

# Localized orthogonal decomposition method for the wave equation with a continuum of scales

Assyr Abdulle<sup>1</sup> and Patrick Henning<sup>1</sup>

## Abstract

In this paper we propose and analyze a new multiscale method for the wave equation. The proposed method does not require any assumptions on space regularity or scale-separation and it is formulated in the framework of the Localized Orthogonal Decomposition (LOD). We derive rigorous a priori error estimates for the  $L^2$ -approximation properties of the method, finding that convergence rates of up to third order can be achieved. The theoretical results are confirmed by various numerical experiments.

**Keywords** finite element, wave equation, numerical homogenization, multiscale method, localized orthogonal decomposition

**AMS subject classifications** 35L05, 65M60, 65N30, 74Q10, 74Q15

## 1 Introduction

This work is devoted to the linear wave equation in a heterogeneous medium with multiple highly varying length-scales. We are looking for an unknown wave function  $u^\epsilon$  that fulfills the equation

$$\begin{aligned} \partial_{tt}u^\epsilon(x, t) - \nabla \cdot (a^\epsilon(x)\nabla u^\epsilon(x, t)) &= F(x, t) && \text{in } \Omega \times (0, T], \\ u^\epsilon(x, t) &= 0 && \text{on } \partial\Omega \times [0, T], \\ u^\epsilon(x, 0) = f(x) \quad \text{and} \quad \partial_t u^\epsilon(x, 0) &= g(x) && \text{in } \Omega. \end{aligned} \tag{1}$$

Here,  $\Omega$  denotes the medium,  $[0, T] \subset \mathbb{R}^+$  the relevant time interval,  $a^\epsilon$  the wave speed,  $F$  a source term and  $f$  and  $g$  the initial conditions for the wave and its time derivative respectively. The parameter  $\epsilon$  is an abstract parameter which simply indicates that a certain quantity is subject to rapid variations on a very fine scale (relative to the extension of  $\Omega$ ).  $\epsilon$  can be seen as a measure for the minimum wave length of these variations. However, we stress that we do not assign a particular value or meaning to  $\epsilon$  in this work. Due to the fast variations in the data functions, which take place at a scale that is very small compared to the total size of the medium, these problems are typically referred to as a *multiscale problems*. Equations such as (1) with a multiscale character arise in various fields such as material sciences, geophysics, seismology or oceanography. For instance, the propagation and reflection of seismic waves can be used to determine the structure and constitution of subsurface formations. In particular, it is necessary in order to locate petroleum reservoirs in earth's crust.

---

<sup>1</sup>ANMC, Section de Mathématiques, École polytechnique fédérale de Lausanne, 1015 Lausanne, Switzerland, Assyr.Abdulle@epfl.ch, Patrick.Henning@epfl.ch

Trying to solve the multiscale wave equation (1) with a direct computation, using e.g. finite elements or finite differences, exceeds typically the possibilities of today's super computers. The reason is that the computational mesh needs to resolve all variations of the coefficient matrix  $a^\varepsilon$ , which leads to extremely high dimensional solution spaces and hence linear systems of tremendous size that need to be solved at every time step.

In order to tackle this issue, numerical homogenization can be applied. The term numerical homogenization refers to a wide set of numerical methods that are based on replacing the multiscale problem (1) by an effective/upscaled/homogenized equation which is of the same type as the original equation, but which has no longer multiscale properties (the fine scale is "averaged out"). Hence, it can be solved in lower dimensional spaces with reduced computational costs. The obtained approximations yield the effective macroscopic properties of  $u^\varepsilon$  (i.e. they are good  $L^2$ -approximations of  $u^\varepsilon$ ). Multiscale methods that were specifically designed for the wave equation, can be e.g. found in [4, 14, 23, 29, 32]. In Section 5.4 we give a detailed overview on these approaches.

In this paper, we will present a new multiscale method for the wave equation which does neither require structural assumptions such as a scale separation nor does it require regularity assumptions on  $a^\varepsilon$ . We will not exploit any higher space regularity than  $H^1$ . Furthermore, it is not necessary to solve expensive global elliptic fine scale problems in a pre-process (sometimes referred to as the 'one-time-overhead', cf. [22, 21, 29]). Our method is based on the following consideration: the  $L^2$ -projection  $P_{L^2}$  of the (unknown) exact solution  $u^\varepsilon$  into a coarse finite element space is assumed to be a good approximation to an (unknown) homogenized solution. Furthermore, the  $L^2$ -projection  $P_{L^2}(u^\varepsilon)$  can be well approximated in a low dimensional finite element space. If we can derive an equation for  $P_{L^2}(u^\varepsilon)$ , all computations can be performed in the low dimensional space and are hence cheap. This approach fits into the framework of the Localized Orthogonal Decomposition (LOD) initially proposed in [27] and further developed in [20, 17]. The idea of the framework is to decrease the dimension of a high dimensional finite element space by splitting it into the direct sum of a low dimensional space with a high  $H^1$ -approximation properties and a high dimensional remainder space with negligible information. The splitting is based on an orthogonal decomposition with respect to an energy scalar product. In this work we will pick up this concept, since the remainder space in the splitting is nothing but the kernel of the  $L^2$ -projection.

The general setting of this paper is established in Section 2. In Section 3 we motivate the method and in Section 4 we introduce the space discretization that is required for formulating the method in a rigorous way. Our main results are stated in Section 5 and proved in Section 6. Finally, numerical experiments confirming our theoretical results are presented in Section 7.

## 2 Wave equation

In the following, we consider the wave equation (1) in weak formulation, i.e. we seek  $u^\varepsilon \in L^2(0, T; H_0^1(\Omega))$  and  $\partial_{tt}u^\varepsilon \in L^2(0, T; H^{-1}(\Omega))$  such that for all  $v \in H_0^1(\Omega)$  and a.e.  $t > 0$

$$\begin{aligned} \langle \partial_{tt}u^\varepsilon(\cdot, t), v \rangle + (a^\varepsilon \nabla u^\varepsilon(\cdot, t), \nabla v)_{L^2(\Omega)} &= (F(\cdot, t), v)_{L^2(\Omega)}, \\ (u^\varepsilon(\cdot, 0), v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)}, \\ (\partial_t u^\varepsilon(\cdot, 0), v)_{L^2(\Omega)} &= (g, v)_{L^2(\Omega)}. \end{aligned} \tag{2}$$

Here, the dual pairing is understood as  $\langle L, v \rangle = L(v)$  for  $L \in H^{-1}(\Omega)$  and  $v \in H_0^1(\Omega)$ . In the following, we make use of the shorthand notation  $L^p(H^s) := L^p(0, T; H^s(\Omega))$  for  $1 \leq p \leq \infty$  and  $0 \leq s \leq 1$ . In order to guarantee the existence of a unique solution of the system (1), we make the following assumptions:

- (H0) •  $\Omega \subset \mathbb{R}^d$ , for  $d = 1, 2, 3$ , denotes a convex bounded Lipschitz domain with a piecewise polygonal boundary;
- the data functions fulfill  $F \in L^2(0, T; L^2(\Omega))$ ,  $f \in H_0^1(\Omega)$  and  $g \in L^2(\Omega)$ ;
  - the matrix-valued function  $a^\varepsilon \in [L^\infty(\Omega)]_{\text{sym}}^{d \times d}$  that describes the propagation field is symmetric and it is uniformly bounded and positive definite, i.e.  $a^\varepsilon \in \mathcal{M}(\alpha, \beta, \Omega)$  for  $\beta \geq \alpha > 0$ . Here, we denote

$$\mathcal{M}(\alpha, \beta, \Omega) := \{a \in [L^\infty(\Omega)]_{\text{sym}}^{d \times d} \mid \alpha |\xi|^2 \leq a(x) \xi \cdot \xi \leq \beta |\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and almost all } x \in \Omega\}. \quad (3)$$

Under assumptions listed in (H0) there exists a unique weak solution  $u^\varepsilon$  of the wave equation (1) with  $\partial_t u^\varepsilon \in L^2(0, T; L^2(\Omega))$ . Furthermore,  $u^\varepsilon$  is regular in time, in the sense that  $u^\varepsilon \in C^0(0, T; H_0^1(\Omega))$  and  $\partial_t u^\varepsilon \in C^0(0, T; L^2(\Omega))$ . This result can be e.g. found in [25, Chapter 3].

In addition to the above assumptions, we also implicitly assume that the wave speed  $a^\varepsilon$  has rapid variations on a very fine scale which need to be resolved by an underlying fine grid. The dimension of the resulting finite element space (for the spatial discretization) is hence very large. The method proposed in the subsequent sections aims to reduce the computational cost that is associated with solving the discretized wave equation in this high dimensional finite element space.

