A symbolic approach to decentralized supervisory control of hybrid systems

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***Abstract*—We suggest a synthetic symbolic approach to decentralized supervisory control for the class of hybrid dynamical systems that can be modelled as hybrid state machines with a finite external signal space. The decentralized computational scheme represents a conjunction of a finite number of subsupervisors, which are invoked by a decomposition of the external signal space. On this basis, we derive sufficiency conditions for the existence of the solution to the decentralized supervisory control problem, which provide a suitable initial point to a constructive approach of appropriate decompositions of the signal space.**

***Index Terms*—decentralized control, hybrid dynamical systems, supervisory control, decomposition**

I. Introduction

Hybrid dynamical control systems arise at heterogeneous systems that exhibit a coupled time-triggered and event-triggered dynamics. A variety of control problems for such systems have been receiving extensive attention (see e.g. [5], [4], etc.). A popular approach has been the abstraction of continuous dynamics in attempting to remove the aforementioned heterogeneity (e.g. see [8]). Such approaches lead necessarily to a discrete event systems framework, where a rich toolset for control and verification exist [2], [7], etc. In this paper, we follow a similar approach in investigating the decentralized supervisory control for the class of systems that can be captured by a hybrid state machine with a finite external signal space [6], [1], etc.

Decentralized control is typically motivated by the inherent distribution of many complex systems, by computational time/space complexity reduction and robustness. Therefore, we initially introduce a decomposition of the original signal space into a finite number of aggregate signal spaces of a lower cardinality by means of, what we call here, the aggregation maps. Note that this concept is different, though closely related to the natural projections [3]. Thereby, each of the introduced aggregate signal spaces invokes a state machine by relabeling the symbols of the original one. Intuitively, this may be interpreted as a setup with a finite number of "coarse" sensors and actuators. Effectively, the information collected from the different discrete-valued sensors is complete, yet coarsely quantized, which represents a subtle difference to the sensory set with partial observations. In this article we derive a few of sufficient conditions which guarantee the existence of the solution to the supervisory decentralized control problem within our framework and also relate them to conditions of signal space decomposition.

The remainder of the paper is organized as follows. In Section II the reader is made familiar with the used notation and the basic preliminary concepts on systems and representation in the context of the behavior theory [9]. Section III represents the core of the work. We define here the conjunctive decentralized supervisory control scheme, and discuss thoroughly the concepts of coobservability, non-conflictness and implementability. The main results are summarized in Section III-E. Plenty of examples are elaborated to clarify and illustrate the concepts and statements.

II. Preliminaries

A. Notation

Capital letters denote signal spaces, e.g. X and W represent the state space and the external symbol space, respectively. The corresponding elements are denoted by Greek lowercase letters, e.g. $W = \{\omega_1, \cdots, \omega_m\}$, $\xi \in X$, etc. We consider the discrete time domain, hence trajectories, which are denoted by lowercase letters, are sequences of symbols from the appropriate symbol space, e.g. $w : \mathbb{Z}_+ \to W$ represents a trajectory. The restriction of a trajectory, or a set of trajectories to an interval $[\tau, t]$, with $0 \leq \tau \leq t \in \mathbb{Z}_+$, is denoted by $.|_{[\tau,t]}$, e.g. $w|_{[\tau,t]} = w(\tau)w(\tau+1)\cdots w(t)$. The space of the finite trajectories (strings) $w|_{[\tau,t]}$ will be denoted as $W^{[\tau,t]} \equiv W^{t-\tau+1}$. Further let $\sim$ be an equivalence relation defined on a set W. The equivalence class of an element $\alpha \in W$ is the subset of all elements in W which are equivalent to α, that is: $[\alpha] = \{\omega \in W; \omega \sim \alpha\}$. The quotient set of W by $\sim$ is defined as $W/\sim = \{[\alpha]; \alpha \in W\}$. The canonical projection map $\pi : W \to W/\sim$ maps elements of W to equivalence classes. Finally, the equivalence classes will be labeled by symbols with the help of an injective function $L : W/\sim \to V = \{\theta_1, \dots, \theta_q\}$, which we refer to as a labeling function. We also use often the shorthand $\boldsymbol{p} := \{1, \dots, p\}$ for $p \in \mathbb{N}$. This is mostly employed in the context of the inclusion $i \in \{1, \dots, p\}$, which we shorthanded by $i \in \boldsymbol{p}$.

B. Systems & Realizations

A dynamical system Σ is defined as a triple $(T, W, \mathcal{B})$, with *time axis* $T \subseteq \mathbb{R}$, the *external signal space* W, and the *behaviour* $\mathcal{B} \subseteq W^T$, where $W^T = \{w : T \to W\}$, see [9].

