Scaled Alexander-Spanier Cohomology and Lqp Cohomology for Metric Spaces

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This thesis is dedicated to the night.

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Abstract

With the aim to generalize the theory of L^p and L^π de Rham cohomology to metric measure spaces, we define the scaled Alexander-Spanier cohomology and L^p and L^π Alexander-Spanier cohomology. We follow the work of Pansu [33], Smale [37] and Hausmann [24].

Alexander-Spanier cohomology at scale $t > 0$ of a metric space (X, ρ) is defined as the simplicial cohomology of the complex given by all simplices $(x_0, \ldots x_k) \in X^{k+1}$ with $\text{diam}\{x_0, \ldots x_k\} < t$. Scaled L^{qp} Alexander-Spanier cohomology is the L^{qp} simplicial cohomology of the same complex. The limit as $t \to \infty$ is the asymptotic Alexander-Spanier cohomology of X.

The asymptotic L^p Alexander-Spanier cohomology is a quasi-isometry invariant for Riemannian manifolds with bounded geometry. We show that the asymptotic L^{qp} Alexander-Spanier cohomology is a quasi-isometry invariant for graphs with bounded degree and that L^∞ L^{qp} Alexander-Spanier cohomology is a quasi-isometry invariant for Riemannian manifolds with bounded geometry.

For Riemannian manifolds with bounded geometry, there exists a number $t_0 > 0$ such that for all scale $t \leq t_0$, the Alexander-Spanier cohomology at scale t is isomorphic to the de Rham cohomology. The same result is true for the L^p Alexander-Spanier cohomology and L^p de Rham cohomology.

We show that for Riemannian manifolds with bounded geometry and non-positive sectional curvature, the L^p Alexander-Spanier cohomology is independant of scale. In this situation, the asymptotic cohomology coincide with the cohomology at any scale. This results in a proof of quasi-isometry invariance for L^p de Rham cohomology on Riemannian manifolds of non-positive sectionnal curvature.

Key words : Alexander-Spanier cohomology, L^π cohomology, Vietoris-Rips complex, quasi-isometry invariance, metric space, bounded geometry, double-complex

Résumé

Nous définissons la cohomologie d'Alexander-Spanier d'échelle t ainsi que la cohomologie L^p et L^π d'Alexander-Spanier d'échelle t, dans l'objectif d'étendre certaines propriétés de la cohomologie L^p et L^π de de Rham au cadre des espaces métriques mesurés. Nous nous basons en particulier sur les travaux de Pansu [33], Hausmann [24] et Smale [37].

La cohomologie d'Alexander-Spanier à l'échelle $t > 0$ d'un espace métrique est la cohomologie simpliciale du complexe défini par l'ensemble des simplexes de la forme $(x_0, \ldots x_k) \in X^{k+1}$ tel que $\mathrm{diam}\{x_0, \ldots x_k\} < t$. La cohomologie L^{qp} d'Alexander-Spanier d'un espace métrique mesuré est la cohomologie L^{qp} du même complexe. En prenant la limite à l'inifini sur le paramètre t, on obtient la cohomologie asymptotique d'Alexander-Spanier.

La cohomologie L^p d'Alexander-Spanier asymptotique est un invariant de quasi-isométrie pour les variétés riemanniennes complète à géométrie bornée. Nous montrons que la cohomologie L^{qp} d'Alexander-Spanier asymptotique est un invariant de quasi-isométrie pour les graphes de degré borné. Nous montrons aussi que la cohomologie L^∞ L^{qp} d'Alexander-Spanier asymptotique est un invariant de quasi-isométrie pour les variétés riemanniennes à géométrie bornées.

Pour les variétés riemanniennes complète à géométrie bornée, il existe un nombre $t_0 > 0$ tel que pour tout $t \leq t_0$, la cohomologie d'Alexander-Spanier d'échelle t est isomorphe à la cohomologie de de Rham. Le même résultat est vérifié pour les cohomologies L^p d'Alexander-Spanier et de de Rham.

Nous montrons que pour les variétés riemanniennes complète à géométrie bornée de courbure sectionnelle non-positive, la cohomologie L^p d'Alexander-Spanier est indépendante de l'échelle $t > 0$ choisie. Dans cette situation, la cohomologie asymptotique coïncide avec la cohomologie d'Alexander-Spanier à n'importe quelle échelle, ce qui prouve que la cohomologie de de Rham L^p est un invariant de quasi-isométrie pour les variétés riemannniennes de courbure sectionnelle non-positive.

Mots clefs : cohomologie L^p, cohomologie L^π, Alexander-Spanier, Vietoris-Rips, espace métrique, quasi-isométrie, géométrie bornée, double-complexe

Contents

Introduction

With the aim of generalizing the L^π de Rham cohomology theory to metric measure spaces, we define a scaled Alexander-Spanier cohomology based on the idea of the Vietoris-Rips complex.

0.1 Metric cohomology

Given a metric space (X,ρ), the Vietoris-Rips complex of X at scale $t>0$ is the abstract simplicial complex X_t whose k-skeleton consists of all $k+1$-tuple of points $\{x_0,...x_k\}$ with diameter smaller than t. The concept was first introduced by L. Vietoris [41] and rediscovered by E. Rips. The name Rips complex was coined by M. Gromov [22], who used it in the study of hyperbolic groups.

J.-C. Hausmann [24] gives the following result for Riemannian manifolds with bounded geometry. For small values of t, the underlying space of the Vietoris-Rips complex $|M_t|$ is homotopy equivalent to the original manifold M. J. Latschev [26] extended this result in 2001 as follows. Assuming M is closed, there exists $t_0>0$ such that for all $t\leq t_0$, there exists a number $\delta>0$ such that for any metric space Y which is at a Gromov-Hausdorff distance less than δ of M, then $|Y_t|$ is homotopy equivalent to M.

Hausmann defines as well the metric cohomology of a metric space X by taking the direct limit of the simplicial cohomology of the Vietoris-Rips complex :

$$\mathscr{H}^*(X)=\varinjlim H^*(X_t),$$

and shows that for compact metric spaces, this cohomology is canonically isomorphic to the Čech cohomology.

The Vietoris-Rips complex is used in computational topology [8] [3] and computational geometry [9], for instance as a way to obtain a good approximation of a shape from a discrete set of points.

We define a slightly different complex, which we call the Alexander-Spanier complex at scale t.

Let X_t^{k+1} be the set of points $(x_0,...x_k) \in X^{k+1}$ such that $\rho(x_i,x_j) < t$ for all i,j. An Alexander-Spanier k-cochain is a function

$$\omega : X_t^{k+1} \to \mathbb{R}$$

and the space of k-cochains of size t is written $AS_t^k(X)$. The Alexander-Spanier differential is

$$\delta\omega(x_0,...x_{k+1}) = \sum_{i=0}^{k+1}(-1)^i\omega(x_0,...\widehat{x}_i,...x_{k+1}).$$

We have $\delta_k \circ \delta_{k+1} = 0$ and so for each $t > 0$ we have a differential complex

$$... \to AS_t^{k-1}(X) \xrightarrow{\delta_{k-1}} AS_t^k(X) \xrightarrow{\delta_k} AS_t^{k+1}(X) \to ...$$

to which are associated the scaled Alexander-Spanier cohomology groups :

$$H^k_{AS,t}(X,\rho) = Z_t^k(X)/B_t^k(X).$$

These cohomology groups are related by the restriction operators $r_{t_1,t_2} : AS^k_{t_1}(X) \to AS^k_{t_2}(X)$, for all $t_1 \geq t_2 > 0$, which induce homomorphisms $r_{t_1,t_2} : H^k_{AS,t_1}(X) \to H^k_{AS,t_2}(X)$. We can study both the direct and inverse limits of this cohomology theory, as well as how it changes depending on the value of the scale. When taking the direct limit on t, as $t \to \infty$, we get the asymptotic Alexander-Spanier cohomology of X :

$$H^k_{AS,\infty}(X,\rho) = \varprojlim H^k_{AS,t}(X,\rho).$$

In the other direction, the inverse limit defines the initial scaled Alexander-Spanier cohomology of X :

$$H^k_{AS,0}(X,\rho) = \varinjlim H^k_{AS,t}(X,\rho).$$

Although the definition of the initial cohomology is different than the definition of metric cohomology given by Hausmann, we can show that both constructions give the same result. Indeed, by choosing an ordering of the points of X, we can uniquely assign an alternating cochains in $AS_k^t(X)$ to each simplicial cochain defined on the Vietoris-Rips complex. Alternating cochains form a subcomplex $AS^k_{t,a}(X)$ of the scaled Alexander-Spanier complex and the projection operator

$$\mathrm{Alt} : AS_t^k(X) \to AS_{t,a}^k(X)$$

is in fact a homotopy equivalence, so both $AS_t^*(X)$ and $AS_{t,a}^*(X)$ yield the same cohomology.

If the diameter of X is finite, then the asymptotic Alexander-Spanier cohomology of X is trivial. More precisely :

Property 0.1.1. *Let X be a metric space of finite diameter. If the scale t is such that $diam(X) \leq t$, then we have*

$$H^k_{AS,t}(X) = \begin{cases} 0 & if\, k \geq 1 \\ \mathbb{R} & if\, k = 0 \end{cases}$$

This gives a notion that details of small size are unseen by the scaled Alexander-Spanier cohomology. We have more interesting results about the asymptotic cohomology in the L^p and L^π cases.

Concerning the initial limit, we give the following "de Rham theorem", which extend the result of Hausmann to the non-compact case.

Theorem 0.1.2. *Assume that M is a complete Riemannian manifold with bounded geometry. There exists $t_0 > 0$ such that for all $t \leq t_0$, the Alexander-Spanier cohomology at scale t is isomorphic to the de Rham cohomology of M :*

$$H^*_{AS,t}(M) = H^*_{DR}(M), \textit{for all } t \leq t_0.$$

In particular, the initial Alexander-Spanier cohomology of M is isomorphic to its de Rham cohomology :

$$H^*_{AS,0}(M) = H^*_{DR}(M).$$

To prove this, we use the method that A. Weil [42] used to give a proof of the original de Rham theorem.

The main tool in this method is the following : a double complex, or bicomplex, is a collection of spaces $(C^{k,l})_{k,l\in\mathbb{N}}$ together with morphisms $d^{k,l} : C^{k,l} \to C^{k+1,l}$ and $\delta^{k,l} : C^{k,l} \to C^{k,l+1}$ such that $d \circ d = 0$, $\delta \circ \delta = 0$ and $(\delta + d) \circ (\delta + d) = 0$. When all rows and all columns are exact, we have the following "Staircase" Lemma.

Lemma 0.1.3. *Let $((C^{k,l})_{k,l\in\mathbb{N}}, d^{k,l}, \delta^{k,l})$ be a double-complex such that every rows and every columns are exact. If we augment exactly the complex with a row $(C^{-1,*}, \delta)$ and a column $(C^{*,-1}, d)$, then the cohomology of the augmented row and column are isomorphic :*

$$H^k(C^{-1,*},d) = H^k(C^{*,-1},\delta)\ , \textit{for all } k \geq 0.$$

The idea is to construct diagrams of this kind relating the different cohomologies we are studying. The archetype for this method is the Čech-de Rham complex. Given a good cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in S}$ of M, the Bicomplex Lemma implies the usual de Rham theorem when applied to the double-complex defined as

$$C^{k,l}_{DR} = \prod_{I\in S_l} \Omega^k(U_I).$$

The differentials are, in one direction, the exterior derivative component by component, and in the other direction, the alternating sum of the components, as in the Čech cohomology. Following this structure, we construct two double-complexes. The Čech-de Rham complex, as described earlier, and the Čech-Alexander-Spanier complex. The first double-complex shows that the de Rham cohomology coincides with the Čech cohomology, and the second shows in turn that the Alexander-Spanier cohomology, at small scales, coincides with the Čech cohomology.

In order to apply Lemma 0.1.3, we need to find the conditions for which the double-complexes are exact. In the direction of the exterior derivative (or the Alexander-Spanier differential), the conditions are given by the Poincaré Lemma and its generalizations. In the direction of the Čech differential, we will look for Mayer-Vietoris sequences. The addition of the hypothesis from both these results gives the hypothesis of Theorem 0.1.2 and are generally met by complete Riemannian manifolds with bounded geometry, at scales that are small in comparison of the curvature and the strong convexity radius.

0.2 L^p cohomology

Assume that (X,ρ,μ) is a metric measure space. The L^p norm of an Alexander-Spanier cochain is defined in the usual way :

$$\|\omega\|_p^p = \left(\int_{X^t_{k+1}} |\omega(x)|^p \, d\mu_{k+1}(x)\right)^{1/p}.$$

The space of L^p Alexander-Spanier cochains is written $L^p AS^k_t(X)$, and so for each $t>0$ and $k\geq 0$ we have unreduced cohomology groups

$$L^p H^k_{AS,t}(X,\rho) = Z^k_{t,p}(X)/B^k_{t,p}(X).$$

Initial and asymptotic cohomologies remain well-defined in this case, and we set :

$$L^p H^k_{AS,\infty}(X) = \varprojlim L^p H^k_{AS,t}(X),$$

$$L^p H^k_{AS,0}(X) = \varinjlim L^p H^k_{AS,t}(X).$$

Throughout our work, we will assume the measure μ on X to be quasi-regular, in order for most of our results to be valid. A measure μ on a metric space (X,ρ) is quasi-regular if there exist positive real functions $v(r)$ and $V(r)$ such that for all $x \in X$ and for all $r > 0$, we have

$$v(r) < \mu(B(x,r)) < V(r).$$

It is generally the case for Riemannian manifolds with bounded geometry.

The method we use for Theorem 0.1.2 works also in the L^p case, and gives the following result for small scales.

Theorem 0.2.1. *Assume that M is a complete Riemannian manifold with bounded geometry. There exists $t_0 > 0$ such that for all $t \leq t_0$, the L^p Alexander-Spanier cohomology at scale t is isomorphic to the L^p de Rham cohomology of M :*

$$L^p H^k_{AS,t}(M) = L^p H^k_{DR}(M), \textit{for all } t \leq t_0.$$

This theorem concerns small scales. For large values of t, we are interested in a different kind of result. We can show that the asymptotic L^p Alexander-Spanier cohomology is a quasi-isometry invariant.

A quasi-isometry between metric spaces $f : X \to Y$ is an application that is close to an isometry in the following sense : there exists positive constants A, B and $C > 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{A}\rho_X(x_1,x_2) - B \leq \rho_Y(f(x_1), f(x_2)) \leq A\rho_X(x_1,x_2) + B$$

and such that the image $f(X)$ is C-dense in Y. The existence of a quasi-isometry is an equivalence relation, and carries the notion that two spaces have the same geometry at large scale. Typically, $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$. To state a more advandced example, one can consider a finiteley generated group and two different sets of generators of this group. The Cayley graphs obtained from each of these sets are quasi-isometric, and thus finitely generated groups can be assigned quasi-isometry classes.

The pullback induced in L^p Alexander-Spanier cohomology by a quasi-isometry between metric measure spaces $f : X \to Y$ is generally not well-defined between L^p classes, and it is generally not possible to evaluate the norm $\|\alpha \circ f\|_p$ given a cochain $\alpha \in L^p AS_t^k(Y)$. This problem is solved by using a kernel and the étalement operator it defines by convolution. Let ϕ be a kernel on X_T^{k+1}, the pullback of f is re-defined as

$$f^*(\alpha)(\Delta) = \alpha(\phi * f(\Delta)).$$

These two concepts are developped with more details in Section 3.2. Given a quasi-isometry $f : X \to Y$ and an inverse quasi-isometry $g : Y \to X$, the pullback of f and g do not necessarily induce isomorphism in cohomology. Because a quasi-isometry generally changes the diameter of simplices, the composition $f^* \circ g^*$ and $g^* \circ f^*$ cannot be compared with the identity $\mathrm{id} : L^p AS_t^k(X) \to L^p AS_t^k(X)$ (and thus are not inverse of each other in cohomology). But they can be compared with, and acts the same way as, the restriction operator. As a consequence, f^* and g^* induces isomorphism between the inverse limits $H^k_{AS,\infty}(X)$ and $H^k_{AS,\infty}(Y)$. This gives us a first invariance theorem :

Theorem 0.2.2. *The asymptotic L^p Alexander-Spanier cohomology is a quasi-isometry invariant on the class of metric measure spaces with quasi-regular measure.*

Theorem 0.2.1 relates Alexander-Spanier cohomology to the de Rham cohomology for small scales and Theorem 0.2.2 states an invariance property for large scales. But it happens that in some cases, the Alexander-Spanier cohomology is constant relatively to its scale, which means that both of these theorems apply at the same time.

A first way to obtain this kind of result is to extend the proof of Poincaré Lemma for the Alexander-Spanier cohomology. The proof of this Lemma relies on building an inverse to the restriction operator, by considering the barycentric subdivision of simplices. Since barycentric subdivision can be extended to CAT(0) space at any scale, we have the following result, which Pansu states as a remark :

Theorem 0.2.3. *Assume that X is a CAT(0)-space. Then the L^p Alexander-Spanier cohomology of X is independant of scale.*

In situations where the scaled L^p Alexander-Spanier cohomology is independant of scale, the quasi-isometry invariance is true for any value of t, with no need to take the limit. In particular, the combinaison of Theorems 0.2.1, 0.2.2 and 0.2.3 results in the following corollary :

Corollary 0.2.4. *The L^p de Rham cohomology is a quasi-isometry invariant for the class of Cartan-Hadamard manifolds, that is, simply connected complete Riemannian manifolds of non-positive sectional curvature.*

Another possibility to obtain this kind of result is the case of uniformly contractible manifolds. The property of double-complex we use to prove de Rham theorems can be used to link the Alexander-Spanier cohomology at different scales instead of linking it to the Čech cohomology.

Theorem 0.2.5. *Let M be a uniformally contractible Riemannian manifold with bounded geometry. The L^p Alexander-Spanier cohomology of M is independant of scale : given any $t, t' > 0$, for all $k \geq 0$,*

$$L^p H^k_{AS,t}(M) = L^p H^k_{AS,t'}(M).$$

We also have this corollary, using Theorem 0.2.5.

Corollary 0.2.6. *The L^p de Rham cohomology is a quasi-isometry invariant for the class of uniformly contractible complete Riemannian manifold.*

This method of proof for the quasi-isometric invariance for L^p de Rham cohomology was sketched by P. Pansu [33] in a preprint. The quasi-isometric invariance was already announced by Gromov [23].

Please note that all the results in this section are initially from the preprint [33] of Pansu. We clarify their proofs, except for Theorem 0.2.5 for which we were not able to do so.

0.3 L^π cohomology

Given a sequence $\pi = \{p_k\}_{k\in\mathbb{N}}$ of numbers $1 \leq p_k \leq \infty$, we define the scaled Alexander-Spanier cohomology in the following way : the space of L^π Alexander-Spanier cochains at scale $t > 0$ is given by

$$L^\pi AS^k_t(X) = \{\alpha \in AS^k_t(X) | \|\alpha\|_{p_k} < \infty \text{ and } \|\delta\alpha\|_{p_{k+1}} < \infty\}.$$

We get the following differential complex :

$$\ldots \to L^\pi AS^{k-1}_t(X) \xrightarrow{\delta_{k-1}} L^\pi AS^k_t(X) \xrightarrow{\delta_k} L^\pi AS^k_t(X) \to \ldots$$

From which we obtain cohomology groups

$$L^\pi H^k_{AS,t}(X) = Z^k_{t,\pi}(X)/B^k_{t,\pi}(X)$$

as well as the initial and asymptotic L^π Alexander-Spanier cohomologies

$$L^\pi H^k_{AS,\infty}(X) = \varprojlim L^\pi H^k_{AS,t}(X) \text{ and } L^\pi H^k_{AS,0}(X) = \varinjlim L^\pi H^k_{AS,t}(X).$$

De Rham L^π cohomology has a some interesting properties that were motivations for this work. M. Troyanov and V. Gold'shtein [17] show that, when setting $p_k = n/k$, the L^π de Rham complex obtained is a quasi-conformal invariant. S. Ducret [13] extends a result by G. Elek [14] by showing that for Riemannian manifolds with bounded geometry with a triangulation, the L^{qp} de Rham cohomology is isomorphic to the L^{qp} simplicial cohomology of the triangulation for q and p such that

$$1 < q, p < \infty \text{ and } \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n} \text{ or}$$

$$1 \leq q, p < \infty \text{ and } \frac{1}{p} - \frac{1}{q} < 1/n.$$

He shows as well that L^{qp} is a quasi-isometric invariant for uniformally contractible Riemannian manifolds for q and p satysfying one of these former inequalities as well as

$$0 \leq \frac{1}{p} - \frac{1}{q}.$$

However, the method of proof we used for the metric and the L^p case does not translate well to the general L^π case. In particular, integrability conditions imply that the Mayer-Vietoris sequence works only when $p_k = p_{k+1}$. By restricting to compact manifolds, we can still state the following result.

Theorem 0.3.1. *Let M be a compact Riemannian manifold. Assume that $\pi = \{\ldots p_k \leq p_{k+1}, \ldots\}$ is a non-decreasing sequence. There exists $t > 0$, such that for all $t \leq t_0$, the L^π Alexander-Spanier cohomology of M at scale t is isomorphic to the de Rham cohomology of M :*

$$L^\pi H^k_{AS,t}(M) = H^k_{DR}(M), \textit{for all } t \leq t_0.$$

In a paper from 2012, S. Smale and N. Smale [37] describe a different complex with the aim of studying L^p cohomology with Alexander-Spanier cochains : an element $x = (x_0, \ldots x_k)$ is a simplex of $\hat{X}^k_t$ if the distance between x and the diagonal is less than t. In this case, we have these inclusions :

$$X^k_t \subset \hat{X}^k_t \subset X^k_{2t}.$$

The main result of this paper is the development of a Hodge theory for compact metric spaces. A de Rham theorem similar to that of Hausmann is given for compact Riemannian manifolds. The cohomology, both simplicial and L^p simplicial, of $\hat{M}_t$ is isomorphic to the Čech cohomology of M for small value of t. The proof also relies on double-complex, but includes a rather technical discussion about the hypothesis.

We extend the result of quasi-isometry invariance in two different ways to L^{π} Alexander-Spanier cohomology. The first way is to consider graphs. If a graph has bounded degree, there is an inclusion of L^p spaces : a cochain which is L^p integrable is also L^q integrable for all $q \geq p$. This allows to state the following result :

Theorem 0.3.2. *Let $\pi = \{...p_k \geq p_{k+1}, ...\}$ be a non-increasing sequence. The asymptotic L^{π} Alexander-Spanier cohomology is a quasi-isometry invariant for graphs with bounded degree.*

Recall that a graph has bounded degree if there exists a uniform bound on the number of neighbours that each vertex has. The counting measure on a graph is quasi-regular if and only if the graph has bounded degree.

The second option is to consider locally bounded cochains. In that case, there is also an inclusion of L^p spaces.

Theorem 0.3.3. *Let $\pi = \{...p_k \geq p_{k+1}, ...\}$ be a non-increasing sequence. The asymptotic L^{π} locally bounded Alexander-Spanier cohomology is a quasi-isometry invariant for metric measure spaces with quasi-regular measure.*

On graphs, the asymptotic L^{π} Alexander-Spanier cohomology coincides with the locally bounded version of it, which can be a motivation to the definition of a coarse L^{π} Alexander-Spanier cohomology. Given a metric space X with a quasi-regular measure, there exists quasi-regular graphs which are quasi-isometric to X. In consequence, given a non-increasing sequence π, the asymptotic L^{π} Alexander-Spanier cohomology of these graphs can be attributed to X.

This thesis is organized as follow : in Chapter 1, we recall some basic properties about cochain complexes and cohomology. In Chapter 2, we introduced the scaled Alexander-Spanier cohomology of a metric space and the L^p Alexander-Spanier cohomology of a metric measure space, as well as some basic properties. In Chapter 3, we show that the asymptotic L^p Alexander-Spanier cohomology is invariant through quasi-isometry and discuss several extensions to L^{qp} cohomology. In Chapter 4, we show the different De Rham-type theorems we mentionned. An important part of the work is to establish a Poincaré Lemma for scaled Alexander-Spanier cohomology. Chapter 5 serves as a conclusion to this thesis, by discussing scale independance and how we can relate the results for Chapter 3 and 4.

1 Preliminaries : differential complexes and cohomology

This thesis presents a number of different cohomology theories. We recall here some basic facts about differential complexes, in order to fix some general notation and terminology. In terms of category, the aim is to work with Banach spaces and bounded operators, but we will also work with abelian groups and vector spaces.

1.1 Differential complexes

Definition 1.1.1. A *cochain complex*, or *differential (co)-complex*, is a collection of abelian groups $\{C^k\}_{k\in\mathbb{N}}$ and homomorphisms $d_k : C^k \to C^{k+1}$ such that $d_{k+1} \circ d_k = 0$ for all k. These homomorphisms are called *coboundary operators* or differentials. Note that the expression *differential complex* will generally be used, even if technically we are working with *cocomplexes*.

A *chain map* between two cochain complexes (C^*, d_*) and (D^*, δ_*) is a collection of homomorphisms $f^k : C^k \to D^k$ such that $f_{k+1} \circ d_k = \delta_k \circ f_k$.