In order to simplify the notation, we define

$$b^\varepsilon(v, w) := \int_{\Omega} a^\varepsilon \nabla v \cdot \nabla w \quad \text{for } v, w \in H_0^1(\Omega). \quad (4)$$

### 3 Motivation - Numerical homogenization by $L^2$ -projection

In this section we motivate a multiscale method for the wave equation and discuss the framework of our approach. All the subsequent discussion will be later rigorously justified by a general convergence proof. We are interested in finding a *homogenized* or *upscaled* approximation of  $u^\varepsilon$ . In engineering applications this can be a function describing the macroscopically measurable properties of  $u^\varepsilon$  and from an analytical perspective it can be defined as a suitable limit of  $u^\varepsilon$  for  $\varepsilon \rightarrow 0$  (see Section 5.3 below for more details).

Since  $u^\varepsilon$  is a continuous function in  $t$ , we restrict our considerations to a fixed time  $t$ . Hence, we leave out the time dependency in the notation and denote e.g.  $u^\varepsilon = u^\varepsilon(\cdot, t)$ .

Let  $\mathcal{T}_H$  denote a given coarse mesh and let  $V_H \subset H_0^1(\Omega)$  denote a corresponding coarse finite dimensional subspace of  $H_0^1(\Omega)$  that is sufficiently accurate to obtain accurate  $L^2$ -approximations. To quantify what we mean by "sufficiently accurate", let  $P_H$  denote the  $L^2$ -projection of  $H_0^1(\Omega)$  on  $V_H$ , i.e. for  $v \in H_0^1(\Omega)$  the projection  $P_H(v) \in V_H$  fulfills

$$\int_{\Omega} P_H(v) w_H = \int_{\Omega} v w_H \quad \text{for all } w_H \in V_H. \quad (5)$$

We assume that

$$\|u^\varepsilon - P_H(u^\varepsilon)\|_{L^2(\Omega)} \leq \delta_H,$$

where  $\delta_H$  is a given small tolerance. Let us denote  $u_H := P_H(u^\varepsilon) \in V_H$ . Obviously, the  $L^2$ -projection will average out all small oscillations that cannot be seen on the coarse grid  $\mathcal{T}_H$  (in this sense the projection homogenizes  $u^\varepsilon$ ). By definition,  $u_H$  is the best approximation of

$u^\varepsilon$  in  $V_H$  with respect to the  $L^2$ -norm. Next, we want to find a macroscopic equation that is fulfilled by  $u_H$ .

Since  $\nabla u_H$  does not approximate  $\nabla u^\varepsilon$ , we are interested in a corrector  $Q(u_H)$ , such that

$$\int_{\Omega} a^\varepsilon (\nabla u_H + \nabla Q(u_H)) \cdot \nabla v_H = \int_{\Omega} a^\varepsilon \nabla u^\varepsilon \cdot \nabla v_H \quad \text{for all } v_H \in V_H,$$

or in a symmetric formulation

$$\int_{\Omega} a^\varepsilon (\nabla u_H + \nabla Q(u_H)) \cdot (\nabla v_H + \nabla Q(v_H)) = \int_{\Omega} a^\varepsilon \nabla u^\varepsilon \cdot (\nabla v_H + \nabla Q(v_H)) \quad (6)$$

for all  $v_H \in V_H$ . A suitable corrector operator  $Q$  needs to fulfill two properties:

1.  $Q(u_H)$  must be in the kernel of the  $L^2$ -projection  $P_H$  in order to preserve the  $L^2$ -best-approximation property

$$\int_{\Omega} u^\varepsilon v_H = \int_{\Omega} u_H v_H = \int_{\Omega} (u_H + Q(u_H)) v_H \quad \text{for all } v_H \in V_H.$$

2. It must incorporate the oscillations of  $a^\varepsilon$ . A natural way to achieve this is to make the ansatz

$$\int_{\Omega} a^\varepsilon (\nabla u_H + \nabla Q(u_H)) \cdot \nabla v_h = 0,$$

where the test function  $v_h$  should be in  $H_0^1(\Omega)$ , but with the constraint  $v_h \in \ker(P_H)$ . The constraint is necessary to make the problem well posed (solution space and test function space are identical).

In summary, we have the following strategy if  $u^\varepsilon$  is a known function: find  $u_H \in V_H$  that fulfills equation (6) and where for a given  $v_H \in V_H$  the corrector  $Q(v_H) \in \ker(P_H)$  solves  $\int_{\Omega} a^\varepsilon (\nabla v_H + \nabla Q(v_H)) \cdot \nabla v_h = 0$  for all  $v_h \in \ker(P_H)$ . Observe that this  $u_H$  fulfills indeed  $u_H = P_H(u^\varepsilon)$  as desired, because (by equation (6)) the function  $e := u^\varepsilon - u_H - Q(u_H)$  is in the kernel of  $P_H$ . Hence for all  $v_H \in V_H$

$$\int_{\Omega} u_H v_H = \int_{\Omega} (u_H + Q(u_H)) v_H = \int_{\Omega} (u_H + Q(u_H) + e) v_H = \int_{\Omega} u^\varepsilon v_H,$$

which means just  $u_H = P_H(u^\varepsilon)$ . Consequently, we also have the estimate

$$\|u^\varepsilon - u_H\|_{L^2(\Omega)} \leq \delta_H. \quad (7)$$

The only remaining problem is that we do not know the term  $\int_{\Omega} a^\varepsilon \nabla u^\varepsilon \nabla v_H$  on the right hand side of (6). However, we know that

$$\int_{\Omega} a^\varepsilon \nabla u^\varepsilon \nabla v = \int_{\Omega} F v - \int_{\Omega} \partial_{tt} u^\varepsilon v.$$

If the solution  $u^\varepsilon$  is sufficiently regular then  $\partial_{tt} u^\varepsilon$  is well approximated by  $\partial_{tt} u_H = P_H(\partial_{tt} u^\varepsilon)$  in the sense of (7).

This suggests to replace  $\partial_{tt} u^\varepsilon$  by  $\partial_{tt} u_H$  and to solve the approximate problem to find  $\bar{u}_H \in V_H$  with

$$\int_{\Omega} a^\varepsilon (\nabla \bar{u}_H + \nabla Q(\bar{u}_H)) \cdot (\nabla v_H + \nabla Q(v_H)) = \int_{\Omega} F v_H - \int_{\Omega} \partial_{tt} \bar{u}_H v_H \approx \int_{\Omega} a^\varepsilon \nabla u^\varepsilon \nabla v_H$$

for all  $v_H \in V_H$ .

Note that the above presented strategy is not yet a ready-to-use method, since the exact computation of the corrector  $Q$  involves global fine scale problems. In order to overcome this difficulty, a localization of  $Q$  is required together with a suitable fine scale discretization. The final method is presented in the Section 5. Before we can formulate the method, we introduce a suitable fully discrete space-discretization.

## 4 Spatial discretization

In this section we propose a space discretization for localizing the fine scale computations in the previously described ansatz. For that purpose, we make use of the tools of the Localized Orthogonal Decomposition (LOD) that were introduced in [27] (see also [13, 17, 20, 18, 26] for related works).

The spatial discretization involves two discretization levels. On the one hand, we have a coarse mesh on  $\Omega$  that is denoted by  $\mathcal{T}_H$ .  $\mathcal{T}_H$  consists either of conforming shape regular simplicial elements or of conforming shape regular quadrilateral elements. The elements are denoted by  $K \in \mathcal{T}_H$  and the coarse mesh size  $H$  is defined as the maximum diameter of an element of  $\mathcal{T}_H$ . On the other hand we have a fine mesh that is denoted by  $\mathcal{T}_h$ . It also consists of conforming and shape regular elements. Furthermore, we assume that  $\mathcal{T}_h$  is obtained from an arbitrary refinement of  $\mathcal{T}_H$ , with the additional requirement that  $h \leq (H/2)$ , where  $h$  denotes the maximum diameter of an element of  $\mathcal{T}_h$ . In practice we usually have  $h \ll H$ . In particular,  $\mathcal{T}_h$  needs to be fine enough to capture all the oscillations of  $a^\varepsilon$ . In contrast, the coarse mesh is only required to provide accurate  $L^2$ -approximations.

For  $\mathcal{T} = \mathcal{T}_H, \mathcal{T}_h$  we denote

$$\begin{aligned} P_1(\mathcal{T}) &:= \{v \in C^0(\omega) \mid \forall K \in \mathcal{T}, v|_K \text{ is a polynomial of total degree } \leq 1\}, \\ Q_1(\mathcal{T}) &:= \{v \in C^0(\omega) \mid \forall K \in \mathcal{T}, v|_K \text{ is a polynomial of partial degree } \leq 1\}. \end{aligned} \quad (8)$$

With this, we define the classical coarse Lagrange finite element space  $V_H$  by  $V_H := P_1(\mathcal{T}_h) \cap H_0^1(\Omega)$  for a simplicial mesh and by  $V_H := Q_1(\mathcal{T}_h) \cap H_0^1(\Omega)$  for a quadrilateral mesh. The fine scale space  $V_h$  is defined in the same way.

