2N}}\, dx\, dy \quad \forall\, g \in C^0\big(\mathbb{S}^N, \mathbb{S}^N\big), \quad (2.15)$$

and there exists $(g_n) \subset \mathbb{C}(\mathbb{S}^N; \mathbb{S}^N)$ such that $\deg g_n = 1$ and

$$\lim_{n\to\infty} \underset{|g_n(x)-g_n(y)|>\ell_N}{\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}} \frac{1}{|x-y|^{2N}}\, dx\, dy = 0.$$

For $g \in C^1(\mathbb{S}^N)$, one has

$$\lim_{\delta\to 0} \underset{|g(x)-g(y)|\geq\delta}{\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}} \frac{\delta^N}{|x-y|^{2N}}\, dx\, dy = C_N \int_{\mathbb{S}^N} |\nabla g|^N$$

for some $C_N > 0$ (see, e.g., Theorem 3.4 for similar settings). In view of this fact, Brezis [26] suggested the following question.

Open question 1. Let $g \in C^0(\mathbb{S}^N, \mathbb{S}^N)$. Is it true that

$$|\deg g| \leq C_N \underset{|g(x)-g(y)|>\delta}{\int_{\mathbb{S}^N}\int_{\mathbb{S}^N}} \frac{\delta^N}{|x-y|^{2N}}\, dx\, dy \quad \forall\, 0 < \delta < 1, \quad (2.16)$$

for some C_N independent of g and δ?

Concerning circle-valued maps, there is an important object associated with them, namely, the lifting which describes the phase of the maps. The lifting of Sobolev maps has attracted much attention in recent decades; see [10, 11, 13, 16, 17] and the references therein. The idea mentioned previously is also used to obtain estimates for liftings in [61, 62, 72] (see also [9, 60]).

We do not discuss here the formula for the winding number based on the Fourier coefficients discovered by Brezis in [25] and related problems. The reader can find an interesting review on these aspects in [25] and recent results due to Bourgain and Kozma in [20] and Bourgain and Kahane in [19].

3. Characterizations of Sobolev spaces

Various definitions of Sobolev spaces and the variants of well-known properties of Sobolev spaces have been studied by many authors, e.g., Ambrosio [3], Korevaar and Schoen [55], Reshetnyak [78], Hajłaz and Koskela [46], Bourgain, Brezis, and Mironescu [15] and the references therein. In this section, we will discuss several characterizations of Sobolev spaces which are motivated by the quantity used in the estimates for the degree in Section 2. Properties of Sobolev spaces related to these characterizations are discussed in Section 5. These characterizations are quite close to the work of Bourgain, Brezis, and Mironescu in [15], however, the connection is not transparent.

We first state the result of Bourgain, Brezis, and Mironescu in [15] (see also [24]). Further studies of their characterization are given in [12, 37, 40, 49, 56, 75, 76] and the references therein.

Theorem 3.1 (Bourgain, Brezis, and Mironescu [15]**).** *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dx\,dy < +\infty.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dx\,dy \\ &= K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx \quad \forall\, g \in W^{1,p}\big(\mathbb{R}^N\big),\end{aligned}$$

where $K_{N,p}$ is defined by

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p\,d\sigma. \tag{3.1}$$

Here $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative radial functions satisfying

$$\lim_{n\to\infty}\int_\tau^\infty \rho_n(r)r^{N-1}\,dr = 0 \quad \forall\,\tau > 0,$$

$$\lim_{n\to\infty}\int_0^{+\infty} \rho_n(r)r^{N-1}\,dr = 1.$$

Remark 3.2. Theorem 3.1 also holds for $p = 1$ if $W^{1,1}$ is replaced by BV and $\int_{\mathbb{R}^N} |\nabla g|$ denotes the total variation of g.

Before stating our results, we introduce the following notation.

Notation 1. *Let $p > 0$ and $\delta > 0$. We denote*

$$I_\delta(g) := \int_{\mathbb{R}^N} \int_{\substack{\mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy. \tag{3.2}$$

Remark 3.3. The quantity I_δ is a variant of the one used in the estimates for the degree in Section 2. It is also related to the definition of seminorm of the fractional Sobolev spaces $W^{s,q}$ for $0 < s < 1$ and $q > 1$.

The following characterization of Sobolev spaces was established by Nguyen in [68], and Bourgain and Nguyen in [21].

Theorem 3.4. *Let $1 < p < +\infty$ and $g \in L^p(\mathbb{R}^N)$. Then*

$$g \in W^{1,p}(\mathbb{R}^N) \quad \textit{if and only if} \quad \varliminf_{\delta \to 0} I_\delta(g) < +\infty. \tag{3.3}$$

Moreover, for $g \in W^{1,p}(\mathbb{R}^N)$,

$$I_\delta(g) \le C_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx \quad \forall\, \delta > 0, \tag{3.4}$$

for some positive constant $C_{N,p}$, and

$$\lim_{\delta \to 0} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx, \tag{3.5}$$

where $K_{N,p}$ is defined by (3.1).

Remark 3.5. Similar results also hold for smooth bounded domains (see [68]).

Property (3.3) does not hold for $p = 1$ (see Proposition 4.1). However, we have the following result.

Theorem 3.6. *Let $p = 1$ and let $g \in L^1(\mathbb{R}^N)$ be such that*

$$\varliminf_{\delta \to 0} I_\delta(g) < +\infty. \tag{3.6}$$

Then $g \in \mathrm{BV}(\mathbb{R}^N)$.

Remark 3.7. In a recent joint work with Brezis [34], we show that a variant of Theorems 3.4 and 3.6 for $0 < p < 1$ does not hold without assuming enough regularity on g. Indeed, let $0 < p < 1$ and $0 < \gamma < 1 - p$. There exists $g \in C^{0,\gamma}(\mathbb{R}) \setminus W^{1,1}(\mathbb{R})$ with compact support such that $\sup_{\delta > 0} I_\delta(g) < +\infty$; see [34, Proposition 5].

Proof of (3.4) *and* (3.5). We only give the proof under the additional assumption $g \in C^1(\mathbb{R}^N)$. The proof in the general case follows similarly but requires some facts on the measure theory. Using polar coordinates and applying Fubini's theorem, we have

$$\begin{aligned} I_\delta(g) &= \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\infty_{|g(x+h)-g(x)|>\delta} \frac{\delta^p}{h^{p+1}}\, dh\, dx \\ &= \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\infty_{\frac{|g(x+\delta h\sigma)-g(x)|}{\delta}>1} \frac{1}{h^{p+1}}\, dh\, dx\, d\sigma. \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \frac{|g(x+\delta h\sigma)-g(x)|}{\delta} &\le \frac{1}{\delta}\int_0^{h\delta} |\langle \nabla g(x+s\sigma), \sigma\rangle|\, ds \\ &\le hM(\nabla g, \sigma)(x), \end{aligned} \tag{3.8}$$

where

$$M(\nabla g, \sigma)(x) := \sup_{t>0} \frac{1}{t}\int_0^t |\langle \nabla g(x+s\sigma), \sigma\rangle|\, ds. \tag{3.9}$$

A combination of (3.7) and (3.9) yields

$$I_\delta(g) \le \frac{1}{p}\int_{\mathbb{S}_{N-1}} \int_{\mathbb{R}^N} |M(\nabla g, \sigma)(x)|^p\, dx\, d\sigma.$$

Decomposing $\mathbb{R}^N = \sigma\mathbb{R} \times (\sigma\mathbb{R})^\perp$ and applying the theory of maximal functions (see, e.g., [87]), we obtain

$$I_\delta(g) \le C_{N,p}\int_{\mathbb{R}^N} |\nabla g|^p\, dx\, d\sigma,$$

which is (3.4).