In words, $\mathcal{B}$ represents a family of sequences $w : \mathbb{N}_0 \to W$ which are compatible with the dynamics of the system Σ. A *state machine* is defined as a tuple $P = (X, W, \Delta, X_0)$ where X denotes the state space, W the external signal space, $\Delta \subseteq X \times W \times X$ the transition relation, and $X_0 \subseteq X$ the initial state set. If $X = \mathbb{R}^n \times D$, where $n \in \mathbb{N}$ and $D \subset \mathbb{N}$ is a finite set, then P is referred to as a *hybrid state machine*; for $n = 0$, P is a *finite state machine*. For systems exhibiting an input/output structure, the external signal space W can be decomposed as $W = U \times Y$, with U and Y being the sets of input and output symbols. A state machine $P = (X, W, \Delta, X_0)$ induces a state space system $\Sigma_\mathrm{S} = (\mathbb{N}_0, W \times X, \mathcal{B}_\mathrm{S})$, where $\mathcal{B}_\mathrm{S}$ is referred to as the *full behaviour*, and is defined as $\mathcal{B}_\mathrm{S} := \{(w,x); (x(t),w(t),x(t+1)) \in \Delta, t \in \mathbb{N}_0, x_0 \in X_0\}$. The *external behaviour* $\mathcal{B}_\mathrm{ex}$ of Σ_S is then defined to be the projection of $\mathcal{B}_\mathrm{S}$ onto $W^{\mathbb{N}_0}$, that is $\mathcal{B}_\mathrm{ex} := \mathcal{P}_W \mathcal{B}_\mathrm{S} = \{w; \exists x \in X^{\mathbb{N}_0}, (w, x) \in \mathcal{B}_\mathrm{S}\}$.A state machine $P = (X, W, \Delta, X_0)$ with induced external behaviour $\mathcal{B}_\mathrm{ex}$ is called a realization of a dynamical system $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ if $\mathcal{B}_\mathrm{ex} = \mathcal{B}$. This will be denoted by $P \cong \Sigma$. Finally, for any sequence $w|_{[0,t]} \in \mathcal{B}|_{[0,t]}$, the set of *feasible* symbol extensions at the next time instant $t + 1$ is given by:

$$\phi_{\mathcal{B}}(w|_{[0,t]}) := \{\sigma \in W; w|_{[0,t]}\sigma \in \mathcal{B}|_{[0,t+1]}\}. \tag{1}$$

Hybrid state machines cover a wide range of system classes including the time-driven (continuous), event-driven and hybrid systems. The synthesis of the transition relation Δ consists in symbolic encoding of the system behaviour $\mathcal{B}$ in terms of the transitions between the states in X. For illustration purposes, consider a time-driven continuous system defined by $\dot{\xi} = a(\xi)$ where $\xi \in X \subseteq \mathbb{R}^n$ and $a : X \to X$. Introduce the external signal space $W = L(X/Q)$, where Q represents a finite equivalence relation in X, and L a labeling function. Unique solutions $\phi(t, \xi)$ can be associated with each initial value $\xi \in X$ if $a : X \to X$ is Lipschitz on X. Then, $(\xi, \omega, \xi') \in \Delta$ if $\xi' = \phi(T_s, \xi)$ with T_s representing the sampling time, and $L(\pi_Q(\xi')) \neq L(\pi_Q(\xi)) = \omega \in W$. If $\phi(t, \xi) \in \pi_Q(\xi)$ for all $t \in \mathbb{R}$, then we adopt $(\xi, \omega, \xi) \in \Delta$. Other encoding scenarios for the transition relation Δ can be utilized alternatively.

C. Signal space decomposition (quantization)

Introduce a finite set of equivalence relations A_k on the external signal space W, with π_k representing the corresponding canonical projection map

$$\pi_k : W \to W/A_k, \quad k \in \boldsymbol{p}. \tag{2}$$

Let further $L_k : W/A_k \to V_k$ be functions which assign to each equivalence class $[\omega]_{A_k}$ a labeling symbol θ_k. The composition $\mathcal{A}_k := L_k \circ A_k$,

$$\mathcal{A}_k : W \to V_k, \quad k \in \boldsymbol{p}, \tag{3}$$

ushers V_k as an "aggregate" space of the external signal space W. Formally, we denote this by $V_k = L_k(W/A_k)$. Then, each symbol $\omega \in W$ is associated with an ordered p-tuple of symbols $(\theta_1, \ldots, \theta_p) \in V_1 \times \ldots \times V_p$.This will be referred to as the *decomposition* or *quantization* of the signal space W and formally designated as

$$W \rightsquigarrow V_1 \times \ldots \times V_p. \tag{4}$$

Next consider the space of infinite sequences $W^{\mathbb{Z}_+}$, and extend the mappings $\mathcal{A}_k : W^{\mathbb{Z}_+} \to V_k^{\mathbb{Z}_+}$ recursively by: $\mathcal{A}_k(w|_{[0,t+1]}) := \mathcal{A}_k(w|_{[0,t]})\mathcal{A}_k(w(t)), \quad t \in \mathbb{Z}_+$. where $\mathcal{A}_k(w|_{[0,0]}) := \mathcal{A}_k(w(0))$. Then, we can introduce the mapping of the behaviour set $\mathcal{B} \subseteq W^{\mathbb{Z}_+}$ defined over the symbol space W as: $\mathcal{A}_k(\mathcal{B}) := \{v \in V_k^{\mathbb{Z}_+}; v = \mathcal{A}_k(w), w \in \mathcal{B}\}$.