Subsequently, we will make use of the notation  $a \lesssim b$  that abbreviates  $a \leq Cb$ , where  $C$  is a constant that can depend on  $d, \Omega, \alpha, \beta$  and interior angles of the coarse mesh, but not on the mesh sizes  $H$  and  $h$ . In particular it does not depend on the possibly rapid oscillations in  $a^\varepsilon$ . We write  $a \lesssim_T b$  if  $C$  is allowed to further depend on  $T$  and the data functions  $F, f$  and  $g$ .

The set of the interior Lagrange points (interior vertices) of the coarse grid  $\mathcal{T}_H$  is denoted by  $\mathcal{N}_H$ . For each node  $z \in \mathcal{N}_H$  we let  $\Phi_z \in V_H$  denote the corresponding nodal basis function that fulfills  $\Phi_z(z) = 1$  and  $\Phi_z(y) = 0$  for all  $y \in \mathcal{N}_H \setminus \{z\}$ .

In the next step, we define the kernel of the  $L^2$ -projection (5) restricted to  $V_h$  in a slightly alternative way. Recall that this kernel was required as the solution space for the corrector problems discussed in Section 3. However, from the computational point of view it is more suitable to not work with the  $L^2$ -projection directly, since it involves to solve a system of equations in order to verify if an element is in the kernel. For that reason, we subsequently express  $\ker(P_H|_{V_h})$  equivalently by means of a weighted Cl  ment-type quasi-interpolation operator  $I_H$  (cf. [9]) that is defined by

$$I_H : H_0^1(\Omega) \rightarrow V_H, \quad v \mapsto I_H(v) := \sum_{z \in \mathcal{N}_H} v_z \Phi_z \quad \text{with } v_z := \frac{(v, \Phi_z)_{L^2(\Omega)}}{(1, \Phi_z)_{L^2(\Omega)}}. \quad (9)$$

With that, we define

$$W_h := \ker(I_H|_{V_h}).$$

Indeed the space  $W_h$  is the previously discussed kernel of the  $L^2$ -projection. This claim can be easily verified: if  $I_H(v_h) = 0$  for an element  $v_h \in V_h$ , then we have by the definition of  $I_H$  that  $(v_h, \Phi_z)_{L^2(\Omega)} = 0$  for all  $z \in \mathcal{N}_H$ . Since  $\Phi_z$  is just the nodal basis of  $V_H$  we have  $(v_h, \Phi_H)_{L^2(\Omega)} = 0$  for any  $\Phi_H \in V_H$ . Hence  $W_h = \ker(I_H|_{V_h}) \subset \ker(P_H|_{V_h})$ . In particular, we have the splitting  $V_h = \text{im}(P_H|_{V_h}) \oplus \ker(P_H|_{V_h}) = V_H \oplus W_h$ .

Recall that the optimal corrector  $Q_{h,\Omega} : V_H \rightarrow W_h$  (in the sense of Section 3) can be now formulated as: find  $Q_{h,\Omega}(v_H) \in W_h$  with

$$b^\varepsilon(v_H + Q_{h,\Omega}(v_H), w_h) = 0 \quad \text{for all } w_h \in W_h, \quad (10)$$

where  $b^\varepsilon(\cdot, \cdot)$  is defined in (4). However, finding  $Q_{h,\Omega}(v_H)$  involves a problem in the whole fine space  $V_h$  and is therefore very expensive. For this purpose, we wish to localize the corrector  $Q_{h,\Omega}$  by element patches.

For  $k \in \mathbb{N}$ , we define patches  $U_k(K)$  that consist of a coarse element  $K \in \mathcal{T}_H$  and  $k$ -layers of coarse elements around it. More precisely  $U_k(K)$  is defined iteratively by

$$\begin{aligned} U_0(K) &:= K, \\ U_k(K) &:= \cup \{T \in \mathcal{T}_H \mid T \cap U_{k-1}(K) \neq \emptyset\} \quad k = 1, 2, \dots \end{aligned} \quad (11)$$

Practically, we will later see that we only require small values of  $k$  (typically  $k = 1, 2, 3$ ).

With that, we define the localized corrector operator in the following way:

**Definition 4.1** (Localized Correctors). For  $k \in \mathbb{N}$ ,  $K \in \mathcal{T}_H$  and  $U_k(K)$  defined according to (11), we define the localized version of  $\text{kern}(P_H|_{V_h})$  by

$$W_h(U_k(K)) := \{w_h \in W_h \mid w_h = 0 \text{ in } \Omega \setminus U_k(K)\}.$$

The localized version of the operator (10) can be constructed in the following way. First, for  $v_H \in V_H$  find  $Q_{h,k}^K(v_H) \in W_h(U_k(K))$  with

$$\int_{U_k(K)} a^\varepsilon \nabla Q_{h,k}^K(v_H) \cdot \nabla w_h = - \int_K a^\varepsilon \nabla v_H \cdot \nabla w_h \quad \text{for all } w_h \in W_h(U_k(K)). \quad (12)$$

Then, the global approximation of  $Q_{h,k}$  is defined by

$$Q_{h,k}(v_H) := \sum_{K \in \mathcal{T}_H} Q_{h,k}^K(v_H). \quad (13)$$

Observe that if  $k$  is large enough so that  $U_k(K) = \Omega$  for all  $K \in \mathcal{T}_H$  (a case that is only useful for the analysis), we have  $Q_{h,k} = Q_{h,\Omega}$ , where  $Q_{h,\Omega}$  is the corrector operator introduced in (10).

**Remark 4.2** (Splittings of  $V_h$ ). The space that is spanned by the image of  $(I + Q_{h,k})$  is given by

$$V_{H,k}^{\text{ms}} := \{v_H + Q_{h,k}(v_H) \mid v_H \in V_H\}. \quad (14)$$

Furthermore, we denote  $V_{H,\Omega}^{\text{ms}} := \{v_H + Q_{h,\Omega}(v_H) \mid v_H \in V_H\}$  for the optimal corrector. This gives us the following splittings of  $V_h$ :

$$\begin{aligned} V_h &= V_H \oplus W_h, & \text{where } V_H &\perp W_h \text{ w.r.t. } (\cdot, \cdot)_{L^2(\Omega)}, \\ V_h &= V_{H,\Omega}^{\text{ms}} \oplus W_h, & \text{where } V_{H,\Omega}^{\text{ms}} &\perp W_h \text{ w.r.t. } b^\varepsilon(\cdot, \cdot), \\ V_h &= V_{H,k}^{\text{ms}} \oplus W_h. \end{aligned}$$

Beside the operator  $I_H$  that we defined in (9) other choices of interpolation operators (such as the classical Clément interpolation) are possible to construct splittings  $V_h = V_{H,k}^{\text{ms}} \oplus W_h$  with  $V_{H,k}^{\text{ms}} \perp_{b^\varepsilon(\cdot, \cdot)} W_h$ . If the operator fulfills various standard properties (like interpolation error estimates,  $H^1$ -stability, etc.; cf. [18] for an axiomatic list) the space  $V_{H,k}^{\text{ms}}$  will have

similar approximation properties as the multiscale space that we use in this contribution. However, we note that the particular Clément-type interpolation operator from (9) yields the  $L^2$ -orthogonality  $V_H \perp W_h$  which is typically not the case for other operators. This is central in our approach. In this paper we particularly exploit this feature to show that we obtain higher order convergence rates under the assumption of additional regularity. We also note that the Lagrange-interpolation fails to yield good approximations (cf. [20]).

**Remark 4.3.** Observe that the solutions  $Q_{h,k}(v_H)$  of (12) are well defined by the Lax-Milgram theorem. Furthermore, it is shown that solutions such as  $Q_{h,k}(v_H)$  (with localized source term) decay with exponential speed to zero outside of the support of the source term (cf. [27, 17]). More precisely, we will later see that we have an estimate of the type  $\|\nabla(Q_{h,k} - Q_{h,\Omega})(v_H)\|_{L^2(\Omega)} \lesssim k^{d/2}\theta^k \|\nabla v_H\|_{L^2(\Omega)}$  for a generic constant  $0 < \theta < 1$ . Hence, we have exponential convergence in  $k$  and small values for  $k$  (typically  $k = 2, 3$ ) can be used to get accurate approximations of  $Q_{h,\Omega}$ . For small values of  $k$ , the local problems (12) are cheap to solve, they can be solved in parallel and  $Q_{h,k}(\Phi_z)$  is only locally supported for every nodal basis function  $\Phi_z \in V_H$ .

## 5 Multiscale Methods and error estimates

Based on the discretization and the correctors defined in Section 4, we present semi-discrete multiscale method for the wave equation and state a corresponding a priori error estimate. As an example of a time-discretization, we also present a Crank-Nicolson realization of the method and state a fully discrete space-time error estimate. In order to abbreviate the notation, we define effective/macroscopic bilinear forms for  $v_H, w_H \in V_H$  by

$$\begin{aligned} b_{H,k}(v_H, w_H) &:= b^\varepsilon(v_H + Q_{h,k}(v_H), w_H + Q_{h,k}(w_H)) \quad \text{and} \\ (v_H, w_H)_{H,k} &:= (v_H + Q_{h,k}(v_H), w_H + Q_{h,k}(w_H))_{L^2(\Omega)}, \end{aligned}$$

where  $b^\varepsilon$  is defined in (4).