We next prove (3.5). We have

$$\lim_{\delta\to 0} \frac{|g(x+\delta h\sigma)-g(x)|}{\delta} = h|\langle \nabla g(x), \sigma\rangle|$$

and

$$\int_{\mathbb{S}_{N-1}} \int_{\mathbb{R}^N} \int_0^\infty_{h|\langle \nabla g(x),\sigma\rangle|>1} \frac{1}{h^{p+1}}\, dh\, dx\, d\sigma = \frac{1}{p}\int_{\mathbb{S}_{N-1}} \int_{\mathbb{R}^N} |\langle \nabla g(x), \sigma\rangle|^p\, dx\, d\sigma.$$

Hence (3.5) follows from the definition of $K_{N,p}$ and the dominated convergence theorem. □

Sketch of the idea of the proof of (3.3). The proof of (3.3) is due to Bourgain and Nguyen in [21]. By (3.4), it suffices to prove that, for $g \in L^p(\mathbb{R}^N)$,

$$\text{if } \varliminf_{\delta\to 0} I_\delta(g) < +\infty, \text{ then } g \in W^{1,p}\big(\mathbb{R}^N\big). \tag{3.10}$$

The proof of (3.10) for $N \ge 2$ is a consequence of the one-dimensional result and the following type of inequalities, whose proof is standard as in the context of fractional Sobolev spaces (see, e.g., [1, Chapter 7]),

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{|g(x',x_N)-g(x',y_N)|>2\delta} & \frac{1}{|x_N - y_N|^{p+1}}\, dx_N\, dy_N\, dx' \\ & \le C_{N,p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{N+p}}\, dx\, dy \end{aligned} \tag{3.11}$$

for all $\delta > 0$. In the one-dimensional case, the proof of (3.3) is essentially based on Lemma 3.8 below which is interesting in itself. In fact, applying Lemma 3.8, one can derive that

$$\int_{\mathbb{R}^d} \frac{|g(x+h) - g(x)|^p}{h^p}\, dx \le C$$

for all $0 < h < 1$ and for some positive constant C independent of h. It follows that $g \in W^{1,p}(\mathbb{R})$ (see, e.g., [23, Chapter 8]). □

The following lemma [21, Lemma 2] due to Bourgain and Nguyen is the key ingredient of the proof of (3.3) and plays an important role in the topics discussed in Sections 4 and 5.

Lemma 3.8. *Let g be a measurable function on a bounded nonempty interval I and $1 \le p < +\infty$. Then*

$$\begin{aligned} \varliminf_{\delta \to 0_+} \iint_{\substack{I \times I \\ |g(x)-g(y)|>\delta}} & \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \\ & \ge c\, \frac{1}{|I|^{p-1}} \Big(\operatorname*{ess\,sup}_I g - \operatorname*{ess\,inf}_I g \Big)^p, \end{aligned} \tag{3.12}$$

where $c = c_p$ is a positive constant depending only on p.

Some words on the proof of Lemma 3.8**.** The proof of Lemma 3.8 is quite technical and based on elementary properties of measurable functions and the mean value theorem. We sketch the idea of the proof in the case g is continuous on $\bar{I}$; which already contains all the ingredients of the proof in the general case. By scaling, without loss of generality, one may assume that

$$I = [0,1], \quad \max_{\bar{I}} g = 1, \quad \min_{\bar{I}} g = 0.$$

Take $0 < \delta \ll 1$ small enough to ensure that there are two points t_+, $t_- \in [40\delta, 1 - 40\delta] \subset [0,1]$ with

$$\begin{cases} |[t_+ - \tau, t_+ + \tau] \cap [g > 3/4]| > 9\,\tau/5, \\ |[t_- - \tau, t_- + \tau] \cap [g < 1/4]| > 9\,\tau/5, \end{cases} \qquad \forall\, 0 < \tau < 40\delta. \tag{3.13}$$

Let $K \in \mathbb{Z}_+$ be such that

$$\delta < 2^{-K} \le 2\delta. \tag{3.14}$$

Denote

$$J = \left\{ j \in \mathbb{Z}_+; \frac{1}{4} < j2^{-K} < \frac{3}{4} \right\}.$$

Then

$$\operatorname{card}(J) \geq 2^{K-1} - 2 \sim \frac{1}{\delta}. \tag{3.15}$$

For each j, define the following sets:

$$A_j = \left\{ x \in [0,1];\ (j-1)2^{-K} \leq g(x) < j2^{-K} \right\},$$

$$B_j = \bigcup_{j' < j} A_{j'}, \qquad C_j = \bigcup_{j' > j} A_{j'},$$

so that $B_j \times C_j \subset \left[|g(x) - g(y)| \geq 2^{-K} \right] \subset [|g(x) - g(y)| > \delta]$. Set

$$G = \left\{ j \in J;\ |A_j| < 2^{-K+2} \right\}. \tag{3.16}$$

Since the collection (A_j) is disjoint, it follows from (3.15) that

$$\operatorname{card}(G) \geq 2^{K-2} - 3 \sim \frac{1}{\delta}. \tag{3.17}$$

We claim that, for each $j \in G$, there exist $t_j \in [40\delta, 1 - 40\delta]$ and $0 < \lambda_j \leq 4\delta$ such that

$$\begin{aligned} \left| [t_j - 4\lambda_j, t_j + 4\lambda_j] \cap B_j \right| &= 4\lambda_j, \\ \frac{\lambda_j}{4} \leq \left| [t_j - 4\lambda_j, t_j + 4\lambda_j] \cap A_j \right| &\leq \lambda_j. \end{aligned} \tag{3.18}$$

Indeed, for $j \in G$, set $\lambda_{1,j} = |A_j|$ and consider

$$\psi_1(t) = \left| [t - 4\lambda_{1,j}, t + 4\lambda_{1,j}] \cap B_j \right| \quad \forall\, t \in [40\delta, 1 - 40\delta].$$

It follows from (3.13) that $\psi_1(t_+) < 4\lambda_{1,j}$ and $\psi_1(t_-) > 4\lambda_{1,j}$. Hence there exists $t_{1,j} \in [40\delta, 1 - 40\delta]$ such that

$$\psi_1(t_{1,j}) = 4\lambda_{1,j}. \tag{3.19}$$

If

$$\left| [t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j \right| \geq \frac{\lambda_{1,j}}{4},$$

then set $\lambda_j = \lambda_{1,j}$. Otherwise, let $\lambda_{2,j} > 0$ be such that $\lambda_{1,j}/\lambda_{2,j} \in \mathbb{Z}_+$ and

$$\frac{\lambda_{2,j}}{2} < |[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j| \leq \lambda_{2,j}$$

and set

$$\psi_2(t) = \left| [t - 4\lambda_{2,j}, t + 4\lambda_{2,j}] \cap B_j \right| \quad \forall\, t \in [t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}].$$

Using (3.19), we have

$$\psi_2(t_{2,j}) = 4\lambda_{2,j} \quad \text{for some } \lambda_{2,j}.$$

If

$$\left| [t_{2,j} - 4\lambda_{2,j}, t_{2,j} + 4\lambda_{2,j}] \cap A_j \right| \geq \frac{\lambda_{2,j}}{4},$$

then set $\lambda_j = \lambda_{2,j}$, etc. This process has to stop by the uniform continuity of g. Claim (3.18) is proved.

We have, as a consequence of the first equality in (3.18),

$$\iint_{\substack{[t_j-4\lambda_j,t_j+4\lambda_j]^2\\ |g(x)-g(y)|>\delta}} \frac{1}{|x-y|^{p+1}} \geq C_p\delta^{p-1}, \tag{3.20}$$

and as a consequence of the second inequality in (3.18),

$$[t_j - 4\lambda_j, t_j + 4\lambda_j] \text{ are almost "disjoint."}$$

By taking the sums of (3.20) with respect to $j \in G$, the conclusion follows.