To simplify the notation, it will be useful to introduce the following *blowing* operators on W:

$$\pi_k := \mathcal{A}_k^{-1}\mathcal{A}_k \quad \text{and} \quad \pi := \cap_{k=1}^{p} \pi_k, \tag{5}$$

which we will apply on symbols, strings, behaviors and sets of behaviors.

A quantization (4) of the symbol space W is said to be *ω-consistent* if

$$\pi(\omega) = \omega \quad (\forall \omega \in W). \tag{6}$$

Similarly, the quantization is said to be *$\mathcal{B}$-consistent* if:

$$\pi(\mathcal{B}) = \mathcal{B} \quad (\mathcal{B} \subseteq W^{\mathbb{Z}_+}). \tag{7}$$

It is obvious that if (6) holds, then so holds $\pi(w) = w \quad (w \in W^{\mathbb{Z}_+})$. The next example illustrates that the latter two consistency concepts differ essentially.

Example 1. *Consider the symbol spaces $W = \{a, b, c, d\}$, $V_1 = \{\theta_1, \theta_2\}$ and $V_2 = \{\theta_3, \theta_4\}$. The quantization introduced by $\mathcal{A}_1, \mathcal{A}_2$ given by:*

$$\begin{aligned} \mathcal{A}_1 &: W \to V_1 : \{a, b\} \mapsto \theta_1, \{c, d\} \mapsto \theta_2, \\ \mathcal{A}_2 &: W \to V_2 : \{a, c\} \mapsto \theta_3, \{b, d\} \mapsto \theta_4 \end{aligned}$$

is $\omega-$consistent, but not $\mathcal{B}-$consistent relative to $\mathcal{B} \subseteq W^{\mathbb{Z}_+}$: $\mathcal{B} = \{(ad)^*, (bd)^*, (cd)^*\}$, since:

$$\pi(\mathcal{B}) = \{(ad)^*, (bd)^*, (cd)^*, (ad)^*\} \neq \mathcal{B}. \quad \square$$

Given two behaviour sets $\mathcal{B}_1, \mathcal{B}_2 \subseteq W^{\mathbb{Z}_+}$, we say that the quantization (4) $W \rightsquigarrow V_1 \times \ldots \times V_p$ is *$\mathcal{B}_2$-consistent relative to $\mathcal{B}_1$* if:

$$\mathcal{B}_1 \cap \pi(\mathcal{B}_2) = \mathcal{B}_1 \cap \mathcal{B}_2. \tag{8}$$

Obviously, if a quantization is $\mathcal{B}_2$-consistent, then it is $\mathcal{B}_2$-consistent relative to any $\mathcal{B}_1 \subseteq W^{\mathbb{Z}_+}$, while the opposite clearly must not hold true.

Two behaviours $\mathcal{B}_1$ and $\mathcal{B}_2$ defined over the same symbol space W are said to be non-conflicting, if:

$$(\mathcal{B}_1 \cap \mathcal{B}_2)|_{[0,t]} = \mathcal{B}_1|_{[0,t]} \cap \mathcal{B}_2|_{[0,t]} \quad (\forall t \in \mathbb{Z}_+). \tag{9}$$

In words, $\mathcal{B}_1$ and $\mathcal{B}_2$ are non-conflicting if the time windowing restriction operator is injective.

Proposition 1. *Consider an $\omega-$consistent decomposition. Then $\pi_k\mathcal{B}$ are non-conflicting for any $\mathcal{B} \subseteq W^{\mathbb{Z}_+}$, that is:*

$$\cap_{k=1}^{p} \pi_k(\mathcal{B})|_{[0,t]} = \left(\cap_{k=1}^{p} \pi_k(\mathcal{B})\right)|_{[0,t]} \quad (\forall t \in \mathbb{Z}_+). \tag{10}$$

In other words, the blowing operator π and the time window restriction are then commutative on $\mathcal{B}$:

$$\pi(\mathcal{B})|_{[0,t]} = \pi(\mathcal{B}|_{[0,t]}) \quad (\forall t \in \mathbb{Z}_+). \tag{11}$$

Hence, for simplicity we drop the brackets in $\pi\mathcal{B}|_{[0,t]}$.
Proof: The right-hand side of (18) is always an subset of the left-hand side. To prove the opposite, let $w \in W^{\mathbb{Z}_+}$, such that $w|_{[0,t]}$ is an element of the left-hand side in (18). Then by $\omega-$consistency, we know that there must exist a w', such that $w'|_{[0,t]} = w|_{[0,t]}$ and $w' \in \mathcal{B}$. Hence, $w|_{[0,t]}$ is element of the right-hand side, as well. □

Finally, $\mathcal{B}_2$ is π-non-conflicting relative to $\mathcal{B}_1$ if

$$(\mathcal{B}_1 \cap \pi(\mathcal{B}_2))\,|_{[0,t]} = \mathcal{B}_1|_{[0,t]} \cap \pi(\mathcal{B}_2|_{[0,t]}) \quad (\forall t \in \mathbb{Z}_+). \tag{12}$$