### 5.1 Semi-discrete multiscale method for the wave equation

We can now formulate the method.

**Definition 5.1** (Semi-discrete multiscale method). Let  $k \in \mathbb{N}$  denote the localization parameter that determines the patch size  $U_k(K)$  for  $K \in \mathcal{T}_H$  (according to (11)) and hence also determines the localized corrector operator  $Q_{h,k}$ . The semi-discrete approximation  $u_{H,k} \in H^2(0, T; V_H)$  (of  $u^\varepsilon$  in  $L^2$ ) solves the following system for all  $v_H \in V_H$  and  $t > 0$

$$\begin{aligned} (\partial_{tt} u_{H,k}(\cdot, t), v_H)_{H,k} + b_{H,k}(u_{H,k}(\cdot, t), v_H) &= (F(\cdot, t), v_H + Q_{h,k}(v_H))_{L^2(\Omega)}, \\ (u_{H,k} + Q_{h,k}(u_{H,k}))(\cdot, 0) &= \pi_{H,k}^{\text{ms}}(f), \\ \partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))(\cdot, 0) &= P_{H,k}^{\text{ms}}(g), \end{aligned} \tag{15}$$

with the projections  $\pi_{H,k}^{\text{ms}}$  and  $P_{H,k}^{\text{ms}}$  defined in (17) and (18) below.

Recall  $V_{H,k}^{\text{ms}}$  be the space defined in (14). We subsequently define two elliptic projections for  $v \in L^2(0, T; H_0^1(\Omega))$  and one  $L^2$ -projection.

1. The projection  $\pi_h : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; V_h)$  is given by:  
find  $\pi_h(v) \in L^2(0, T; V_h)$  with

$$b^\varepsilon(\pi_h(v)(\cdot, t), w) = b^\varepsilon(v(\cdot, t), w) \quad \text{for all } w \in V_h, \quad \text{for almost every } t \in (0, T). \tag{16}$$

2. The projection  $\pi_{H,k}^{\text{ms}} : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; V_{H,k}^{\text{ms}})$  is given by:  
find  $\pi_{H,k}^{\text{ms}}(v)(\cdot, t) \in V_{H,k}^{\text{ms}}$  with

$$b^\varepsilon(\pi_{H,k}^{\text{ms}}(v)(\cdot, t), w) = b^\varepsilon(v(\cdot, t), w) \quad \text{for all } w \in V_{H,k}^{\text{ms}}, \quad \text{for almost every } t \in (0, T). \quad (17)$$

For  $U_k(K) = \Omega$ , we denote by  $\pi_{H,\Omega}^{\text{ms}}$  the above projection mapping from  $L^2(0, T; H_0^1(\Omega))$  to  $L^2(0, T; V_{H,\Omega}^{\text{ms}})$ .

3. The  $L^2$ -projection  $P_{H,k}^{\text{ms}} : L^2(0, T; H_0^1(\Omega)) \rightarrow L^2(0, T; V_{H,k}^{\text{ms}})$  is given by:

find  $P_{H,k}^{\text{ms}}(v)(\cdot, t) \in V_{H,k}^{\text{ms}}$  with

$$(P_{H,k}^{\text{ms}}(v)(\cdot, t), w)_{L^2(\Omega)} = (v(\cdot, t), w)_{L^2(\Omega)} \quad \text{for all } w \in V_{H,k}^{\text{ms}}, \quad \text{for a.e. } t \in (0, T). \quad (18)$$

**Remark 5.2** (Existence and uniqueness). If assumptions (H0) are fulfilled, the system (15) has a unique solution. This result directly follows from standard ODE theory, after a reformulation of (15) into a (finite) system of first order ODE's with constant coefficients and applying Duhamel's formula. Due to the Sobolev embedding theorems, we have  $u_{H,k} \in C^1([0, T]; V_H)$ . If additionally  $f \in C^0(0, T; L^2(\Omega))$ , we even get  $u_{H,k} \in C^2([0, T]; V_H)$ . The corrector  $Q_{h,k}(u_{H,k})$  inherits this time regularity.

Note that the time-regularity does not allow any conclusions on uniform bounds for e.g.  $\|\partial_t u_{H,k}\|_{L^\infty(0,T;H^1(\Omega))}$ . Bounding this term by the data functions requires more assumptions. In order to guarantee optimal convergence rates, we need more assumptions on the regularity of the data functions. Our assumptions will slightly differ from the regularity assumptions for the standard finite element method for the wave equation, which requires higher space regularity. Our assumptions on the other hand involve a higher time regularity to obtain optimal convergence rates. The following (graded) regularity assumptions refer to the solution  $u^\varepsilon$  of (2). The parameter  $s \in \{0, 1\}$  in (H2) and (H3) below is specified when one of the assumptions is used. Exploiting the notation  $L^p(H_0^s) := L^p(0, T; H_0^s(\Omega))$  (with  $H_0^0 := L^2$ ), we introduce the following (graded) assumptions.

(H1) We have  $\partial_t u^\varepsilon \in L^1(H_0^1)$ .

(H2) We have  $\partial_t u^\varepsilon \in L^1(H_0^1)$ ;  $\partial_{tt} u^\varepsilon, F \in L^\infty(H_0^s)$  and  $\partial_{ttt} u^\varepsilon, \partial_t F \in L^1(H_0^s)$ .

(H3) We have  $\partial_t u^\varepsilon \in L^\infty(H_0^1)$ ;  $\partial_{ttt} u^\varepsilon, \partial_t F \in L^\infty(L^2)$ ;  $\partial_{tt} u^\varepsilon, F \in L^\infty(H_0^s)$ ;  $\partial_{tt} u^\varepsilon \in L^1(H_0^1)$ ;  $\partial_t^4 u^\varepsilon, \partial_{tt} F \in L^1(L^2)$  and  $g \in H_0^1(\Omega)$ .

Assumption (H1) is the minimum assumption for deriving explicit convergence rates in  $H$ . Observe that we do not make any high order space regularity assumptions (i.e. involving  $H^s(\Omega)$  for  $s > 1$ ). This guarantees that the following estimates are independent of the oscillations in  $a^\varepsilon$ . We are now prepared to formulate the first main result of this contribution. In the following,  $C_d$  denotes a (generic) constant that can depend on  $F, f, g, \Omega, \alpha$  and  $\beta$ , but not on  $\varepsilon$ .

**Theorem 5.3** (A priori error estimates for the semi-discrete method).

Assume that (H0) and (H1) with  $\|\partial_t u^\varepsilon\|_{L^1(0,T;H^1(\Omega))} \leq C_d$  (i.e. there holds a uniform bound in  $\varepsilon$ ). By  $\pi_h$  we denote the elliptic projection of  $H_0^1(\Omega)$  on  $V_h$  (cf. (16)). Let  $u^\varepsilon$  denote the exact solution of the wave equation (2) and let  $u_{H,k}$  be the numerically homogenized solution



defined by (15). We let  $s \in \{0, 1\}$ . Then there exists a generic constant  $C_\theta$  (i.e. independent of  $H$ ,  $h$  and  $\varepsilon$ ) such that if  $k \geq (1/2)(2 + s)C_\theta |\ln(H)|$  we have the following a priori error estimates:

If (H2) is fulfilled with  $s = 0$  and if  $\|\partial_t u^\varepsilon\|_{L^1(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(L^2)} + \|\partial_{ttt} u^\varepsilon\|_{L^1(L^2)} \leq C_d$ , we obtain the corrector estimate

$$\|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(0,T;L^2(\Omega))} \lesssim_T H^2 + \delta_{h,1}. \quad (19)$$

If (H2) is fulfilled with  $s = 1$  and if  $\|\partial_t u^\varepsilon\|_{L^1(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_{ttt} u^\varepsilon\|_{L^1(H^1)} \leq C_d$ , we get

$$\|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(0,T;L^2(\Omega))} \lesssim_T H^3 + \delta_{h,1}. \quad (20)$$

If (H3) holds with  $s = 0$  and if

$$\|\partial_t u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(L^2)} + \|\partial_{ttt} u^\varepsilon\|_{L^1(H^1)} + \|\partial_{ttt} u^\varepsilon\|_{L^\infty(L^2)} + \|\partial_t^4 u^\varepsilon\|_{L^1(L^2)} \leq C_d,$$

we get

$$\|\partial_t u^\varepsilon - \partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(L^2)} + \|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(H^1)} \lesssim_T H + \delta_{h,2}. \quad (21)$$

If it holds (H3) with  $s = 1$ ; if the initial value in (15) is picked such that we have  $\partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))(\cdot, 0) = \pi_{H,k}^{\text{ms}}(g)$  and if

$$\|\partial_t u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_{ttt} u^\varepsilon\|_{L^1(H^1)} + \|\partial_{ttt} u^\varepsilon\|_{L^\infty(L^2)} + \|\partial_t^4 u^\varepsilon\|_{L^1(L^2)} \leq C_d,$$

we get

$$\|\partial_t u^\varepsilon - \partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(L^2)} + \|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(H^1)} \lesssim_T H^2 + \delta_{h,2}. \quad (22)$$

Here, the fine scale errors  $\delta_{h,1}$  and  $\delta_{h,2}$  are given by

$$\delta_{h,1} := \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(L^2)} + \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^1(L^2)}$$

and

$$\delta_{h,2} := \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^\infty(L^2)} + \|\partial_{tt} u^\varepsilon - \pi_h(\partial_{tt} u^\varepsilon)\|_{L^1(L^2)} + \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(H^1)},$$

where  $\pi_h$  is the elliptic-projection on  $V_h$ . Note that  $\delta_{h,i}$  will yield optimal orders in  $h$ , if  $u^\varepsilon$  is sufficiently regular.