Under the stronger assumption

$$\overline{\lim_{\delta\to 0}}\, I_\delta(g) < +\infty, \tag{3.21}$$

the proof of (3.3) had been obtained previously by Nguyen in [68]. The proof is simpler and interesting in itself. It is based on a convex property derived from I_δ when (3.21) is assumed. The proof is as follows. We first assume that $g \in L^\infty$. Set $A = \|g\|_{L^\infty}$ and define

$$J_\varepsilon(g) = \int_0^{2A} \varepsilon\delta^{\varepsilon-1} I_\delta(g)\, d\delta.$$

Then

$$\overline{\lim_{\varepsilon\to 0}}\, J_\varepsilon(g) \leq \overline{\lim_{\delta\to 0}}\, I_\delta(g). \tag{3.22}$$

We have

$$J_\varepsilon(g) = \frac{\varepsilon}{p+\varepsilon} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy.$$

We now follow the method in the work of Bougain, Brezis, and Mironescu with the suggestion of Stein presented in [24]. Let (ρ_n) be a sequence of smooth mollifiers. Set $g_n = \rho_n * g$. Then

$$J_\varepsilon(g_n) \leq J_\varepsilon(g),$$

since $J_\varepsilon(\cdot)$ is convex. Letting ε go to 0, we have, since g_n is smooth,

$$\int_{\mathbb{R}^N} |\nabla g_n|^p\, dx \leq C \lim_{\varepsilon\to 0} J_\varepsilon(g_n) \leq C\, \overline{\lim_{\delta\to 0}}\, I_\delta(g),$$

by (3.22). The proof under the additional assumption that $g \in L^\infty$ is complete. The proof in the general case can be derived from the previous one by noting that

$$I_\delta(g_K) \leq I_\delta(g) \quad \text{for } K > 0,$$

where g_K is the truncation of g with respect to K; i.e.,

$$g_K = \min\big\{K, \max\{g, -K\}\big\}.$$

The characterizations in Theorems 3.4 and 3.6 have been extended in [71] (see [71, Theorem 1]) to the functionals of the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p} \int_0^\infty F_n(t)t^{-(p+1)}\, dt \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx,$$

under the following assumptions on F_n:

(i) $F_n(t)$ is a nondecreasing function with respect to t on $[0, +\infty)$ for all $n \in \mathbb{N}$;

(ii) $\int_0^1 F_n(t) t^{-(p+1)}\, dt = 1$ for all $n \in \mathbb{N}$;

(iii) $F_n(t)$ converges uniformly to 0 on every compact subset of $(0, +\infty)$ as n goes to infinity.

The idea of the proof of Lemma 3.8 is also developed in [71] to obtain the following lemma [71, Lemma 2] which is interesting in itself.

Lemma 3.9. *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Assume that*

$$\iint\limits_{\substack{K \times K \\ |g(x)-g(y)|>\varepsilon}} \frac{1}{|x-y|^{N+1}}\, dx\, dy < +\infty \quad \forall K \Subset \mathbb{R}^N, \forall \varepsilon > 0 \tag{3.23}$$

and

$$\varliminf_{\varepsilon \to 0_+} \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ \varepsilon < |g(x)-g(y)| < 10\varepsilon}} \frac{\varepsilon^p}{|x-y|^{N+p}}\, dx\, dy < +\infty. \tag{3.24}$$

Then $g \in W^{1,p}(\mathbb{R}^N)$.

The case $p = 1$ is unknown, the following question remains open.

Open question 2. Let $g \in L^1(\mathbb{R}^N)$ be such that

$$\iint\limits_{\substack{K \times K \\ |g(x)-g(y)|>\varepsilon}} \frac{1}{|x-y|^{N+1}}\, dx\, dy < +\infty \quad \forall K \Subset \mathbb{R}^N, \forall \varepsilon > 0$$

and

$$\varliminf_{\varepsilon \to 0_+} \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ \varepsilon < |g(x)-g(y)| < 10\varepsilon}} \frac{\varepsilon}{|x-y|^{N+1}}\, dx\, dy < +\infty.$$

Is it true that $g \in \mathrm{BV}(\mathbb{R}^N)$?

4. Γ-convergence and Sobolev norms

The characterizations of Sobolev spaces mentioned in Section 3 are complete in the case $p > 1$. However, in the case $p = 1$, one has the following scenario, observed by Ponce (the proof is presented in [71]).

Proposition 4.1 (Ponce). *There exists $g \in W^{1,1}(\mathbb{R}^N)$ with compact support such that*

$$\lim_{\delta \to 0} I_\delta(g) = +\infty. \tag{4.1}$$

The situation in the case $p = 1$ is quite complicated. The Γ-convergence is more flexible and suitable than the pointwise convergence in this case. We first recall the definition of Γ-convergence. A family of functionals $(I_\delta)_{\delta \in (0,1)}$

defined in $L^p(\mathbb{R}^N)$ Γ-converges in $L^p(\mathbb{R}^N)$ ($p \geq 1$), as $\delta \to 0$, to a functional I defined in $L^p(\mathbb{R}^N)$ if and only if the following two conditions hold.

(G1) For each $g \in L^p(\mathbb{R}^N)$ and for every family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ as $\delta \to 0$, one has

$$\varliminf_{\delta\to 0} I_\delta(g_\delta) \geq I(g).$$

(G2) For each $g \in L^p(\mathbb{R}^N)$, there exists a family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ as $\delta \to 0$, and

$$\varlimsup_{\delta\to 0} I_\delta(g_\delta) \leq I(g).$$

In [69, 73], we proved that I_δ Γ-converges in $L^p(\mathbb{R}^N)$. Surprisingly, the Γ-limit is *strictly less* than the pointwise limit *for all* $p \geq 1$. More precisely, we have the following result.

Theorem 4.2. *Let $p \geq 1$ and $N \geq 1$. Then (I_δ) Γ-converges in $L^p(\mathbb{R}^N)$, as $\delta \to 0$, to the functional I defined in $L^p(\mathbb{R}^N)$ by*

$$I(g) = \begin{cases} \widehat{K}_{N,p} \displaystyle\int_{\mathbb{R}^N} |\nabla g|^p \, dx & \text{if } p > 1 \text{ and } g \in W^{1,p}\big(\mathbb{R}^N\big) \\ & \big(\text{resp., } p = 1 \text{ and } g \in \mathrm{BV}\left(\mathbb{R}^N\right)\big), \\ +\infty & \text{otherwise,} \end{cases}$$

for some constant $\widehat{K}_{N,p}$ which satisfies

$$0 < \widehat{K}_{N,p} < \frac{1}{p} K_{N,p}. \tag{4.2}$$

The quantity I_δ is *nonconvex* and very sensitive to small perturbations. In a *convex* setting related to Theorem 3.1, the corresponding Γ-convergence result is proved by Ponce in [75]; his proof uses essentially the convexity in that context and is much simpler than ours, which is described below.

The proof of Theorem 4.2 is as follows. Set

$$\widehat{K}_{N,p} = \inf \varliminf_{\delta\to 0} \iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy, \tag{4.3}$$

where the infimum is taken over all families of measurable functions $(h_\delta)_{\delta\in(0,1)}$ defined in the unit open cube $Q = (0,1)^N$ of $\mathbb{R}^N$ such that h_δ converges to $h(x) \equiv \frac{x_1+\cdots+x_N}{\sqrt{N}}$ in (Lebesgue) measure in Q as $\delta \to 0$ (using Lemma 4.3 stated later in this section, one can replace the convergence in measure of h_δ by the convergence in L^p or L^∞; the monotonicity of I_δ plays an important role here). We then establish Theorem 4.2 where $\widehat{K}_{N,p}$ is given by (4.3).