The identity on a cochain complex is a chain map and the composition of two chain maps is a chain map as well. Thus, the collection of cochain complexes on a pointed category together with the collection of chain maps is also a category.

Given a cochain complex (C^*, d_*), it is usual to define the following notations :

- $Z^k(C,d) = \ker d_k$;
- $B^k(C,d) = \mathrm{im}\, d_{k-1}$.

The elements of $Z^k(C,d)$ are called *cocycles*. The k^{th}*-group of cohomology* of (C^*, d_*) is then defined as the following quotient :

$$H^k(C,d) = Z_k(C,d)/B_k(C,d).$$

By ease of langage, we called the sequence of groups $\{H^k(C,d)\}_{k\in\mathbb{N}}$ the cohomology of (C^*,d^*).

Remark 1.1.2. In the case where (C^*,d^*) is a complex of Banach spaces, its cohomology itself will not always be complete and hence not consists of Banach spaces. Indeed, $\ker d_k$ is a closed subspace of C^k, as the kernel of a continuous mapping. It is thus a Banach space itself. However, $\mathrm{im} d_{k-1}$ is not closed in general. It can be useful to considerate the *reduced cohomology groups* defined by :

$$\overline{H}^k(C,d) = Z_k(C,d)/\overline{B}_k(C,d),$$

with $\overline{B}_k(C,d) = \overline{\mathrm{im} d_{k-1}}$. In this case, $\overline{H}^k(C,d)$ is a Banach space.

The following property is the reason why we defined chain maps in the first place.

Property 1.1.3. *A chain map $f : (C^*,d_*) \to (D^*,\delta_*)$ induces an homomorphism $f^* : H^k(C,d) \to H^k(D,\delta)$, for all $k \in \mathbb{N}$, in cohomology.*

Proof. Let α and $\beta \in Z_k(C,d)$ be cocycles that are in the same cohomology class. That is, there exists $\omega \in B_k(C,d)$ such that $\alpha - \beta = d\omega$. In this case, $f(\alpha)$ and $f(\beta)$ are cocycles as well :

$$\delta \circ f(\alpha) = f \circ d(\alpha) = 0.$$

And $f(\alpha)$ and $f(\beta)$ are still in the same cohomology class :

$$f(\alpha) - f(\beta) = f(\alpha - \beta) = f(d\omega) = \delta(f(\omega)).$$

□

A consequence of Property 1.1.3 is that the cohomology of differential complex is a covariant functor from the category of differential complexes (of a given category) to the category of abelian groups. From that viewpoint, a cohomology theory is defined by the way we construct a differential complex.

1.2 Chain homotopy

We recall now a tool of very general use in this thesis. A *chain homotopy* between two chain maps f and $g : (C^*,d_*) \to (D^*,\delta_*)$ is given by a collection of mappings

$$E_k : C^k \to D^{k-1}$$

such that

$$f_k - g_k = E_{k+1} \circ \delta_k + d_{k-1} \circ E_k.$$

The homomorphisms E_k are called *homotopy operators*. The situation is shown in Figure 1.1.

Property 1.2.1. ***(Chain homotopy)*** *Let (C^*, δ_*) and (D^*, d_*) two differential complexes and $f, g : C \to D$ two chain maps. Assume there exists a chain homotopy $E_k : C^k \to D^{k-1}$ between f and g. Then f and g induce the same homomorphism in cohomology.*

Figure 1.1: Chain homotopy

$$\begin{tikzcd}
\dots \arrow[r, "\delta_{k-2}"] & C^{k-1} \arrow[r, "\delta_{k-1}"] \arrow[d, shift right, "f_{k-1}"'] \arrow[d, shift left, "g_{k-1}"] & C^k \arrow[r, "\delta_k"] \arrow[d, shift right, "f_k"'] \arrow[d, shift left, "g_k"] \arrow[dl, "E_k"'] & C \arrow[r, "\delta_{k+1}"] \arrow[d, shift right, "f_{k+1}"'] \arrow[d, shift left, "g_{k+1}"] \arrow[dl, "E_{k+1}"] & \dots \\
\dots \arrow[r, "d_{k-2}"] & D^{k-1} \arrow[r, "d_{k-1}"] & D^k \arrow[r, "d_k"] & D^{k+1} \arrow[r, "d_{k+1}"] & \dots
\end{tikzcd}$$

Proof. Let $c \in \ker \delta_k$ and compute :

$$(f - g)(c) = E \circ \delta(c) + d \circ E(c).$$

We have $\delta(c) = 0$ by hypothesis and $d \circ E(c) \in \operatorname{im} d$ by definition, and so $(f - g)(c)$ is a coboundary and thus the difference, in cohomology, between f and g is always 0. □

Furthermore, if $f : C \to D$ and $g : D \to C$ are two chain maps such that there exist a chain homotopy between $g \circ f$ and the identity as well as between and $f \circ g$ and the identity, then the cohomology groups of C and D are isomorphic. In that situation, we call g and f a *homotopy equivalence* between the complexes C and D.

2 Alexander-Spanier Cohomology on metric spaces

In this section, we discuss the definitions of the classical and scaled Alexander-Spanier cohomology and recall some basic results about these objects. The aim is to define a cohomology theory that is suited to study the geometry of metric spaces.

2.1 Classical definition : Alexander-Spanier cohomology for topological spaces

Definition 2.1.1. Let X be a topological space. A k-function on X is a function of $k+1$ variables $f : X^{k+1} \to \mathbb{R}$. The space of all k-functions on X is denoted by $\Phi^k(X,\mathbb{R})$. We define a differential $\delta_k : \Phi^k(X,\mathbb{R}) \to \Phi^{k+1}(X,\mathbb{R})$ by :

$$\delta_k f(x_0,...,x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(x_0,...,\hat{x}_i,...,x_{k+1}).$$

A direct computation shows that $\delta_{k+1} \circ \delta_k = 0$ and thus $(\Phi^k(X),\delta_k)$ is a differential complex. We say that two k-functions f and g are *locally equal* if for each $x \in X$, there exists an open neighborhood $V \subset X$ of x such that for all $(x_0,...,x_k) \in V$ we have

$$f(x_0,...,x_k) = g(x_0,...,x_k).$$

In other word, there exists an open neighbourhood of the diagonal on which f and g coincide. This is an equivalence relation, and the quotient of $\Phi^k(X,\mathbb{R})$ by this relation is denoted by $C^k_{AS}(X,\mathbb{R})$. The differential δ_k induces a map

$$\delta_k : C^k_{AS}(X,\mathbb{R}) \to C^{k+1}_{AS}(X,\mathbb{R})$$

which is a differential as well. This gives us a differential complex $(C^k_{AS}(X,\mathbb{R}),\delta_k)$, and its cohomology is the *Alexander-Spanier cohomology* of X.

There are a number of variants to this construction that give different cohomology theories. For instance, we can restrict to the subcomplex of finitely valued cochains or to compactly supported cochains, depending on the class of spaces we want to study.

For sufficiently nice spaces, the Alexander-Spanier cohomology is isomorphic to the Čech cohomology and the singular cohomology (paracompact spaces for Čech, cell complexes for singular cohomology). For a discussion of this, see Spanier [38] and Massey [27].

2.2 Metric Alexander-Spanier cohomology

We want to construct an L^p version of this cohomology for metric measure spaces. We start by defining a scaled Alexander-Spanier cohomology for metric spaces. The main idea is to fix a specific neighborhood of the diagonal, depending on a parameter t which we call the scale of the cohomology. Varying the scale will give different results depending on the features of the space. Intuitively, a large scale will result in a rough approximation and a small scale will capture local features.

Let (X,ρ) be a metric measure space.

Definition 2.2.1. Let $t > 0$. We denote by X^k_t the set of points $(x_0,...,x_{k-1}) \in X^k$ such that $\text{diam}\{x_0,...,x_{k-1}\} < t$. One can consider X^k_t to be the space of ordered $k-1$-simplices in X of diameter at most t. Elements of X^{k+1}_t will often be written $\Delta = (x_0,...x_k)$.

The distance between points $x = (x_0,...x_k)$ and $y = (y_0,...y_k) \in X^{k+1}_t$ is given by

$$\rho_{k+1}(x,y) = \max_i \rho(x_i,y_i).$$

In the situation when X is a measure space, with measure μ, we will use the product measure on X^{k+1}_t, and write $d\mu_{k+1}(\Delta)$ for $d\mu(x_0)...d\mu(x_k)$, or even $d\mu(\Delta)$ when the index is obvious.

Definition 2.2.2. An *Alexander-Spanier cochain* of degree k and size $t > 0$ on X is a real valued function

$$f : X^{k+1}_t \to \mathbb{R}.$$

The space of such cochains is denoted by $AS^k_t(X)$. We use the same differential as in the topological settings. Let $\delta_k : AS^k_t(X) \to AS^{k+1}_t(X)$ be defined by

$$\delta_k f(x_0,...,x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(x_0,...,\hat{x}_i,...,x_{k+1}).$$

The index will be dropped when unnecessary. Again, $\delta_{k+1} \circ \delta_k = 0$ and $(AS_t^k(X), d_k)_k$ is a differential complex, called the *Alexander-Spanier complex* of size, or scale, t.

Remark 2.2.3. Without any size restriction on the simplices, the cohomology of the Alexander-Spanier complex is always trivial. Let $\alpha : X^{k+1} \to \mathbb{R}$ be such that $\delta\alpha = 0$ and a an arbitrary element of X. Then the cochain $\theta : X^k \to \mathbb{R}$ defined by

$$\theta(x_0,...,x_{k-1}) = \alpha(a, x_0,...,x_{k-1})$$

is such that $\delta\theta = \alpha$. This remark is also valid for classical Alexander-Spanier cochains : if we do not use the local equivalence relation, the same method shows that every cocyle has a δ preimage, and thus the cohomology is trivial.

Remark 2.2.4. In the litterature, there are sometimes variations in the definitions for this cohomology. The Vietoris-Rips of size t is usually defined as the complex with simplices given by finite subsets $\{x_0,...,x_k\}$ of diameter smaller than t. We give a definition using points of X^{k+1}. We show in section 2.5 that the subcomplex of $AS_t^k(X)$ defined by alternating cochains is equivalent to $AS_t^k(X)$. The consequence of this is that both definitions are equivalent when considering the cohomology of these complexes. Pansu [33] uses the usual definition of the Vietoris-Rips complex and calls its cohomology the (scaled) Alexander-Spanier cohomology.

Hausmann [24] shows that the underlying topological space of the Vietoris-Rips complex (using the usual definition) of a closed Riemannian manifold is homotopy equivalent to that manifold for small values of t. Latschev [26] expands this result : if a metric space Y is sufficiently close, regarding to the Gromov-Hausdorff distance, to a closed Riemannian manifold M, then $|Y_t|$ is homotopy equivalent to M for small values of t. Hausmann calls the cohomology of the Vietoris-Rips complex the metric cohomology and gives a de Rham Theorem for this cohomology, for compact manifolds. We shall prove a stronger result using the method of double-complex.

N. Smale and S. Smale [37] define a scaled Alexander-Spanier cohomology on the subspace $\hat{X}_t^{k+1}$ defined by points of X^{k+1} which are at a distance smaller than t from the diagonal, that is, a point $(x_0,...x_k) \in X^{k+1}$ is in $\hat{X}_t^{k+1}$ if there exists a point $y \in X$ such that $\max_{0\leq i\leq k} \rho(x_i, y) < t$. We have in particular the following inclusions :

$$X_t^{k+1} \subset \hat{X}_t^{k+1} \subset X_{2t}^{k+1}.$$

Those authors show in the same paper that the L^p cohomology they define this way coincide with the de Rham cohomology for compact Riemannian manifolds. We find the same result and extend it to L^π cohomogy and to some non-compact cases.

2.3 L^p and L^π scaled Alexander-Spanier cohomology

We define the L^p version of our Alexander-Spanier cohomology. Let (X,ρ,μ) be a metric measure space.

Definition 2.3.1. We denote by $MAS_t^k(X)$ the space of measurable Alexander-Spanier cochains of size t on X. The L^p norm of a cochain $f \in MAS_t^k(X)$ is defined by :

$$\|f\|_p = \left(\int_{X_t^{k+1}} |f(x_0,...,x_k)|^p d\mu(x_0,...,x_k)\right)^{1/p}$$

for $p \in [1,\infty)$. When $p=\infty$, the definition is :

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in X_t^{k+1}} |f(x)|.$$

We denote by $L^pAS_t^k(X)$ the L^p class of cochains such that $\|f\|_p < \infty$, that is :

$$L^pAS_t^k(X) = \left\{f \in MAS_t^k(X) \mid \|f\|_p < \infty\right\} / \left\{f \in MAS_t^k(X) \mid \|f\|_p = 0\right\}.$$

In Lemma 2.3.2 and Proposition 2.3.3, we check that δ_k is well-defined on $L^pAS_t^k(X)$: if $f=g$ almost everywhere, so do δf and δg, and if $\|f\|_p<\infty$, then $\|\delta f\|_p < \infty$. Thus $(L^pAS_t^k(X), \delta_k)_{k\in\mathbb{N}}$ defines a differential complex as well. Note that Proposition 2.3.3 includes a condition of regularity on the measure of X. We will see in our main results that this is an important hypothesis.

Lemma 2.3.2. *Let (X,ρ,μ) be a metric measured space. Let $f : X^{k+1} \to \mathbb{R}$ be a real function such that*

$$\int_{X^{k+1}} |f(x_0,...x_k)| d\mu(x_0)...d\mu(x_k) = 0.$$

Then we also have :

$$\int_{X^{k+2}} |\delta f(x_0,...x_{k+1})| d\mu(x_0)...d\mu(x_{k+1}) = 0.$$

Proof. In a very straightforward manner, we have :

$$\begin{aligned}\int_{X^{k+2}} |\delta f(\Delta)| d\mu_{k+1}(\Delta) &\leq \int_{X^{k+2}} \sum_{i=0}^{k+1} |f(x_0,...\hat{x}_i,...x_{k+1})| d\mu(x_0)...d\mu(x_{k+1}) \\ &\leq (k+2)\int_{X^{k+2}} |f(x_0,...x_k)| d\mu(x_0)...d\mu(x_k)d\mu(x_{k+1}).\end{aligned}$$

In the last line, we use Fubini theorem to conclude the proof :

$$\int_{X^{k+2}} |\delta f(\Delta)| d\mu_{k+1}(\Delta) \leq \int_X 0 d\mu(x_{k+1}) = 0.$$

□

Proposition 2.3.3. *Assume there exists $V(t)$ such that $\mu(B(x,t)) < V(t)$ for any $x \in X$. Let $p \in [1,\infty]$. Then $\delta_k : L^p AS_t^k(X) \to L^p AS_t^{k+1}(X)$ is bounded for the L^p-norm.*

Proof. Set $\Delta = (x_0,...x_{k+1})$, and let $f \in L^p AS_t^k(X)$. If $p < \infty$, we have :

$$\begin{aligned}\|\delta_k f\|_p^p &= \int_{X_t^{k+1}} \left| \sum_{i=0}^{k+1} (-1)^i f(x_0,...\hat{x}_i,...x_{k+1}) \right|^p d\mu_{k+1}(\Delta) \\ &\leq \int_{X_t^{k+1}} \left(\sum_{i=0}^{k+1} |f(x_0,...\hat{x}_i,...x_{k+1})| \right)^p d\mu(x_0)...d\mu(x_{k+1}) \\ &\leq \int_{X_t^{k+1}} (k+1)^{p-1} \sum_{i=0}^{k+1} \left| f(x_0,...\hat{x}_i,...x_{k+1}) \right|^p d\mu(x_0)...d\mu(x_{k+1})\end{aligned}$$

The last step is Jensen's inequality. Now we can exchange the sum and the integral.

$$\begin{aligned}\|\delta_k f\|_p^p &\leq (k+1)^{p-1} \sum_{i=0}^{k+1} \int_{X_t^{k+1}} \left| f(x_0,...\hat{x}_i,...x_{k+1}) \right|^p d\mu(x_0)...d\mu(x_{k+1}) \\ &\leq (k+1)^p \int_{X_t^{k+1}} \left| f(x_0,...x_k) \right|^p d\mu(x_0)...d\mu(x_{k+1}).\end{aligned}$$

In this last line, x_{k+1} does not appear in the integrand. This has as a consequence that the volume of the acceptable domain for x_{k+1} factors in. This domain is precisely

$$\left\{ x_{k+1} \in X \mid (x_0,...x_{k+1}) \in X_t^{k+1} \right\} = \bigcap_{i=0}^{k} B(x_i,t).$$

This is always contained in some ball $B(x_i, t)$. Because we assume that balls of a given radius t has a uniformly bounded volume, we have :

$$\begin{aligned}\|\delta_k f\|_p^p &\leq V(t)\cdot(k+1)^p \cdot \int_{X_t^k} \left|f(x_0,...x_k)\right|^p d\mu(x_0)...d\mu(x_k)\\ &\leq V(t)(k+1)^p \|f\|_p^p.\end{aligned}$$

For $p = \infty$, we have :

$$\|\delta f\|_\infty \leq (k+1)\cdot \operatorname*{ess\,sup}_{x\in X_t^{k+1}} |f(x)| \leq (k+1)\|f\|_\infty.$$

Note that we do not need to bound the measure of balls for this case. □

Example 2.3.4. We give an example where there is no bound on the volume of balls. Consider $\mathbb{N}$ with distance $\rho(n,m) = |n-m|$ and $\mu(\{n\}) = n$ as a measure. In this case, the volume of balls of fixed radius has no upper bound, and the Alexander-Spanier differential is not bounded at all scale. Indeed, let $f : \mathbb{N} \to \mathbb{R}$ be defined by

$$f(n) = \begin{cases} 1/n^3 & \text{if } n \text{ is even ;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

It is bounded in L^1 norm :

$$\|f\|_1 = \sum_{n=1}^{\infty} f(n)\mu(n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

We then have $\delta f(n,m) = f(m) - f(n)$. Choose a scale t such that $1 < t < 2$. Then an element of $\mathbb{N}_t^2$ is of the form $(n, n+1)$ or $(n, n-1)$. The norm of δf is the following :

$$\|\delta f\|_1 = \sum_{(n,m)\in\mathbb{N}_t^2} |f(n) - f(m)|\cdot\mu(n)\mu(m).$$

If we split the sum between terms of the form $(n, n+1)$ and $(n, n-1)$, we get twice the same result, because of the absolute value. We can thus write :

$$\|\delta f\|_1 = 2\sum_{n\in\mathbb{N}} |f(n) - f(n+1)|\cdot\mu(n)\mu(n+1).$$

Here again, for each pair $(n, n+1)$, either n or $n+1$ is odd, and thus either $f(n) = 0$ or $f(n+1) = 0$.

$$\begin{aligned}\|\delta f\|_1 &= 4 \sum_{n \in 2\mathbb{N}} |f(n)| \cdot \mu(n)\mu(n+1) \\ &= \sum_{n \in 2\mathbb{N}} \frac{1}{n^3} \cdot n(n+1).\end{aligned}$$

This last series does not converge and hence δ is not a bounded operator in this case.

With all of this in place, we can define L^{qp} cohomology for the Alexander-Spanier scaled cochains.

Definition 2.3.5. Given $q, p > 1$, we define the space of L^{qp} Alexander-Spanier cochains of size t :

$$L^{qp}AS_t^k(X) = \{f \in MAS_t^k(X) | \|f\|_q < \infty \text{ and } \|\delta f\|_p < \infty\}.$$

The space of cocycles and closed cochains are defined as usual :

- $Z_{p,t}^k(X) = \ker \delta_k$;
- $B_{qp,t}^k(X) = \mathrm{im}\delta_{k-1} = \delta_{k-1}(L^{qp}AS_t^{k-1}(X)) = \delta_{k-1}(L^q AS_t^{k-1}) \cap L^p AS_t^k$.

This allows to define cohomology spaces that we write :

$$L^{qp}H_{AS,t}^k(X) = Z_{p,t}^k(X)/B_{qp,t}^k(X).$$

As we noted before, the normed space $Z_{p,t}^k(X)$ is a closed subspace of $L^{qp}AS_t^k(X)$, and thus it is a Banach space. However, $B_{qp,t}^k(X)$ is not necessarly closed, which prevents the cohomology space $L^{qp}H_{AS,t}^k(X)$ to be a Banach space as well. We can however consider the *reduced cohomology space*

$$L^{qp}\overline{H}_{AS,t}^k(X) = Z_p^k(X)/\overline{B}_{q,p}^k(X),$$

where $\overline{B}_{qp,t}^k(X)$ is the closure of $B_{qp,t}^k(X)$. These spaces are then Banach spaces.

We can organize L^{qp} cochains as a differential complex by proceeding as follow. Let $\pi = \{p_k \geq 1 | k \in \mathbb{N}\}$, with $1 \leq p_k \leq \infty$ be a sequence of numbers. We note

$$L^{\pi} AS_t^k(X) = L^{p_k p_{k+1}} AS_t^k(X).$$

We then have a differential complex :

$$\dots \to L^{\pi} AS_t^{k-1}(X) \xrightarrow{\delta_{k-1}} L^{\pi} AS_t^k(X) \xrightarrow{\delta_k} L^{\pi} AS_t^{k+1}(X) \to \dots$$

We write the associated cohomology groups $L^{\pi} H^k_{AS,t}(X)$.

Definition 2.3.6. Given $t' \geq t > 0$, the *restriction operator*

$$r_{t't} : L^{qp} AS_{t'}^k(X) \to L^{qp} AS_t^k(X)$$

is defined by the restriction of cochains defined on simplices of size t' to simplices of size t : given $f \in L^{qp} AS^k_{t'}(X)$, we have

$$r_{t't} f = f|_{X_t^{k+1}}.$$

The restriction is a chain map :

$$r_{t't}\delta = \delta r_{t't}$$

This means that $r_{t't}$ induces a map in cohomology, for all $t' \geq t$. These maps have the following properties :

1. $r_{tt} : H^*_{AS,t}(X) \to H^*_{AS,t}(X)$ is the identity ;
2. $r_{t't} = r_{st} \circ r_{t's}$ for all $t' \geq s \geq t$.

These two properties make $(H^*_{AS,t}(X), r_{t't})$ a direct system with index set $((0,\infty),\geq)$ as well as an inverse system, with index set $((0,\infty),\leq)$. The limit of such systems always exist for abelian groups, and we define the following objects.

Definition 2.3.7. The *asymptotic Alexander-Spanier cohomology* is the inverse limit, as $t \to \infty$:

$$L^{qp} H^k_{AS,\infty}(X) = \varprojlim L^{qp} H^k_{AS,t}(X).$$

The *initial Alexander-Spanier cohomology* is direct limit, as $t \to 0$:

$$L^{qp} H^k_{AS,0}(X) = \varinjlim L^{qp} H^k_{AS,t}(X).$$

Concretely, an element of the inverse limit $L^{qp}H^k_{AS,\infty}(X)$ is a collection $f = (f_t)$, where $f_t \in L^{qp}H^k_{AS,t}(X)$ for any $t > 0$, that satisfies $r_{st} f_t = f_s$ for any $t \geq s > 0$.

On the other hand an element of the direct limit $L^{qp}H^k_{AS,0}(X)$ is a germ of Alexander-Spanier scaled cochain: it is represented by an element f_t in the disjoint union $\sqcup_{t>0} L^{qp}H^k_{AS,t}(X)$ modulo the equivalence relation defined by $f_s \sim f_t$ if and only if there exists $0 < u \leq \min(s,t)$ such that $r_{ut} f_t = r_{us} f_s$.

Observe that we have natural maps

$$r_t : L^{qp}H^k_{AS,\infty}(X) \to L^{qp}H^k_{AS,t}(X) \text{ and } r_t : L^{qp}H^k_{AS,t}(X) \to L^{qp}H^k_{AS,0}(X)$$

commuting with the restriction operators in the following way : if $s \geq t > 0$, then

$$r_{st} \circ r_s = r_t.$$

The spaces $L^{qp}H^k_{AS,0}(X)$ and $L^{qp}H^k_{AS,\infty}(X)$ can be endowed with a natural topology, this is not a trivial task and we shall not be concerned with this question in this thesis. The interested reader can consult the book *Topological Vector Spaces* by A. P. Robertson, Wendy Robertson [34].

2.4 Cohomology in degree 0

We treat the case of $AS^0_t(X)$ and $L^{qp}AS^0_t(X)$. The behavior is somewhat different from other cohomologies, such as the de Rham cohomology. Note first that for any $1 \leq q, p \leq \infty$,

$$L^{qp}H^0_{AS,t}(X) = L^p H^0_{AS,t}(X).$$

Definition 2.4.1. Two connected components A and B of a metric space (X,ρ) are *t-separated* if $\rho(A,B) > t$. The space X is *t-connected* if none of its components are t-separated. We say that a subset A of X is a *t-component* or a *t-cluster* if for all $x, y \in A$, there is a sequence $x = x_0, x_1, ...x_n = y$ in A such that for $\rho(x_i, x_{i+1}) < t$ for all $0 \leq i \leq n-1$, and if for all $x \in X \setminus A$, we have $\rho(x, A) \geq t$.