III. Decentralized control

A. Problem formulation

Consider a plant denoted by $\Sigma_\text{p} = (\mathbb{Z}_+, W, \mathcal{B}_\text{p}) \cong P_\text{p}$ and a specification denoted by $\Sigma_\text{spec} = (\mathbb{Z}_+, W, \mathcal{B}_\text{spec}) \cong P_\text{spec}$, where the behaviour of the specification $\mathcal{B}_\text{spec}$ contains all the allowed trajectories. The task of the supervisory control is to design a supervisor denoted by $\Sigma_\text{sup} = (\mathbb{Z}_+, W, \mathcal{B}_\text{sup}) \cong P_\text{sup}$, which prevents the trajectories in $\mathcal{B}_\text{p}$ that do not belong to $\mathcal{B}_\text{spec}$, that is:

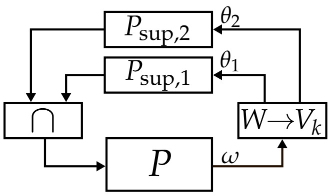


Fig. 1. Decentralized control struc.

$$\emptyset \subset \mathcal{B}_\text{cl} := \mathcal{B}_p \cap \mathcal{B}_\text{sup} \subseteq \mathcal{B}_\text{spec}, \tag{13}$$

where $\mathcal{B}_\text{cl}$ stands for the closed-loop behaviour.

With reference to the above figure, we define analogously the decentralized supervisory controller as the tupple $P_\text{sup,dec} = (P_{\text{sup},1}, \ldots, P_{\text{sup},p})$, where $P_\text{sup,dec} \cong \Sigma_\text{sup,dec} := (W, \mathbb{Z}_+, \mathcal{B}_\text{sup,dec})$, $P_{\text{sup},k} \cong \Sigma_{\text{sup},k} := (V_k, \mathbb{Z}_+, \mathcal{B}_{\text{sup},k})$. By taking the intersection as the fusion rule, $\mathcal{B}_\text{sup,dec}$ is defined by: $\mathcal{B}_\text{sup,dec} = \cap_{k=1}^{p} \mathcal{A}_k^{-1}(\mathcal{B}_{\text{sup},k})$, where we define

$$\mathcal{B}_{\text{sup},k} = \mathcal{A}_k(\mathcal{B}_\text{p}) \cap \mathcal{A}_k(\mathcal{B}_\text{sup}), \tag{14}$$

or explicitly:

$$\mathcal{B}_\text{sup,dec} = \pi(\mathcal{B}_\text{p}) \cap \pi(\mathcal{B}_\text{spec}). \tag{15}$$

In words, all local controllers $P_{\text{sup},k}$ have to agree on the control action at each time sample. The subsupervisors $P_{\text{sup},k}$ receive the information $\mathcal{A}_k(\omega)$ (namely, θ_1 and θ_2), which can be interpreted as a partial observation of the plant output ω. Thereby a number of additional constraints have to be guaranteed, as discussed in the sequel.

B. Coobservability

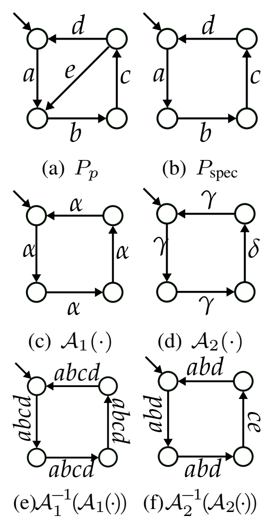
Definition 1. *Given a plant $\Sigma_p = (\mathbb{Z}_+, W, \mathcal{B}_p)$, a specification denoted by $\Sigma_{spec} = (\mathbb{Z}_+, W, \mathcal{B}_{spec})$ and a decomposition* (4) *$W \rightsquigarrow V_1 \times \ldots \times V_p$, the specification $\mathcal{B}_{spec}$ is said to be π-coobservable relative to the plant $\mathcal{B}_p$, if $\forall t \in \mathbb{Z}_+$ and $\forall w \in W^{\mathbb{Z}_+}$, where $w|_{[0,t]} \in \mathcal{B}_p \cap \mathcal{B}_{spec}|_{[0,t]}, w|_{[0,t+1]} \in B_p$, but $w|_{[0,t+1]} \notin B_{spec}|_{[0,t+1]}$, there exists at least one $k \in \{1, \ldots, p\}$, such that $\mathcal{A}_k(w|_{[0,t+1]}) \notin \mathcal{A}_k(B_{spec}|_{[0,t+1]})$.*

In words, coobservability guarantees that blocking of the symbol $w(t+1)$ is inherited by at least one of the local supervisors P_k, i.e. at least one of the subsupervisors knows that it has to block the monolithic symbol $w(t+1)$.