The proof of the fact $\widehat{K}_{N,p} < K_{N,p}/p$ is essentially based on the following observation in one dimension. Let g and g_δ be defined in $[0,1]$ by

$$g(x) = x$$

and

$$g_\delta(x) = (k+1)\delta \quad \text{if } k\delta \leq x < (k+1)\delta \text{ for } 0 \leq k \leq [1/\delta].$$

Here $[1/\delta]$ denotes the largest integer less than $1/\delta$. Then g_δ converges to g in $L^\infty(0,1)$ as $\delta \to 0$ and

$$\varlimsup_{\delta\to 0} \underset{|g_\delta(x)-g_\delta(y)|>\delta}{\int_0^1\int_0^1} \frac{\delta^p}{|x-y|^{1+p}}\,dx\,dy < \frac{1}{p}K_{1,p}\int_0^1 |g'|^p\,dx. \tag{4.4}$$

Indeed, (4.4) is a consequence of the following facts:

$$\varliminf_{\delta\to 0}\left(\underset{|g(x)-g(y)|>\delta}{\int_0^1\int_0^1} \frac{\delta^p}{|x-y|^{1+p}}\,dx\,dy - \underset{|g_\delta(x)-g_\delta(y)|>\delta}{\int_0^1\int_0^1} \frac{\delta^p}{|x-y|^{1+p}}\,dx\,dy\right)$$
$$\geq \varliminf_{\delta\to 0}\sum_{k=0}^{[1/\delta]} \underset{\substack{k\delta\leq x\leq (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta\leq y\in (k+2)\delta}}{\iint} \frac{\delta^p}{|x-y|^{1+p}}\,dx\,dy,$$

$$\lim_{\delta\to 0} \underset{|g(x)-g(y)|>\delta}{\int_0^1\int_0^1} \frac{\delta^p}{|x-y|^{1+p}}\,dx\,dy = \frac{1}{p}K_{1,p}\int_0^1 |g'|^p\,dx = \frac{2}{p},$$

and

$$\varliminf_{\delta\to 0}\sum_{k=0}^{[1/\delta]} \underset{\substack{k\delta\leq x\leq (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta\leq y\leq (k+2)\delta}}{\iint} \frac{\delta^p}{|x-y|^{1+p}}\,dx\,dy > 0.$$

The proof of the fact $\widehat{K}_{N,p} > 0$ is based on (4.3) and essentially uses the idea in the proof of Lemma 3.8. We first consider the case $N = 1$. Let $h_\delta \to h$ in measure in $I := [0,1]$. It follows that, for δ small, (3.13) holds for h and h_δ with $t_+ \approx 1$, $t_- \approx 0$. Hence, using the same arguments as in the proof of Lemma 3.8, we have

$$\underset{\substack{I\times I \\ |h_\delta(x)-h_\delta(y)|>\delta}}{\iint} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \geq c \tag{4.5}$$

for small δ, where $c = c_p$ is a positive constant depending only on p. This implies that $\widehat{K}_{1,p} > 0$. The proof in the general case makes use of (3.11) and is a consequence of the one in one dimension.

We next discuss the ideas of the proof of properties (G1) and (G2). The proof of (G1) in the case $p > 1$ somewhat follows from the definition of $\widehat{K}_{N,p}$ in (4.3) and the fact that any function in $W^{1,p}(\mathbb{R}^N)$ is locally approximately linear in the sense of measure; see, e.g., [42, Theorem 4 on page 223]. The proof in the case $p = 1$ is more complicated since the behaviour of functions of bounded variations, whose derivatives contain the jump part and the Cantor part as well, is more complicated than the one in $W^{1,p}$. In this case, the proof involves another representation for $\widehat{K}_{N,1}$ (the one given in (4.3) does not reflect the jumps of BV functions). More precisely, we have (see [73, Proposition 4])

$$\widehat{K}_{N,1} = \inf \varliminf_{\delta\to 0} \iint_{\substack{Q^2\\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy, \tag{4.6}$$

where the infimum is taken over all families of measurable functions $(h_\delta)_{\delta\in(0,1)}$ such that h_δ converges to $H_{\frac{1}{2}}$ in measure as $\delta\to 0$. Here $H_c(x) := H(x_1-c, x')$ for any $c\in\mathbb{R}$, where H is the function defined in $\mathbb{R}^N$ by

$$H(x) = \begin{cases} 0 & \text{if } x_1 < 0,\\ 1 & \text{otherwise.}\end{cases}$$

We next turn to the ideas of the proof of property (G2). The proof of property (G2) is the same for both cases $p>1$ and $p=1$. It suffices to establish (G2) under the assumption that g is continuous piecewise linear with compact support; the general case follows by a standard density argument. The first step is to show that there exists a family

$$(h_\delta)\to h(x) = \frac{x_1+\cdots+x_N}{\sqrt{N}}$$

in $L^p(Q)$, as $\delta\to 0$, such that

$$\varlimsup_{\delta\to 0} \iint_{\substack{Q^2\\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \widehat{K}_{N,p}. \tag{4.7}$$

The second step is to rescale functions obtained from (4.7) and glue them together. These two steps are delicate and essentially based on the following monotonicity property of I_δ [73, Lemma 1], whose proof is quite elementary.

Lemma 4.3. *Let $N\ge 1$, $p\ge 1$, let A be a measurable subset of $\mathbb{R}^N$, and let f and g be two measurable functions defined in A. Define*

$$h_1 = \min(f,g) \quad \textit{and} \quad h_2 = \max(f,g).$$

Assume g is Lipschitz in A with a Lipschitz constant L. Then

$$\begin{aligned}\iint_{\substack{A^2\\ |h_1(x)-h_1(y)|>\delta}} &\frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy\\ &\le \iint_{\substack{A^2\\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + CL^p|A\setminus B_1|\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\iint_{\substack{A^2\\ |h_2(x)-h_2(y)|>\delta}} &\frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy\\ &\le \iint_{\substack{A^2\\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + CL^p|A\setminus B_2|.\end{aligned} \tag{4.9}$$

Here

$$B_1 = \{x \in A; f(x) \le g(x)\} \quad \text{and} \quad B_2 = \{x \in A;\, f(x) \ge g(x)\}.$$

Proof of Lemma 4.3. We only prove (4.8); the proof of (4.9) follows similarly. Note that if $x, y \in B_1$, then $|h_1(x) - h_1(y)| = |f(x) - f(y)|$. Otherwise, $x \notin B_1$ or $y \notin B_1$, which implies

$$|h_1(x) - h_1(y)| \le \max\big(|f(x) - f(y)|, |g(x) - g(y)|\big).$$

It follows that

$$\begin{aligned} \iint_{\substack{A^2 \\ |h_1(x)-h_1(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \le & \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \\ & + \iint_{\substack{A^2 \setminus B_1^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy. \end{aligned} \tag{4.10}$$

Since g is Lipschitz with a Lipschitz constant L, it follows that

$$\iint_{\substack{A^2 \setminus B_1^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \le CL^p |A \setminus B_1|. \tag{4.11}$$

A combination of (4.10) and (4.11) yields (4.8). $\square$

We next concentrate on the case $N = 1$ and explain very briefly how to use Lemma 4.3 to reach the conclusions of the two steps. Applying (4.8) with $f = f_1 := h_\delta$ and $g = g_1 := x + \varepsilon_1$, $f = f_2 := \min\{f_1, g_1\}$ and $g = g_2 := 1 - \varepsilon_2$ and applying (4.9) with $f = f_3 := \min\{f_2, g_2\}$ and $g = g_3 = x - \varepsilon_1$, and $f = f_4 := \max\{f_3, g_3\}$ and $g = g_4 := \varepsilon_2$ for appropriate choices of ε_1 and ε_2 (these choices depend on δ), we can assume that h_δ given in (4.7) is constant near 0 and 1, the endpoints of $Q = [0, 1]$ ($N = 1$). In fact, by using g_1 and g_3, one can first assume that h_δ converges to x uniformly in $[0, 1]$ and by using g_2 and g_4, one can then assume that h_δ is constant near 0 and 1. Knowing that h_δ can be chosen to be constant near 0 and 1 in (4.7), we then can rescale these functions and glue them together to reach the conclusion of the first step. The proof of the second step is more complicated but uses the same observation.

5. Inequalities related to Sobolev norms

This section is devoted to variants of the Poincaré inequality, the Sobolev inequality, and the Rellich–Kondrachov compactness criterion which are related to the characterizations of Sobolev spaces in Section 3. Concerning the Poincaré inequality, we have the following result [74, Theorem 1].