A proof including refined estimates (i.e. estimates where all dependencies on  $u^\varepsilon$  and  $k$  are worked out in detail) is given in Section 6.

**Remark 5.4.** Concerning Theorem 5.3, we note that the uniform bounds in  $\varepsilon$  are not very restrictive, since they do *not* involve higher order space derivatives (i.e.  $H^s(\Omega)$  with  $s > 1$ ). For instance for  $m \in \mathbb{N}$  and  $s > 1$ , it is well known that typically  $\|\partial_t^m u^\varepsilon\|_{L^2(0,T;H^s(\Omega))} \rightarrow \infty$  for  $\varepsilon \rightarrow 0$  (cf. [10]), whereas (if it exists)  $\|\partial_t^m u^\varepsilon\|_{L^2(0,T;H^1(\Omega))}$  remains uniformly bounded, if the initial data does not trigger rapid time oscillations. In Remark 6.5 we present an example how to guarantee the uniform bounds by making assumptions on the initial data.

## 5.2 Time discrete multiscale method for the wave equation

In this section, for  $J \in \mathbb{N}$ , we let  $\Delta t := \frac{T}{J} > 0$  denote the time step size and we define  $t^n := n\Delta t$  for  $n \in \mathbb{N}$ . In order to propose a time discretization of (15), we introduce some simplifying notation. First, recall that for every coarse interior node  $z \in \mathcal{N}_H$ , we denote the corresponding nodal basis function by  $\Phi_z$ . The total number of these coarse nodes shall be denoted by  $N_H$  and we assume that they are order by some index set, i.e.  $\mathcal{N}_H = \{z_1, \dots, z_N\}$ . With that we define the corresponding stiffness matrix  $S_k \in \mathbb{R}^{N \times N}$  by the entries

$$(S_k)_{ij} := b_{H,k}(\Phi_{z_j}, \Phi_{z_i})$$

and the entries of the corrected mass matrix  $M_k \in \mathbb{R}^{N \times N}$  by

$$(M_k)_{ij} := (\Phi_{z_j}, \Phi_{z_i})_{H,k}.$$

The load vectors  $G_k, \bar{f}_k, \bar{g}_k \in \mathbb{R}^N$  arising in (15) are defined by

$$(G_k)_i(t) := (F(\cdot, t), \Phi_{z_i} + Q_{h,k}(\Phi_{z_i}))_{L^2(\Omega)},$$

$$(\bar{f}_k)_i \text{ is such that } \pi_{H,k}^{\text{ms}}(f) = \sum_{i=1}^N (\bar{f}_k)_i (\Phi_{z_i} + Q_{h,k}(\Phi_{z_i})), \quad (23)$$

$$(\bar{g}_k)_i \text{ is such that } P_{H,k}^{\text{ms}}(g) = \sum_{i=1}^N (\bar{g}_k)_i (\Phi_{z_i} + Q_{h,k}(\Phi_{z_i})), \quad (24)$$

with  $\pi_{H,k}^{\text{ms}}$  being the elliptic projection on  $V_{H,k}^{\text{ms}}$  (see (17)) and  $P_{H,k}^{\text{ms}}$  being the  $L^2$ -projection on  $V_{H,k}^{\text{ms}}$  (see (17)). Hence, we can write (15) as the system: find  $\xi_k(t) \in \mathbb{R}^N$  with

$$M_k \ddot{\xi}_k(t) + S_k \xi_k(t) = G_k(t), \quad \text{for } 0 < t < T$$

and  $\xi_k(0) = \bar{f}_k$  and  $\dot{\xi}_k(0) = \bar{g}_k$ . This yields  $u_{H,k}(\cdot, t) = \sum_{i=1}^N (\xi_k(t))_i \Phi_{z_i}$ .

In order to solve this system we can apply the Newmark scheme.

**Definition 5.5** (Newmark scheme). For  $n \geq 1$ , given initial values  $\xi_k^{(0)} \in \mathbb{R}^N$  and  $\xi_k^{(1)} \in \mathbb{R}^N$ , and given load vectors  $G_k^{(n)} \in \mathbb{R}^N$ , we define the Newmark approximation  $\xi_k^{(n+1)}$  of  $\xi_k(t^{(n+1)})$  iteratively as the solution  $\xi_k^{(n+1)} \in \mathbb{R}^N$  of

$$\begin{aligned} & (\Delta t)^{-2} M_k \left( \xi_k^{(n+1)} - 2\xi_k^{(n)} + \xi_k^{(n-1)} \right) \\ & + \frac{1}{2} S_k \left( 2\hat{\beta} \xi_k^{(n+1)} + (1 - 4\hat{\beta} + 2\hat{\gamma}) \xi_k^{(n)} + (1 + 2\hat{\beta} - 2\hat{\gamma}) \xi_k^{(n-1)} \right) = G_k^{(n)}. \end{aligned}$$

Here,  $\hat{\beta}$  and  $\hat{\gamma}$  are given parameters.

An example for an implicit method is given by the choice  $\hat{\beta} = 1/4$  and  $\hat{\gamma} = 1/2$ , which leads to the classical Crank-Nicolson scheme. Another example is the leap-frog scheme that is obtained for  $\hat{\beta} = 0$  and  $\hat{\gamma} = 1/2$ . The leap-frog scheme is explicit (up to a diagonal mass matrix which can be obtained by mass lumping).

As one possible realization, we subsequently consider the case  $\hat{\beta} = 1/4$  and  $\hat{\gamma} = 1/2$ , i.e. the Crank-Nicolson scheme (see Definition 5.6 below). We state a corresponding a priori error estimate and the numerical experiments in Section 7 are also performed with this method. Before we present the main theorem of this section, let us detail the method by specifying the initial values and the load vectors for the Crank-Nicolson method.

**Definition 5.6** (Fully-discrete Crank-Nicolson multiscale method). As before, let  $k \in \mathbb{N}$  denote the localization parameter that determines the patch size  $U_k(K)$  for  $K \in \mathcal{T}_H$  (according to (11)). The load functions  $G_k, \bar{f}_k, \bar{g}_k \in \mathbb{R}^N$  are defined according to (23) and we denote  $G_k^{(n)} := \frac{1}{2}(G_k t^n + G_k(t^{n-1}))$  for  $t^n = n\Delta t$ ,  $n \geq 1$ . Defining  $\xi_k^{(0)} := \bar{f}_k$  and  $\eta_k^{(0)} := \bar{g}_k$ , the approximation  $(\xi_k^{(n)}, \eta_k^{(n)}) \in \mathbb{R}^N \times \mathbb{R}^N$  in the  $n$ 'th time step is given as the solution of the linear system

$$\left(\frac{\Delta t^2}{4}S_k + M_k\right)\eta_k^{(n)} = \left(M_k - \frac{\Delta t^2}{4}S_k\right)\eta_k^{(n-1)} - \Delta t S_k \xi_k^{(n-1)} + \Delta t G_k^{(n)}$$

and  $\xi_k^{(n)} := \frac{\Delta t}{2}\eta_k^{(n)} + \frac{\Delta t}{2}\eta_k^{(n-1)} + \xi_k^{(n-1)}$ .

With that, the Crank-Nicolson approximation of (15) is defined as the piecewise linear function  $u_{H,\Delta t,k}$  with

$$u_{H,\Delta t,k}(\cdot, t) := \sum_{i=1}^N \left( \frac{t^{n+1} - t}{\Delta t} (\xi_k^{(n)})_i + \frac{t - t^n}{\Delta t} (\xi_k^{(n+1)})_i \right) \Phi_{z_i} \quad \text{for } t \in [t^n, t^{n+1}]. \quad (25)$$

**Remark 5.7.** Existence and uniqueness of  $(\xi_k^{(n)}, \eta_k^{(n)})$  in Definition 5.6 is obvious since the system matrix  $(\frac{\Delta t^2}{4}S_k + M_k)$  has only positive eigenvalues.

In order to obtain the optimal convergence rates with regard to the time step size, we require an additional regularity assumption.