Recall that in the de Rham cohomology, the 0-cocyles are locally constant functions : for all $x \in X$, there exists a neighbourhood U of x such that f is constant on U. For Alexander-Spanier 0-cocycles, we have a stronger variant of this property : not only a 0-cocyle is locally constant, but it is also constant on each t-component.

Property 2.4.2. *If $f \in AS_t^0(X)$, we have :*

$$\delta f(x_0, x_1) = f(x_1) - f(x_0),\ \forall x_0, x_1 \in X_t^2.$$

If two connected components X_0 and X_1 of X are close enough, we can find $x_0 \in X_0$ and $x_1 \in X_1$ such that $\rho(x_0, x_1) < t$, and so f has to take the same value on X_0 and X_1. So the first group of scaled Alexander-Spanier cohomology will "count" the number of t-components rather than connected components :

Property 2.4.3. *If X has n t-components, then one has :*

$$H^0_{AS,t}(X) = \mathbb{R}^n.$$

Finally, the L^p case is determined by the measure of each t-component :

Property 2.4.4. *If $p < \infty$ and X is t-connected, we have :*

$$L^p H^0_{AS,t}(X) = \begin{cases} \mathbb{R} & \text{if } \mu(X) < \infty; \\ 0 & \text{if } \mu(X) = \infty. \end{cases}$$

Thus, the L^p Alexander-Spanier 0-cohomology at scale t counts the number of t-components of finite measure.

This property allows the construction of examples to illustrate the initial and asymptotic cohomologies as well as some properties inherent to cohomologies based on the Vietoris-Rips complex.

Consider $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. For any $t > 0$, the Alexander-Spanier cohomology of degree 0 at scale t of $\mathbb{R}^*$ is $\mathbb{R}$, because $\mathbb{R}^*$ is t-connected for all $t > 0$. Note that the cohomology in degree 0 of $\mathbb{R}^*$ is stable through the different values of t.

Consider now the sequence of points $A = \{1/2^n\}_{n\in\mathbb{N}}$ with the euclidian distance. The largest distance between any two points of this sequences is $1/2$, so for any $t > 1/2$, we have $H^0_{AS,t}(A) = \mathbb{R}$. If $1/2^{n+1} < t \le 1/2^n$, then A has n different t-components, and so $H^0_{AS,t}(A) = \mathbb{R}^n$. Note that the restriction operator will be injective for any choice of scale. In consequence, we have

$$H^0_{AS,\infty}(A) = \mathbb{R} \text{ and } H^0_{AS,0}(A) = \mathbb{R}^{\mathbb{N}}.$$

In particular, there is no stability as $t \to 0$.

2.5 Alternating Cochains

The scaled Alexander-Spanier cohomology can be computed using only alternatig cochains, both in the classical and the L^p case.

Definition 2.5.1. A cochain $f : X_t^{k+1} \to \mathbb{R}$ is said to be *alternating* or *antisymmetrical* if for any permutation $\tau \in \mathfrak{S}_{k+1}$, we have

$$f(x_{\tau(1)}, \ldots x_{\tau(k)}) = \mathrm{sgn}(\tau) \cdot f(x_0, \ldots x_k).$$

The subset of $AS_t^k(X)$ and $L^\pi AS_t^k(X)$ of alternating cochains are written $AS_{t,a}^k(X)$ and $L^\pi AS_{t,a}(X)$. If f is alternating, δf is also alternating and thus $(AS_{t,a}^k(X), \delta_k)$ and $(L^\pi AS_{t,a}(X), \delta_k)$ are subcomplexes. The corresponding cohomology groups are written $H_{AS,t,a}^k(X)$ and $L^\pi H_{AS,t,a}^k(X)$.

Definition 2.5.2. Let $\Delta = (x_0, \ldots x_k) \in X^{k+1}$ be a k-simplex. We define the chain $\mathrm{Alt}(\Delta)$ by

$$\mathrm{Alt}(\Delta) = \frac{1}{(k+1)!} \sum_{\tau \in \mathfrak{S}_{k+1}} \mathrm{sgn}(\tau)(x_{\tau(0)}, \ldots x_{\tau(k)}).$$

Given a cochain $f \in AS_t^k(X)$, we also define

$$\mathrm{Alt} : AS_t^k(X) \to AS_t^k(X)$$

by $\mathrm{Alt}(f)(\Delta) = f(\mathrm{Alt}(\Delta))$.

Property 2.5.3. *Let $f \in AS_t^k(X)$.*

1. *Alt(f) is alternating ;*
2. *If f is alternating, Alt(f) $= f$;*
3. *$Alt \circ \delta = \delta \circ Alt$;*
4. *If $f \in L^p AS_t^k(X)$, then Alt(f) $\in L^p AS_t^k(X)$;*
5. *If $\delta f \in L^q AS_t^{k+1}(X)$, then δAlt(f) $\in L^q AS_t^{k+1}(X)$.*

Proof. 1. Let $\sigma \in \mathfrak{S}_{k+1}$ and compute :

$$\begin{aligned}
\mathrm{Alt} f(x_{\sigma(0)}, \ldots x_{\sigma(k)}) &= \frac{1}{(k+1)!} \sum_{\tau \in \mathfrak{S}_{k+1}} \mathrm{sgn}(\tau) \cdot f(x_{\tau\sigma(0)}, \ldots x_{\tau\sigma(k)}) \\
&= \frac{1}{(k+1)!} \sum_{\tau \in \mathfrak{S}_{k+1}} \mathrm{sgn}(\tau) \cdot (\mathrm{sgn}(\sigma))^2 \cdot f(x_{\tau\sigma(0)}, \ldots x_{\tau\sigma(k)})
\end{aligned}$$

Now write $\tau' = \tau\sigma$ and note that $\mathrm{sgn}(\tau') = \mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\sigma)$.

$$\begin{aligned}
\mathrm{Alt} f(x_{\sigma(0)}, \ldots x_{\sigma(k)}) &= \frac{1}{(k+1)!} \sum_{\tau' \in \mathfrak{S}_{k+1}} \mathrm{sgn}(\tau') \cdot \mathrm{sgn}(\sigma) \cdot f(x_{\tau'(0)}, \ldots x_{\tau'(k)}) \\
&= \mathrm{sgn}(\sigma) \cdot \mathrm{Alt} f.
\end{aligned}$$

2. This is straightforward :

$$\begin{aligned}
\mathrm{Alt} f(x_0, \ldots x_k) &= \frac{1}{(k+1)!} \sum_{\tau \in \mathfrak{S}_{k+1}} \mathrm{sgn}\tau f(x_{\tau(0)}, \ldots x_{\tau(k)}) \\
&= \frac{1}{(k+1)!} \sum_{\mathfrak{S}_{k+1}} f(x_{\tau(0)}, \ldots x_{\tau(k)}) = f(x_0, \ldots x_k).
\end{aligned}$$

3. The proof of this consists in re-arranging the terms. Let $\Delta = (x_0, \ldots x_{k+1}) \in X^{k+2}$.

$$\begin{aligned}
\mathrm{Alt}\delta f(\Delta) &= \delta f(\mathrm{Alt}(\Delta)) \\
&= \frac{1}{(k+2)!} \sum_{\tau \in \sum_{k+2}} \mathrm{sgn}(\tau) \delta f(x_{\tau(0)}, \ldots x_{\tau(k+1)}) \\
&= \frac{1}{(k+2)!} \sum_{\tau \in \sum_{k+2}} \mathrm{sgn}(\tau) \sum_{i=0}^{k+1} (-1)^i f(x_{\tau(0)}, \ldots \hat{x}_{\tau(i)}, \ldots x_{\tau(x_{k+1})}).
\end{aligned}$$

The term $f(x_{\tau(0)}, \ldots \hat{x}_i, \ldots x_{\tau(x_{k+1})})$ appears $k+2$ times in the summation, each time with the same sign. Indeed, consider

$$(x_{\tau(0)}, \ldots \hat{x}_{\tau(i)}, \ldots x_{\tau(x_{k+1})})$$

and suppose we add $x_{\tau(i)}$ at the j^{th} place in the array. The simplex we get is of the form

$$(x_{\tau'(0)}, \ldots x_{\tau'(k+1)})$$

with $\tau' = \sigma \circ \tau$, and such that

$$\partial_j (x_{\tau'(0)}, \ldots x_{\tau'(k+1)}) = (x_{\tau(0)}, \ldots \hat{x}_{\tau(i)}, \ldots x_{\tau(x_{k+1})}).$$

The sign for $f(x_{\tau(0)}, \ldots \hat{x}_i, \ldots x_{\tau(x_{k+1})})$ is $\mathrm{sgn}(\tau) \cdot (-1)^i$.

4. This is rather direct : $\|\mathrm{Alt}(f)\|_p^p \leq \frac{1}{(k+1)!} \sum_{\tau \in \mathfrak{S}_{k+1}} \|f\|_p^p \leq \|f\|_p^p$.

5. This is a consequence of the two preceding points.

□

This list of properties means that Alt is a projector and induces an homomorphism in L^π cohomology. We can show that it is an isomorphism.

Proposition 2.5.4. *There exist operators* $B_{Alt} : AS_t^k(X) \to AS_t^{k-1}(X)$ *such that*

$$id - Alt = B_{Alt} \circ \delta + \delta \circ B_{Alt}.$$

This property is verified as well for L^p Alexander-Spanier cohomology.

3 Invariance under quasi-isometry

In this chapter, we discuss results of quasi-isometry invariance for L^p and L^{qp} Alexander-Spanier cohomology. The main results are Theorem 3.3.5 and its corollaries. We first define a suitable action of quasi-isometries on the L^{qp} complex : we need to be able to compare the norms of a cochain and of its image through the pullback induced by a quasi-isometry. In order to do so we define the étalement of a chain. We then prove that the action of quasi-isometries on scaled L^p Alexander-Spanier complexes is equivalent to the restriction of the size of the cochains, and thus deduce that the L^p asymptotic cohomology is invariant through quasi-isometries. This result does not extend effortlessly to the L^{qp} cohomology. However, we state two generalizations. The asymptotic L^π cohomology is a quasi-isometry invariant for graphs with bounded geometry, and the locally bounded asymptotic L^π cohomology is a quasi-isometry invariant for Riemannian manifolds with bounded geometry.

3.1 Quasi-isometries

We recall the notion of quasi-isometry, which capture the idea that an application is close to an isometry and that two metric spaces have a comparable geometry when looked at from far away. For further references, see [6] and [31].

Definition 3.1.1. Let (X,ρ) and (Y,ρ') be metric spaces. A mapping $f : X \to Y$ is a *quasi-isometry* if there exist constants $A,B,C > 0$ such that

1. $\forall x_1, x_2 \in X, \frac{1}{A}\rho(x_1,x_2) - B \leq \rho'(f(x_1), f(x_2)) \leq A\rho(x_1,x_2) + B$
2. $\forall y \in Y, \exists x \in X$ such that $\rho'(f(x), y) \leq C$.

The first condition means that f is close to a local isometry, in the sense that for $B = 0$, a quasi-isometry is a bilipschitz map, and if $B = 0$ and $A = 1$ it is an isometry. The second condition means that f almost surjective in the sense that its range is C-dense in Y. In paticular if $C = 0$, f is surjective. We call A,B,C the *coefficients* of the quasi-isometry f.

A former terminology was to call surjective bilipschitz maps quasi-isometries and to call quasi-isometries in the sense of Definition 3.1.1 *coarse isometries.*

Property 3.1.2. *If there exists a quasi-isometry $f : X \to Y$, then there exists a quasi-isometry $g : Y \to X$.*

Proof. Let $y \in Y$. There exists $y' \in Y$ such that $f^{-1}(y') \neq \emptyset$ and $d(y, y') \leq C$, by definition. Choose $g(y) \in f^{-1}(y')$. Then g is a quasi-isometry. $\square$

Property 3.1.3. *The composition of two quasi-isometries is a quasi-isometry.*

Proof. Let $f : X \to Y$ be a quasi-isometry with coefficients A, B, C and $g : Y \to Z$ a quasi-isometry with coefficient D, E, F. Then for any $x, y \in X$, we have :

$$\rho_Z(g \circ f(x), g \circ f(y)) \leq D\rho_Y(f(x), f(y)) + E \leq AD\rho_X(x, y) + E + B.$$

The other inequality works in the same fashion. For the density, consider a point z_0 in Z. There is a point of the form $g(y)$ which has $\rho_Z(z_0, g(y)) \leq F$, and then there is a point $x \in X$ such that $\rho_Y(y, f(x)) \leq C$. We can estimate the distance between $g \circ f(x)$ and z_0 using the triangle inequality :

$$\rho_Z(g \circ f(x), z_0) \leq \rho_Z(g \circ f(x), g(y)) + \rho_Z(g(y), z_0).$$

The terms on the right can be estimated as follow :

$$\rho_Z(g \circ f(x), z_0) \leq D \cdot C + E + F.$$

The composition $g \circ f$ is thus a quasi-isometry with coefficients $A \cdot D, E + B$ and $D \cdot C + E + F$. $\square$

These two properties imply that the existence of a quasi-isometry between two metric spaces is an equivalence relationship.

Definition 3.1.4. We then say that two spaces are *quasi-isometric* if there exists a quasi-isometry between them.

We cite the following alternate definition, often given in litterature :

Definition 3.1.5. Two metric spaces (X, ρ) and (Y, ρ') are *quasi-isometric* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that

1. $d(f(a), f(b)) \leq Ad(a, b) + B$, $d(g(x), g(y)) \leq Cd(x, y) + D$;

2. $f \circ g$ and $g \circ f$ are at a bounded distance from the identity : there exists a constant $K > 0$ such that for all $x \in X, y \in Y$, we have $d(f \circ g(y), y) \leq K$, $d(g \circ f(x), x) \leq K$.

Proposition 3.1.6. *The definition 3.1.4 and 3.1.5 are equivalent.*

Proof. We only show that $g \circ f$, with g as constructed in the proof of Property 3.1.2, is close to identity, as we will need it later.

The construction of g implies that $g \circ f(x) \in f^{-1}(y)$ for some y such that $\rho'(f(x), y) < C$. In consequence $f \circ g \circ f(x) = y$, and we have :

$$\begin{aligned}
\rho(x, g \circ f(x)) &\leq A\rho'(f(x), f \circ g \circ f(x)) + AB \\
&\leq A\rho(f(x), y) + AB \\
&\leq A \cdot C + AB.
\end{aligned}$$

□

Examples 3.1.7.

- Any mapping between two metric spaces with finite diameter is a quasi-isometry.
- $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry and more generally if a subset N of X is a net in X, or C-dense for some constant C, then the injection of N in X is a quasi-isometry.
- Given a metric space X, any bounded perturbation (like removing any bounded subset) will not change the quasi-isometry class of X.
- Assume V and W are two acyclic graphs, each with a finite number of branches. If they do not have the same number of infinite branches, they are not quasi-isometrics.

We recall the notion of Cayley graph of a group, which will give us an important example of quasi-isometry.

Definition 3.1.8. Let G be a group and $S \subset G$ be a set of generators of G. The Cayley graph $\Gamma(G, S)$ is obtained by attributing a vertex to each element $g \in G$ and an edge to each couple (g, gs) with $s \in S$.

We use the word metric on $\Gamma(G, S)$. Different choices of generating set will result in different graphs, but for finitely generated groups, the Calyey graph defines a quasi-isometry class :

Proposition 3.1.9. *Let G be a finitely generated group and S, S', two generating subsets of G. Then $\Gamma(G, S)$ and $\Gamma(G, S')$ are quasi-isometric.*

We cite also this result from [40] to illustrate the notion of quasi-isometry class for groups.

Proposition 3.1.10. *Let M be a compact Riemannian manifold. Then its universal cover $\tilde{M}$ and its fundamental group $\pi_1(M)$ are quasi-isometric metric spaces.*

3.2 Etalement

Pansu define the etalement of simplex in order to be able to measure the effect of quasi-isometries on the norm of cochains. Assume that $f : X \to Y$ is a quasi-isometry and let $\phi \in AS_t^k(Y)$. We want to compute the L^p norm of the pullback of ϕ through f :

$$\|f^*\phi\|_p^p = \int_{X^{k+1}} |\phi(f(\Delta))|^p d\mu_{k+1}(\Delta)$$

This might cause some troubles to compute. Consider the following example. Assume that a cochain $\phi \in \mathrm{AS}_t^0(\mathbb{R})$ is defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

Consider the inclusion $f : \mathbb{Z} \to \mathbb{R}$. Then $\|f^*\phi\|_1 = \infty$ whereas $\|\phi\|_1 = 0$. Etalement consists in using kernels to smooth out such discrespancies.

Definition 3.2.1. A *kernel* on a metric measure space (X, ρ, μ) is a non-negative measurable function $\phi : X \times X \to \mathbb{R}$ such that

1. ϕ is bounded : specifically, there exists $K > 0$ such that $\phi(x, y) \leq K$ for all $(x, y) \in X \times X$;
2. For almost all $x \in X$, $\int_X \phi(x, y) d\mu(y) = 1$;
3. $\phi \equiv 0$ outside of a bounded neighbourhood of the diagonal $D = \{(x, x) \in X \times X | x \in X\}$: there exists $s > 0$ such that $\rho((x, y), D) > s$ implies that $\phi(x, y) = 0$.

Example 3.2.2. Assume that for $r > 0$, we have $\inf_{x\in X} \mu(B(x, r)) > 0$. The following defines a kernel :

$$\phi(x, y) = \begin{cases} (\mu(B(x, r)))^{-1} & \text{if } \rho(x, y) < r \\ 0 & \text{else.} \end{cases}$$

Property 3.2.3. *Given a kernel ϕ on X, we can define a kernel Φ_{k+1} on X^{k+1} :*

$$\Phi_{k+1}(x_0, \dots x_k; y_0, \dots y_k) = \prod_{i=0}^{k} \phi(x_i, y_i).$$

Remark 3.2.4. Assuming an hypothesis of uniformity on μ, these two last points allow us to define a kernel on X_t^{k+1}, by choosing a radius $r > 0$ such that

$$\left\{(x_0, \dots, x_k) \in X^k \mid \max_i \rho(x_i, D) < r\right\} \subset X_t^{k+1}.$$

Note that the same radius can be used for all $k \geq 0$, and so we can deduce from a kernel on X a kernel on X_t^{k+1} for each $k \geq 0$. Whenever we use a kernel in this chapter, we will assume that it is constructed in this way. The hypothesis that $\inf_{x \in X} \mu(B(x,r)) > 0$ for any $r > 0$ is one of the main hypothesis of the proof of quasi-isometry invariance we develop in this chapter.

Now that we have well-defined kernels, we can define the étalement of a simplex and its action on Alexander-Spanier cochains. The notion of étalement is an idea from P. Pansu. The definitions we give here are a variant of his construction that we choose in order to have well-defined objects with minimal technical background.

Definition 3.2.5. Let ϕ be a kernel on X_t^{k+1}. Given a simplex $\Delta = (x_0, \ldots x_k)$, its *étalement* is the measure

$$\Delta * \phi = \phi(\Delta, \cdot) d\mu_{k+1}(\cdot).$$

The étalement is extended linearly to simplicial chains :

$$\left(\sum_i \lambda_i \Delta_i \right) * \phi = \sum_i \lambda_i (\Delta_i * \phi).$$

In the next proof, we assume that $B_r(x)$ is bounded for x above and below by positive constants. We already used the assumption that $\inf_{x \in X} \mu(B(x,r)) > 0$ for all $r > 0$ to ensure the existence of kernels. We also need the existence of a positive function $V(r)$ such that $\sup_{x \in X} \mu(B(x,r)) \leq V(r)$ for all $r > 0$ to have a well-defined differential $\delta_k : L^p AS_t^k(X) \to L^p AS_t^k(X)$. We will use the following terminology :

Definition 3.2.6. Let (X, ρ, μ) be a metric measure space. The measure μ is said to be *quasi-regular* if there exist postive functions $v, V : \mathbb{R} \to \mathbb{R}$ such that for all $x \in X$ and all $r > 0$ we have :

$$v(r) \leq \mu(B(x,r)) \leq V(r).$$

In analysis on metric spaces, we often find the case when the function v and V are of the form $c \cdot r^s$ and $C \cdot r^s$ for some $s > 0$. The metric space X is then called Ahlfors regular.

The étalement of a simplex gives a mapping on Alexander-Spanier cochains :

Property 3.2.7. *Let X be a quasi-regular metric measure space and let $\alpha \in L^p AS_t^k(X)$. Then the function defined by*

$$\alpha * \phi(\Delta) = \int_{X_t^{k+1}} \alpha(\Delta') \phi(\Delta, \Delta') d\mu_{k+1}(\Delta')$$

is a well-defined Alexander-Spanier cochain.

Proof. The cochain α and the kernel ϕ are measurable, so what we have to check is that $\alpha * \phi(\Delta)$ is finite a finite number for any $\Delta \in X_t^{k+1}$ and that $\|\alpha * \phi\|_p < \infty$.

For the first part, note first that since α is L^p integrable, it is in particular L^1 integrable on any set of finite measure. Recall that $\phi(\Delta, \Delta') \leq K$ for all $\Delta, \Delta' \in X_t^{k+1}$ and that if we fix Δ, the support of $\phi(\Delta, \Delta')$ is contained in a ball of radius $s > 0$, independant of the choice of Δ. We can thus make the following estimate :

$$\int_{X_t^{k+1}} |\alpha(\Delta')\phi(\Delta, \Delta')| d\mu_{k+1}(\Delta') \leq \int_{X_t^{k+1}} |\alpha(\Delta')| \cdot \mathbb{1}_{\mathrm{supp}(\phi(\Delta,\cdot))} \cdot K d\mu_{k+1}(\Delta').$$

As we mentioned earlier, $\mathrm{supp}(\phi(\Delta, \cdot))$ has finite measure because μ is quasi-regular, and thus, $\alpha * \phi(\Delta)$ is well-defined. We now show that $\alpha * \phi(\Delta)$ is L^p. Consider first the following :

$$\begin{aligned} \|\alpha * \phi\|_p^p &= \int_{X_t^{k+1}} |\alpha * \phi(\Delta)|^p d\mu_{k+1}(\Delta) \\ &= \int_{X_t^{k+1}} \left| \int_{X_t^{k+1}} \alpha(\Delta')\phi(\Delta, \Delta') d\mu_{k+1}(\Delta') \right|^p d\mu_{k+1}(\Delta). \end{aligned}$$

Note that with the general construction we gave for ϕ, $\phi(\Delta, \cdot) d\mu_{k+1}(\Delta')$ is a probability measure. We thus have the following, using Jensen's inequality :

$$\left| \int_{X_t^{k+1}} \alpha(\Delta')\phi(\Delta, \Delta') d\mu_{k+1}(\Delta') \right|^p \leq \int_{X_t^{k+1}} |\alpha(\Delta')|^p \phi(\Delta, \Delta') d\mu_{k+1}(\Delta').$$

We thus have :

$$\begin{aligned} \|\alpha * \phi\|_p^p &\leq \int_{X_t^{k+1}} \int_{X_t^{k+1}} |\alpha(\Delta')|^p \phi(\Delta, \Delta') d\mu_{k+1}(\Delta') d\mu_{k+1}(\Delta) \\ &\leq \int_{X_t^{k+1}} |\alpha(\Delta')|^p \int_{X_t^{k+1}} \phi(\Delta, \Delta') d\mu_{k+1}(\Delta) d\mu_{k+1}(\Delta'). \end{aligned}$$

We can use the same estimate on ϕ as we did before :

$$\|\alpha * \phi\|_p^p \leq \int_{X_t^{k+1}} |\alpha(\Delta')|^p \cdot K \cdot \mu_{k+1}(\mathrm{supp}(\phi(\Delta, \cdot))) d\mu_{k+1}(\Delta').$$

By definition of a kernel, the support of $\phi(\Delta, \cdot)$ in X_t^{k+1} is contained in a ball of radius $s > 0$, independantly of the choice of Δ. Because μ is quasi-regular, the volume of this ball is bounded by a constant that depends only on $V(s)$. We can conclude :

$$\begin{aligned}\|\alpha * \phi\|_p^p &\leq \text{Cste} \int_{X_t^{k+1}} |\alpha(\Delta')|^p d\mu_{k+1}(\Delta) \\ &\leq \text{Cste}\|\alpha\|_p^p.\end{aligned}$$

□

This property defines the étalement of a cochain :

$$\begin{aligned} \cdot * \phi : \quad L^p AS_t^k(X) &\to L^p AS_t^k(X) \\ \alpha &\mapsto \alpha * \phi \end{aligned}$$

We can verify that the étalement is a chain map.

Recall 3.2.8. The *boundary* of a simplex $\Delta \in X^{k+1}$, $\Delta = (x_0, ..., x_k)$, is given by the simplicial chain :

$$\partial(x_0, ... x_k) = \sum_{i=0}^{k} (-1)^i (x_0, ... \hat{x}_i, ... x_k)$$

It is defined on simplicial chains by linear extension. We denote the i^{th} *face* of Δ by :

$$\partial_i (x_0, ... x_k) = (x_0, ... \hat{x}_i, ... x_k)$$

Using this terminology, the boundary of Δ is the alternating sum of its faces :

$$\partial \Delta = \sum_{i=0}^{k} (-1)^i \partial_i \Delta.$$

Note that the Alexander-Spanier differential δ_k is the adjoint of the boundary. Given a cochain $f : X^{k+2} \to \mathbb{R}$, we have :

$$(\delta_k f)(\Delta) = f(\partial \Delta)$$

where f is extended linearly to simplicial chains.