Theorem 5.1. *Let $N \geq 1$, $p \geq 1$, and let g be a real measurable function defined in a ball $B \subset \mathbb{R}^N$. We have*

$$\int_B \int_B |g(x) - g(y)|^p \, dx \, dy \leq C_{N,p}\left(|B|^{\frac{N+p}{N}} \underset{|g(x)-g(y)|>\delta}{\int_B \int_B} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |B|^2 \right). \tag{5.1}$$

Applying Theorem 5.1, we have $g \in \mathrm{BMO}(\mathbb{R}^N)$, the space of all functions of bounded mean oscillation defined in $\mathbb{R}^N$, if $g \in L^1(\mathbb{R}^N)$ and $I_\delta(g) < +\infty$ for some $\delta > 0$. Moreover, there exists a positive constant C, depending only on N, such that

$$|g|_{\mathrm{BMO}} := \sup_B \fint_B \fint_B |g(x) - g(y)| \, dx \, dy \leq C \left(I_\delta^{\frac{1}{N}}(g) + \delta \right),$$

where the supremum is taken over all balls of $\mathbb{R}^N$. In a joint work with Brezis [31] we also showed that if $g \in L^1(\mathbb{R}^N)$ and $I_\delta(g) < +\infty$ with $p = N$ for all $\delta > 0$,[2] then $g \in \mathrm{VMO}(\mathbb{R}^N)$, the spaces of all functions of vanishing mean oscillation. More properties in the case $p = N$ can be found in [31].

Here is a variant of the Sobolev inequality [74, Theorem 3].

Theorem 5.2. *Let $1 < p < N$, $\delta > 0$, and let g be a real measurable function defined in $\mathbb{R}^N$. There exist two positive constants C and λ, depending only on N and p, such that*

$$\left(\int_{|g|>\lambda\delta} |g|^q \, dx \right)^{\frac{1}{q}} \leq C \left[I_\delta(g) \right]^{\frac{1}{p}} \quad \text{with } q = \frac{Np}{N-p}. \tag{5.2}$$

In the case $p = 1$, the following question remains open.

Open question 3. Let $p = 1$, $N > 1$, $\delta > 0$, and let g be a real measurable function defined in $\mathbb{R}^N$. Is it true that, for some positive constants C and λ, depending only on N,

$$\left(\int_{|g|>\lambda\delta} |g|^{\frac{N}{N-1}} \, dx \right)^{\frac{N-1}{N}} \leq C I_\delta(g). \tag{5.3}$$

A variant of Rellich–Kondrachov theorem [74, Theorem 2] is the following.

Theorem 5.3. *Let $N \geq 1$, $p \geq 1$, (g_n) a bounded sequence in $L^p(\mathbb{R}^N)$, and let (δ_n) be a sequence of positive numbers converging to 0 such that*

$$\varliminf_{n \to \infty} I_{\delta_n}(g_n) < +\infty. \tag{5.4}$$

There exist a subsequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that (g_{n_k}) converges to g in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, $g \in W^{1,p}(\mathbb{R}^N)$ for $p > 1$ (resp., $g \in$

[2] One does not require any blow-up rate of $I_\delta(g)$ as $\delta \to 0$.

$\mathrm{BV}(\mathbb{R}^N)$ *for* $p = 1$*) and there exists a positive constant* C*, depending only on* N *and* p*, such that*

$$\int_{\mathbb{R}^N} |\nabla g|^p \, dx \le C \varliminf_{n \to \infty} I_{\delta_n}(g_n). \tag{5.5}$$

Letting δ go to 0 in (5.1), (5.2), and (5.4), we rediscover the Poincaré inequality, the Sobolev inequality, and the Rellich–Kondrachov compactness criterion. In the case $p > 1$, by Theorem 3.4, we improve these classical results. Since $I_\delta(g) \le (\delta^p/\delta'^p) I_{\delta'}(g)$ for $\delta \ge \delta'$, Theorems 5.1 and 5.2 are more interesting when they are used for large δ.

The proof of Theorem 5.1 essentially uses the following lemma [74, Lemma 5] which is based on Lemma 3.8.

Lemma 5.4. *Let* $p \ge 1$*,* $0 < \tau_0 < 1/2$*, and let* g *be a real measurable function defined in a bounded interval* I*. Suppose that there exist* $0 < \tau_0 < \tau < 1/2$*,* $c_1 < c_2$*, and two nonempty subintervals* I_1 *and* I_2 *of* I *such that*

$$\big|\{x \in I_1; g(x) < c_1\}\big| \ge \tau |I_1| \quad \textit{and} \quad \big|\{x \in I_2; g(x) > c_2\}\big| \ge \tau |I_2|. \tag{5.6}$$

There exists some positive constant C*, depending only on* p *and* τ_0*, such that*

$$\underset{|g(x)-g(y)|>\delta}{\int_I \int_I} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy \ge C_{p,\tau_0} (c_2 - c_1)^p |I|^{1-p} \quad \forall \, \delta \in (0, \delta_0), \tag{5.7}$$

where $\delta_0 = \tau(c_2 - c_1)/200$*.*

Admitting Lemma 5.4, we give a sketch of the proof of Theorem 5.1.

Sketch of the proof of Theorem 5.1. We first consider the case $N = 1$. We assume in addition that $g \in L^\infty$. By scaling, one may assume that $I = [0, 1]$ and[3]

$$|g|_{\mathrm{BMO}(I)} = 2. \tag{5.8}$$

We recall the following fact due to John and Nirenberg [54]: There exist two universal constants c_1 and c_2 such that if $-\infty < a < b < +\infty$ and $u \in \mathrm{BMO}([a, b])$, then

$$\begin{aligned} &\left| \{x \in (a, b); |u - \fint_a^b u(s) \, ds| > t\} \right| \\ &\qquad \le c_1 (b - a) \exp\left(- \frac{c_2 t}{|u|_{\mathrm{BMO}([a,b])}} \right) \quad \forall \, t > 0. \end{aligned} \tag{5.9}$$

Let $0 < a < b < 1$ be such that

$$\fint_a^b \left| g(x) - \fint_a^b g(s) \, ds \right| dx \ge 1. \tag{5.10}$$

[3] In this paper, we use the following definition of the BMO seminorm:

$$|f|_{\mathrm{BMO}(\Omega)} := \sup_{B(x,r) \Subset \Omega} \fint_{B(x,r)} \left| f(\xi) - \fint_{B(x,r)} f(\eta) \, d\eta \right| d\xi \quad \forall \, f \in \mathrm{BMO}(\Omega),$$

where $B(x, r)$ denotes the ball in Ω of radius r centered at x.

The existence of a and b follow from (5.8). Without loss of generality, one may assume that

$$\fint_a^b g\,dx = 0. \tag{5.11}$$

Using (5.9), we derive from (5.8), (5.10), and (5.11) that there exist two universal constants $\tau_1 < 0$ and $\tau_2 > 0$ such that

$$\frac{1}{b-a}\big|\{x \in (a,b); g(x) < \tau_1\}\big| \geq C,$$
$$\frac{1}{b-a}\big|\{x \in (a,b); g(x) > \tau_2\}\big| \geq C.$$

Applying Lemma 5.4, we have

$$\underset{|g(x)-g(y)|>\delta}{\int_a^b\int_a^b} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p \geq C \quad \forall\,\delta > 0.$$

The conclusion follows.