(H4) Let either  $s = 0$  or  $s = 1$ . We have  $\partial_t u^\varepsilon \in L^2(H_0^1)$ ;  $\partial_{tt} u^\varepsilon, F \in L^\infty(H_0^s)$ ;  $\partial_{ttt} u^\varepsilon, \partial_t F \in L^2(H_0^s)$  and  $\partial_t^4 u^\varepsilon \in L^2(L^2)$ . Furthermore, there exists a generic constant  $C_d$  that can depend on  $T, F, f, g, \Omega, \alpha$  and  $\beta$  but not on  $\varepsilon$ , such that

$$\|\partial_t u^\varepsilon\|_{L^2(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(H^s)} + \|\partial_{ttt} u^\varepsilon\|_{L^2(H^s)} + \|\partial_t^4 u^\varepsilon\|_{L^2(L^2)} \leq C_d.$$

With this assumption, we can formulate the optimale error estimate for the Crank-Nicolson version of the multiscale method.

**Theorem 5.8** (A priori error estimates for the Crank-Nicolson fully-discrete method).

Assume that (H0) is fulfilled and let  $s \in \{0, 1\}$ . Beside this, let the notation from Theorem 5.3 hold true and let  $u_{H,\Delta t} := u_{H,\Delta t,k}$  be the fully discrete numerically homogenized approximation as in Definition 5.6. Then there exists a generic constant  $C_\theta$  (i.e. independent of  $H, h$  and  $\varepsilon$ ) such that if  $k \geq (1/2)(s+2)C_\theta |\ln(H)|$  we have the following a priori error estimates:

If (H4) holds with  $s = 0$  we obtain the corrector estimate

$$\max_{0 \leq n \leq J} \|(u^\varepsilon - (u_{H,\Delta t} + Q_{h,k}(u_{H,\Delta t})))(\cdot, t^n)\|_{L^2(\Omega)} \lesssim_T H^2 + \Delta t^2 + \delta_h. \quad (26)$$

If (H4) holds with  $s = 1$  we have

$$\max_{0 \leq n \leq J} \|(u^\varepsilon - (u_{H,\Delta t} + Q_{h,k}(u_{H,\Delta t})))(\cdot, t^n)\|_{L^2(\Omega)} \lesssim_T H^3 + \Delta t^2 + \delta_h. \quad (27)$$

Here, the fine scale error  $\delta_h$  is given by

$$\delta_h := \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^2(0,T;L^2(\Omega))}.$$

The theorem is proved at the end of Section 6.

### 5.3 Relation to the homogenization of the wave equation

In this section we recall some fundamental results concerning the homogenization of the wave equation and relate it to our multiscale method. Furthermore, we present an estimate for the error between homogenized solution and the coarse part of our multiscale approximation. In the following, we let  $Y := [0, 1]^d$  denote the unit cube in  $\mathbb{R}^d$  and we let  $C_{\#}^0(Y)$  denote the set of continuous functions on  $\mathbb{R}^d$  that are  $Y$ -periodic. Furthermore, we denote the space of periodic  $H^1$ -functions by

$$H_{\#}^1(Y) := \overline{C_{\#}^0(Y)}^{\|\cdot\|_{H^1(Y)}}.$$

The essential question of classical homogenization is the following: if  $(a^\varepsilon)_{\varepsilon>0}$  represents a sequence of coefficients and if we consider the corresponding sequence of solutions  $(u^\varepsilon)_{\varepsilon>0}$  (2), does  $u^\varepsilon$  converge in some sense to a  $u^0$  that we can characterize in a simple way. The hope is that  $u^0$  fulfills some equation (the homogenized equation) which is cheap to solve since it does no longer involve multiscale features (which were averaged out in the limit process for  $\varepsilon \rightarrow 0$ ). With the abstract tool of  $G$ -convergence it is possible to answer this question:

**Definition 5.9** ( $G$ -convergence). A sequence  $(a^\varepsilon)_{\varepsilon>0} \subset \mathcal{M}(\alpha, \beta, \Omega)$  (i.e. with uniform spectral bounds in  $\varepsilon$ ) is said to be  $G$ -convergent to  $a^0 \in \mathcal{M}(\alpha, \beta, \Omega)$  if for all  $F \in H^{-1}(\Omega)$  the sequence of solutions  $v^\varepsilon \in H_0^1(\Omega)$  of

$$\int_{\Omega} a^\varepsilon \nabla v^\varepsilon \cdot \nabla v = F(v) \quad \text{for all } v \in H_0^1(\Omega)$$

satisfies  $v^\varepsilon \rightharpoonup v^0$  weakly in  $H_0^1(\Omega)$ , where  $v^0 \in H_0^1(\Omega)$  solves

$$\int_{\Omega} a^0 \nabla v^0 \cdot \nabla v = F(v) \quad \text{for all } v \in H_0^1(\Omega).$$

The following result was obtained in [8, Theorem 3.2]:

**Theorem 5.10** (Homogenization of the wave equation). *Let assumptions (H0) be fulfilled and let the sequence of symmetric matrices  $(a^\varepsilon)_{\varepsilon>0} \subset \mathcal{M}(\alpha, \beta, \Omega)$  be  $G$ -convergent to some  $a^0 \in \mathcal{M}(\alpha, \beta, \Omega)$ . Let  $u^\varepsilon \in L^\infty(0, T; H_0^1(\Omega))$  denote the solution of the wave equation (2). Then it holds*

$$\begin{aligned} u^\varepsilon &\rightharpoonup u^0 \quad \text{weak-}^* \text{ in } L^\infty(0, T, H_0^1(\Omega)), \\ \partial_t u^\varepsilon &\rightharpoonup \partial_t u^0 \quad \text{weak-}^* \text{ in } L^\infty(0, T, L^2(\Omega)) \end{aligned}$$

and where  $u^0 \in L^2(0, T; H_0^1(\Omega))$  is the unique weak solution of the homogenized problem

$$\begin{aligned} \langle \partial_{tt} u^0(\cdot, t), v \rangle + (a^0 \nabla u^0(\cdot, t), \nabla v)_{L^2(\Omega)} &= (F(\cdot, t), v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega) \text{ and } t > 0, \\ (u^0(\cdot, 0), v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega), \\ (\partial_t u^0(\cdot, 0), v)_{L^2(\Omega)} &= (g, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega). \end{aligned} \tag{28}$$

Theorem 5.10 implies that it is sufficient to identify the  $G$ -limit of a sequence of matrices  $(a^\varepsilon)_{\varepsilon>0} \subset \mathcal{M}(\alpha, \beta, \Omega)$  in order to formulate the homogenized problem. Hence, by the definition of  $G$ -convergence, the problem of finding  $a^0$  reduces to a standard linear elliptic problem. However, in many cases it is not possible to construct  $a^0$  explicitly. Explicit formulas are for instance available for locally periodic coefficients  $a^\varepsilon$ , i.e.  $a^\varepsilon(x) = a(x, \frac{x}{\varepsilon})$  for a Caratheodory type matrix-valued function  $a \in [L^\infty(\Omega, C_{\#}^0(Y))]^{d \times d}$  (hence,  $a^\varepsilon$  is  $\varepsilon$ -periodic on a fine scale). By standard theory for elliptic problems (see e.g. [11, 28]), it is well known that  $(a^\varepsilon)_{\varepsilon>0}$  is  $G$ -convergent to a limit  $a^0 \in \mathcal{M}(\alpha, \beta, \Omega)$  that can be expressed by the solutions of so called cell problems. More precisely, the following holds true:

**Conclusion 5.11** (Homogenization for locally periodic structures). Let assumptions (H0) be fulfilled and let  $a^\varepsilon(x) = a(x, \frac{x}{\varepsilon})$  for a function  $a \in [L^\infty(\Omega, C_\#^0(Y))]^{d \times d}$ . Then  $a^\varepsilon$  is  $G$ -convergent to  $a^0 \in \mathcal{M}(\alpha, \beta, \Omega)$  that is given by

$$a_{ij}^0(x) := \int_Y a(x, y)(e_j + \nabla_y w_j(x, y)) \cdot (e_i + \nabla_y w_i(x, y)) dy,$$

and where  $w_i \in L^2(\Omega, \tilde{H}^1(Y))$  is a solution of

$$\int_Y a(x, \cdot) \nabla_y w_i(x, \cdot) \cdot \nabla v = - \int_Y a(x, \cdot) e_i \cdot \nabla v \quad \forall \phi \in \tilde{H}^1(Y)$$

for almost every  $x \in \Omega$ . Hence, after computing/approximating  $a^0$  with the above strategy, problem (28) can be straight forwardly solved in a coarse finite element space.

The next theorem shows that in scenarios where it is not possible to compute or approximate  $a^0$ , we can still approximate  $u^0$  by using our multiscale method.

**Theorem 5.12** (A priori error estimate for the homogenized solution).

Let  $u^\varepsilon$  denote the solution of (2) and assume that (H0) holds. Consider the setting and the notation of Theorem 5.3 and 5.8. In particular, we assume  $k \geq C_\theta |\ln(H)|$  and the fine scale error  $\delta_h$  is given by

$$\delta_h := \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^2(0,T;L^2(\Omega))}.$$

Furthermore, by  $u^0$  we denote the homogenized solution given by (28), by  $u_H := u_{H,k}$  we denote the semi-discrete numerically homogenized solution and by  $u_{H,\Delta t} := u_{H,\Delta t,k}$  the fully-discrete one.