Proposition 3.2.9. *The étalement is a chain map. For $\alpha \in L^p AS_t^k(X)$ and $\Delta \in X^{k+1}$, we have :*

$$\delta_k(\alpha * \phi_{k+1}) = (\delta_k \alpha) * \phi_{k+2}.$$

Proof. Let $\Delta_{k+2} = (x_0, ... x_{k+1})$ be a simplex in X_t^{k+2}. We develop the left hand side of the equation :

$$\delta_k(\alpha * \phi_{k+1})(\Delta_{k+2}) = \sum_{i=0}^{k+1} (-1)^i \alpha * \phi_{k+1}(\partial_i \Delta_{k+2}).$$

On the righthand side we have :

$$\begin{aligned}(\delta_k \alpha) * \phi_{k+2}(\Delta_{k+2}) &= \int_{X_t^{k+2}} \delta_k \alpha(\Delta')\phi_{k+2}(\Delta_{k+2}, \Delta') d\mu_{k+2}(\Delta') \\ &= \sum_{i=0}^{k+1} (-1)^i \int_{X_t^{k+2}} \alpha(\partial_i \Delta')\phi_{k+2}(\Delta_{k+2}, \Delta') d\mu_{k+2}(\Delta')\end{aligned}$$

At this point, we use the construction of ϕ_{k+2}. If we write $\Delta' = (y_0, ... y_{k+1})$, we have :

$$\int_{X_t^{k+2}} \alpha(\partial_i \Delta')\phi_{k+2}(\Delta_{k+2}, \Delta') d\mu_{k+2}(\Delta') = \int_{X_t^{k+2}} \alpha(\partial_i \Delta') \prod_{j=0}^{k+1} \phi(x_j, y_j) d\mu(y_0)...d\mu(y_k)$$

We observe that in each term, the i^{th} component of Δ' appears in the argument of ϕ but not in those of α. We thus have a factor on which the following property applies :

$$\int_X \phi(x_i, y_i) d\mu(y_i) = 1.$$

In consequence, we have :

$$\int_{X_t^{k+2}} \alpha(\partial_i \Delta')\phi_{k+2}(\Delta_{k+2}, \Delta') d\mu_{k+2}(\Delta') = \int_{X_t^{k+1}} \alpha(\partial_i \Delta') \prod_{j \neq i} \phi(x_j, y_j) d\mu(y_0)...\widehat{d\mu(y_i)}...d\mu(y_k).$$

On the righthand side, $\partial_i \Delta'$ is the integration variable, and thus we can rename it Δ''. We have the following :

$$\begin{aligned}\int_{X_t^{k+2}} \alpha(\partial_i \Delta')\phi_{k+2}(\Delta_{k+2}, \Delta') d\mu_{k+2}(\Delta') &= \int_{X_t^{k+1}} \alpha(\Delta'')\phi_{k+1}(\partial_i \Delta_{k+2}, \Delta'') d\mu_{k+1}(\Delta'') \\ &= \alpha * \phi_{k+1}(\partial_i \Delta_{k+2}).\end{aligned}$$

We can conclude by writing everything together.

$$(\delta_k \alpha) * \phi_{k+2}(\Delta_{k+2}) = \sum_{i=0}^{k+1} (-1)^i \alpha * \phi_{k+1}(\partial_i \Delta_{k+2})$$
$$= \delta_k(\alpha * \phi_{k+1})(\Delta_{k+2}).$$

□

Definition 3.2.10. Let $f : X \to Y$ be a quasi-isometry. By definition, if $\mathrm{diam}(\Delta) \leq t$, then $\mathrm{diam}(f(\Delta)) \leq At + B$. In other words, $f(X_t^{k+1}) \subset Y_T^{k+1}$ for all $T > A \cdot t + B$. Thus, given a cochain $\alpha : Y_T^{k+1} \to \mathbb{R}$, with $T > At + B$, $\alpha(f(\Delta))$ is well-defined for any $\Delta \in X_t^{k+1}$. Let ϕ' a kernel on Y_T^{k+1}. We define the pullback

$$f^* : AS_T^k(Y) \to AS_t^k(X)$$

in the following way :

$$f^* \alpha(\Delta) = \alpha * \phi'(f(\Delta)) \text{ , for all } \Delta \in X_t^{k+1}.$$

We can write it explicitely :

$$f^* \alpha(\Delta) = \int_{Y_T^{k+1}} \alpha(\Delta') \phi'(f(\Delta), \Delta') d\mu_{k+1}(\Delta').$$

We can first verify that f^* is a bounded operator. The proof is similar to the one we gave to show that the étalement is bounded.

Proposition 3.2.11. *Let X and Y be metric measure spaces with quasi-regular measures μ_X and μ_Y. Let $p, q \geq 1$ and let $f : X \to Y$ be a quasi-isometry. Then f^* is a bounded operator for the L^{qp} norm.*

Proof. The case where $p = \infty$ is straightforward. Assume $p < \infty$. Given $\Delta \in X_t^{k+1}$, $\alpha \in L^{qp} AS_T^k(Y)$ and ϕ' a kernel on Y_T^{k+1}, we have :

$$|f^* \alpha(\Delta)|^p = |\alpha * \phi'(f(\Delta))|^p$$
$$= \left| \int_{Y_T^{k+1}} \alpha(\Delta') \phi'(f(\Delta), \Delta') d\mu(\Delta') \right|^p$$

As in the proof of Proposition 3.2.9, we can use the fact that $\phi'(f(\Delta),\Delta')d\Delta'$ is a probability measure on Y_T^{k+1}, and apply Jensen's inequality :

$$\left|\int_{Y_T^{k+1}} \alpha(\Delta')\phi'(f(\Delta),\Delta')d\mu(\Delta')\right|^p \leq \int_{Y_T^{k+1}} |\alpha(\Delta')|^p\phi'(f(\Delta),\Delta')d\mu(\Delta').$$

This allows us to compute :

$$\begin{aligned}
\|f^*\alpha\|_p^p &= \int_{X_t^{k+1}} |f^*\alpha(\Delta)|^p d\mu(\Delta) \\
&\leq \int_{X_t^{k+1}}\int_{Y_T^{k+1}} |\alpha(\Delta')|^p\phi'(f(\Delta),\Delta')d\mu(\Delta')d\mu(\Delta) \\
&= \int_{Y_T^{k+1}} |\alpha(\Delta')|^p\int_{X_t^{k+1}} \phi'(f(\Delta),\Delta')d\mu(\Delta)d\mu(\Delta')
\end{aligned}$$

Observe that the quantity $\int_{X_t^{k+1}} \phi'(f(\Delta),\Delta')d\Delta$ is uniformally bounded. Indeed, ϕ' is bounded and its support is contained in a bounded neighborhood of the diagonal, and since we assumed the volume of balls to be bounded as well, we have :

$$\int_{X_t^{k+1}} \phi'(f(\Delta),\Delta')d\mu(\Delta) \leq \int_{B_R(\Delta')} K d\mu(\Delta)$$

We can then conclude :

$$\begin{aligned}
\|f^*\alpha\|^p &\leq \int_{Y_T^{k+1}} |\alpha(\Delta')|^p\int_{X_t^{k+1}} \phi'(f(\Delta),\Delta')d\mu(\Delta)d\mu(\Delta') \\
&\leq K'\int_{Y_T^{k+1}} |\alpha(\Delta')|^p d\mu(\Delta')
\end{aligned}$$

In a similar fashion, we have

$$\|f^*\delta\alpha\|_q^q = \|\delta f^*\alpha\|_q^q \leq K\|\delta\alpha\|_q^q.$$

The adjoint f^* is thus a bounded operator for the L^{qp} norm. □

We can check that f^* is a chain map as well.

Property 3.2.12. *Let X and Y be metric measure spaces with quasi-regular measures μ_X and μ_Y. Let $p,q \geq 1$ and let $f: X \to Y$ be a quasi-isometry. Then f^* is a chain map.*

Proof. It is a simple computation that relies on the following equation

$$\partial_i f(\Delta) = f(\partial_i \Delta)$$

as well as the fact that étalement is a chain map on cochains. We have :

$$\begin{aligned} f^*(\delta\alpha)(\Delta) &= (\delta\alpha) * \phi'(f(\Delta)) \\ &= \delta(\alpha * \phi')(f(\Delta)) \\ &= \sum_{i=0}^{k+2} (\alpha * \phi')(\partial_i f(\Delta)) \\ &= \sum_{i=0}^{k+2} (\alpha * \phi')(f(\partial_i \Delta)) \\ &= \delta(f^*\alpha)(\Delta). \end{aligned}$$

□

3.3 Quasi-isometry invariance

We now have a way to associate a homomorphism $f^* : L^p H^k_{AS,T}(Y) \to L^p H^k_{AS,t}(X)$ to a quasi-isometry $f : X \to Y$. In this part, we will show that given a quasi-isometry f and its quasi-inverse g, the composition $f^* \circ g^* : L^p H^k_{AS,T'}(X) \to L^p H^k_{AS,t}(X)$ is equal to the restriction $r_{T't} : L^p H^k_{AS,T'}(X) \to L^p H^k_{AS,t}(X)$. The same result holds for $g^* \circ f^*$. As a consequence, f^* and g^* induce inverse homomorphisms in the direct limit, when $t \to \infty$, and the asymptotic L^p cohomologies of X and Y are isomorphic. The proof relies on the following proprerty of étalement.

Proposition 3.3.1. *Let X be a metric space with a quasi-regular measure μ. Let ϕ be a kernel on X_t^{k+1}. The étalement is a homotopy equivalence for simplicial chains, that is, for all $k > 0$ there exist operators*

$$B_k : L^p AS_t^k(X) \to L^p AS_t^{k-1}(X)$$

such that for all $\alpha \in L^p AS_t^k(X)$, we have :

$$\alpha * \phi - \alpha = \delta B_k(\alpha) + B_{k+1}\delta(\alpha).$$

To demonstrate this proposition, we use a property of the prism defined on two simplices. We recall the definition of the prism and its property.

Definition 3.3.2. Let $\Delta = (x_0, ..., x_k)$ and $\Delta' = (y_0, ... y_k)$ two k-simplices. The prism of basis Δ, Δ' is the $k+1$-chain

$$b(\Delta, \Delta') = \sum_{i=0}^{k} (-1)^i (x_0, ... x_{i-1}, x_i, y_i, ... y_k).$$

Lemma 3.3.3. *Let Δ and Δ' be simplices in X^{k+1}. We have :*

$$\partial b(\Delta, \Delta') = \Delta' - \Delta - \sum_{i=0}^{k} (-1)^i b(\partial_i \Delta, \partial_i \Delta').$$

Proof. We first compute

$$\begin{aligned} \partial b(x_0, ... x_k; y_0, ... y_k) &= \sum_i (-1)^i \sum_j (-1)^j \partial_i (x_0, ... x_j, y_j, ... y_k) \\ &= \Delta' - \Delta + \sum_{\substack{(i,j) \neq (k,k+1) \\ (i,j) \neq (0,0)}} (-1)^i (-1)^j \partial_i (x_0, ... x_j, y_j, ... y_k) \end{aligned}$$

This last summation can be splitted as follow :

$$\sum_j (-1)^j \left(\sum_{i=0}^{j} (-1)^i (x_0, ... \hat{x}_i, ... x_j, y_j, ... y_k) + \sum_{i=j}^{k} (-1)^{i+1} (x_0, ... x_j, y_j, ... \hat{y}_i, ... y_k) \right)$$

On the other, one can observe that :

$$\begin{aligned} \sum_{l=0}^{k} (-1)^l b(\partial_l \Delta, \partial_l \Delta') = \sum_{l=0}^{k} (-1)^l & \left(\sum_{m=0}^{l-1} (x_0, ... x_m, y_m, ... \hat{y}_l, ... y_k) \right. \\ & \left. + \sum_{m=l+1}^{k} (x_0, ... \hat{x}_l, ... x_m, y_m, ... y_k) \right) \end{aligned}$$

This decomposition is the same as the former, which allows to write :

$$\partial b(\Delta, \Delta') = \Delta' - \Delta - \sum_{i=0}^{k} (-1)^i b(\partial_i \Delta, \partial_i \Delta').$$

□

We can now prove Proposition 3.3.1.

Proof. We define the operator B_k as follow. Let $\alpha \in L^p AS_t^{k+1}(X)$ and $\Delta \in X_t^{k+1}$.

$$B_k f(\Delta) = \int_{X^{k+1}} f(b(\Delta, \Delta'))\phi(\Delta, \Delta') d\mu_{k+1}(\Delta').$$

Note here that $f(b(\Delta, \Delta'))$ is well-defined if Δ' is close enough to Δ so that the simplices in $b(\Delta, \Delta')$ are in X_t^{k+2}. We can choose ϕ so that $\phi(\Delta, \Delta') = 0$ if $\rho(\Delta, \Delta') \geq t$.

We can do the following computation :

$$\begin{aligned}
B(\delta\alpha)(\Delta) &= \int_{X^{k+1}} \delta\alpha(b(\Delta, \Delta'))\phi(\Delta, \Delta') d\mu(\Delta') \\
&= \int_{X^{k+1}} \alpha(\partial b(\Delta, \Delta'))\phi(\Delta, \Delta') d\mu(\Delta') \\
&= \int_{X^{k+1}} \alpha\left(\Delta' - \Delta - \sum_{i=0}^{k} (-1)^i b(\partial_i \Delta, \partial_i \Delta')\right)\phi(\Delta, \Delta') d\mu(\Delta') \\
&= \alpha * \phi(\Delta) - \alpha(\Delta) - \sum_{i=0}^{k} (-1)^i \int_{X^{k+1}} \alpha(b(\partial_i \Delta, \partial_i \Delta'))\phi(\Delta, \Delta') d\mu(\Delta')
\end{aligned}$$

This last integral can be rewritten as :

$$\begin{aligned}
\int_{X^{k+1}} \alpha(b(\partial_i \Delta, \partial_i \Delta'))\phi(\Delta, \Delta') d\mu(\Delta') &= \int_{X^{k+1}} b(\partial_i \Delta, \partial_i \Delta') \prod_{j=0}^{k} \phi(x_j, y_j) d\mu(y_0)...d\mu(y_k) \\
&= \int_{X^k} \alpha(b(\partial_i \Delta, \partial_i \Delta')) \prod_{j \neq i} \phi(x_j, y_j) d\mu(y_0)...\widehat{d\mu(y_i)}...d\mu(y_k) \\
&= \int_{X^k} \alpha(b(\partial_i \Delta, \Delta_k))\phi(\partial_i \Delta, \Delta_k) d\mu(y_0)...d\mu(y_{k-1})
\end{aligned}$$

And thus

$$\sum_{i=0}^{k} (-1)^i \int_{X^{k+1}} \alpha(b(\partial_i \Delta, \partial_i \Delta'))\phi(\Delta, \Delta') d\mu(\Delta') = \delta B(\alpha)(\Delta).$$

We can check that B_k is a bounded operator to conclude this proof :

$$\begin{aligned}
\|B(\alpha)\|_p^p &= \int_{X^{k+1}} \left| \int_{X^{k+1}} \alpha(b(\Delta,\Delta'))\phi(\Delta,\Delta')d\mu(\Delta') \right|^p d\mu(\Delta) \\
&\leq \int_{X^{k+1}} \int_{X^{k+1}} \left|\alpha(b(\Delta,\Delta'))\right|^p \phi(\Delta,\Delta')d\mu(\Delta')d\mu(\Delta) \\
&\leq \int_{X^{k+1}} \int_{X^{k+1}} (k+1)\,|\alpha(\Delta)|^p \phi(\Delta,\Delta')d\mu(\Delta')d\mu(\Delta) \\
&\leq \int_{X^{k+1}} \int_{\mathrm{supp}(\phi''(\Delta,\ldots))} \sup\phi\cdot(k+1)\,|\alpha(\Delta)|^p\, d\mu(\Delta')d\mu(\Delta) \\
&\leq \mathrm{cste}\cdot\|\alpha\|_p^p.
\end{aligned}$$

□

We now state the main technical result of the proof of quasi-isometry invariance for the L^p Alexander-Spanier cohomology. We have to choose scales T', T and t such that we can compose the mappings f^* and g^* induced by quasi-isometries f and g.

Lemma 3.3.4. *Let X and Y be metric measure spaces with quasi-regular metrics. Assume $f: X \to Y$ and $g: Y \to X$ are quasi-isometries with coefficients A and B for f and C and D for g, and g constructed as in the proof of Property 3.1.2. Fix $t > 0$, $T \geq At + B$ and $T' \geq CT + D$. Let ϕ be a kernel on X_T^{k+1} and ϕ' a kernel on $Y_{T'}^{k+1}$.*

Then for all $k > 0$, the composition $f^* \circ g^* : L^p AS_{T'}^k(X) \to L^p AS_t^k(X)$ is homotopy equivalent to the restriction operator $r_{T't} : L^p AS_{T'}^k(X) \to L^p AS_t^k(X)$.

Proof. Let $\Delta \in X_t^{k+1}$ and $\alpha \in L^p AS_{T'}^k(X)$. Compute directly :

$$\alpha(g(f(\Delta)*\phi')*\phi) = \int_{X^{k+1}} \int_{Y^{k+1}} \alpha(\Delta_X)\phi'(f(\Delta),\Delta_Y)\phi(g(\Delta_Y),\Delta_X)d\mu(\Delta_Y)d\mu(\Delta_X).$$

If we write, for $\Delta_1, \Delta_2 \in X^{k+1}$,

$$\phi''(\Delta_1,\Delta_2) = \int_{Y^{k+1}} \phi(f(\Delta_1),\Delta_Y)\cdot\phi'(g(\Delta_Y),\Delta_2)d\mu(\Delta_Y),$$

the first equation becomes :

$$\alpha(g(f(\Delta)*\phi)*\phi') = \alpha(\Delta*\phi'').$$

We can check that ϕ'' is a kernel on X^{k+1} :

- If we fix Δ_1, we have :

$$\begin{aligned}\int_{X^{k+1}} \phi''(\Delta_1,\Delta_2)d\mu(\Delta_2) &= \int_{X^{k+1}} \left(\int_{Y^{k+1}} \phi'(f(\Delta_1),\Delta_Y)\phi(g(\Delta_Y),\Delta_2)d\mu(\Delta_Y)\right)d\mu(\Delta_2)\\ &= \int_{Y^{k+1}} \phi'(f(\Delta_1),\Delta_Y)\left(\int_{X^{k+1}} \phi(g(\Delta_Y),\Delta_2)d\mu(\Delta_2)\right)d\mu(\Delta_Y)\\ &= \int_{Y^{k+1}} \phi'(f(\Delta_1),\Delta_Y)d\mu(\Delta_Y) = 1\end{aligned}$$

- There exists $D_{\phi''}$ such that $\phi''(x_1,x_2) = 0$ whenever $\rho(x_1,x_2) > D_{\phi''}$: assume that $\phi(a,b) = 0$ when $\rho(a,b) > D_\phi$ and $\phi'(a,b) = 0$ when $\rho(a,b) > D_{\phi'}$.

  Assume that $\rho'(f(x_1),y) < D_{\phi'}$ and $\rho(g(y),x_2) < D_\phi$. The first inequality gives $\rho(g\circ f(x_1),g(y)) < A\cdot D_{\phi'} + B$. Together with the second inequality, we have :

$$\rho(g\circ f(x_1),x_2) < D_\phi + A\cdot D_{\phi'} + C$$

  Because $g\circ f$ is close to the identity, we also have :

$$\begin{aligned}\rho(x_1,x_2) &\le d(x_1,g\circ f(x_1),) + \rho(g\circ f(x_1),x_2)\\ &\le K + \rho(g\circ f(x_1),x_2)\\ &\le K + D_\phi + A\cdot D_{\phi'} + C\end{aligned}$$

  So $\phi''(x_1,x_2)$ is non-zero only on a bounded neighborhood of the diagonal.

- It is bounded, because ϕ and ϕ' are bounded and because the support of $\phi\cdot\phi'$ is bounded as well, as seen just before.

Thus, we can apply Proposition 3.3.1 :

$$g^*f^*\alpha(\Delta) - \alpha(\Delta) = B\delta\alpha(\Delta) + \delta B\alpha(\Delta).$$

Because $\Delta \in X_t^{k+1}$, we have in fact the following :

$$g^*f^*\alpha(\Delta) - r_{T't}\alpha(\Delta) = B\delta\alpha(\Delta) + \delta B\alpha(\Delta).$$

We can conclude that

$$g^*f^* : L^pAS^k_{T'}(Y) \to L^pAS^k_t(Y)$$

and

$$r_{T't} : L^p AS^k_{T'}(Y) \to L^p AS^k_t(Y)$$

induce the same homomorphism in cohomology. □

Using this result, we can show that the asymptotic L^p Alexander-Spanier cohomology is a quasi-isometry invariant for metric space with quasi-regular measure.

Theorem 3.3.5. *Let (X, ρ_1, μ_1) and (Y, ρ_2, μ_2) two quasi-isometric metric measure spaces such that both μ_1 and μ_2 are quasi-regular. For any $1 \leq p \leq \infty$, the asymptotic L^p Alexander-Spanier cohomology of X and Y are isometric, that is, for all $k \geq 0$:*

$$L^p H^k_{AS}(X) = L^p H^k_{AS}(Y).$$

Proof. In cohomology, theorem 3.3.4 implies that $f^* \circ g^*$ and $g^* \circ f^*$ are equivalent to the respective restriction operators. In turn, this implies that the mappings induced in the asymptotic cohomology

$$f^* \circ g^* : L^p H^k_{AS}(X) \to L^p H^k_{AS}(X)$$

and

$$g^* \circ f^* : L^p H^k_{AS}(Y) \to L^p H^k_{AS}(Y)$$

are equivalent to the mapping induced by the restrictions, which are the identities on $L^p H^k_{AS}(X)$ and $L^p H^k_{AS}(Y)$. Thus, in asymptotic cohomology, f^* and g^* are inverse homomorphisms. □

3.4 Quasi-isometry invariance for graphs

In the case where we conside graphs, we can extend the result of quasi-isometry invariance to the L^{qp} asymptotic cohomology. The proof remains the same, but we can discuss what the hypothesis of quasi-regularity becomes when we have a metric space with counting measure. For graphs, there is an equivalent definition to quasi-regularity in term of the number of edges linking each vertex.

Definition 3.4.1. Assume that X is a graph, with length metric and counting measure. We say that X has *bounded degree* if there exists a constant $K > 0$ such that for any vertex $x \in X$, the number of neighbours of x is less than K.

Property 3.4.2. *The measure of a graph X is quasi-regular if and only if X has bounded degree.*

Proof. The lower bound exists automatically, since any ball $B(x,r)$ contains its center x and thus always has a volume bigger than 1. The upper bound implies that each vertex has a finite number of neighbours : if $\mu(B(x,3/2)) < V(3/2)$, for all x, it implies that any given point x has less than $V(3/2)$ neighbouring points, and that X is a graph with bounded geometry. Reciproqually, assume that X has bounded geometry and let $x \in X$ and $r > 0$. Because any point in X has less than K neighbours, $B(x,r)$ contains less than K^r points (the maximum being the case of a regular tree), and thus $\mu(B(x,r)) < K^r$. □

Corollary 3.4.3. *In the settings of theorem 3.3.4, assume that X and Y are graphs with bounded geometry. Let $1 \le p \le q \le \infty$. Then there exists $T' \ge T \ge t > 0$ such that $g^* \circ f^* : L^{qp}AS^k_{T'}(Y) \to L^{qp}AS^k_t(Y)$ is homotopy equivalent to the restriction operator $r_{T't} : L^{qp}AS^k_{T'}(Y) \to L^{qp}AS^k_t(Y)$.*

Proof. We can rewrite the last part of the proof of theorem 3.3.4. We have to check that $\|B(\alpha)\|_q < \infty$. We already know that

$$\|B(\alpha)\|_q^q \le \text{cste} \cdot \|\alpha\|_q^q.$$

Since X and Y are graphs, if $q \ge p$, then $\|\alpha\|_p < \infty$ implies $\|\alpha\|_q < \infty$, which concludes the proof. □

Corollary 3.4.4. *Let X and Y be graphs with bounded geometry. Assume X and Y are quasi-isometric. Then, for $1 \le p \le q \le \infty$, their asymptotic L^{qp} cohomologies are isomorphic :*

$$L^{qp}H^k_{AS}(X) = L^{qp}H^k_{AS}(Y).$$

Proof. The corollary 3.4.3 allows to use the proof of theorem 3.3.5 for L^{qp} Alexander-Spanier cohomology. Indeed, since $f^* \circ g^*$ and $g^* \circ f^*$ are equivalent to the restriction in L^{qp} cohomology, f^* and g^* induce inverse homomorphism in L^{qp} asymptotic cohomology. □

3.5 Bounded cohomology

Another way to extend the result of invariance through quasi-isometry toward L^{qp} cohomology is to consider L^{qp} cochains that are globally bounded, or L^∞. Although the bounded cohomology is already used for a different construction, we will use the term *bounded L^{qp} cohomology.*

Definition 3.5.1. Let $f \in L^pAS^k_t(X)$. We say that f is *bounded* if $\|f\|_\infty < \infty$. We denote the space of bounded L^{qp} Alexander-Spanier cochains by $L^{qp}_bAS^k_t(X)$. The k^{th} group of cohomology associated is denoted by $L^{qp}_bH^k_{AS,t}(X)$. The *asymptotic bounded L^{qp} Alexander-Spanier cohomology of X* is simply written

$$L_b^{qp} H_{AS}^k(X) = \varprojlim L_b^{qp} H_{AS,t}^k(X)$$

Property 3.5.2. *Let $f \in L_b^p AS_t^k(X)$ and assume that $q \geq p$. Then $\|f\|_q < \infty$.*

Proof. Let us consider $A = \{\Delta \in X_t^{k+1} \mid f(\Delta) \geq 1\}$ and $B = X_t^{k+1} \setminus A$. Then we have :

$$\|f\|_q^q = \|f\restriction_A\|_q^q + \|f\restriction_B\|_q^q.$$

Since $f(\Delta) < 1$ for $\Delta \in B$, $|f(\Delta)|^q \leq |f(\Delta)|^p$, and thus

$$\|f\restriction_B\|_q^q \leq \|f\restriction_A\|_p^p \text{, for all } q \geq p.$$

Because $\|f\|_\infty < \infty$ and $\|f\restriction_A\|_p < \infty$, we deduce that $\mu(A) < \infty$, and so $\|f\restriction_A\|_q < \infty$ for all q. □

The bounded L^{qp} Alexander-Spanier cohomology allows us to formulate a result similar to the one we stated for graphs, but for quasi-regular metric measure spaces. We restate Lemma 3.3.4 in this situation.