Lemma 1. *$\mathcal{B}_{spec}$ is π-coobservable realtive to $\mathcal{B}_p$ if and only if the decomposition* (4) *is $\mathcal{B}_{spec}$-consistent realtive to $\mathcal{B}_p$, that is, if:*

$$\mathcal{B}_p \cap \pi(\mathcal{B}_{spec}) = \mathcal{B}_p \cap \mathcal{B}_{spec}. \tag{16}$$

Proof: This equation is equivalent to stating that every trajectory $w \in W^{\mathbb{Z}_+}$ which belongs to its left-hand side, must belong to its right-hand side, as well. Hence, if it does not hold true, then there exists a sequence $w \in W^{\mathbb{Z}_+}$ and a $t \in \mathbb{Z}_+$ such that $w|_{[0,t]} \in \mathcal{B}_p \cap \mathcal{B}_\text{spec}|_{[0,t]}$ and $w|_{[0,t+1]} \notin \mathcal{B}_\text{spec}|_{[0,t+1]}$, but $w|_{[0,t+1]} \in \pi_k(w|_{[0,t+1]})$. This is precisely the negation of the coobservability condition. This completes the proof. □



(a) P_p (b) P_spec (c) $\mathcal{A}_1(\cdot)$ (d) $\mathcal{A}_2(\cdot)$ (e)$\mathcal{A}_1^{-1}(\mathcal{A}_1(\cdot))$ (f)$\mathcal{A}_2^{-1}(\mathcal{A}_2(\cdot))$

Fig. 2. Example 2

If $\mathcal{B}_\text{spec}$ is $\pi-$coobservable, then $\mathcal{B}_\text{spec}$ is obviously $\pi-$coobservable with respect to $\mathcal{B}_p$ for any plant $\mathcal{B}_p \subseteq W^{\mathbb{Z}_+}$. Next example illustrates that the opposite must not necessarily hold true, i.e., that coobservability does not imply behaviour consistency.

Example 2: Consider a plant $P_p \cong (\mathbb{Z}_+, W, \mathcal{B}_p)$ and a specification $P_\text{spec} \cong (\mathbb{Z}_+, W, \mathcal{B}_\text{spec})$ with the realizations as specified in Figure 2, with $W = \{a, b, c, d, e\}$. Introduce next $\mathcal{A}_1 : W \to V_1$ and $\mathcal{A}_2 : W \to V_2$, with $V_1 = \{\alpha, \beta\}$ and $V_2 = \{\gamma, \delta\}$ and

$$\mathcal{A}_1 : \{a, b, c, d\} \mapsto \alpha, \{e\} \mapsto \beta,$$
$$\mathcal{A}_2 : \{a, b, d\} \mapsto \gamma, \{c, e\} \mapsto \delta.$$

Then, the mappings $\mathcal{A}_1$ and $\mathcal{A}_2$ are not $\mathcal{B}_\text{spec}$-consistent as $\cap_{k=1}^{2} \mathcal{A}_k^{-1}(\mathcal{A}_k(\mathcal{B}_\text{spec})) \supseteq \mathcal{B}_\text{spec} = \{(abcd)^*\}$, but it is easily checked that they are $\mathcal{B}_\text{spec}$-consistent relative to $\mathcal{B}_\text{p}$.

Proposition 2. *Let $\mathcal{B}_{spec}$ be π-coobservable w.r.t. $\mathcal{B}_p$. Then, the decentralized closed loop guarantees $\mathcal{B}_{cl,dec} \subseteq \mathcal{B}_{spec}$.*

Proof: This follows directly from (15) and (16):

$$\begin{aligned}\mathcal{B}_\text{cl,dec} &= \mathcal{B}_p \cap \mathcal{B}_\text{sup,dec} = \mathcal{B}_p \cap \pi\mathcal{B}_\text{spec} \\ &= \mathcal{B}_p \cap \mathcal{B}_\text{spec} \subseteq \mathcal{B}_\text{spec}. \quad \square\end{aligned}$$

C. Non-conflictness

Non-conflictnes is mandatory to avoid the blocking of any closed-loop structure. For the decentralized control structure employing the set of aggregation functions $\mathcal{A}_k$, $k \in \boldsymbol{p}$, the non-conflictness reads:

$$\begin{aligned}&\left(\mathcal{B}_p \cap \left(\cap_{k=1}^{p} \mathcal{A}_k^{-1}(\mathcal{B}_{\text{sup},k})\right)\right)|_{[0,t]} \\ &\quad = \mathcal{B}_p|_{[0,t]} \cap \left(\cap_{k=1}^{p} \mathcal{A}_k^{-1}(\mathcal{B}_{\text{sup},k})|_{[0,t]}\right) \quad (t \in \mathbb{Z}_+)\end{aligned} \tag{17}$$

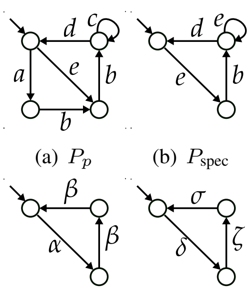
Proposition 3. *A decentralized control structure (i.e. $\mathcal{B}_{sup,dec}$ and $\mathcal{B}_p$) is non-conflicting if $\mathcal{B}_{spec}$ is π-non-conflicting relative to $\mathcal{B}_p$, that is*

$$(\mathcal{B}_p \cap \pi\mathcal{B}_{spec})\,|_{[0,t]} = \mathcal{B}_p|_{[0,t]} \cap \pi(\mathcal{B}_{spec}|_{[0,t]}) \quad (t \in \mathbb{Z}_+). \quad (18)$$