If (H0) and (H1) are fulfilled and if  $\|\partial_t u^\varepsilon\|_{L^1(0,T;H^1(\Omega))} \leq C_d$  for some generic constant  $C_d$  that can depend on  $F, f, g, \Omega, \alpha$  and  $\beta$ , but not on  $\varepsilon$ , then we have the error estimates:

$$\|u^0 - u_H\|_{L^\infty(0,T;L^2(\Omega))} \lesssim_T H + \delta_h + \|u^0 - u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \quad (29)$$

and

$$\max_{0 \leq n \leq J} \|u^0(\cdot, t^n) - u_{H,\Delta t}(\cdot, t^n)\|_{L^2(\Omega)} \lesssim_T H + \Delta t^2 + \delta_h + \|u^0 - u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}. \quad (30)$$

If we replace the elliptic projection  $\pi_{H,k}^{\text{ms}}(f)$  in (15) by the  $L^2$ -projection  $P_{H,k}^{\text{ms}}(f)$ , estimate (29) still remains valid.

The theorem is proved in Section 6. Observe that if we are in the homogenization setting of  $G$ -convergence, i.e. if it holds  $\|u^\varepsilon - u^0\|_{L^\infty(0,T;L^2(\Omega))} \lesssim_T C(\varepsilon) \lesssim H$  with  $C(\varepsilon) \rightarrow 0$ , then estimate (29) reads  $\|u^0 - u_H\|_{L^\infty(0,T;L^2(\Omega))} \lesssim_T H + \delta_h$ . The assumption  $C(\varepsilon) \lesssim H$  excludes that the coarse grid already resolves all microscopic structures (which is an unpractical case that is of no relevant interest).

## 5.4 Survey on other multiscale methods for the wave equation

The number of existing multiscale methods for the wave equation is rather small, compared to the number of multiscale methods that exist for other types of equations. Subsequently we give a short survey on existing strategies to put our method into perspective.

One way of realizing numerical homogenization is to use the framework of the Heterogeneous Multiscale Method (HMM) (cf. [1, 2, 3, 12, 19]). The method is based on the idea

to predict an effective limit problem of (1) for  $\varepsilon \rightarrow 0$ . This can be achieved by solving local problems in sampling cells (typically called cell problems) and to extract effective macroscopic properties from the corresponding cell solutions. In some cases it can be explicitly shown that this strategy in fact yields the correct limit problem for  $\varepsilon \rightarrow 0$ . The central point of the method is that the cell problems are very small and systematically distributed in  $\Omega$ , but do not cover  $\Omega$ . This makes the method very cheap. For the wave equation, an HMM based on Finite Elements was proposed and analyzed in [4]. An HMM based on Finite Differences can be found in [14]. Since the classical homogenized model is known to fail to capture long time dispersive effects (cf. [24]) another effective model is needed for longer times. Solutions for this problem by a suitable model adaptation in the HMM context can be found in [5, 6, 15]. The advantage of the HMM framework is that it allows to construct methods that do not have to resolve the fine scale globally, allowing for a computational cost proportional to the degrees of freedom of the macroscopic mesh. But it requires scale separation and the cell problems must sample the microstructure sufficiently well. In many applications, especially in material sciences, these assumptions are typically well justified, in geophysical applications on the other hand, they might be often problematic. In this work, we hence focus on the latter case, where the HMM might not be applicable.

Beside the multiscale character of the problem, one of the biggest issues is the typically missing space regularity of the solution. In realistic applications, the propagation field  $a^\varepsilon$  is discontinuous. For instance in geophysics or seismology, the waves propagate through a medium that consists of different, heterogeneously distributed types of material (e.g. different soil or rock types). Hence, the properties of the propagation field cannot change continuously. This typically also involves a high contrast. The missing smoothness of  $a^\varepsilon$  directly influences the space regularity of the solution  $u^\varepsilon$  which is often not higher than  $L^\infty(0, T; H^1(\Omega))$ . As a consequence, the convergence rates of standard Finite Element methods deteriorate besides being very costly.

To overcome these issues (multiscale character and missing regularity of  $u^\varepsilon$ ), Owhadi and Zhang [29] proposed an interesting multiscale method based on a harmonic coordinate transformation  $G$ . The method is only analyzed for  $d = 2$ , but it is also applicable for higher dimensions. The components of  $G = (G_1, \dots, G_d)$  are defined as the weak solutions of an elliptic boundary value problem  $\nabla(a^\varepsilon \nabla G_i) = 0$  in  $\Omega$  and  $G_i(x) = x_i$  on  $\partial\Omega$ . Under a so called Cordes-type condition (cf. [29, Condition 2.1]) the authors managed to prove a compensation theorem saying that the solution in harmonic coordinates yields in fact the desired space-regularity. More precisely, they could show that  $(u^\varepsilon \circ G) \in L^\infty(0, T; H^2(\Omega))$  and furthermore that

$$\|u^\varepsilon \circ G\|_{L^\infty(0, T; H^2(\Omega))} \leq C(F, g) + C\|\partial_{tt}u(\cdot, 0)\|_{L^2(\Omega)},$$

where  $C(F, g)$  and  $C$  are constants depending on the data functions, but not on the variations of  $a^\varepsilon$ . Consequently, by using the equality  $\partial_{tt}u(\cdot, 0) = \nabla \cdot (a^\varepsilon \nabla f) - F(\cdot, 0)$ , the  $L^\infty(H^2)$ -norm of  $u^\varepsilon \circ G$  can be bounded *independently of the oscillations* of  $a^\varepsilon$  if the choice of the initial value is such that  $\|\nabla \cdot (a^\varepsilon \nabla f)\|_{L^2(\Omega)}$  can be bounded independent of  $\varepsilon$ . Note that  $\|u^\varepsilon\|_{L^\infty(0, T; H^2(\Omega))}$  (if it even exists) is normally proportional to the  $W^{1, \infty}$ -norm of  $a^\varepsilon$  (if it exists), which is the reason why classical finite elements cannot converge unless this frequency is resolved by the mesh. The harmonically transformed solution of the wave equation does not suffer from this anymore. With this key feature, an adequate analysis (and corresponding numerics) can be performed in an harmonically transformed finite element space, allowing optimal orders of convergence. The method has only two drawbacks: the approximation of the harmonic coordinate transformation  $G$  and the validity of the Cordes-type condition. Even though the Cordes-type condition can be hard to verify in practice, the numerical experiments given in [29] indicate that the condition might not be necessary for a good behavior of the method.

The approximation of the harmonic coordinate transformation  $G$  on the other hand can become a real issue, since it involves the solution of  $d$  global fine-scale problems. This is an expensive one-time overhead. Furthermore, spline spaces are needed and it is not clear how the analytically predicted results change, when  $G$  is replaced by a numerical approximation  $G_h$ . Compared to [29], our method has therefore the advantage that it does not involve to solve global fine scale problems and relies on localized classical P1-finite element spaces.

Another multiscale method applicable to the wave equation was also presented by Owhadi and Zhang in [30]. Here a multiscale basis is assembled by localizing a certain transfer property (which can be seen as an alternative to the aforementioned harmonic coordinate transformation). In this approach, the number of local problems to solve is basically the same as for our method. However, the local problems require finite element spaces consisting of certain  $C^1$ -continuous functions. Furthermore, the diameter of the localization patches must at least be of order  $\sqrt{H}|\ln(H)|$  to guarantee an optimal linear convergence rate for the  $H^1$ -error, whereas our approach only requires  $H|\ln(H)|$ .

The Multiscale Finite Element Method using Limited Global Information by Jiang et al. [22, 21] can be seen as a general framework that also covers the harmonic coordinate transformation approach by Owhadi and Zhang. The central assumption for this method is the existence of a number of *known* global fields  $G_1, \dots, G_N$  and an unknown smooth function  $H = H(G_1, \dots, G_N, t)$  such that the error  $e = u^\varepsilon - H(G_1, \dots, G_N, t)$  has a small energy. Based on the size of this energy, an a priori error analysis can be performed. The components of the harmonic coordinate transformation  $G$  are an example for global fields that fit into the framework. Other (more heuristic) choices are possible (cf. [22, 21]), but equally expensive as computing the harmonic transformation  $G$ . The drawback of the method is hence the same as for the Owhadi-Zhang approach: the basic assumption on the existence of global fields can be hard to verify and even if it is known to be valid, there is an expensive one-time overhead in computing them with a global fine scale computation.

With regard to the previous discussion, our multiscale method proposed in Definition 5.6 has two benefits: the method does not require additional assumptions on scale separation or regularity of  $a^\varepsilon$  and it does not involve one-time-overhead computations on the full fine scale. In particular, our method is independent of the homogenization setting and does not exploit any higher space regularity than  $H^1$ .