Corollary 3.5.3. *In the settings of theorem 3.3.4, assume that $q \geq p$. Then there exists $T', T, t > 0$ such that $g^* \circ f^* : L_b^{qp} AS_{T'}^k(Y) \to L_b^{qp} AS_t^k(Y)$ is homotopy equivalent to the restriction operator $r_{T't} : L_b^{qp} AS_{T'}^k(Y) \to L_b^{qp} AS_t^k(Y)$.*

Proof. In this case again, we gain an inclusion : if α is globally bounded and $q \geq p$, then $\|\alpha\|_p < \infty$ implies $\|\alpha\|_q < \infty$. Thus, we do have $\|B(\alpha)\|_q < \infty$. □

Corollary 3.5.3 allows to state quasi-isometry invariance for bounded L^{qp} cohomology :

Corollary 3.5.4. *Let (X, ρ_1, μ_1) and (Y, ρ_2, μ_2) two quasi-isometric metric measure spaces such that both μ_1 and μ_2 are quasi-regular. For any $1 \leq p \leq q \leq \infty$, the asymptotic bounded L^{qp} Alexander-Spanier cohomology of X and Y are isometric, that is, for all $k \geq 0$:*

$$L_b^{qp} H_{AS}^k(X) = L_b^{qp} H_{AS}^k(Y).$$

Proof. The proof is the same as for Theorem 3.3.5, using Corollary 3.5.3. □

For graphs with bounded degree, L^p Alexander-Spanier cochains are bounded. We thus have the following property :

Property 3.5.5. *Let X be a graph with bounded degree. We have the following equality :*

$$L_b^{qp} AS_t^k(X) = L^{qp} H_t^k(X).$$

And so we have the following isomorphism in cohomology :

$$L_o^{qp} H_{AS}^k(X) = L^{qp} H_{AS}^k(X).$$

3.6 L^{qp} Coarse cohomology

Corollary 3.4.4 allows to define the following cohomology for metric measure spaces that are quasi-isometric to a graph with bounded geometry.

Definition 3.6.1. Let $1 \le p \le q \le \infty$. Assume that Y is a graph with bounded degree which is quasi-isometric to X. The L^{qp} *coarse Alexande-Spanier cohomology* of X is the asymptotic L^{qp} Alexander-Spanier cohomology of Y. This is well-defined, since any other choice of Y would yield the same cohomology for X, since the asymptotic L^{qp} cohomology is invariant through quasi-isometry for graphs. We denote the k^{th} group of coarse L^{qp} Alexander-Spanier cohomology of X by

$$L^{qp} H_{AS,c}^k(X) = L^{qp} H_{AS}^k(Y).$$

We call this the coarse cohomology in reference to the work of J. Roe [35] and P. Fan [15]. Although this is not the same construction, the idea is close.

As a consequence of Property 3.5.5, we have the following isomorphism.

Property 3.6.2. *Let X be a metric space with a quasi-regular measure. Let $1 \le p \le q \le \infty$. Then the coarse L^{qp} cohomology of X is isomorphic to its bounded L^{qp} cohomology :*

$$L^{qp} H_{AS,c}^k(X) = L_b^{qp} H_{AS}^k(X).$$

Proof. Assume that Y is a graph with bounded degree that is quasi-isometric to X. Then we have this chain of isomorphism :

$$L_b^{qp} H_{AS}^k(X) = L_b^{qp} H_{AS}^k(Y) = L^{qp} H_{AS}^k(Y).$$

The first isomorphism results from quasi-isometry invariance of the L^{qp} bounded cohomology and the second one from Property 3.5.5. By definition we have

$$L^{qp}H^k_{AS}(Y) = L^{qp}H^k_{AS,c}(X)$$

which conclude the proof. □

This leaves us wondering under which conditions we can build a graph with bounded degree Y that is quasi-isometric to a given metric measure space X. We can give an answer by considering nets, which gives an example of quasi-isometric approximation to metric space.

Definition 3.6.3. Let X be a metric space and let $\epsilon > 0$. A discrete subset $X_0 \subset X$ is an ϵ*-net* if

1. for any $x_0, x_1 \in X_0$, we have $\rho(x_0, x_1) \geq \epsilon$;
2. X_0 is maximal for the first condition.

Maximality implies that you can not add a point to X_0 without breaching the first hypothesis. In particular, we have

$$\bigcup_{x \in X_0} B(x, \epsilon) = X.$$

The inclusion of X_0 in X is a quasi-isometry, since it is a isometry and X_0 is ϵ dense in X.

Proposition 3.6.4. *Let X be a metric space with a quasi-regular measure μ. Then there exists a graph with bounded degree Y which is quasi-isometric to X.*

Proof. Fix $\epsilon > 0$ and let X_0 be a ϵ-net in X. Consider Y to be the graph that has X_0 as vertices and an edge for each couple of points $x_0, x_1 \in X_0$ such that $\rho(x_0, x_1) \leq 2\epsilon$. In other words, Y is the 0-skeleton of the Vietoris-Rips complex of X_0 for 2ϵ. We use the word metric and the counting measure on Y. Note that Y is quasi-isometric to X for the inclusion : given two points y_0, y_1 linked by an edge in Y, we have by construction

$$\epsilon \leq \rho_X(y_0, y_1) \leq 2\epsilon.$$

So given arbitrary points $y_0, y_1 \in Y$, we have

$$\epsilon \rho_Y(y_0, y_1) \leq \rho_X(y_0, y_1) \leq 2\epsilon \rho_Y(y_0, y_1).$$

Because X_0 is an ϵ-net, we also have

$$\bigcup_{x \in Y} B(x, \epsilon) = X.$$

We will now show that Y has bounded degree. Let $x \in X_0$. The measure μ is quasi-regular, and so we have the following bound :

$$\mu(B(x, 2\epsilon)) \leq V(2\epsilon).$$

Let x_0 and x_1 be two points of X_0 lying in $B(x, 2\epsilon)$. Because $\rho(x_0, x_1) \geq \epsilon$, we have

$$B(x_0, \frac{\epsilon}{2}) \cap B(x_1, \frac{\epsilon}{2}) = \emptyset.$$

As a consequence, we have :

$$\sum_{x_i \in B(x,2\epsilon)} \mu(B(x_i, \frac{\epsilon}{2})) \leq \mu(B(x, 2\epsilon)).$$

We use the lower bound for the volume of balls :

$$\sum_{x_i \in B(x,2\epsilon)} v(\frac{\epsilon}{2}) \leq V(2\epsilon).$$

If we denote the number of neighbours of x in Y by $K(x)$, we find the following upper bound :

$$K(x) \leq \frac{v(\frac{\epsilon}{2})}{V(2\epsilon)}.$$

Since the choice of x is arbitrary, this bound is uniform, and thus Y has bounded degree. $\square$

4 De Rham Theorems

In this chapter, the main results are different variations of the de Rham Theorem for the metric and L^p Alexander-Spanier cohomology. We will show that for complete Riemannian manifolds with bounded geometry, the scaled Alexander-Spanier cohomology is isomorphic to the de Rham cohomology, for small value of the scale t. This result is also true for the L^p Alexander-Spanier cohomology and the L^p de Rham cohomology. In the compact case, we show that the L^{qp} Alexander-Spanier cohomology is isomorphic to the de Rham cohomology. The proof of these results relies on a property of bicomplexes, which we detail in Section 4.4. A bicomplex is a diagram of spaces and homomorphisms forming a two dimentional matrix, such that each column and each row is a differential complex. We can build bicomplexes that link the Alexander-Spanier cohomology or the de Rham cohomology to the Čech cohomology. When the rows and columns of these bicomplexes are exact, then the cohomologies they links are isomorphic. In order to apply this property, we need to discuss the Poincaré Lemma and the Mayer-Vietoris sequence for the different cohomologies we consider.

We first recall the definitions of L^π de Rham cohomology and L^π Čech cohomology.

4.1 De Rham cohomology

Definition 4.1.1. Let $\Omega^k(M)$ be the set of smooth differential forms of degree k on M. The exterior differential $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ has the property that

$$d_{k+1} \circ d_k = 0 \text{ , for all } k \geq 0$$

and thus we get de Rham complex :

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M)\ldots$$

The *de Rham cohomology* is the cohomology of this complex. We note

$$Z^k_{DR}(M) = \ker d_k$$

$$B^k_{DR}(M) = \mathrm{im} d_{k-1}$$

and

$$H^k_{DR}(M) = Z^k_{DR}(M)/B^k_{DR}(M).$$

A form in $Z^k_{DR}(M)$ is *closed* and it is *exact* if it is in $B^k_{DR}(M)$.

It is rather straightforward to see that the de Rham cohomology is invariant through diffeomorphism. It is in fact a homotopy invariant, and de Rham [10] showed in his thesis that it is isomorphic to the singular cohomology.We can define an L^p version of the de Rham cohomology, which leads to the construction of geometric invariants.

Definition 4.1.2. Let M be an orientable Riemannian manifold of dimension n without boundary. We denote by $L^1_{\mathrm{loc}}(M,\Lambda^k)$ the set of forms α which norm is locally integrable, that is, for any compact set $K \subset M$, we have :

$$\int_K |\alpha(x)| d\mu(x) < \infty.$$

Let $\alpha \in L^1_{\mathrm{loc}}(M,\Lambda^k)$. A form $\theta \in L^1_{\mathrm{loc}}(M,\Lambda^{k+1})$ is a *weak derivative* of α when the following equality holds for any compactly supported smooth form $\omega \in C^\infty_c(M,\Lambda^{n-k-1})$:

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \alpha \wedge d\omega$$

where d is the usual exterior differential.

Remark 4.1.3.

- If α is a smooth form, then its usual derivative is a weak derivative.
- For a given form, two weak derivatives will differ only on a set of measure 0.
- If $\theta = d\alpha$ in the weak sense, then, as in the usual case, we have $d\theta = 0$.

Definition 4.1.4. Given p, q with $1 \le p, q \le \infty$, $\Omega^k_{q,p}(M)$ is the space of k-forms $\omega \in L^q(M,\Lambda^k)$ such that $d\omega \in L^p(M,\Lambda^{k+1})$

The k^{th} group of de Rham $L^{q,p}$ de Rham cohomology of M is

$$L^{qp}H^k_{DR}(M) = Z^k_p(M)/d\Omega^{k-1}_{q,p}(M)$$

where $Z^k_p(M) = \ker d_k$, the space of closed L^p $k-$forms.

We can construct L^{qp} cohomology as the cohomology of a differential complex. Fix a sequence $\pi = \{p_k \geq 1 | k \in \mathbb{N}\}$. By writing

$$\Omega^k_\pi(M) = \Omega^k_{p_k,p_{k+1}}(M),$$

we obtain a differential complex which defines the L^π *de Rham cohomology* of M :

$$L^\pi H^k_{DR}(M) = Z^k_{p_k}(M)/d\Omega^{k-1}_{p_{k-1},p_k}(M)$$

As for L^{qp} Alexander-Spanier cohomology, we defined the reduced L^{qp} de Rham cohomology in order to have Banach spaces :

$$L^{qp}\bar{H}^k_{DR}(M) = Z^k_p(M)/\overline{d\Omega^{k-1}_{q,p}(M)}.$$

Property 4.1.5. *The $L^{q,p}$ de Rham cohomology is invariant through bilipschitz diffeomorphism.*

4.2 Čech cohomology

Notation 4.2.1. Given a countable, ordered set of indices S, a multi-index is a finite increasing sequence $I = (\alpha_0, ..., \alpha_k)$. The set of all multi-indices of length $k+1$ is written S_k.

Definition 4.2.2. Let X be a topological space and let $\mathscr{U} = \{U_\alpha | \alpha \in S\}$ be an open, countable cover of X. For any multi-index $I = (\alpha_0, \alpha_1, ..., \alpha_k) \in S_k$, we write

$$U_I = U_{\alpha_0} \cap ... \cap U_{\alpha_k}.$$

We define $\check{C}^k(X, \mathscr{U})$ as the space of all maps

$$c = \prod_{I \in S_k} c_I$$

such that $c_I : U_I \to \mathbb{R}$ is a locally constant function.

We define the Čech coboundary operator

$$\check{\delta}_k : \check{C}^k(X, \mathscr{U}) \to \check{C}^{k+1}(X, \mathscr{U})$$

as the alternating difference : for $x \in U_{\alpha_0 \dots \alpha_k+1}$,

$$\check{\delta}_k c(x) = \sum_i (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{k+1}}(x).$$

The index k will generally be dropped, unless necessary.

Proposition 4.2.3. *The coboundary operator is a differential :*

$$\check{\delta} \circ \check{\delta} = 0.$$

Proof. It is a direct calculation :

$$\begin{aligned}
\check{\delta}^2(c_I) &= \sum (-1)^i \check{\delta} c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{k+2}} \\
&= \sum_{j<i} (-1)^i (-1)^j c_{\alpha_0 \dots \hat{\alpha}_j \dots \hat{\alpha}_i \dots \alpha_{k+2}} \\
&+ \sum_{i>j} (-1)^i (-1)^{j-1} c_{\alpha_0 \dots \hat{\alpha}_i \dots \hat{\alpha}_j \dots \alpha_{k+2}} \\
&= 0
\end{aligned}$$

□

The spaces $\check{C}^k(X, \mathscr{U})$ together with the Čech differential $\check{\delta}$ form the *Čech complex* of X and $\mathscr{U}$.

The Čech complex defines cohomology groups which are called the *Čech cohomology*. The notation is as usual :

- $\check{Z}^k(X, \mathscr{U}) = \ker(\check{\delta}_k)$;
- $\check{B}^k(X, \mathscr{U}) = \operatorname{im}(\check{\delta}_{k-1})$;
- $\check{H}^k(X, \mathscr{U}) = \check{Z}^k(X, \mathscr{U}) / \check{B}^k(X, \mathscr{U})$.

With this definition, the Čech cohomology is dependant on the cover $\mathscr{U}$. We can remove this dependance by considering the limit as the open sets become, in a way, smaller and smaller.

Definition 4.2.4. Given two covers of X, $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathscr{V} = \{V_\beta\}_{\beta \in B}$, we say that $\mathscr{V}$ is a *refinement* of $\mathscr{U}$ if any V_β is contained is some U_α. In other words, there exist a map between the set of indices

$$\phi : B \to A$$

such that for all $\beta \in B$, V_β is a subset of $U_{\phi(\beta)}$. We call such a map a *refinement map*. This map induces a map on cochains :

$$\phi^\sharp : \check{C}^k(X,\mathscr{U}) \to \check{C}^k(X,\mathscr{V}).$$

Its value on a cochain $\omega \in \check{C}^k(X,\mathscr{U})$ is given by

$$\phi^\sharp\omega(V_{\beta_0\ldots\beta_k}) = \omega(U_{\phi(\beta_0)\ldots\phi(\beta_k)}).$$

The map induced on cochains is actually a chain map, and thus induces an homomorphism in cohomology, for every $k \geq 0$:

$$\phi^\sharp : \check{H}^k(X,\mathscr{U}) \to \check{H}^k(X,\mathscr{V}).$$

Moreover, two different refinement maps $\phi_1, \phi_2 : B \to A$ will induces homotopy equivalent chain maps, and thus the same homomorphism in cohomology :

$$\phi_1^\sharp = \phi_2^\#.$$

This means that the homomorphism depends only on the covers and not on the refinement maps. In consequence, we will denote the refinement homomorphism by

$$\phi^\sharp_{\mathscr{U},\mathscr{V}} : \check{H}^k(X,\mathscr{U}) \to \check{H}^k(X,\mathscr{V}).$$

Proposition 4.2.5. *The collection of all cohomology groups $\check{H}^k(X,\mathscr{U})$ and all refinement homomorphisms $\phi^\sharp_{\mathscr{U},\mathscr{V}}$ forms a direct system : for three covers $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$, such that $\mathscr{V}$ is a refinement of $\mathscr{U}$ and $\mathscr{W}$ is a refinement of $\mathscr{V}$, we have :*

1. $\phi^\sharp_{\mathscr{U},\mathscr{U}} = id$;
2. $\phi^\sharp_{\mathscr{V},\mathscr{W}} \circ \phi^\sharp_{\mathscr{U},\mathscr{V}} = \phi^\sharp_{\mathscr{U},\mathscr{W}}$.

Definition 4.2.6. This last proposition allows us to define the *Čech cohomology* of a topological space X as the direct limit of the Čech cohomology of X taken on all the open covers :

$$\check{H}^k(X) = \varinjlim \check{H}^k(X,\mathscr{U}).$$

Assume now that a measure μ is given on X.

Definition 4.2.7. We define also an L^p version of Čech cohomology using the following $p-$norm. For $c \in \check{C}^k(X, \mathscr{U})$, set :

$$\|c\|_p^p = \sum_I \int_{U_I} |c(x)|^p d\mu(x).$$

The space of Čech cochains $c \in \check{C}^k(X, \mathscr{U})$ such that $\|c\|_p < \infty$ is denoted by $L^p\check{C}^k(X, \mathscr{U})$ and the space of cochains such that $\|c\|_q < \infty$ and $\|\check{\delta}\|_p < \infty$ is denoted by $L^{qp}\check{C}^k(X, \mathscr{U})$. We then use the following notations :

- $\check{Z}_p^k(X, \mathscr{U}) = \{c \in L^p\check{C}^k(X, \mathscr{U}) \mid \check{\delta}c = 0\}$;
- $\check{B}_{qp}^k(X, \mathscr{U}) = \check{\delta}(L^{qp}\check{C}^{k-1}(X, \mathscr{U})$;
- $L^{qp}\check{H}^k(X, \mathscr{U}) = \check{Z}_p^k(X, \mathscr{U}) / \check{B}_{qp}^k(X, \mathscr{U})$.

If $q = p$, we obtain the L^p Čech cohomology group $L^p\check{H}^k(X, \mathscr{U})$. Given a sequence of numbers $\pi = \{p_k\}_{k \in \mathbb{N}}$, with $1 \leq p_k \leq \infty$, we write $(\check{C}_\pi^k(X, \mathscr{U}), \check{\delta})$ to denote the Čech L^π complex and $\check{H}_\pi^k(X, \mathscr{U})$ the k^{th} cohomology group associated.

For covers that are regular enough, we can simplify the norm we use and instead of integrating on each U_I, use a counting measure that sums the value that c takes on each intersections U_I.

Property 4.2.8. *Let $\mathscr{U}$ be a locally finite cover of X. If there exist numbers A and B such that $0 < A \leq \mu(U_I) \leq B$ for every multi-index I and if each U_I is connected, then for any choice of $x_I \in U_I$, we have*

$$A^p \sum_I |c(x_I)|^p \leq \sum_I \int_{U_I} |c(x)|^p d\mu(x) \leq B^p \sum_I |c(x_I)|^p.$$

Note that since c is locally constant, if we assume each U_I to be connected, we might as well write c_I for the constant value of c on U_I.

This property means that for a cover of X with the properties listed above, a Čech cochain is bounded for the original norm if and only if it is also bounded for the discrete one, defined by ony by a summation. A consequence of that is that for covers which sets have volume that are bounded above and below uniformly, the L^p Čech cohomology is isomorphic to the L^p simplicial cohomology of the nerve of the cover.

4.3 The Poincaré Lemmas

In this section, we state two similar results for de Rham and scaled Alexander-Spanier cohomologies : for convex subsets of $\mathbb{R}^n$, these two cohomologies are trivial. In the L^{qp} case, we have some additional hypothesis to make. For the L^{qp} de Rham cohomology, we need to have

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$$

and for L^{qp} Alexander-Spanier cohomology, the result is true when $q \leq p$.

We first recall the Poincaré Lemma for de Rham and L^{qp} de Rham cohomology in $\mathbb{R}^n$.

Proposition 4.3.1. *Let U be a contractible open subset of $\mathbb{R}^n$. The de Rham cohomology of U is trivial :*

$$H^k_{DR}(U) = \begin{cases} 0 & if\, k \geq 1 \\ \mathbb{R} & if\, k = 0 \end{cases}$$

Proposition 4.3.2. *Let $p, q > 1$ such that*

$$1/p - 1/q \leq 1/n,$$

and U a convex and bounded open subset of $\mathbb{R}^n$, then :

$$L^{qp} H^k_{DR}(U) = \begin{cases} 0 & if\, k \geq 1 \\ \mathbb{R} & if\, k = 0 \end{cases}$$

This result is from Troyanov and Gol'dshtein [16].

For the scaled Alexander-Spanier cohomology, we start by an easy result : if the scale t is large compared to the diameter of the metric space X, then the Alexander-Spanier cohomology of X at scale t is trivial.

Lemma 4.3.3. *Let U be a measurable subset of (X, ρ, μ), such that $diam(U) < t_0$. For $t > t_0$, the Alexander-Spanier cohomology of U at scale t is trivial :*

$$H^k_{AS,t}(U) = \begin{cases} 0 & if\, k \geq 1 \\ \mathbb{R} & if\, k = 0 \end{cases}$$

Moreover, if $0 < \mu(U) < \infty$ and given $1 < q \leq p < \infty$, the Alexander-Spanier L^{qp} cohomology at scale t is trivial as well :

$$L^{qp} H^k_{AS,t}(U) = \begin{cases} 0 & if\, k \geq 1 \\ \mathbb{R} & if\, k = 0 \end{cases}$$

Proof. We use the idea of remark 2.2.3 : let $f : U_t^{k+1} \to \mathbb{R}$ a k-cochain with $\|f\|_p < \infty$ and $\delta f = 0$. We define a δ-preimage to f :

$$Af : U_t^k \to \mathbb{R}$$

by setting

$$Af(x_0,...,x_{k-1}) = \frac{1}{\mu(U)} \int_U f(x,x_0,...,x_{k-1})d\mu(x).$$

Because $t > \cdot \text{diam}(U)$, $f(x,x_0,...,x_{k-1})$ is defined for any $x \in U$, so Af is well-defined.

Using Fubini's Theorem and Jensen's inequality, we can check that $A : f \mapsto Af$ is bounded :

$$\left| \frac{1}{\text{vol}(U)} \int_U f(x,x_0,...,x_{k-1})d\mu(x) \right|^p \leq \text{vol}(U)^{p-2} \int_U \left| f(x,x_0,...,x_{k-1}) \right|^p d\mu(x).$$

Thus

$$\|Af\|_p^p \leq \text{vol}(U)^{p-2} \|f\|_p^p.$$

Because U has finite measure, $\|Af\|_p < \infty$ implies that $\|Af\|_q < \infty$ if $q \leq p$. □

This result is elementary, but it gives the intuition that the Alexander-Spanier cohomology, both classical and L^p, ignores features with diameter smaller than the scale considered. We complete this result for smaller scales : for bounded, convex subset of $\mathbb{R}^n$, the Alexander-Spanier cohomology is trivial for any scale. We show this by constructing a homotopy inverse to the restriction operator, using barycentric subdivision. We need a few definitions.

Definition 4.3.4. Let $\Delta = (x_0,...,x_k) \in X^{k+1}$. Recall the boundary of Δ is the chain :

$$\partial\Delta = \sum_i (-1)^i (x_0,...,\hat{x}_i,...x_k).$$

The *cone* of base Δ and vertex x is the simplex of degree $k+1$ with first vertex is x and first face is Δ :

$$\text{Cone}_x(\Delta) = (x,x_0,...,x_k)$$

We will need the following formula about the boundary of the cone.