Proof: Consider the right-hand side of the latter equation (17) and substitute therein equation (15), as well as our proposition assumption to yield:

$$\begin{aligned}
&\mathcal{B}_p|_{[0,t]} \cap \left(\cap_{k=1}^{p} \mathcal{A}_k^{-1}(\mathcal{B}_{\text{sup},k})|_{[0,t]}\right) = \\
&\qquad = \mathcal{B}_p|_{[0,t]} \cap \left(\cap_{k=1}^{p} \mathcal{A}_k^{-1}\left(\mathcal{A}_k(\mathcal{B}_{\text{p}}) \cap \mathcal{A}_k(\mathcal{B}_{\text{spec}})\right)|_{[0,t]}\right) \\
&\qquad \subseteq \mathcal{B}_{\text{p}}|_{[0,t]} \cap \pi\mathcal{B}_p|_{[0,t]} \cap \pi\mathcal{B}_{\text{spec}}|_{[0,t]} \\
&\qquad = \mathcal{B}_{\text{p}}|_{[0,t]} \cap \pi\mathcal{B}_{\text{spec}}|_{[0,t]} \\
&\qquad = (\mathcal{B}_{\text{p}} \cap \pi\mathcal{B}_{\text{spec}})\,|_{[0,t]}.
\end{aligned}$$

Note that the left-hand side of (17) is precisely equal to this outcome, which completes the proof. □



(a) P_p (b) P_{spec} (c) $P_{\text{sup},1}$ (d) $P_{\text{sup},2}$

Fig. 3. Example 3

The non-conflicting condition is independent of the coobservability condition as illustrated by the next example.

Example 3: Consider the decentralized control structure in Figure 1, and let $\mathcal{B}_p$ and $\mathcal{B}_{\text{spec}}$ over $W = \{a, b, c, d, e\}$ be defined by the realizations P_{p} and P_{spec}, respectively, as depicted in Figure 3. Consider the maps $\mathcal{A}_1$ and $\mathcal{A}_2$ given by:

$$\begin{aligned}
\mathcal{A}_1 &: \{a, e\} \mapsto \alpha, \{b, d\} \mapsto \beta, \{c\} \mapsto \gamma, \\
\mathcal{A}_2 &: \{a, e\} \mapsto \delta, \{b\} \mapsto \zeta, \{c\} \mapsto \theta, \{d\} \mapsto \sigma.
\end{aligned}$$

It can be then easily checked by means of (8) that $\mathcal{B}_{\text{spec}}$ is coobservable w.r.t. $\{\mathcal{A}_1, \mathcal{A}_2\}$ and the plant $\mathcal{B}_p$. However, with the subsupervisors defined by

$$\mathcal{B}_{\text{sup},k} = \mathcal{A}_k(\mathcal{B}_p) \cap \mathcal{A}_k(\mathcal{B}_{\text{spec}}),$$

we find that for any sequence w satisfying $w|_{[0,1]} = ab$, the non-conflictness condition (17) is violated. □

The statement condition in the Proposition 3 can be made simpler, yet stronger, if the result of Proposition 1 is regarded.

Lemma 2. *$\mathcal{B}_{sup,dec}$ and $\mathcal{B}_p$ are non-conflicting if the symbol space decomposition $W \rightsquigarrow V_1 \times \ldots \times V_p$ is*

(i) ω-consistent;

(ii) $\mathcal{B}_{spec}$-consistent; and

(iii) $\mathcal{B}_p$ and $\mathcal{B}_{spec}$ are non-conflicting.

Proof: It is sufficient to recover the conditions of the previous proposition:

$$\begin{aligned}
(\mathcal{B}_{\text{p}} \cap \pi\mathcal{B}_{\text{spec}})\,|_{[0,t]} &= \\
\mathcal{B}_{\text{spec}}\text{-consistency} &= (\mathcal{B}_{\text{p}} \cap \mathcal{B}_{\text{spec}})\,|_{[0,t]} \\
(\mathcal{B}_{\text{spec}}, \mathcal{B}_{\text{p}})\text{-non-conflictness} &= \mathcal{B}_{\text{p}}|_{[0,t]} \cap \mathcal{B}_{\text{spec}}|_{[0,t]} \\
\mathcal{B}_{\text{spec}}\text{-consistency} &= \mathcal{B}_{\text{p}}|_{[0,t]} \cap (\pi\mathcal{B}_{\text{spec}})|_{[0,t]} \\
\text{commutativity} &= \mathcal{B}_{\text{p}}|_{[0,t]} \cap \pi(\mathcal{B}_{\text{spec}}|_{[0,t]}).
\end{aligned}$$
□

D. Implementability

The following notion of correlated inputs is motivated by the physical actuation structure of the plant.

Definition 2. *Given a system $\Sigma_p \cong (\mathbb{Z}_+, W, \mathcal{B}_p)$, a (controllable) symbol $\sigma' \in W$ is said to be* correlated *to the (controllable) symbol $\sigma \in W$ if σ' is disabled whenever σ is disabled.*

We include the set of correlated symbols to σ in $\text{cor}(\sigma)$, and, for simplicity, we will assume that $\sigma' \in \text{cor}(\sigma)$ implies $\sigma \in \text{cor}(\sigma')$. Equivalently, σ is accepted *iff* σ' is accepted in Σ_p, and the correlation relation is an equivalence relation. A motivation for considering the implementability is, for instance, revealed by input/output symbol spaces $W = U \times Y$, since all symbols sharing the same input event are clearly correlated.