To drop the L^∞ assumption, one just needs to apply the result for g_K, where $g_K := \min\{\max\{g,-K\},K\}$ and let $K \to +\infty$. The proof in higher dimensions is based on the one-dimensional result. The proof is quite standard and uses (3.11). $\square$

Ideas of the proof of Theorem 5.2. The proof of Theorem 5.2 is based on Theorem 5.1. Using the Poincaré inequality to prove the Sobolev inequality is not new; see, e.g., [46]. Nevertheless, the standard way to do it is to use the Riesz potential theory. This approach does not seem to apply in the context of Theorem 5.2 where (5.1) is known. Note that (5.1) gives an estimate for the sharp function $g^\sharp$ of g from $I_\delta(g)$. From the famous result of Fefferman and Stein (see, e.g., [87]), it is known that if $g \in L^p(\mathbb{R}^d)$ and $p > 1$, then

$$\|g\|_{L^p} \sim \|g^\sharp\|_{L^p}.$$

Our proof of Theorem 5.2 is in the same spirit. We first derive some estimate for the sharp function of g (more precisely, the dyadic sharp function of g) from $I_\delta(g)$ and then establish the estimate for g. Indeed, using Vitali's covering lemma, it is not difficult to derive from (5.1) that $g^\sharp \in L^q_w$ (weak L^q) with $q = Np/(N-p)$. To obtain the information for L^q, we apply the truncation method due to Maz'ya [58] and use an inequality on sharp functions [87, Estimate (22) on page 153].[4] $\square$

[4]In fact, we use the following variant of (3.24), for $p > 1$

$$\int_B\int_B |g(x)-g(y)|^p\,dx\,dy \leq C_{N,p}\left(|B|^{\frac{N+p}{N}} \underset{|g(x)-g(y)|>\delta}{\int_B\int_B} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + \delta^p|B|^2\right),$$

if the right-hand side is finite (see Lemma 3.9 for its variant). This is due to the truncation method.

Proof of Theorem 5.3. Proving Rellich–Kondrachov compactness criterion using the Poincaré inequality is quite standard; see, e.g., [15, 76]. Applying Theorem 5.1, we have, for each ball B of $\mathbb{R}^N$,

$$\int_B |g_n(x) - g_{n,\varepsilon}(x)|^p \, dx \\ \leq C_{N,p} \left(\varepsilon^p \underset{|g_n(x)-g_n(y)|>\delta_n}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\delta_n^p}{|x-y|^{N+p}} \, dx \, dy + \delta_n^p |B| \right),$$

where

$$g_{n,\varepsilon} = \frac{1}{|\varepsilon B_1|} g_n * \chi_\varepsilon.$$

We recall that B_1 is the unit ball centered at the origin. Here χ_ε is the characteristic function of εB_1. Hence

$$\lim_{\varepsilon \to 0} \left(\varlimsup_{n \to \infty} \int_B |g_n(x) - g_{n,\varepsilon}(x)|^p \, dx \right) = 0.$$

Since (g_n) is bounded in $L^p(\mathbb{R}^N)$, it follows from a standard argument (see, e.g., the proof of the theorem of Riesz–Frechet–Kolmogorov in [23, Theorem IV.25]) that there exist a subsequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that g_{n_k} converges to g in $L^p_{\text{loc}}(\mathbb{R}^N)$. The second assertion of Theorem 5.3 is in the same spirit of the fact that $\widehat{K}_{N,p} > 0$, where $\widehat{K}_{N,p}$ is the constant in Theorem 4.2. $\square$

A variant of the Poincaré inequality related to the setting of Bourgain, Brezis, and Mironescu in Theorem 3.1 was established by Ponce in [76]. He first established a compactness criterion, which extended a compactness result in [15] (see [15, Theorem 4]), and then obtained a variant of the Poincaré inequality from the compactness. Our proof is completely different from his. In fact, we use a variant of the Poincaré inequality to prove the compactness criterion.

6. The Jacobian distributional of maps from a sphere into itself

Brezis and Nirenberg in [35] proved that if $(g_k) \subset C^0(\mathbb{S}^N, \mathbb{S}^N)$ and $g \in C^0(\mathbb{S}^N, \mathbb{S}^N)$ $(N \geq 1)$ are such that $\lim_{k \to 0} |g_k - g|_{\text{BMO}} = 0$, then

$$\lim_{k \to \infty} \deg g_k = \deg g. \tag{6.1}$$

On the other hand, the well-known Kronecker formula asserts that

$$\deg g = \frac{1}{|\mathbb{S}^N|} \int_{\mathbb{S}^N} \det(\nabla g) \, d\sigma \tag{6.2}$$

for any $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$. In this integral, "det" denotes the determinant of an $N \times N$ matrix, once an orientation has been chosen on $\mathbb{S}^N$. Consequently, $\int_{\mathbb{S}^N} \det(\nabla g_k) \, d\sigma$ converges to $\int_{\mathbb{S}^N} \det(\nabla g) \, d\sigma$ provided $g_k \to g$ in $\text{BMO}(\mathbb{S}^N)$.

In joint work with Brezis [32], we considered the quantity

$$\mathbf{J}(g,\psi) := \int_{\mathbb{S}^N} \psi \det(\nabla g)\, d\sigma \quad \forall\, \psi \in C^1(\mathbb{S}^N,\mathbb{R})$$

and studied the convergence of $\mathbf{J}(g_k,\psi)$ for fixed $\psi \in C^\infty(\mathbb{S}^N,\mathbb{R})$ under various assumptions on the convergence of (g_k). We proved the following result [32, Theorem 1].

Theorem 6.1. *Let $N \geq 1$, $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$, and $g \in C^1(\mathbb{S}^N,\mathbb{S}^N)$ be such that the following two conditions hold:*

(i) $\overline{\lim}_{k\to\infty} \|g_k - g\|_{\mathrm{BMO}(\mathbb{S}^N)} < 1$;
(ii) $\lim_{k\to\infty} \|g_k - g\|_W = 0$.

Then

$$\lim_{k\to\infty} \mathbf{J}(g_k,\psi) = \mathbf{J}(g,\psi) \quad \forall\, \psi \in C^1(\mathbb{S}^N,\mathbb{R}).$$

Here,

$$\|g\|_W^N := |g|_W^N + \|g\|_{L^N}^N = \int_\Omega\int_\Omega \frac{|g(x)-g(y)|^N}{|x-y|^{2N-1}}\, dx\, dy + \|g\|_{L^N}^N. \tag{6.3}$$

Note that when $N \geq 2$, $\|\cdot\|_W$ corresponds to the norm in the fractional Sobolev space $W^{s,p}$ with $s = \frac{N-1}{N}$ and $p = N$.

If one of the assumptions of Theorem 6.1 fails, the conclusion need not be true. More precisely, we can construct the following examples (see [32, Propositions 1 and 5]).

(a) There exists a sequence $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$ $(N \geq 2)$ such that

$$(g_k) \to g := (0,\ldots,0,1) \quad \text{in } C^{0,\frac{N-1}{N}},$$
$$\sup_k \|g_k\|_W < +\infty,$$
$$\underline{\lim}_{k\to\infty} \mathbf{J}(g_k, x_{N+1}) > 0 = \mathbf{J}(g, x_{N+1}).$$

(b) There exists a sequence $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$ $(N \geq 1)$ such that

$$g_k \to g := (0,\ldots,0,1) \text{ a.e.}, \quad \|g_k - g\|_W \to 0,$$
$$\sup_k \|\nabla g_k\|_{L^N} < +\infty, \quad \lim_{k\to\infty} \|g_k - g\|_{\mathrm{BMO}} = 1,$$
$$\deg g_k = 1 > 0 = \deg g \quad \text{for all } k.$$

As a consequence of Theorem 6.1, we obtain the following result, which is optimal by statement (a) mentioned above.

Corollary 6.2. *Let $N \geq 2$, $\frac{N-1}{N} < \alpha < 1$, $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$, and $g \in C^1(\mathbb{S}^N,\mathbb{S}^N)$ be such that g_k converges to g in $C^{0,\alpha}(\mathbb{S}^N)$. Then*

$$\lim_{k\to\infty} \mathbf{J}(g_k,\psi) = \mathbf{J}(g,\psi) \quad \forall\, \psi \in C^1(\mathbb{S}^N,\mathbb{R}).$$

We also establish an estimate for $\mathbf{J}(g,\psi)$ in the spirit of Section 2 (see [32, Theorem 2]).