Property 4.3.5. *Let $x \in X$ and $\Delta \in X^{k+1}$. We have :*

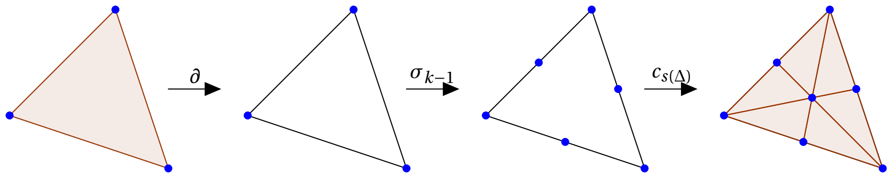
$$\begin{aligned} \partial Cone_x(\Delta) &= \partial(x,x_0,...x_k) \\ &= \Delta + \sum_i (-1)^{i+1}(x,x_0,...\hat{x}_i,...x_k) \\ &= \Delta - Cone_x(\partial\Delta) \end{aligned}$$

Definition 4.3.6. The boundary and the cone operators allow us to define the *generalized barycentric subdivision.* Let $s : \Delta \mapsto s(\Delta)$ be a function which associates a point in X to each simplex $(x_0,...x_k)$, for any dimension $k > 0$. We define σ_k recursively on k : for $k = 0$, σ_k is the identity. Then, for any $k > 0$ and any $\Delta \in X^{k+1}$, we set :

$$\sigma_k(\Delta) = \text{Cone}_{s(\Delta)}\sigma_{k-1}(\partial\Delta)$$

We then extend σ_k linearly to simplicial chains.

Figure 4.1: The barycentric subdivision of a 2-simplex.



Remark 4.3.7. We have to note here that $\sigma_k(\partial\Delta) \neq 0$ for all $\Delta \in X^{k+2}$, eventhough the definition seems to include $\sigma_{k-1}(\partial\partial\Delta)$. Indeed, because we define σ_k on chains by extended linearly, we have :

$$\sigma_k(\partial\Delta) := \sum_i (-1)^i \sigma_k(\partial_i\Delta).$$

Remark 4.3.8. From the definition of the subdivision, we can observe that the general form of a simplex in the chain $\sigma(\Delta)$ is

$$(s(\tau_0), \ldots s(\tau_k))$$

where τ_{i+1} is a face of τ_i for all $0 \leq i \leq k$. In particular, every simplex in $\sigma(\Delta)$ has $s(\Delta)$ as its first vertex and some vertex of Δ as its $k+1^{\text{th}}$ vertex.

Definition 4.3.9. In $\mathbb{R}^n$, the *barycenter* of a simplex $\Delta = (x_0, \ldots x_k)$, $x_i \in \mathbb{R}^n$, is the point

$$s_b(\Delta) = \sum_{i=0}^{k} \frac{1}{k+1} x_i.$$

If $s(\Delta)$ is chosen to be the barycenter of Δ, for every $\Delta \in (\mathbb{R}^n)^{k+1}$, then σ_k is the usual barycentric subdivision. The barycentric subdivision is particularly useful because it contracts the diameter of the simplices :

Property 4.3.10. *Let $\Delta = (x_0, \ldots x_k) \in (\mathbb{R}^n)^{k+1}$ and let τ be a simplex of the barycentric subdivision of Δ. Then*

$$diam(\tau) \leq \frac{k}{k+1} diam(\Delta)$$

Proof. First recall that the diameter of Δ is given by

$$\max_{i \neq j} \rho(x_i, x_j).$$

For all x_j, we have

$$\rho(x_j, s_b(\Delta)) \leq \frac{k}{k+1} \text{diam}(\Delta).$$

The computation is rather straightforward :

$$\begin{aligned}
\rho(x_j, s_b(\Delta)) &= \|x_j - s_b(\Delta)\| \\
&= \|x_j - \sum_{i=0}^{k} \frac{1}{(k+1)} x_i\| \\
&\leq \sum_{i=0}^{k} \frac{1}{k+1} \|x_j - x_i\| \\
&\leq \frac{k}{k+1} \max_{i \neq j} \|x_j - x_i\| \\
&= \frac{k}{k+1} \mathrm{diam}(\Delta).
\end{aligned}$$

As a consequence, if x is any point in the convex hull of $\{x_0, ...x_k\}$, the same inequality remains true :

$$\rho(x, s_b(\Delta)) \leq \frac{k}{k+1} \mathrm{diam}(\Delta).$$

We can now prove the property by induction on the dimension of the simplex. If Δ is a point, then its diameter is 0 and the inequality is verified. Fix $k > 0$ and assume that for any simplex of dimension smaller than k, the property holds. Let τ be a simplex of $\sigma(\Delta)$. It is of the following form :

$$\tau = (s_b(\tau_0), ...s_b(\tau_k))$$

where τ_{i+1} is a face of τ_i for all i. Now, for any $0 \leq i < j \leq k$, either $s(\tau_i) = s(\Delta)$ and we have

$$\rho(s_b(\tau_j), s_b(\Delta)) \leq \frac{k}{k+1} \mathrm{diam}(\Delta),$$

either τ_i is of dimension l with $l < k$ and the induction hypothesis applies :

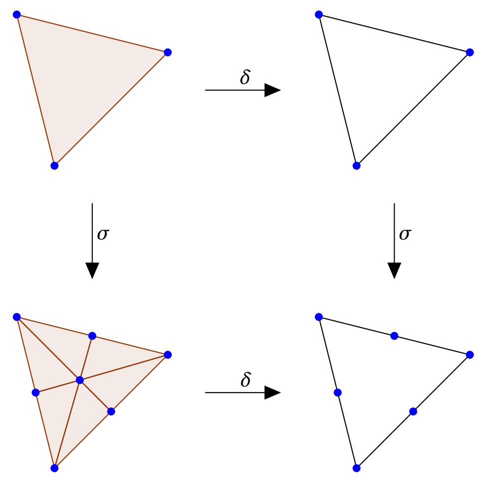
$$\rho(s_b(\tau_i), s_b(\tau_j)) \leq \frac{l}{l+1} \mathrm{diam}(\tau_i) \leq \frac{k}{k+1} \mathrm{diam}(\Delta)$$

□

Property 4.3.11. *The subdivision is a chain map, that is :*

$$\partial \sigma_k = \sigma_{k-1} \partial$$

Figure 4.2: The subdivision is a chain map.



Proof. This is quite natural intuitively speaking : the boundary of the subdivision of a simplex is equal to the subdivision of the boundary of this simplex (see figure 4.2). We show this formally using induction. We can easily verify for small values of k that $\partial_l \sigma_l = \sigma_{l-1}\partial_l$ as well as

$$\partial_l \sigma_{l-1} \partial_l = 0.$$

Assume now these two equalities are true for any $l < k$. We then have

$$\partial_l \sigma_l \partial_{l+1} = \sigma_{l-1}\partial_l \partial_{l+1} = 0.$$

Let $\Delta \in X^{k+1}$ and consider the following, where we apply Property 4.3.5 to the cone operator $c_{s(\Delta)}$:

$$\begin{aligned}\partial_k \sigma_k(\Delta) &= \partial_k(\mathrm{Cone}_{s(\Delta)} \sigma_{k-1}(\partial_k \Delta)) \\ &= \sigma_{k-1}(\partial_k \Delta) - \mathrm{Cone}_{s(\Delta)} \partial_{k-1}\sigma_{k-1}(\partial_k \Delta).\end{aligned}$$

We do have here $\partial_{k-1}\sigma_{k-1}\partial_k = 0$ by the argument before and thus, $\partial_k \sigma_k = \sigma_{k-1}\partial_k$. □

Proposition 4.3.12. *The generalized barycentric subdivision is a homotopy operator on chain : there exists operators b such that*

$$1 - \sigma = \partial b + b\partial$$

Proof. We define b inductively. Given a 0-simplex x, $b_0(x) = 0$. This is to be understood as the empty simplicial chain : $f(b_0(x)) = 0$ for all 0-cochain f. In this case, we have

$$x - \sigma(x) = 0 = \partial b_0(x) + b_0\partial(x).$$

For $\Delta \in X^{k+1}$, we set

$$b_k(\Delta) = \mathrm{Cone}_{s(\Delta)}(\Delta - b_{k-1}\partial\Delta).$$

Assume that the result is true for all $l < k$ and verify it for k :

$$\begin{aligned}\partial b_k(\Delta) &= \partial(\mathrm{Cone}_{s(\Delta)}(\Delta - b_{k-1}\partial\Delta)) \\ &= (\Delta - b_{k-1}\partial\Delta) - \mathrm{Cone}_{s(\Delta)}(\partial\Delta - \partial b_{k-1}\partial\Delta).\end{aligned}$$

We can rewrite this as :

$$\partial b_k(\Delta) + b_{k-1}\partial(\Delta) = \Delta - \mathrm{Cone}_{s(\Delta)}(\partial\Delta - \partial b_{k-1}\partial\Delta).$$

The induction hypothesis implies the following :

$$\begin{aligned}\mathrm{Cone}_{s(\Delta)}[\partial b_{k-1}(\partial\Delta) + b_{k-2}\partial(\partial\Delta)] &= \mathrm{Cone}_{s(\Delta)}(\partial\Delta - \sigma_{k-1}\partial\Delta) \\ &= \mathrm{Cone}_{s(\Delta)}\partial\Delta - \mathrm{Cone}_{s(\Delta)}\sigma_{k-1}\partial\Delta \\ &= \mathrm{Cone}_{s(\Delta)}\partial\Delta - \sigma_k(\Delta).\end{aligned}$$

Since $b_{k-2}\partial\partial(\Delta) = 0$, we have

$$\sigma_k(\Delta) = \mathrm{Cone}_{s(\Delta)}(\partial\Delta - \partial b_{k-1}(\partial\Delta))$$

which allows us to conclude. □

We can now apply these two last properties in order to prove the Poincaré Lemma for the scaled Alexander-Spanier cohomology.

Proposition 4.3.13. *Let U be a convex, bounded subset of $\mathbb{R}^n$. Then for any $t > 0$ we have :*

$$H^k_{AS,t}(U) = \begin{cases} 0 & \text{if } k \geq 1 \\ \mathbb{R} & \text{if } k = 0 \end{cases}$$

Proof. The operators b and σ defined earlier are well-defined here since U is bounded and convex, and they induce operators on the Alexander-Spanier cochains. We write

$$\sigma^* : AS^k_t(U) \to AS^k_t(U)$$

and

$$B : AS^k_t(U) \to AS^{k-1}_t(U)$$

for the respective adjoints

$$\sigma^* f(\Delta) = f \circ \sigma(\Delta)$$

and

$$Bf(\Delta) = f \circ b(\Delta).$$

Because σ contracts the size of cochains, σ^* is well-defined : if $\mathrm{diam}(\Delta) < t$, then $\mathrm{diam}(\tau) < t$ for any τ in $\sigma(\Delta)$ and thus $f(\sigma(\Delta))$ can be computed. The same holds for B : if $\mathrm{diam}(\Delta) < t$, then $\mathrm{diam}(b(\Delta)) < t$, and thus B is well-defined.

The homotopy equation remains valid :

$$1 - \sigma^* = B \circ \delta + \delta \circ B.$$

This is close to what we want, but it does not use the fact that σ diminishes the size of cochains. We define a slightly different pullback. Fix $T = \frac{k+1}{k} \cdot t$ and set :

$$E : AS^k_t(U) \to AS^k_T(U)$$

with $Ef(\Delta) = f \circ \sigma(\Delta)$. It is well-defined by the same argument than before : if $\Delta \in X^{k+1}_T$, then any τ from $\sigma(\Delta)$ has $\mathrm{diam}(\tau) \leq \frac{k}{k+1}\mathrm{diam}(\Delta) = t$. Thus any $f \in AS^k_t(U)$ can be evaluated on

$\sigma(\Delta)$.

Note now that if we compose E with the restriction operator $r_{T,t} : AS^k_T(U) \to AS^k_t(U)$, we get :

$$r_{T,t} \circ E = \sigma^* : AS^k_t(U) \to AS^k_t(U)$$

and

$$E \circ r_{T,t} = \sigma^* : AS^k_T(U) \to AS^k_T(U).$$

In consequence, we have the following equations :

$$1 - E \circ r_{T,t} = B \circ \delta + \delta \circ B$$

and

$$1 - r_{T,t} \circ E = B \circ \delta + \delta \circ B.$$

This means that $r_{T,t}$ and E induce inverse homomorphisms in cohomology and that the Alexander-Spanier cohomology of U does not depend on the scale. Since at large scale, U has the cohomology of the point, it follows that it is the case for any scale. □

This results remains valid in L^p and L^{qp} cohomology. We need to redefine the operators E and B a bit differently so that they are well-defined bounded operators on L^p classes.

Proposition 4.3.14. *Let $1 < q \leq p < \infty$ and U be a convex, bounded open subset of $\mathbb{R}^n$. Then for any $t > 0$ we have,*

$$L^{qp} H^k_{AS,t}(U) = \begin{cases} 0 & \text{if } k \geq 1 \\ \mathbb{R} & \text{if } k = 0 \end{cases}$$

Proof. We need to redefine E and B from the last proof. They would still be well-defined in the sense that for f_1 and $f_2 \in AS^k_t(U)$ that differ only on a set of measure 0, then $E(f_1)$ and $E(f_2)$ as well as $B(f_1)$ and $B(f_2)$ are also equal almost everywhere. However, because B sends simplices of degree k to simplices of degree $k-1$, we can not control $\|B(f)\|_p$ with $\|f\|_p$.

In order to counter that, we first note that given a simplex $\Delta \in U^{k+1}$, there exists a number $\epsilon(\Delta) > 0$ such that for any choice of $y \in B(s(\Delta), \epsilon(\Delta))$ the subdivision

$$\sigma_k(\Delta, y) = \mathrm{Cone}_y \sigma_{k-1}(\Delta)$$

keeps the property of diameter contraction : there exists $0 < k < 1$ such that for any simplex Δ' which appears in the chain $\sigma_k(\Delta, y)$, $\mathrm{diam}(\Delta') \leq k \cdot \mathrm{diam}(\Delta)$.

We denote by B_Δ the open ball $B(s(\Delta), \epsilon(s(\Delta)))$. Given $f \in L^p AS_t^k(U)$, we set :

$$Ef(\Delta) = \frac{1}{\mu(B_\Delta)} \int_{B_\Delta} f(\mathrm{Cone}_y \sigma_{k-1}(\Delta)) d\mu(y).$$

We also set

$$Bf(\Delta) = \frac{1}{\mu(B_\Delta)} \int_{B_\Delta} f(\mathrm{Cone}_y(\Delta - b_{k-1}\partial\Delta)) d\mu(y).$$

If f is a cocycle, we then have :

$$f - E(f) = \delta B(f).$$

The only thing left to do is to check that E and B are bounded. We shows the calculation for B, which is more complicated than for E :

$$\begin{aligned}
\|Bf\|_p^p &= \int_{U_t^k} \left| \frac{1}{\mu(B_\Delta)} \int_{B_\Delta} f(\mathrm{Cone}_y(\Delta - b_{k-1}\partial\Delta)) d\mu(y) \right|^p d\mu_k(\Delta) \\
&\leq \int_{U_t^k} (\mu(B_\Delta))^{p-2} \int_{B_\Delta} \left| f(\mathrm{Cone}_y(\Delta - b_{k-1}\partial\Delta)) \right|^p d\mu(y) d\mu_k(\Delta) \\
&\leq V(t)^{p-2} \cdot \int_{U_t^k} \int_{B_\Delta} \left| f(\mathrm{Cone}_y(\Delta - b_{k-1}\partial\Delta)) \right|^p d\mu(y) d\mu_k(\Delta)
\end{aligned}$$

Recall here that $V(t)$ is such that $\mu(B(x,t)) \leq V(t)$ for all $x \in U$. Observe that for any $\Delta = (x_0, \ldots, x_{k-1}) \in U_t^k$ and any $y \in B_\Delta$, the simplice $(y, x_0, \ldots, x_{k-1})$ is in U_t^{k+1}, by choice of B_Δ. We can thus estimate :

$$\begin{aligned}
\|Bf\|_p^p &\le V(t)^{p-2}\cdot\int_{U_t^k}\int_{B_\Delta}\left|f(\mathrm{Cone}_y(\Delta-b_{k-1}\partial\Delta))\right|^p d\mu(y)d\mu_k(\Delta)\\
&\le V(t)^{p-2}\cdot\int_{U_t^{k+1}}\left|f(\mathrm{Cone}_y(\Delta-b_{k-1}\partial\Delta))\right|^p d\mu(y)d\mu_k(\Delta)\\
&\le V(t)^{p-2}\cdot\int_{U_t^{k+1}}\left|\sum_{\Delta_i\in Bf(\Delta)} f(\Delta_i)\right|^p d\mu(y)d\mu_k(\Delta)\\
&\le V(t)^{p-2}\cdot\int_{U_t^{k+1}}\mathrm{Cste}(k)\sum_{\Delta_i\in Bf(\Delta)}\left|f(\Delta_i)\right|^p d\mu(y)d\mu_k(\Delta)\\
&\le V(t)^{p-2}\mathrm{Cste}'(k)\cdot\int_{U_t^{k+1}}\left|f(\Delta')\right|^p d\mu_{k+1}(\Delta')\\
&\le V(t)^{p-2}\mathrm{Cste}'(k)\cdot\|f\|_p^p
\end{aligned}$$

Since U has finite measure, if $1\le q\le p$, then $\|f\|_p<\infty$ implies $\|f\|_q<\infty$, which in turn implies $\|Bf\|_q<\infty$.

□

We can extend both Propositions 4.3.13 and 4.3.14 using bilipschitz mappings.

Definition 4.3.15. Let (X,ρ_X) and (Y,ρ_Y) be metric spaces. A map $\phi: X\to Y$ is *bilipschitz* if it is bijective and there exists a constant $L\ge 1$ such that for every $x_1,x_2\in X$, we have

$$\frac{1}{L}\rho_X(x_1,x_2)\le\rho_Y(f(x_1),f(x_2))\le L\rho_X(x_1,x_2).$$

The inverse map ϕ^{-1} is also bilipschitz, for the same constant K.

Proposition 4.3.16. *Let M be a Riemannian manifold and let U be a convex, bounded subset of $\mathbb{R}^n$. Assume there exists a bilipschitz map $\phi: U\to V$, with $V\subset M$. Then for every $t>0$, V and U have the same Alexander-Spanier cohomology :*

$$H^k_{AS,t}(V)=\begin{cases}0 & \text{if } k\ge 1\\ \mathbb{R} & \text{if } k=0\end{cases}$$

If V has finite measure, this result is also true for L^p and L^{qp} cohomology, for $q\le p$:

$$L^{qp}H^k_{AS,t}(V)=\begin{cases}0 & \text{if } k\ge 1\\ \mathbb{R} & \text{if } k=0\end{cases}$$

Proof. Let $\Delta\in V_t^{k+1}$. We can use ϕ to define a subdivision of Δ. Let $\sigma_\phi(\Delta)=\phi\circ\sigma_b\circ\phi^{-1}(\Delta)$, where σ_b is the usual barycentric subdivision. An equivalent definition for σ_ϕ is to consider the center s_ϕ given by

$$s_\phi(\Delta) = \phi \circ s_b \circ \phi^{-1}(\Delta)$$

where s_b is the barycenter. The subdivision σ_ϕ is then a homotopy equivalence, by Proposition 4.3.12. If Δ_i is a simplex that appears in $\sigma_\phi(\Delta)$, then the diameter can be estimated as follow :

$$\mathrm{diam}(\Delta_i) \leq \frac{k}{k+1} L^2 t.$$

Of course, depending on L, this will generally not suffice for σ_ϕ to contract the size of simplices. But we can subdivide further : consider σ_ϕ^m. For m sufficiently large, we have

$$\left(\frac{k}{k+1}\right)^m L^2 < 1.$$

This means that we can define, by precomposition, an application

$$\phi^*_{\sigma,m} : AS^k_T(V) \to AS^k_t(V)$$

with $T = \left(\frac{k}{k+1}\right)^m L^2 t$. The same reasoning as in the proof of Proposition 4.3.13, $\phi^*_{\sigma,m}$ induces an inverse to $r_{T,t}$ in cohomology, which allows to conclude for the metric case. For the L^p version, we can refer to the proof of Proposition 4.3.14 which applies as well here. □

The idea of using subdivision to show that the cohomology is stable with regard to the scale t can be used in other situations. We develop this idea in Chapter 5. Note that the above proposition is a restrictive case for bilipschitz invariance. We state it that way, since it is sufficient for the use we will make of it and allow for a much simpler proof.

4.4 Bicomplexes

In this section, we develop the tool we use to prove de Rham Theorems. One can read [4] for an exposition of the concept and its application to the original de Rham theorem.

Definition 4.4.1. A collection of Banach spaces $C^{i,j}$, $i, j \geq 0$ together with bounded operators $d : C^{i,j} \to C^{i+1,j}$ and $\delta : C^{i,j} \to C^{i,j+1}$ form a *bicomplex of Banach spaces* if $d \circ d = \delta \circ \delta = (d+\delta) \circ (d+\delta) = 0$.

A bicomplex is a differential complex on its own, with $k-$cochains of the form $w \in \sum_{i+j=k} C^{i,j}$

and differential operator $D = d + \delta$.

Note also that $D \circ D = (d+\delta)\circ(d+\delta) = d\circ\delta + \delta\circ d$, which implies that $d\delta = -\delta d$.

The subcomplexes $(C^{*,j}, d)$ are called the *columns* and the subcomplexes $(C^{i,*}, \delta)$ are called the *rows.*

Lemma 4.4.2. *Let $(C^{i,j}, d, \delta)$, with $k \geq 0, l \geq 0$, be a bicomplex such that for all $l \geq 0$, the complex $(C^{*,j}, d)$ is exact, that is for all $k \geq 0$, there exist operators $h : C^{i+1,j} \to C^{i,j}$ such that*

$$1 = hd + dh.$$

Then each cocycle $\omega = \sum_{i+j=m} \omega_{i,j}$ of the bicomplex can be represented by a cochain ω' in $C^{0,n}$ that has $D\omega' = 0$.

Proof. Consider the application

$$b : C^{*,*} \to C^{*,*}$$

defined by

$$b(c) = -\delta hc - h\delta c.$$

It is a homotopy operator :

$$1 - b = (d+\delta)h + h(d+\delta).$$

Let $c = c_{i,j}$ be cochain in $C^{*,*}$ which only non-zero component lies in $C^{i,j}$. Then $bc \in C^{i-1,j+1}$. Upon iteration, we find that c is cohomologous, for some m, to $b^m(c)$ which lies only in $C^{0,k}$, where $k = i + j$. $\square$

This lemma amounts to say that, when all the rows are exact, the cohomology of the complex

$$0 \longrightarrow C^{0,0} \xrightarrow{D} C^{1,0} \times C^{0,1} \xrightarrow{D} \prod_{i+j=2} C^{i,j} \xrightarrow{D} \prod_{i+j=3} C^{i,j} \xrightarrow{D} \ldots$$

is isomorphic to the cohomology of

$$0 \longrightarrow C^{0,0} \cap \ker\delta \xrightarrow{d} C^{1,0} \cap \ker\delta \xrightarrow{d} C^{2,0} \cap \ker\delta \xrightarrow{d} \ldots$$

Lemma 4.4.2 is sometimes called a staircase argument. The reason is clear if we formulate the proof in the following way. If $\omega \in \sum_{i+j=k} C^{i,j}$ is a cocycle, we have in particular that

$$D\omega_{0,k} = 0$$

which implies that $d\omega_{0,k} = 0$. Since the columns are exact, there exists a preimage $h\omega_{0,k} \in C^{0,k-1}$. Consider the cochain

$$\omega' = \omega - Dh\omega_{0,k}.$$

It is also a cocycle and we have $\omega'_{0,k} = 0$. Starting from an arbitrary cocycle ω, we constructed a cocycle ω' which has no component of degree $(0,k)$. We can iterate this idea k times and obtain a cocycle ω'' whose only non-null component is in degree $(k,0)$.

The next lemma describes the way we will use this fact.

Lemma 4.4.3. *Consider a bicomplex* $(C^{i,j}, d, \delta)$ *to which we add a column* $(C^{*,-1}, d)$, *using homomorphisms* $r_k : C^{k,-1} \to C^{k,0}$. *Assume that* $rd = dr$. *If all the rows*

$$0 \to C^{i,-1} \xrightarrow{r} C^{i,0} \xrightarrow{\delta} \ldots$$

are exact, then the cohomology of the complex $C^{*,-1}$ *is isomorphic to the cohomology of the bicomplex* $(C^{*,*}, D)$.

Figure 4.3: The bottom left part of a bicomplex, augmented with a column.

$$\begin{tikzcd}
0 \arrow[r] & C^{2,-1} \arrow[r, "r"] & C^{2,0} \arrow[r, "\delta"] & C^{2,1} \arrow[r, "\delta"] & C^{2,2} \\
0 \arrow[r] & C^{1,-1} \arrow[r, "r"] \arrow[u, "d"] & C^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,1} \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,2} \arrow[u, "d"] \\
0 \arrow[r] & C^{0,-1} \arrow[r, "r"] \arrow[u, "d"] & C^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & C^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & C^{0,2} \arrow[u, "d"]
\end{tikzcd}$$

Proof. Let α be a D-cocycle of degree k of the bicomplex. By lemma 4.4.2, it is equivalent to a cocycle β supported only in the first column. More precisely, since α is of degree k, $\beta = \beta_{k,0}$. More over, since β is a cocyle, we have $\delta(\beta) = 0$ and $d(\beta) = 0$.