Definition 3. *Consider a plant $\Sigma_p = (\mathbb{Z}_+, W, \mathcal{B}_p)$ and a supervisor $\Sigma_{sup} = (\mathbb{Z}_+, W, \mathcal{B}_{sup})$. Then, Σ_{sup} is said to be implementable w.r.t. to the plant $\mathcal{B}_p$, if $\forall t \in \mathbb{Z}_+$ and $\forall w \in W^{\mathbb{Z}_+}$, such that $w|_{[0,t]} \in \mathcal{B}_p|_{[0,t]} \cap \mathcal{B}_{sup}|_{[0,t]}, w|_{[0,t+1]} \in \mathcal{B}_p|_{[0,t+1]}$ but $w|_{[0,t+1]} \notin \mathcal{B}_{sup}|_{[0,t+1]}$, then*

$$\phi_{\mathcal{B}_{sup}}(w|_{[0,t]}) \cap cor(w(t+1)) = \emptyset. \quad (19)$$

$\mathcal{B}_{sup}$ is said to be essentially implementable if it is implementable w.r.t. to $W^{\mathbb{Z}_+}$. In this sense, introduce the operator "ess", such that $ess(\mathcal{B}_{sup})$ is essentially implementable. Then, $\mathcal{B}_{sup}$ is implementable w.r.t. $\mathcal{B}_p$ if and only if

$$\mathcal{B}_p \cap ess(\mathcal{B}_{sup}) = \mathcal{B}_p \cap \mathcal{B}_{sup}. \quad (20)$$

In words, if $\mathcal{B}_{\text{sup}}$ is essentially implementable, then at any time $t \in \mathbb{Z}_+$, $\mathcal{B}_{\text{sup}}$ either accepts the whole correlated symbols or denies them all. Likewise, we extend the definition of the implementability to the decentralized control structure.

Definition 4. *Consider the decentralized control system as introduced in Section III-A with the decentralized supervisor $P_{sup,dec} = (P_{sup,k}; k \in \boldsymbol{p})$ and $P_{sup,k} \cong \Sigma_{sup,k} := (V_k, \mathbb{Z}_+, \mathcal{B}_{sup,k})$. We say that $P_{sup,dec}$ is* implementable *w.r.t. the plant $\mathcal{B}_p$ if $\forall t \in \mathbb{Z}_+$ and $\forall w \in W^{\mathbb{Z}_+}$, such that if $w|_{[0,t]} \in \mathcal{B}_p|_{[0,t]} \cap \mathcal{B}_{sup,dec}|_{[0,t]}, w|_{[0,t+1]} \in \mathcal{B}_p|_{[0,t+1]}$, but $w|_{[0,t+1]} \notin \mathcal{B}_{sup,dec}|_{[0,t+1]}$, the following holds true:*

$$\cap_{k=1}^{p} \mathcal{A}_k^{-1} \phi_{\mathcal{B}_{sup,k}} \mathcal{A}_k(w|_{[0,t]}) \cap \pi\, cor(w(t+1)) = \emptyset. \quad (21)$$

Lemma 3. *Consider $\mathcal{B}_{spec} \subseteq W^{\mathbb{Z}_+}$ and the decomposition mappings $\mathcal{A}_k$, $k \in \boldsymbol{p}$. Then, $\mathcal{B}_{sup,dec}$ is implementable w.r.t. $\mathcal{B}_p$ if:*

(i) $\mathcal{B}_{spec}$ is essentially implementable, and

(ii) $\pi_k(\sigma) \subseteq cor(\sigma)$, $\forall \sigma \in W$ and $k \in \boldsymbol{p}$.

Proof: In the following lines of proof we shall use (15) and the fact that the condition *(ii)* in the lemma guarantees the operator identity $\pi\phi_{\mathcal{B}_{\text{spec}}} = \phi_{\mathcal{B}_{\text{spec}}}$:

$$\begin{aligned}
\cap_{k=1}^{p} \mathcal{A}_k^{-1} \phi_{\mathcal{B}_{\text{sup},k}} \mathcal{A}_k(w|_{[0,t]}) = & \\
&= \cap_{k=1}^{p} \mathcal{A}_k^{-1} \phi_{\mathcal{A}_k \mathcal{B}_{\text{p}} \cap \mathcal{A}_k \mathcal{B}_{\text{spec}}} \mathcal{A}_k(w|_{[0,t]}) \\
&\subseteq \cap_{k=1}^{p} \mathcal{A}_k^{-1} \phi_{\mathcal{A}_k \mathcal{B}_{\text{spec}}} \mathcal{A}_k(w|_{[0,t]}) \\
&= \cap_{k=1}^{p} \mathcal{A}_k^{-1} \mathcal{A}_k \phi_{\mathcal{B}_{\text{spec}}}(w|_{[0,t]}) \\
&= \pi\phi_{\mathcal{B}_{\text{spec}}}(w|_{[0,t]}) \\
&= \phi_{\mathcal{B}_{\text{spec}}}(w|_{[0,t]}).
\end{aligned}$$

Note that in the fourth line we used the identity $\phi_{\mathcal{A}_k \mathcal{B}_{\text{spec}}} \mathcal{A}_k = \mathcal{A}_k \phi_{\mathcal{B}_{\text{spec}}}$. But, due to the condition *(i)*, we further have $\phi_{\mathcal{B}_{\text{spec}}}(w|_{[0,t]}) \cap \text{cor}(w(t+1)) = \emptyset$, yielding:

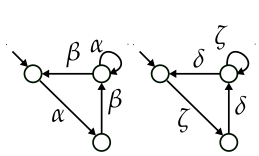
$$\begin{aligned}
\cap_{k=1}^{p} \mathcal{A}_k^{-1} \phi_{\mathcal{B}_{\text{sup},k}} \mathcal{A}_k(w|_{[0,t]}) \cap \pi\,\text{cor}(w(t+1)) = & \\
&\subseteq \phi_{\mathcal{B}_{\text{spec}}}(w|_{[0,t]}) \cap \pi\,\text{cor}(w(t+1)) \\
&\subseteq \phi_{\mathcal{B}_{\text{spec}}}(w|_{[0,t]}) \cap \text{cor}(w(t+1)) \\
&= \emptyset. \quad \square
\end{aligned}$$

E. Main statement

Now, we are ready to state our main result of this paper, which provides a sufficiency condition for the existence of the decentralized supervisor $P_{\text{sup,dec}} = (P_{\text{sup},1}, P_{\text{sup},2}, \ldots, P_{\text{sup},p})$ which solves the problem as stated in Section III-A. Note that the proof of the result has been tacitly conducted along the body of the article.

Theorem 1. *Consider a given plant $\Sigma_p = (\mathbb{Z}_+, W, \mathcal{B}_p)$, a specification $\Sigma_{spec} = (\mathbb{Z}_+, W, \mathcal{B}_{spec})$ and a given quantization $W \rightsquigarrow V_1 \times \ldots \times V_p$. Then, there exists a non-blocking and implementable decentralized supervisor $P_{sup,dec} = (P_{sup,1}, \ldots, P_{sup,p})$ such that the closed-loop behaviour $\mathcal{B}_{cl,dec}$ can be restricted to $\emptyset \subset \mathcal{B}_{cl,dec} = \mathcal{B}_p \cap \mathcal{B}_{spec} \subseteq \mathcal{B}_{spec}$ if:*

(i) the symbol space decomposition is $\mathcal{B}_{spec}-$consistent relative to $\mathcal{B}_p$;
(ii) $\mathcal{B}_{spec}$ is π-non-conflicting relative to $\mathcal{B}_p$;
(iii) $\mathcal{B}_{spec}$ is essentially implementable; and
(iv) $\pi(\sigma) \subseteq cor(\sigma)$ for all $\sigma \in W$.



(a) $P_{\text{sup},1}$ (b) $P_{\text{sup},2}$

Fig. 4. Example 4

Example 4: Reuse the problem data from *Example 3* with redefined quantization $\mathcal{A}_2$:

$$\mathcal{A}_2 : \{a, b, d\} \mapsto \delta, \{e, c\} \mapsto \zeta,$$

It can be checked that the conditions in Theorem 1 are satisfied with the subsupervisors as shown left in the figure. $\square$

Again, the conditions of Theorem 1 can be simplified if one regards the stronger non-conflicteness result of Lemma 1.

Corollary 1. *Consider the data of the latter theorem. Then, there exists a solution to the decentralized supervisory control problem if the symbol space decomposition* (4) *is:*

(i) $\omega-$consistent;
(ii) $\mathcal{B}_{spec}-$consistent;
(iii) $\mathcal{B}_{spec}$ and $\mathcal{B}_p$ are non-conflicting;
(iv) $\mathcal{B}_{spec}$ is essentially implementable; and
(v) $\pi_k(\sigma) \subseteq cor(\sigma)$ for all $\sigma \in W$, $k \in \boldsymbol{p}$.

IV. Conclusions

In this paper, we introduced a symbolic approach to decentralized supervisory control of hybrid state machines. The decentralized supervisor consists of a set of individual distributed state machines equipped with an intersection fusion rule, whereby each of them is constructed in a transformed signal space. This transformation is introduced by means of the abstract "quantization" or "aggregation" maps that lead to a decomposition of the external signal space. In practice, such maps may refer to a set of coarse sensors.

By using the coarse information, the subsupervisors provide control actions to a fusion block, which then produces the final control signal to the plant. There are several advantages of this decentralized control method. For instance, a multiple of coarse sensors can be still employed to meet the control objectives without a need for signal re-constructors and observers. Moreover, as the system is jointly controlled by relatively simple subsupervisors which are constructed in "smaller" external signal spaces (hence, implying reduced transition plant relations in the transformed signal space), significant reduction in the overall space/time computational complexity may be expected. In addition, major advantages in terms of robustness and reliability are gained due to the redundancy of computation units.

We have introduced a few of sufficient conditions for the existence of the solution to the decentralized supervisory control problem. These conditions heavily depend on the decomposition of the external signal space. Hence, it is very appealing to develop methods for the design of the proper "aggregation" maps which guarantee the underlying sufficiency existence conditions. It turns out that under some mild conditions, the so-called $\omega-$ and the $\mathcal{B}-$consistency are critical. While the first one is rather trivial to fulfill, the strategies for the second one are a matter of a further research.

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