Theorem 6.3. *Let $N \geq 1$, $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$. Then*

$$|\mathbf{J}(g, \psi)| \leq C \left(\|\psi\|_{L^\infty} T_{\ell_N}(g) + \|\nabla\psi\|_{L^\infty} |g|_W^N \right) \tag{6.4}$$

for some positive constant $C = C(N)$.

Here

$$T_\delta(g) := \int_{\mathbb{S}^N} \int_{\mathbb{S}^N}_{|g(x)-g(y)| \geq \delta} \frac{1}{|x-y|^{2N}} \, dx \, dy,$$

ℓ_N is defined by (2.14), and $| \cdot |_W$ is defined in (6.3). Clearly, Theorem 6.3 implies (2.15). One cannot derive Theorem 6.3 from (2.15). The proof of Theorem 6.3 borrows many ideas from the proof of (2.15) by Nguyen in [70] and also from the earlier papers of Bourgain, Brezis, and Mironescu [16, 17], and Bourgain, Brezis, and Nguyen [18].

An immediate consequence of Theorem 6.3 is the following.

Corollary 6.4. *Let $N \geq 1$, $\frac{N-1}{N} < \alpha < 1$, $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$. Then*

$$|\mathbf{J}(g, \psi)| \leq C \left(\|\psi\|_{L^\infty} |g|_{0,\alpha}^{\frac{N}{\alpha}} + \|\nabla\psi\|_{L^\infty} |g|_{0,\alpha}^N \right)$$

for some positive constant $C = C(\alpha, N)$, depending only on α and N.

Corollary 6.4 is optimal in the following sense: Let $N \geq 2$ and $g = (0, \ldots, 0, 1) \in \mathbb{S}^N$. There exist a sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ such that

$$\lim_{k \to \infty} \|g_k - g\|_{0, \frac{N-1}{N}} = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathbf{J}(g_k, \psi) = +\infty.$$

The construction is given in [32, Section 3].

The Jacobian distributional has a special structure when $N = 1$, which is considered with great details in [32]. We only present here an open question [32, Open question 2].[5]

Open question 4. Let $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ and $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ satisfy the following two conditions:

(i) $\overline{\lim}_{k \to \infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^1)} < 1$,

(ii) g_k converges to g a.e. on $\mathbb{S}^1$.

Is it true that

$$\lim_{k \to \infty} \int_{\mathbb{S}^1} \det(\nabla g_k) \psi \, dx = \int_{\mathbb{S}^1} \det(\nabla g) \psi \, dx \quad \forall \psi \in C^1(\mathbb{S}^1, \mathbb{R})? \tag{6.5}$$

The results in [32] are reported by Mironescu in [63]. These results are related to the work of Bourgain, Brezis, and Mironescu [16, 17], Jerrard and Soner [53], Hang and Lin [47], Brezis, Mironescu, and Ponce [30]. The reader is warmly invited to consult the original paper [32] for the detailed discussions. Other types of results concerning $\mathbb{S}^N$-valued maps can be found in, e.g., [2, 6, 7, 8, 16, 22, 27, 29, 35, 41, 43, 45, 47, 48, 50, 57, 59, 77, 82] and the references therein.

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[5] See [32, Remark 10] for partial results.

7. The Jacobian determinant

The study of the Jacobian determinant was initiated by the seminal works of Morrey [64], Reshetnyak [79], and Ball [4]. It has been known that one can define the distributional Jacobian determinant $\mathrm{Det}(\nabla g)$ under fairly weak assumptions on g; in particular, it is defined for all maps

$$g \in W^{1,\frac{N^2}{N+1}}(\Omega)$$

and also for all maps $g \in L^\infty(\Omega) \cap W^{1,N-1}(\Omega)$ (see, e.g., [4, 5, 39, 44]). Moreover,

$$\begin{aligned}&\Big|\langle \mathrm{Det}(\nabla g), \psi\rangle\Big| \\ &\quad\le C \min\Big\{ \|\nabla g\|^N_{L^{\frac{N^2}{N+1}}}, \|g\|_{L^\infty} \|\nabla g\|^{N-1}_{L^{N-1}} \Big\} \|\nabla \psi\|_{L^\infty} \quad \forall\, \psi \in C^1_{\mathrm{c}}(\Omega).\end{aligned} \tag{7.1}$$

Estimate (7.1) follows from the divergence structure of the Jacobian determinant which is originally due to Morrey [64, Lemma 4.4.6]. Namely, if g is smooth, we have

$$\det(\nabla g) = \sum_{j=1}^N \frac{\partial g_i}{\partial x_j} C_{i,j} = \sum_{j=1}^N \frac{\partial}{\partial x_j}[g_i C_{i,j}] \quad \forall\, i = 1, \ldots, N, \tag{7.2}$$

since

$$\sum_{j=1}^N \frac{\partial C_{i,j}}{\partial x_j} = 0 \quad \forall\, i = 1, 2, \ldots, N.$$

Here (∇g) is the matrix whose components are $(\nabla g)_{i,j} = \frac{\partial g_i}{\partial x_j}$, and $C = (C_{i,j})$ is the matrix of cofactors of matrix (∇g).

As a natural continuation of the study of the Jacobian distributional discussed in Section 6, with Brezis in [33], we deal with the Jacobian determinant. One of our goals is devoted to the search of an "*optimal*" space (containing all the above cases) in which the Jacobian determinant is well defined (note, for example, that neither $W^{1,\frac{N^2}{N+1}}(\Omega)$ nor $W^{1,N-1}(\Omega) \cap L^\infty(\Omega)$ is a subset of the other). In what follows, we will only concentrate on this aspect. For this purpose it is convenient to work in the fractional Sobolev spaces $W^{s,p}(\Omega)$. We prove the following result [33, Theorem 3].

Theorem 7.1. *Let $N \ge 2$.*

(i) $\mathrm{Det}(\nabla g)$ *is well defined for g in $W^{\frac{N-1}{N},N}(\Omega)$ in the distributional sense.*

(ii) *For all f and $g \in W^{\frac{N-1}{N},N}(\Omega, \mathbb{R}^N)$, and for all $\psi \in C^1_{\mathrm{c}}(\Omega, \mathbb{R})$, we have*

$$\begin{aligned}&\big|\langle \mathrm{Det}(\nabla f), \psi\rangle - \langle \mathrm{Det}(\nabla g), \psi\rangle\big| \\ &\quad\le C_{N,\Omega} |f - g|_{W^{\frac{N-1}{N},N}} \Big(|f|^{N-1}_{W^{\frac{N-1}{N},N}} + |g|^{N-1}_{W^{\frac{N-1}{N},N}} \Big) \|\nabla \psi\|_{L^\infty}.\end{aligned}$$

(iii) *Let $s \in (0,1)$ and $p \in (1, +\infty)$ be such that $W^{s,p}(\Omega) \not\subset W^{\frac{N-1}{N},N}(\Omega)$. There exist a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ and a function $\psi \in C^1_{\mathrm{c}}(\Omega, \mathbb{R})$*

such that

$$\lim_{k\to\infty} \|g^{(k)}\|_{W^{s,p}} = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega \det\left(\nabla g^{(k)}\right)\psi = +\infty. \tag{7.3}$$

The proofs of parts (i) and (ii) are based on Lemma 7.3 below and a standard density argument. The proof of part (iii) is more complicated using a suggestion of Mironescu. We are grateful for his suggestion. Lemma 7.3 is a consequence of the following useful lemma which is inspired from the work of Bourgain, Brezis, and Mironescu in [16, Lemma 3] (see also [47]).