Recall that exactness implies that $\ker(r) = \{0\}$ and $\mathrm{im}(r) = \ker\delta$. This means that the complex

$$0 \longrightarrow C^{0,0} \cap \ker\delta \xrightarrow{d} C^{1,0} \cap \ker\delta \xrightarrow{d} C^{2,0} \cap \ker\delta \xrightarrow{d} \ldots$$

is isomorphic to the complex given by

$$0 \longrightarrow C^{0,-1} \xrightarrow{d} C^{1,-1} \xrightarrow{d} C^{2,-1} \xrightarrow{d} \ldots$$

□

We can now formulate the form that we will use later.

Theorem 4.4.4. *Assume that $(C^{i,j}, d, \delta)$ with $-1 \leq i, j < \infty$, is a bicomplex, augmented by a row and a column. Assume as well that each row $(C^{i,*}, d)$ and each column $(C^{*,j}, \delta)$ are exact for $0 \leq i, j < \infty$. Then, the cohomology groups of $(C^{-1,*}, d)$ and $(C^{*,-1}, \delta)$ are isomorphic.*

Proof. We apply lemma 4.4.3 twice. Once to show that $(C^{-1,*}, d)$ has the same cohomology as the bicomplex as a whole, and once to show that the bicomplex has the same cohomology as $(C^{*,-1}, \delta)$. □

4.5 Mayer-Vietoris Sequences

For the main results of this chapter, we need the following versions of the Mayer-Vietoris sequence in order to apply lemma 4.4.3. The operator $\check{\delta}$ we use in this section is an extension of the Čech differential defined earlier.

Proposition 4.5.1. *(Generalized Mayer-Vietoris Sequence for de Rham complex) Let M be a Riemannian manifold and let $\mathscr{U} = \{U_\alpha\}_{\alpha\in S}$ be a locally finite open cover of M. Let $r : \Omega^k(X) \to \prod_{\alpha\in S}\Omega^k(U_\alpha)$ be the component-wise restriction. The following sequence is exact :*

$$0 \to \Omega^k(X) \xrightarrow{r} \prod_{\alpha\in S}\Omega^k(U_\alpha) \xrightarrow{\check{\delta}} \prod_{I\in S^1}\Omega^k(U_I) \xrightarrow{\check{\delta}} \prod_{I\in S^2}\Omega^k(U_I) \to \ldots$$

Recall that the operator

$$\check{\delta} : \prod_{I\in S^l}\Omega^k(U_I) \xrightarrow{\check{\delta}} \prod_{I\in S^{l+1}}\Omega^k(U_I)$$

is defined as follow, given $x \in U_I$ with $I = \alpha_0 \ldots \alpha_l$, and $\omega \in \prod_{I\in S^l}\Omega^k(U_I)$:

$$(\check{\delta}\omega)_I(x) = \sum_{i=0}^{l} \omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_l}(x).$$

Proof. We have obviously $\ker(r) = \{0\}$. Since $\check{\delta}_0$ is the difference on each intersection, we also have $\ker(\check{\delta}_0) = \mathrm{im}(r)$. For the remaining cases, we construct a right inverse to $\check{\delta}_k$.

Let $\{\eta_\alpha\}_{\alpha\in S}$ be a partition of unity subordinated to $\mathscr{U}$. Consider a cocycle $\omega \in \prod_J \Omega^k(U_J)$ where $J = \alpha_0...\alpha_k \in S_k$. If $I = \alpha_0...\alpha_{k-1}$, write $\alpha I = \alpha\alpha_0...\alpha_{k-1}$ and consider the following :

$$K\omega_I = \sum_\alpha \eta_\alpha \omega_{I\alpha}.$$

Then we have $\check{\delta}K\omega = \omega$. Indeed :

$$\begin{aligned}\check{\delta}K\omega_I &= \sum_{i=0}^{k}(-1)^i (K\omega)_{\alpha_0...\hat{\alpha}_i...\alpha_k}\\ &= \sum_{\alpha\in S}\sum_{i=0}^{k}(-1)^i \eta_\alpha \omega_{\alpha\alpha_0...\hat{\alpha}_i...\alpha_k}\end{aligned}$$

Because ω is a cocycle, we have :

$$\check{\delta}(\omega)_{\alpha\alpha_0...\alpha_k} = \omega_{\alpha_0...\alpha_k} + \sum (-1)^{i+1}\omega_{\alpha\alpha_0...\hat{\alpha}_i...\alpha_k} = 0.$$

So back to the former equation :

$$\begin{aligned}\check{\delta}K\omega_I &= \sum_{\alpha\in S}\eta_\alpha \sum_{i=0}^{k}(-1)^i \omega_{\alpha\alpha_0...\hat{\alpha}_i...\alpha_k}\\ &= \sum_{\alpha\in S}\eta_\alpha \omega_{\alpha_0...\alpha_k}\\ &= \omega_{\alpha_0...\alpha_k}\end{aligned}$$

□

In order to describe the L^p version of this result, we need to introduce some notations.

Definition 4.5.2. Let M be a Riemannian manifold of dimension n and $\mathscr{U} = \{U_\alpha\}_{\alpha\in S}$ be an open cover of M. An element of $\mathscr{M}C^{k,l}_{DR}(M,\mathscr{U})$ is a measurable function

$$\omega = \prod_{I\in S_l} \omega_I \text{ with } \omega_I : U_I \to \Lambda^k(\mathbb{R}^n).$$

The L^p norm of such a function is defined by :

$$\|\omega\|_p = \left(\sum_{I \in S_l} \int_{U_I} |\omega_I(x)|^p \, d\mu(x) \right)^{1/p}$$

The space $L^p C^{k,l}_{DR}(M, \mathcal{U})$ consists of the equivalence classes of $\mathcal{M} C^{k,l}_{DR}(M, \mathcal{U})$ for this L^p norm.

Proposition 4.5.3. *(Generalized Mayer-Vietoris Sequences for L^p de Rham complex) Let M be a Riemannian manifold with an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$. Assume that there exists a positive constant B such that for any $I = \alpha_0 ... \alpha_k$, $\mu(U_I) \leq B$. Assume as well that there exists $N > 0$ such that for all $\alpha \in S$, U_α intersects less than N other open of the cover. Then the following sequence is exact :*

$$0 \to L^p \Omega^k_p(M) \xrightarrow{r} L^p C^{k,0}_{DR}(M, \mathcal{U}) \xrightarrow{\check{\delta}} L^p C^{k,1}_{DR}(M, \mathcal{U}) \to ...$$

Proof. We use the same proof as before, but we have to prove that K is a bounded operator for the L^p norm, that is, for any $\omega \in L^p C^{k,l}_{DR}(M, \mathcal{U})$, we have $\|K\omega\|_p \leq \text{cste} \cdot \|\omega\|_p$.

Indeed :

$$\begin{aligned}
\|K\omega\|_p^p &= \sum_I \int_{U_I} \left| \sum_\alpha \eta_\alpha \omega_{I\alpha}(x) \right|^p d\mu(x) \\
&\leq \sum_I \int_{U_I} \left| \sum_\alpha 1_\alpha \omega_{I\alpha}(x) \right|^p d\mu(x) \\
&\leq \sum_I \int_{U_I} \Lambda \cdot \sum_\alpha |1_\alpha \omega_{I\alpha}(x)|^p \, d\mu(x).
\end{aligned}$$

To obtain the last line, we used Jensen's inequality. The coefficient Λ depends on the number of terms that appears in the summation. Because we assume that each U_α intersect at most N other sets of the cover, we can choose a unique Λ for all U_I.

$$\begin{aligned}
\|K\omega\|_p^p &\leq \sum_I \int_{U_I} \Lambda \sum_\alpha |1_\alpha \omega_{I\alpha}(x)|^p \, d\mu(x) \\
&\leq \Lambda \sum_I \sum_\alpha \int_{U_{I\alpha}} |\omega_{I\alpha}(x)|^p \, d\mu(x) \\
&\leq \Lambda N \sum_J \int_{U_J} \left|\omega_J(x)\right|^p d\mu(x) \\
&\leq \Lambda N \|\omega\|_p^p.
\end{aligned}$$

The bound N is also used in the passage from a double summation to a simple one. When we sum through all multi-indices αI, with $\alpha \in S$, the number of α such that $U_{\alpha I}$ is non-empty is smaller than N. In the double summation on $I \in S_k$ and $\alpha \in S$, a given mult-indice $J \in S_{k+1}$ can appears at most N times, which gives us the last estimation we make.

□

We now give the same results for the scaled Alexander-Spanier complex. We first state the non-L^p case. In this situation, we need to make adjustement to take into account that the cochains are not pointwise, but defined on simplices.

Notation 4.5.4. Given a collection of sets $\mathscr{U} = \{U_\alpha\}_{\alpha\in S}$, we denote by $\mathscr{U}^k$ the collection of sets $\{U_\alpha^k\}_{\alpha\in S}$.

Proposition 4.5.5. *(Generalized Mayer-Vietoris Sequence for the scaled Alexander-Spanier complex) Let X be a metric space and let $\mathscr{U}$ be a locally finite open cover of X. Fix $t_0 > 0$. Assume that, for all $k > 0$, $\mathscr{U}^k$ is a cover of $X_{t_0}^k$. Then for all $t \le t_0$, the following sequence is exact :*

$$0 \to AS_t^k(X) \xrightarrow{r} \prod_{\alpha\in S} AS_t^k(U_\alpha) \xrightarrow{\check{\delta}} \prod_{I\in S_1} AS_t^k(U_I) \to \dots$$

Proof. The proof is basically the same than for proposition 4.5.1. There is two steps where we need the additional hypothesis. The first one is the injectivity of the restriction operator. Let $f \in \prod_\alpha AS_t^k(U_\alpha)$ with $\check{\delta} f = 0$. The hypothesis that $\mathscr{U}^{k+1}$ is a cover of X_t^{k+1} ensure that f defines a value for every $(x_0,\dots,x_k) \in X_t^{k+1}$, and thus that f has a well-defined pre-image in $AS_t^k(X)$.

The second step is in the definition of the inverse to $\check{\delta}$. We need to be able to define a partition of unity on X_t^{k+1} subordinate to $\mathscr{U}^{k+1}$, which would be ill-defined if $\mathscr{U}^{k+1}$ were not a cover of X_t^{k+1}. □

The generalization to the L^p requires some additional notations, as well as the same additional hypothesis that we add to make for the L^p de Rham versions of these results.

Definition 4.5.6. Consider the following L^p norm on $\prod_{I\in S_l} AS_t^k(U_I)$:

$$\|f\|_p = \left(\sum_{I\in S_l}\int_{U_I} |f_I|^p \, d\mu^{k+1}\right)^{1/p}$$

The equivalence classes on $\prod_{I\in S_l} AS_t^k(U_I)$ for this norm form the space $L^p C_{AS,t}^{k,l}(X)$.

Proposition 4.5.7. *(Generalized Mayer-Vietoris Sequence for L^p-Alexander-Spanier complex)Let X be a metric measure space. Fix $t_0 > 0$. Let $\mathscr{U}$ be a cover of X such that, for all $k > 0$, $\mathscr{U}^k$ is a cover of $X_{t_0}^k$. Assume that there exists $B > 0$ such that $\mu(U_\alpha) < B$ for all α and assume there exists $N > 0$ such that for all $\alpha \in S$, U_α intersects less than N other sets of the cover.*

The following sequence is then exact for all $t \le t_0$:

$$0 \to L^p AS_t^k(X,\mathscr{U}) \xrightarrow{r} L^p C_{AS,t}^{k,0}(X,\mathscr{U}) \xrightarrow{\check{\delta}} L^p C_{AS,t}^{k,1}(X,\mathscr{U}) \to \dots$$

Proof. Apart from the hypothesis we had to make in order to prove the metric case, the proof of this version is analoguous to the proof of Proposition 4.5.3. □

4.6 De Rham theorems

In the following sections, we state several de Rham type Theorems. They all use the same principle, which is, given a metric space or a Riemannian manifold, to construct a cover of the space considered such that both the Poincarré Lemma and the Mayer-Vietoris sequence hold, and then deduce an isomorphism between the Čech cohomology and the de Rham or Alexander-Spanier cohomology, in an application of Theorem 4.4.4. We begin by recalling the proof for the classical de Rham Theorem, as given by Bott and Tu [4]. The other results are refinements of this one.

We start by defining the Čech-de Rham complex, which links the Čech cohomology of a Riemannian manifold to its de Rham cohomology.

Definition 4.6.1. Let M be a Riemannian manifold and $\mathscr{U} = \{U_l\}_{l\in\mathbb{N}}$ be an open cover of M. Let $I \in S_l$ be a multi-indice $(i_0, i_1, ...i_l)$ with $i_0 < i_1 < ... < i_l$ and let $U_I = \cap_{n=0}^{l} U_{i_n}$. The *Čech-de Rham complex* associated to M and $\mathscr{U}$ is the bicomplex $(C_{DR}^{k,l}(M,\mathscr{U}), d, \check{\delta})$ where the spaces are :

$$C_{DR}^{k,l}(M,\mathscr{U}) = \prod_{I\in S_l} \Omega^k(U_I)$$

for all $0 \leq k, l < \infty$. Thus the general form of an element ω of $C_{DR}^{k,l}(M,\mathscr{U})$ is given by

$$\omega = \prod_{I\in S_l} \omega_I \text{ , with } \omega_I \in \Omega^k(U_I).$$

The morphisms of this bicomplex are the usual Čech differential

$$\check{\delta} : C_{DR}^{k,l}(M,\mathscr{U}) \to C_{DR}^{k,l+1}(M,\mathscr{U})$$

given by

$$(\check{\delta}\omega)_{(\alpha_0,...,\alpha_{l+1})} = \sum_{i=0}^{l+1} (-1)^i \omega_{(\alpha_0,...\hat{\alpha}_i,...\alpha_{l+1})}$$

and a modified version of the derivative on forms :

$$d : C_{DR}^{k,l}(M, \mathscr{U}) \to C_{DR}^{k+1,l}(M, \mathscr{U}).$$

It is the classical derivative components by components, to which we add a sign depending on the degree of the cochain :

$$d(\omega) = (-1)^{k+l} \prod_{I \in S_l} d\omega_I.$$

The sign is needed in order to have

$$D \circ D = (d + \check{\delta}) \circ (d + \check{\delta}) = 0.$$

Indeed, without the factor $(-1)^{k+l}$, the two differentials commutes, and we have

$$(d + \check{\delta}) \circ (d + \check{\delta}) = 2d \circ \delta$$

whereas with it, since the operator d appearing in the sum $d \circ \delta + \delta \circ d$ differ by one degree, they have opposite signs and cancel each others.

Definition 4.6.2. A cover $\mathscr{U}$ of a manifold M is a *good cover* if every intersection U_I is diffeomorphic to $\mathbb{R}^n$.

Theorem 4.6.3. *Let M be a Riemannian manifold with a good cover $\mathscr{U}$. Then the Čech cohomology of M for the cover $\mathscr{U}$ is isomorphic to the de Rham cohomology of M.*

Proof. Because each open of the cover and each intersection $U_I = U_{\alpha_0} \cap \ldots \cap U_{\alpha_l}$ are diffeomorphic to an open ball of $\mathbb{R}^n$, the Poincaré Lemma 4.3.2 applies, and thus the columns of the bicomplexes are exact. The Mayer-Vietoris sequence 4.5.1 applies as well and so the rows are exact. We can thus apply theorem 4.4.4 on the Čech-de Rham complex. □

This result can be extended to the Čech cohomology of the constant presheaf over M, without dependance on the cover. Not only does every Riemannian manifold have a good cover, but good covers allow to compute the Čech cohomology.

Proposition 4.6.4. *Riemannian manifolds have good cover.*

Proof. For any point $p \in M$, there exists a strongly convex neighborhood. These strongly convex neighbourhoods are diffeomorphic to $\mathbb{R}^n$ and intersections of such neighbourhoods are again strongly convex. Thus, in order to build a good cover, one can choose a convex neighbourhood for each point of M. This already gives a good cover, from which we can choose a locally finite subcover. Each intersections of sets from this cover is diffeomorphic to $\mathbb{R}^n$. □

Proposition 4.6.5. *Let M be a Riemannian manifold. Good covers are cofinal in the set of all covers of M, that is, for every cover $\mathscr{V}$ of M, there exist a refinement $\mathscr{U}$ of $\mathscr{V}$ which is a good cover.*

Proof. We can use the same idea than in the proof of proposition 4.6.4. Around each point, we can build a strongly convex neighbourhood. This neighbourhood can in particular be chosen to be included in an open of $\mathscr{V}$. The resulting cover is thus a refinement of $\mathscr{V}$. □

This last proposition has the following straightforward consequence :

Corollary 4.6.6. *Let M be a Riemannian manifold. The Čech cohomology and the de Rham cohomology of M are isomorphic.*

Proof. Since good covers are cofinal to the family of all covers, the limit can be computed using only good covers. For any good cover $\mathscr{U}$ of M, the Čech cohomology $\check{H}^*(M,\mathscr{U})$ is isomorphic to the de Rham cohomology of M, and so is the limit $\check{H}^*(M) = \varinjlim \check{H}^*(M,\mathscr{U})$. □

This idea of proof works as well for the metric Alexander-Spanier cohomology. Let us define the Čech-Alexander-Spanier complex.

Definition 4.6.7. Let X be a metric space and $\mathscr{U}$ an open cover of X. The *Čech-Alexander-Spanier* complex of X of size t for the cover $\mathscr{U}$ is given by $(C^{k,l}_{AS,t}(X,\mathscr{U}), \delta_k, \check{\delta}_l)_{k,l\in\mathbb{N}}$, where

$$C^{k,l}_{AS,t}(X,\mathscr{U}) = \prod_{I\in S_l} AS^k_t(U_I)$$

and the differentials are

$$\delta_k : C^{k,l}_{AS,t}(X,\mathscr{U}) \to C^{k+1,l}_{AS,t}(X,\mathscr{U})$$

and

$$\check{\delta}_l : C^{k,l}_{AS,t}(X,\mathscr{U}) \to C^{k,l+1}_{AS,t}(X,\mathscr{U}).$$

The differential δ_k is the usual Alexander-Spanier differential, component by component, with a corrected sign :

$$\delta(f) = (-1)^{k+l} \prod_{I\in S_l} \delta(f_I).$$

The differential $\check{\delta}_l$ is the alternating difference on the components, as defined earlier for the Čech complex and the Mayer-Vietoris sequence.

Recall that the injectivity radius $r_i(p)$ of a point p in a Riemannian manifold is the supremum of all radius $r > 0$ such that the exponential map is a diffeomorphism from $B(p,r)$ to its image in $\mathbb{R}^n$. The convexity radius, in turn, is the maximal radius for which the ball $B(p,r)$ is geodesically convex, that is, for any two points $x, y \in B(p,r)$, there exists a unique minimizing geodesic which links x and y and is cointained in $B(p,r)$. At any given point, the convexity radius is smaller than the injectivity radius.

Definition 4.6.8. In the context of our work, we will say that a complete, orientable Riemannian manifold M has *bounded geometry* if

1. it has a lower bound on its convexity radius : $r_c(p) > c_M$ for all $p \in M$;
2. for any $r \leq c_M$ and any point $p \in M$, there is a bilipschitz diffeomorphism between $B(p,r)$ and the unit ball of $\mathbb{R}^n$;
3. the measure Vol_g is quasi-regular.

Concerning the convexity radius of a Riemannian manifold, we have the following result :

Proposition 4.6.9. *Let (M,g) be a complete Riemannian manifold. Assume that its sectional curvature satisfies $K \leq a^2$. Then for any $\epsilon > 0$, we have :*

$$c_M \geq \begin{cases} \frac{i_M}{2} - \epsilon & , \textit{if } a = 0 \\ \min\{\frac{i_M}{2}, \frac{\pi}{2a}\} - \epsilon & , \textit{if } a > 0, \end{cases}$$

where c_M is the convexity radius and i_M is the injectivity radius.

A proof of this proposition is given in [36], Theorem 5.3, p. 169.

We mainly use the properties of a manifold with bounded geometry to build uniform covers, using open, convex balls.

Definition 4.6.10. An open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in S}$ of M is *uniform* if

1. there exists $t > 0$ such that $\mathscr{U}^{k+1}$ is also a cover of M_t^{k+1} for all $k > 0$;
2. each U_α meets only a bounded number of other sets in the cover;
3. the diameter and measure of U_α is bounded above and below by strictly positive numbers.

Property 4.6.11. *Let (M, g) be an orientable Riemannian manifold with bounded geometry. Then there exists a uniform cover $\mathscr{U}$ of M which use only convex sets.*

Proof. The proof relies on the same idea as Proposition 3.6.4. Fix $t > 0$ and $\epsilon > 0$ such that $\epsilon + t$ is smaller than the convexity radius of M. Choose an ϵ-net M_0 of M. Let $k \in \mathbb{N}$ such that $k \cdot \epsilon > \epsilon + t$. We will now prove that the cover

$$\mathscr{U} = \{B(x, \epsilon + t)\}_{x \in M_0}$$

is a uniform cover of M. First, since M_0 is an ϵ-net, $\mathscr{U}$ is a cover of M and $\mathscr{U}^{k+1}$ is a cover of M_t^{k+1}. To show that $\mathscr{U}^k$ is a cover of M^k, consider $\Delta = (x_0, \ldots x_k) \in M_t^{k+1}$. Since $\mathscr{U}$ is a cover of M, there exists $x \in M_0$ such that the first vertex of Δ, x_0, is contained in $B(x, \epsilon)$. Because $\mathrm{diam}(\Delta) < t$, we have

$$x_i \in B(x, \epsilon + r) \text{ , for all} i \leq k + 1.$$

Secondly, we show that $\mathscr{U}$ is uniformly locally finite. Let $m > 0$ such that $m \cdot \epsilon \geq \epsilon + t$. Let $p \in M_0$ and consider $B(p, 2m\epsilon) \subset M$. Since Vol_g is quasi-regular, we have

$$\mathrm{Vol}(B(p, 2m\epsilon)) \leq V(2m\epsilon).$$

On another hand, the collection of balls $B(x, \epsilon/2)$ with $x \in M_0$ is disjoint. As a consequence,

$$\sum_{\substack{x \in M_0 \\ x \in B(p, 2m\epsilon)}} \mathrm{Vol}(B(x, \epsilon/2)) \leq V(2m\epsilon).$$

Also, this sum has this lower bound :

$$\mathrm{card}(\{x \in M_0 \mid x \in B(p, 2m\epsilon)\} \cdot v(\epsilon/2) \leq \sum_{\substack{x \in M_0 \\ x \in B(p, 2m\epsilon)}} \mathrm{Vol}(B(x, \epsilon/2)).$$

All of this add to this estimation :

$$\mathrm{card}(\{x \in M_0 \mid x \in B(p, 2m\epsilon)\}) \leq \frac{v(\epsilon/2)}{V(2m\epsilon)}.$$

Now, if for some $x \in M_0$, $B(x, m\epsilon) \cap B(p, m\epsilon)$ is non-empty, then $B(x, m\epsilon)$ is contained in $B(p, 2m\epsilon)$. Thus we have the following bound, which is independant of p :

$$\mathrm{card}(\{x \in M_0 \mid B(x, m\epsilon) \cap B(p, m\epsilon) \neq \emptyset\}) \leq \frac{\nu(\epsilon/2)}{V(2m\epsilon)}.$$

Since we chose m such that this bound applies also to balls of radius $\epsilon + t$, this concludes the proof that $\mathscr{U}$ is uniformly locally finite. The balls of $\mathscr{U}$ are convex since their radius are smaller than the convexity radius. □

Note that in this proof, we use only the fact that the measure is quasi-regular in order to build a uniform cover. The fact that the cover uses only convex sets depends on the fact that the convexity radius is bounded.

We can now state one of the main results of this thesis.

Theorem 4.6.12. *Let M be a Riemannian manifold with bounded geometry. Then there exists a uniform cover $\mathscr{U}$ of M and $t_0 > 0$ such that for all $t \leq t_0$, the Alexander-Spanier cohomology of X at scale t is isomorphic to the Čech cohomology X for the cover $\mathscr{U}$.*

Proof. Since M has bounded geometry, we can construct a uniform cover M. Choose $\epsilon > 0$ and $t_0 > 0$ such that $\epsilon + t_0$ is smaller than the radius of convexity of M. The construction in the proof of Property 4.6.11 gives us a uniform cover $\mathscr{U}$ consisting of balls of radius $\epsilon + t_0$.

Consider now the Čech-Alexander-Spanier complex of $(M, \mathscr{U})$ for some scale $t > 0$. Since the sets of $\mathscr{U}$ are bilipschitz to an open ball of $\mathbb{R}^n$, the Poincaré Lemma 4.3.13 applies to each U_I for any value of t. This ensures that the columns of the Čech-Alexander-Spanier complex are exact.

Since $\mathscr{U}$ is uniformly locally finite and since $\mathscr{U}^{k+1}$ is a cover of M_t^{k+1} for all $t \leq t_0$, the Mayer-Vietoris sequence applies and the rows are exact. The theorem 4.4.4 thus says that, if $t \leq t_0$, the Alexander-Spanier cohomology of M is isomorphic to the Čech cohomology of $(M, \mathscr{U})$. □

We can sum up this section by formulating the following corollary, which does not need a particular cover in its statement.