Lemma 7.2. *Let $N \ge 2$, $g \in C^1(\Omega, \mathbb{R}^N)$, and $\psi \in C^1_{\rm c}(\Omega, \mathbb{R})$. Then*

$$\int_\Omega \det(\nabla g)\psi = \sum_{i=1}^{N+1} \int_{\Omega\times(0,1)} \mathtt{D}_i(u)\partial_i\varphi\, dx \tag{7.4}$$

for any extensions $u \in C^1(\Omega \times [0,1), \mathbb{R}^N) \cap C^2(\Omega \times (0,1), \mathbb{R}^N)$ and $\varphi \in C^1_{\rm c}(\Omega \times [0,1), \mathbb{R})$ of g and ψ. Here

$$\mathtt{D}_i(u) = (-1)^{N-i} \det(\partial_1 u, \dots, \partial_{i-1}u, \partial_{i+1}u, \dots, \partial_{N+1}u) \quad \forall\, 1 \le i \le N,$$

and

$$\mathtt{D}_{N+1}(u) = -\det(\partial_1 u, \dots, \partial_N u).$$

Proof. Note that

$$\operatorname{div} \mathtt{D} = 0 \quad \text{in } \Omega \times (0,1).$$

This implies

$$\sum_{i=1}^{N+1} \int_{\Omega\times(0,1)} \mathtt{D}_i\, \partial_i\varphi = \int_{\partial\left(\Omega\times(0,1)\right)} \varphi(\mathtt{D}\cdot n).$$

Since $\varphi = 0$ for $x \in \partial(\Omega \times (0,1)) \setminus (\Omega \times \{0\})$, the conclusion follows. $\square$

Using Lemma 7.2, the trace theory, and the multilinear structure of the determinant, we can reach the following important estimate.

Lemma 7.3. *Let $N \ge 2$ and $g \in C^1(\bar\Omega, \mathbb{R}^N)$. For any $f, g \in C^1(\bar\Omega, \mathbb{R}^N)$ and $\psi \in C^1_{\rm c}(\Omega)$, we have*

$$\begin{aligned} &\left| \int_\Omega \det(\nabla f)\psi - \int_\Omega \det(\nabla g)\psi \right| \\ &\quad \le C_{N,\Omega} |f-g|_{W^{\frac{N-1}{N},N}} \left(|f|^{N-1}_{W^{\frac{N-1}{N},N}} + |g|^{N-1}_{W^{\frac{N-1}{N},N}} \right) \|\nabla\psi\|_{L^\infty}. \end{aligned} \tag{7.5}$$

We just discuss here a few results in [33], reported by Mironescu in [63]. We finally mention that the Jacobian determinant was extensively studied in the literature; see, e.g., [4, 5, 28, 38, 39, 44, 51, 52, 64, 65, 66, 67, 80, 81, 85, 86] and the references therein.

8. Further results and their applications in Image Processing

Recently in a joint work with Brezis [34], we extended the characterizations of Sobolev spaces mentioned in Section 3 to a more general setting. We were also able to establish the Γ-convergence for a class of functionals for which the monotonicity is not required. We as well applied our results for Image Processing. We will present here only two results corresponding to the case $p = 1$. The reader can find more results in [34].

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$ where it admits a limit from the left and from the right. Assume that $\varphi(0) = 0$, $\varphi(t) = \min\{\varphi(t+), \varphi(t-)\}$ for all $t > 0$, so that φ is lower semicontinuous,

$$\varphi(t) \le at^2 \quad \text{in } [0,1] \text{ for some positive constant } a, \tag{8.1}$$

$$\varphi(t) \le b \quad \text{in } \mathbb{R}_+ \text{ for some positive constant } b, \tag{8.2}$$

and the normalization condition

$$K_{N,1} \int_0^\infty \varphi(t) t^{-2}\, dt = 1. \tag{8.3}$$

Recall that $K_{N,1}$ is given in (3.1). In this section, we assume that the domain $\Omega \subset \mathbb{R}^N$ is either bounded and smooth, or that $\Omega = \mathbb{R}^N$. Given a measurable function u on Ω and a small parameter $\delta > 0$, we define

$$\Lambda(u) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x-y|^{N+1}}\, dx\, dy \quad \text{and} \quad \Lambda_\delta(u) := \delta \Lambda\left(\frac{u}{\delta}\right).$$

Under conditions (8.1), (8.2), and (8.3), we can prove that (see [34, Proposition 1])

$$\Lambda_\delta(u) \to \int_\Omega |\nabla u| \quad \text{as } \delta \to 0, \quad \text{for } u \in C^1(\bar{\Omega}). \tag{8.4}$$

Note that Λ is *never convex* when (8.2) and (8.3) hold. Here are some examples that we have in mind.

Example 1.

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ 1 & \text{if } t > 1. \end{cases}$$

Example 2.

$$\varphi(t) = 1 - e^{-t^2}.$$

Example 3.

$$\varphi(t) = \begin{cases} t^2 & \text{if } t \le 1, \\ t^{-\gamma} & \text{if } t > 1 \end{cases}$$

for some $\gamma > 0$.

In these examples, we ignore the normalization condition (8.3). Example 1 is extensively discussed previously. Example 2 is motivated by Image Processing and used in the Yaroslavsky filter [88].

Concerning Γ-convergence, we prove the following result (see [34, Theorem 1]).

Theorem 8.1. *Assume that* (8.1), (8.2), *and* (8.3) *hold. There exists a constant* $k \in [0,1]$, *which depends only on* d *and* φ *but is independent of* Ω *such that, as* $\delta \to 0$,

$$\Lambda_\delta \xrightarrow{\Gamma} k\Lambda_0 \quad \text{in } L^1(\Omega), \tag{8.5}$$

where

$$\Lambda_0(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|\, dx & \text{for } u \in \mathrm{BV}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, if $\underline{\lim}_{t\to\infty} \varphi(t) > 0$, *then* $k > 0$.

Remark 8.2. The conclusion of Theorem 8.1 is ambiguous when $k = 0$. The precise meaning in this case is that

$$\Lambda_\delta \xrightarrow{\Gamma} 0 \quad \text{in } L^1(\Omega).$$

The novelty in Theorem 8.1 is that no assumption on monotonicity on φ is required. This assumption is crucial in the proof of Theorem 4.2 given in [73] (in particular in the proof of Lemma 4.3). Our proof is inspired by and borrows ideas from the approach in [73].

Here is an application to Image Processing. Let $\lambda > 0$ and define

$$E_\delta(u) = \lambda \int_\Omega |u - f|^q\, dx + \Lambda_\delta(u). \tag{8.6}$$

We have the following theory (see [34, Theorem 3 and Corollary 6]).

Theorem 8.3. *Let* $N \geq 1$, $q \geq 1$, $\lambda > 0$, Ω *be bounded, and let* $f \in L^q(\Omega)$. *Assume that* φ *satisfies* (8.1), (8.2), *and* (8.3). *Let* $(\delta_n) \to 0$ *as* $n \to \infty$. *There exists* $u_n \in L^q(\Omega)$ *such that*

$$E_{\delta_n}(u_n) = \inf_{u \in L^q(\Omega)} E_{\delta_n}(u).$$

Assume in addition that φ *is nondecreasing. Then* $u_n \to u_0$ *in* $L^q(\Omega)$ *where* u_0 *is the unique minimizer of the functional* E_0 *defined on* $L^q(\Omega) \cap \mathrm{BV}(\Omega)$ *by*

$$E_0(u) := \lambda \int_\Omega |u - f|^q + k \int_\Omega |\nabla u|.$$

Here k *is the constant in Theorem* 8.1.

As explained in [34], E_δ and E_0 are closely related to functionals used in Image Processing for the purpose of denoising the image f. In fact, E_0 corresponds to the celebrated ROF filter originally introduced by Rudin, Osher, and Fatemi in [83]. While, E_δ (with φ as in Example 2) is reminiscent of a Yaroslavsky filter (see [36, 88, 89]), once it has been expressed in the variational framework, as explained in the paper by Kindermann, Osher, and Jones [84]. Theorem 8.3 says that the Yaroslavsky filter converges to the ROF filter—a fact which seems to be new to the experts in Image Processing.

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