Corollary 4.6.13. *Let M be a Riemannian manifold with bounded geometry.Then there exist $t_0 > 0$ such that for all $t \leq t_0$ the Alexander-Spanier cohomology of M at scale t is isomorphic to the de Rham cohomology of M.*

Proof. This result is simply the combination of the theorems 4.6.3 and 4.6.12. Indeed, if M has bounded geometry, we can construct a uniform cover using convex balls. Note that the convex balls are diffeomorphic to a ball of $\mathbb{R}^n$, so in particular, the cover is a good cover, which means both theorems applies to the same cover. □

4.7 L^p de Rham theorems

We can extend Corollary 4.6.13 to the L^p versions of Alexander-Spanier cohomology and de Rham cohomology. For that, we need to include integrability conditions to the corresponding bicomplexes. In the L^p Čech-de Rham complex, instead of looking at cochains whose components are smooth, we consider cochains which are bounded for the L^p norm. Fortunately, the definitions of the spaces have already been given before, and the differentials are only slightly modified versions of those from the classical case.

Definition 4.7.1. Let M be a Riemannian manifold and $\mathscr{U}$ an open cover of M. The spaces $L^p C_{DR}^{k,l}(M,\mathscr{U})$ we consider have already been defined in order to describe the Mayer-Vietoris sequence. The differential

$$\check{\delta} : L^p C_{DR}^{k,l}(M,\mathscr{U}) \to L^p C_{DR}^{k,l+1}(M,\mathscr{U})$$

is the alternating difference, exactly as before. In the other direction, the only difference with the classical case is that we consider the weak differential :

$$d_k : L^p C_{DR}^{k,l}(M,\mathscr{U}) \to L^p C_{DR}^{k+1,l}(M,\mathscr{U}).$$

The spaces $L^p C_{DR}^{k,l}(M,\mathscr{U})$ together with the differentials δ and d as described above form the *L^p Čech-de Rham bicomplex.*

Theorem 4.7.2. *Let M be a Riemannian manifold with bounded geometry. Then there exists a cover $\mathscr{U}$ of M such that the L^p de Rham cohomology of M and the L^p Čech cohomology of M and $\mathscr{U}$ are isomorphic.*

Proof. There are two differences with the classical version of this result. We need the sets of the cover and their intersections to be bilipschitz diffeomorphic to an open ball of $\mathbb{R}^n$, and we need the cover to be uniformly locally finite. This is achieved by the type of uniform cover that we used for Theorem 4.6.12. □

For the L^p version of the Čech-Alexander-Spanier complex, we only have to restrict the type of cochain that we consider. The definition of the differentials stay the same.

Definition 4.7.3. The *L^p Čech-Alexander-Spanier bicomplex* of a metric measure space (X,ρ,μ) for a cover $\mathscr{U}$ of X is given by the spaces $L^p C_{AS}^{k,l}(X,\mathscr{U})$ together with the differentials $\check{\delta}$ and δ described earlier.

Theorem 4.7.4. *Let M be a Riemannian manifold with bounded geometry. Then there exist a cover $\mathscr{U}$ and a scale $t_0 > 0$ such that for all $t < t_0$, the L^p Alexander-Spanier cohomology at scale t of M is isomorphic to the L^p Čech cohomology of M and $\mathscr{U}$.*

Theorem 4.7.2 and 4.7.4 are both the combination of the corresponding Poincaré Lemma and Mayer-Vietoris Sequence. These two theorems allow to formulate the following result.

Corollary 4.7.5. *Let M be a orientable Riemannian manifold with bounded geometry. Then there exist t_0 such that for all $t \le t_0$, $L^p H^*_{DR}(M)$ and $L^p H^*_{AS,t}(M)$ are isomorphic.*

Proof. This is a direct consequence of theorem 4.7.2 and theorem 4.7.4.

□

$$\begin{tikzcd}
0 \arrow[r] & L^p\Omega^2(M) \arrow[r, "r"] & L^p C^{2,0}_{DR}(M,\mathcal{U}) \arrow[r, "\check{\delta}"] & L^p C^{2,1}_{DR}(M,\mathcal{U}) \arrow[r, "\check{\delta}"] & L^p C^{2,2}_{DR}(M,\mathcal{U}) \\
0 \arrow[r] & L^p\Omega^1(M) \arrow[r, "r"] \arrow[u, "d"] & L^p C^{1,0}_{DR}(M,\mathcal{U}) \arrow[r, "\check{\delta}"] \arrow[u, "d"] & L^p C^{1,1}_{DR}(M,\mathcal{U}) \arrow[r, "\check{\delta}"] \arrow[u, "d"] & L^p C^{1,2}_{DR}(M,\mathcal{U}) \arrow[u, "d"] \\
0 \arrow[r] & L^p\Omega^0(M) \arrow[r, "r"] \arrow[u, "d"] & L^p C^{0,0}_{DR}(M,\mathcal{U}) \arrow[r, "\check{\delta}"] \arrow[u, "d"] & L^p C^{0,1}_{DR}(M,\mathcal{U}) \arrow[r, "\check{\delta}"] \arrow[u, "d"] & L^p C^{0,2}_{DR}(M,\mathcal{U}) \arrow[u, "d"] \\
 & & L^p\check{C}^0(M,\mathcal{U}) \arrow[r, "\check{\delta}"] \arrow[u, "i"] & L^p\check{C}^1(M,\mathcal{U}) \arrow[r, "\check{\delta}"] \arrow[u, "i"] & L^p\check{C}^2(M,\mathcal{U}) \arrow[u, "i"] \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Figure 4.4: Bicomplex for the Čech and de Rham L_p-cohomology

Example 4.7.6. We give a simple counter-example for Corollary 4.6.13. Consider $\mathbb{R} \setminus \{0\}$. Its de Rham cohomology is isomorphic to the cohomology of two points. However, for any $t > 0$, the Alexander-Spanier cohomology at scale t is the cohomology of a point. This does however not contradict our result, since $\mathbb{R} \setminus \{0\}$ is not complete. In order for Corollary 4.6.13 to apply, we would need to find a cover $\mathcal{U}$ of $\mathbb{R} \setminus \{0\}$ such that each element of the cover has the de Rham and scaled Alexander-Spanier cohomology of a point, and such that $\mathcal{U}^k$ is a cover of $(\mathbb{R} \setminus \{0\})^k_t$. Because $(t/3, -t/3)$ is in $(\mathbb{R} \setminus \{0\})^2_t$, this means that for some $U \in \mathcal{U}$, we would have $(t/3, -t/3) \in U^2$, and thus $t/3$ and $-t/3 \in U$, which would be in contradiction with the requirement that U has trivial de Rham cohomology.

4.8 The Compact case

For compact Riemannian manifolds, it is well-known that the L^p and L^π cohomologies coincide with the usual cohomologies, assuming $1/p_{k+1} - 1/p_k \le 1/n$ for the L^π case. The theorems we described it the previous sections gives an easy proof of this fact. For the L^p case, we can consider a finite uniform cover of M. The Čech cohomology of this cover is isomorphic to its L^p Čech cohomology. Applying the Theorem 4.7.4 to the same cover shows that the L^p Alexander-Spanier is isomorphic to the Čech cohomology.

Corollary 4.8.1. *Let M be a compact Riemannian manifold. The L^p de Rham cohomology and the classical de Rham cohomology of M are isomorphic. There exists $t_0 > 0$ such that for all $t \leq t_0$, the metric and the L^p Alexander-Spanier cohomology at scale t are isomorphic to the de Rham cohomology of M.*

Proof. If M is a compact Riemannian manifold, we can find a finite cover satisfying the conditions of Corollary 4.7.5. Indeed, the convexity radius of M has a lower bound and we can choose ϵ and t such that $\epsilon + t$ is smaller than the convexity radius at any point of M. Consider the cover of M given by all balls of radius ϵ. By compactness, we can choose a finite subcover of balls centered at points $\{x_0, x_1, ...x_m\}$. The cover $\{B(x_i, \epsilon + t)\}_{0 \leq i \leq m}$ is uniform. Thus, the L^p de Rham and the L^p Alexander-Spanier cohomologies are isomorphic to the L^p Čech cohomology. However, for a finite cover consisting of sets of finite measure, any Čech cochain is L^p integrable, and thus the L^p Čech cohomology is the same as the usual Čech cohomology. □

This situation extends further to the L^π case, both for the de Rham and Alexander-Spanier cohomology. We give a proof specifically for the Alexander-Spanier cohomology by defining the L^π *Čech-Alexander-Spanier complex.* Let $\pi = \{p_k\}_{k \in \mathbb{N}}$ be a sequence of numbers with $1 \leq p_k \leq \infty$ and let $\mathscr{U}$ be an open cover of M. The differentials of the L^π Čech-de Rham bicomplex of $(M, \mathscr{U})$ are the same as those of the L^p version and the spaces are defined as follows, where $k = i + j$:

$$L^\pi C^{i,j}_{AS,t}(M,\mathscr{U}) = \{\alpha \in L^{p_k} C^{i,j}_{AS,t}(M,\mathscr{U}) \mid \delta\alpha \in L^{p_{k+1}} C^{i+1,j}_{AS,t}(M,\mathscr{U}) \text{ and } \check{\delta}\alpha \in L^{p_{k+1}} C^{i,j+1}_{AS,t}(M,\mathscr{U})\}$$

With this definition, a cochain in the L^π Čech-Alexander-Spanier complex an element

$$\alpha \in \prod_{i+j=k} C^{i,j}_{AS,t}(M,\mathscr{U})$$

such that $\|\alpha\|_{p_k} < \infty$ and $\|D\alpha\|_{p_{k+1}} < \infty$.

Theorem 4.8.2. *Let M be a compact Riemannian manifold and let π be a non-decreasing sequence. There exists t_0 such that for all $t \leq t_0$, the L^π Alexander-Spanier cohomology of M at scale t is isomorphic to the Čech cohomology of M.*

Proof. In this situation, we can build a finite uniform cover $\mathscr{U}$ of M such that $M^k_{t_0}$ is covered by $\mathscr{U}^k$. In consequence, any cochain $\alpha \in L^\pi C^{i,j}_{AS,t}(M,\mathscr{U})$ is a finite direct sum of components

$\alpha_I \in L^\pi AS_t^i(U_I)$, with $I \in S_j$. This means that we can check the integrability conditions on the δ and $\check{\delta}$ preimages of α component by component. Assume that $\check{\delta}(\alpha) = 0$. Proposition 4.5.7 gives a cochain $K\alpha$ such that

$$\check{\delta}(K\alpha) = \alpha \text{ and } \|K\alpha\|_{p_k} < \infty.$$

Since each U_I has finite measure, $\|(K\alpha)_I\|_{p_k} < \infty$ implies that $\|(K\alpha)_I\|_{p_{k-1}} < \infty$ if $p_{k-1} \leq p_k$, and thus, that $\|K\alpha\|_{p_{k-1}} < \infty$ as well. The reasoning for the δ preimage of α is similar. This means that for all $t \leq t_0$, the rows and columns of the L^π Čech-Alexander-Spanier bicomplex are exact, if π is a non-decreasing sequence. This means that the L^π Alexander-Spanier cohomology of M is isomorphic to the L^π Čech cohomology of $(M, \mathscr{U})$, which is in turn isomorphic to the Čech cohomology of M.

□

5 Scale independance and consequences

We discuss results of scale independance. This type of result is important, since it relates the behavior of scaled Alexander-Spanier cohomology at large and small scales. In particular, if the restriction operator $r_{t_0 t_1} : H^*_{AS,t_0}(X) \to H^*_{AS,t_1}(X)$ is an isomorphism for all value of t_0 and t_1, the initial and asymptotic L^p cohomologies coincide. In the context of this thesis, for a class of Riemannian manifolds with such a property, it means that, no matter the scale considered, the L^p Alexander-Spanier is a quasi-isometry invariant and isomorphic to the L^p de Rham cohomology. This is a way to give of proof of the quasi-isometry invariance of L^p de Rham cohomology.

5.1 Self-similar spaces

We have a first naive example by considering $\mathbb{R}^n$. Let $t_0, t_1 > 0$. There is a bijection between $AS^k_{t_0}(\mathbb{R}^n)$ and $AS^k_{t_1}(\mathbb{R}^n)$: given a cochain $\alpha \in AS^k_{t_0}(\mathbb{R}^n)$, we define a cochain $F_{t_0 t_1}(\alpha) \in AS^k_{t_0}(\mathbb{R}^n)$ by setting

$$F_{t_0 t_1}\alpha(\Delta) = \alpha\left(\frac{t_0}{t_1}\Delta\right).$$

The inverse $F_{t_1 t_0}$ is given by

$$F_{t_1 t_0}\beta(\Delta) = \beta\left(\frac{t_1}{t_0}\Delta\right).$$

These are chain maps, and thus induce mappings in cohomology :

$$F_{t_0 t_1} : H^*_{t_0}(\mathbb{R}^n) \to H^*_{t_1}(\mathbb{R}^n).$$

These homomorphisms verify the following :

$$\mathrm{id} = F_{t_1 t_0} \circ F_{t_0 t_1} \text{ and } \mathrm{id} = F_{t_0 t_1} \circ F_{t_1 t_0}.$$

This means that F_{t_0} and F_{t_1} are isomorphisms, and thus that $\mathbb{R}^n$ has constant scaled Alexander-Spanier cohomology. We can also check that F_{t_0} and F_{t_1} are bounded operators in the L^p and L^{qp} cohomology, and thus the result of scale independance extends to these cases.

We can make this example a litte more interesting by considering subsets of $\mathbb{R}^n$ which are self-similar. Let $C \subset [0,1]$ be the usual ternary Cantor set. It has the property that $C = f_0(C) \cup f_1(C)$, with $f_0(x) = 1/3 \cdot x$ and $f_1(x) = f_0(x) + 2/3$. We define a extended version of the Cantor set as follows :

$$C^* = \bigcup_{k \in \mathbb{N}} 3^k \cdot C.$$

With this definition, C^* has the following property :

$$C^* = 3^k \cdot C^* \text{ , for all } k \in \mathbb{Z}.$$

Using the same reasoning as for $\mathbb{R}^n$, we deduce that the scaled Alexander-Spanier cohomology of C^* is periodic with respect to the scale :

$$H^*_{AS,t}(C^*) = H^*_{AS,3t}(C^*).$$

However, we cannot deduce that the restriction operator $r_{3t,t} : H^*_{AS,3t}(C^*) \to H^*_{AS,t}(C^*)$ is an isomorphism, and thus this does not give us a mean to compute the initial and asymptotic cohomology of C^*.

This kind of example can be built using many different subsets of $\mathbb{R}^n$, independantly of the dimension. The main hypothesis here is the existence of a bijective contraction $f : X \to X$, with $f(x) = \lambda \cdot x$ for all $x \in X$, with $|\lambda| < 1$.

5.2 Scale independance for CAT(0) spaces

In the proof of Propositions 4.3.13 and 4.3.14, we use the hypothesis that U is bounded in order for it to have the cohomology of a point at large scales. Without this condition, we cannot state that it has the cohomology of a point, but we still have the following result :

Proposition 5.2.1. *Let U be a convex subset of $\mathbb{R}^n$. For all $t_0 \geq t_1$, and for all $k \geq 0$, the restriction operator $r_{t_0 t_1} : AS^k_{t_0}(U) \to AS^k_{t_1}(U)$ induces an isomorphism in scaled Alexander-Spanier cohomology and we have :*

$$H^*_{AS,t_0}(U) = H^*_{AS,t_1}(U).$$

This is also true for L^p cohomology, but if the measure of U is not finite, it is not necessary verified for L^{qp} cohomology.

Proposition 5.2.2. *Let U be a convex subset of $\mathbb{R}^n$. For all $t_0 \geq t_1$, and for all $k \geq 0$ and $1 \leq p \leq \infty$, the restriction operator $r_{t_0 t_1} : L^p AS^k_{t_0}(U) \to L^p AS^k_{t_1}(U)$ induces an isomorphism in L^p Alexander-Spanier cohomology and we have :*

$$L^p H^*_{AS,t_0}(U) = L^p H^*_{AS,t_1}(U).$$

These two propositions imply in particular that $\mathbb{R}^n$ has constant metric and L^p Alexander-Spanier cohomology, and that in consequence, its initial cohomologies $H^*_{AS,0}(\mathbb{R}^n)$ and $L^p H^*_{AS,0}(\mathbb{R}^n)$ coincide with its asymptotic cohomology $H^*_{AS,\infty}(\mathbb{R}^n)$ and $L^p H^*_{AS,\infty}(\mathbb{R}^n)$.

The barycenter is also well-defined in the hyperbolic space $\mathbb{H}^n$. Consider the hyperboloid model of $\mathbb{H}^n$ and let $\Delta = (x_0, \dots x_k)$ be a simplex of $\mathbb{H}^n$. The euclidean barycenter of Δ is given by $s_b(\Delta) \in \mathbb{R}^{n+1}$ and the barycenter of Δ is the projection of $s_b(\Delta)$ on $\mathbb{H}^n$ from 0. In consequence, just as $\mathbb{R}^n$, the Alexander-Spanier cohomology of the hyperbolic space $\mathbb{H}^n$ is constant with regard to the scale.

In the more general case where a Riemannian manifold M is also a CAT(0), we use the following notion of center for an arbitrary subset $U \subset M$.

Definition 5.2.3. The *radius* of a set U is defined as follow :

$$r_U = \inf\{r > 0 \mid \text{ there exists } x \in M \text{ such that } U \subset B(x, r)\}.$$

If $c \in M$ is such that $U \subset B(c, r_U)$, c is called a *center* of U.

We have the following property, which is particularly useful in our situation :

Property 5.2.4. *Let X be a CAT(0) space and let $U \subset X$ be a bounded subset. Then there exists a unique center c_U.*

This center is not necessary in U, but it is not a concern for us. A proof of this property can be read in Bridson and Haefliger [6].

Recall that a complete, simply connected Riemannian manifold with non-positive sectional curvature is CAT(0). This sort of manifold is called Hadamard manifold.

Theorem 5.2.5. *Let M be a Hadamard manifold with quasi-regular measure. The L^p Alexander-Spanier cohomology of M is independant of the scale :*

$$L^p H^*_{AS,t_0}(M) = L^p H^*_{AS,t_1}(M) \text{ for all } t_0, t_1 \in (0,\infty).$$

Proof. The proof relies on the construction of a center for simplices of any size in M, with the property that the associated subdivision contracts the size of simplices. Let $(x_0, \dots x_k) \in M^{k+1}$ be a k-simplex in M. The unique center of Δ given by Property 5.2.4 is written $s_r(\Delta)$. This gives us a subdivision of Δ :

$$\sigma_r(\Delta) = \mathrm{Cone}_{s_r(\Delta)} \sigma_r(\partial\Delta).$$

Consider the following inequalities, where $\Delta' = (y_0, \dots y_k)$ is a simplex that appears in the chain $\sigma_r(\Delta)$:

$$\mathrm{diam}(\Delta') \leq r_\Delta \leq \sqrt{\frac{n}{2(n+1)}}\mathrm{diam}(\Delta).$$

The second inequality is Theorem 1.3 in [7]. We demonstrate the first inequality by induction on the dimension of Δ.The diameter of a simplex is equal to the maximum of the distance between any two of its vertices :

$$\mathrm{diam}(\Delta') = \max\{\rho(y_i, y_j) \mid 0 \leq i, j \leq k\}.$$

In the case of a simplex obtained by subdivision, we can distinguish two different sets of edges :

$$\mathrm{diam}(\Delta') = \max\{\rho(s_r(\Delta), y_i) \mid 0 \leq i \leq k\} \cup \{\rho(y_i, y_j) \mid 0 \leq i, j \leq k \text{ and } y_i, y_j \neq s_r(\Delta)\}.$$

If an edge contains $s_r(\Delta)$, then by definition, $\rho(s_r(\Delta, y_i)) \leq r_\Delta$. We use the induction hypothesis to show that the edges in the second set verify the inequality : let τ be a face of Δ. Then $\mathrm{diam}(\tau_i) \leq r_\tau$ for any τ_i in the subdivision of τ. On another hand, the radius of $\{x_0, \dots \hat{x}_i, \dots x_k\}$ is smaller, by definition, than the radius of $\{x_0, \dots x_k\}$. This implies that $\mathrm{diam}(\tau_i) \leq r_\Delta$. This concludes the proof that the subdivision based on the center of the simplex is a contraction in term of diameter. From this point, we can refer to the proof of Proposition 4.3.14 to obtain a homotopy equivalence between $L^p AS^*_{t_0}(M)$ and $L^p AS^*_{t_1}(M)$. □

This theorem implies that the initial and asymptotic L^p Alexander-Spanier cohomology coincide. We can sum everything up in the following corollary :

Corollary 5.2.6. *The L^p de Rham cohomology is a quasi-isometry invariant for simply connected Riemannian manifolds with bounded geometry and non-positive curvature.*

5.3 Uniformly contractible spaces

Another class of manifolds for which a result of scale independance is possible is uniformly contractible Riemannian manifolds with bounded geometry.

Definition 5.3.1. A metric space is *uniformly contractible* if there exists a function $R : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for every $x_0 \in X$, there exists a homotopy $F : [0,1] \times B(x_0, r) \to B(x_0, R(r))$ between the identity and the constant map x_0.

Remark 5.3.2. Uniformly contractible spaces are contractible, but the converse is not true.

As a motivation, we recall the result quasi-isometry invariance given by S. Ducret [13] :

Theorem 5.3.3. *Assume that M and N are uniformly contractible Riemannian manifolds with bounded geometry and of dimension n. Assume that M and N are quasi-isometric. Then if q and p satisfy either*

$$1 < q, p < \infty \text{ and } 0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$$

or

$$1 \leq q, p < \infty \text{ and } 0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{n},$$

we have the following isomorphisms for L^{qp} de Rham cohomology and reduced L^{qp} de Rham cohomology :

$$L^{qp} H^*_{DR}(M) = L^{qp} H^*_{DR}(N) \text{ and } L^{qp} \overline{H}^*_{DR}(M) = L^{qp} \overline{H}^*_{DR}(N).$$

P. Pansu sketches the following result of scale invariance in his preprint [33].

Theorem 5.3.4. *Let M be a uniformly contractible Riemannian manifold with bounded geometry. Then for any $t, t' > 0$, there is a homotopy equivalence between $AS^*_t(M)$ and $AS^*_{t'}(M)$, i.e. the Alexander-Spanier L_p cohomology does not depend on the size of the cochains.*

The consequence of this result is the same as for CAT(0) manifolds : as the restriction is an isomorphism for all scales, the asymptotic L^p Alexander-Spanier cohomology and the initial L^p Alexander-Spanier cohomology of a uniformly contractible Riemannian manifold with bounded geometry coincide. This gives an alernative proof that the L^p de Rham cohomology is a quasi-isometry invariant for this class of Riemannian manifold.

A Open questions

There are a number of questions that are open to further investigations.

Considering Theorem 0.3.1, we can consider the case of a complete non-compact Riemannian manifold with finite volume.

Question A.0.5. *Let (M,g) be a complete non-compact Riemannian manifold such that $Vol_g(M) < \infty$, and let π be a non-decreasing sequence. Do we have*

$$H^*_{DR}(M) = L^\pi H^*_{AS,0}(M) \text{ ?}$$

Because it has finite volume, there are inclusions of L^p cochain spaces we would not be able to use in the general case. In turn, we do not have a bound on the (strong) injectivity radius, which we use for Theorem 0.2.1. What could happen would be to have an isomorphism between initial L^π Alexander-Spanier cohomoloy and de Rham cohomology, but not in general for scaled L^π Alexander-Spanier cohomology.

The general case remains also open :

Question A.0.6. *Let (M,g) be a uniformly contractible Riemannian manifold with bounded geometry and let π be a non-increasing sequence. Do we have*

$$L^\pi H^*_{AS,t}(M) = L^\pi H^*_{DR}(M)$$

for all $t \in (0,\infty)$?

This is true in the case where π is a constant sequence, which is the L^p case. This question could be solved by using a different method of proof, for example by defining a pairing between Alexander-Spanier cochains and de Rham cochains. Another possibility would be to restrict to

L^{∞} L^{π} cohomology and compare to simplicial L^{π} cohomology in the case where M accepts a triangulation with bounded geometry.

Recall that Ducret shows that the $L^{q,p}$ de Rham cohomology is a quasi-isometry invariant for Riemannian manifolds with bounded geometry that are uniformly contractible, for $1 < q \leq p < \infty$ and $1/p - 1/p \leq 1/n$.

Question A.0.7. *In the situation of Ducret's result, can we show that the $L^{q,p}$ Alexander-Spanier cohomology is a quasi-isometry invariant?*

The proof given by Ducret relies on showing that the $L^{q,p}$ de Rham cohomology coincides with the simplicial $L^{q,p}$ cohomology and the coarse $L^{q,p}$ cohomology defined by Roe, the latter being itself quasi-invariant through quasi-isometry. A positive answer to Question A.0.7 would give a more direct proof of Ducret's result.

A last question can be considered, following the discussion on Ducret's work :

Question A.0.8. *Given a graph X, is there a link between the asymptotic L^{π} Alexander-Spanier cohomology and the coarse L^{π} cohomology as defined by Roe?*

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Curriculum Vitae

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