## 6 Proofs of the main results

This section is devoted to the proof of Theorem 5.3. Before we can start with proving the a priori error estimates, we present two lemmata. The first result can be found in [9, 27]:

**Lemma 6.1** (Properties of the interpolation operator). *The interpolation operator  $I_H : H_0^1(\Omega) \rightarrow V_H$  from (9) has the following properties:*

$$\|v - I_H(v)\|_{L^2(\Omega)} + H\|v - I_H(v)\|_{H^1(\Omega)} \leq C_{I_H} H\|v\|_{H^1(\Omega)}, \quad (31)$$

for all  $v \in H_0^1(\Omega)$ . Here,  $C_{I_H}$  denotes a generic constant, that only depends on the shape regularity of the elements of  $\mathcal{T}_H$ . Furthermore, the restriction  $I_H|_{V_H} : V_H \rightarrow V_H$  is an isomorphism on  $V_H$ , with  $(I_H|_{V_H})^{-1}$  being  $H^1$ -stable.

Observe that  $((I_H|_{V_H})^{-1} \circ I_H)|_{V_H} = \text{Id}$ . On the other hand  $((I_H|_{V_H})^{-1} \circ I_H)|_{W_h} = 0$ . Hence, for any  $v_h = v_H + w_h \in V_h = V_H \oplus W_h$  with  $v_H \in V_H$  and  $w_h \in W_h$  we have  $((I_H|_{V_H})^{-1} \circ I_H)(v_h) = v_H = P_H(v_h)$  and therefore

$$((I_H|_{V_H})^{-1} \circ I_H)|_{V_h} = P_H|_{V_h}. \quad (32)$$

Furthermore we have the equation

$$\pi_{H,k}^{\text{ms}}(v) = (P_H \circ \pi_{H,k}^{\text{ms}})(v) + (Q_{h,k} \circ P_H \circ \pi_{H,k}^{\text{ms}})(v) \quad \text{for all } v \in H_0^1(\Omega). \quad (33)$$

The next lemma was proved in [17]:

**Lemma 6.2** (Decay of local correctors). *Assume that (H0) hold true and that  $k \in \mathbb{N}_{>0}$ . For  $K \in \mathcal{T}_h$  let  $p_h^K \in W_h$  be the solution of*

$$\int_{\Omega} a^\varepsilon \nabla p_h^K \cdot \nabla w_h = F_K(w_h) \quad \text{for all } w_h \in W_h \quad (34)$$

where  $F_K \in W_h'$  is such that  $F_K(w_h) = 0$  for all  $w_h$  with  $\text{supp}(w_h) \subset (\Omega \setminus K)$ . Furthermore, let  $p_h^{K,k} \in W_h(U_k(K))$  be the solution of

$$\int_{U_k(K)} a^\varepsilon \nabla p_h^{K,k} \cdot \nabla w_h = F_K(w_h) \quad \text{for all } w_h \in W_h(U_k(K)). \quad (35)$$

Then there exists a generic constant  $0 < \theta < 1$  (independent of  $H$ ,  $h$  or  $\varepsilon$ ) such that

$$\left\| \sum_{K \in \mathcal{T}_H} \nabla(p_h^K - p_h^{K,k}) \right\|_{L^2(\Omega)}^2 \lesssim k^d \theta^{2k} \sum_{K \in \mathcal{T}_H} \|\nabla p_h^K\|_{L^2(\Omega)}^2. \quad (36)$$

**Proposition 6.3.** *Let assumptions (H0) be fulfilled and corrector operators defined according to Definition 4.1 for some  $k \in \mathbb{N}_{>0}$ . Then we have the estimate*

$$\|\nabla(Q_{h,k} - Q_{h,\Omega})(v_H)\|_{L^2(\Omega)} \lesssim k^{d/2} \theta^k \|\nabla v_H\|_{L^2(\Omega)}. \quad (37)$$

for all  $v_H \in V_H$  and with  $\theta$  from Lemma 6.2. Furthermore, the operator  $Q_{h,k}$  is  $H^1$ -stable on  $V_H$  and the operator  $(P_H \circ \pi_{H,k}^{\text{ms}})$  is  $H^1$ -stable on  $H_0^1(\Omega)$ , i.e.

$$\begin{aligned} \forall v_H \in V_H : \quad & \|Q_{h,k}(v_H)\|_{H^1(\Omega)} \lesssim \|v_H\|_{H^1(\Omega)} \quad \text{and} \\ \forall v \in H_0^1(\Omega) : \quad & \|(P_H \circ \pi_{H,k}^{\text{ms}})(v)\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}. \end{aligned} \quad (38)$$

*Proof.* Let  $v_H \in V_H$  be arbitrary. In view of (36) for  $p_h^{K,k} = Q_{h,k}^K(v_H)$  and  $p_h^K = Q_{h,\Omega}^K(v_H)$  we get

$$\begin{aligned} \|\nabla(Q_{h,k} - Q_{h,\Omega})(v_H)\|_{L^2(\Omega)} & \lesssim k^{d/2} \theta^k \left( \sum_{K \in \mathcal{T}_H} \|\nabla Q_{h,\Omega}^K(v_H)\|_{L^2(\Omega)}^2 \right)^{1/2} \\ & \lesssim k^{d/2} \theta^k \left( \sum_{K \in \mathcal{T}_H} \|\nabla v_H\|_{L^2(K)}^2 \right)^{1/2}, \end{aligned}$$

where we used that for any  $K \in \mathcal{T}_H$

$$\begin{aligned} \|Q_{h,k}^K(v_H)\|_{H^1(\Omega)}^2 & \lesssim b^\varepsilon(Q_{h,k}^K(v_H), Q_{h,k}^K(v_H)) = - \int_K a^\varepsilon \nabla v_H \cdot \nabla Q_{h,k}^K(v_H) \\ & \lesssim \|\nabla v_H\|_{L^2(K)} \|Q_{h,k}^K(v_H)\|_{H^1(\Omega)}. \end{aligned}$$

This proves (37). Since  $Q_{h,\Omega}$  is obviously  $H^1$ -stable on  $V_H$  and since  $k^{d/2} \theta^k$  is monotonically decreasing for growing  $k$ , the  $H^1$ -stability of  $Q_{h,k}$  follows directly from (37). The elliptic



projection  $\pi_{H,k}^{\text{ms}}$  is also obviously  $H^1$ -stable. Finally, the  $H^1$ -stability of  $(P_H \circ \pi_{H,k}^{\text{ms}})$  follows from the identity  $P_H|_{V_h} = (I_H|_{V_H}^{-1} \circ I_H)|_{V_h}$  (see (32)), the  $H^1$ -stability of  $I_H|_{V_H}^{-1}$  on  $V_H$  and the  $H^1$ -stability of  $I_H$  on  $H_0^1(\Omega)$  (see Lemma 6.1). Hence:

$$\|(P_H \circ \pi_{H,k}^{\text{ms}})(v)\|_{H^1(\Omega)} = \|(I_H|_{V_H}^{-1} \circ I_H \circ \pi_{H,k}^{\text{ms}})(v)\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}$$

for arbitrary  $v \in H_0^1(\Omega)$ .  $\square$

The next lemma gives explicit error estimates for the elliptic projections on  $V_{H,k}^{\text{ms}}$ .

**Lemma 6.4.** *Let  $u^\varepsilon$  be the solution of (2) and let the corrector operator  $Q_{h,k}$  be given as in Definition 4.1 for some  $k \in \mathbb{N}_{>0}$ . Furthermore, let  $\pi_{H,k}^{\text{ms}}$  and  $\pi_h$  denote the elliptic projections according to (17) and (16). We further denote the  $L^2$ -projection of  $V_h$  on  $V_H$  by  $P_H$ . The following estimates hold for almost every  $t \in [0, T]$ .*

*If  $\partial_t^i u^\varepsilon \in L^1(0, T; H^1(\Omega))$  for  $i \in \{0, 1, 2\}$ , then it holds*

$$\begin{aligned} & \|(P_H \circ \pi_{H,k}^{\text{ms}})(\partial_t^i u^\varepsilon(\cdot, t)) - \partial_t^i u^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \\ & \leq \|\partial_t^i u^\varepsilon(\cdot, t) - \pi_h(\partial_t^i u^\varepsilon(\cdot, t))\|_{L^2(\Omega)} + (H + \theta^k k^{d/2}) \|\partial_t^i u^\varepsilon(\cdot, t)\|_{H^1(\Omega)}. \end{aligned} \quad (39)$$

*Assume that  $i \in \{0, 1, 2\}$  and  $s, m \in \{0, 1\}$ . If  $\partial_t^i u^\varepsilon \in L^1(0, T; H^1(\Omega))$  and  $\partial_t^{2+i} u^\varepsilon, \partial_t^i F \in L^1(0, T; H^s(\Omega))$  it holds*

$$\begin{aligned} & \|\pi_{H,k}^{\text{ms}}(\partial_t^i u^\varepsilon(\cdot, t)) - \partial_t^i u^\varepsilon(\cdot, t)\|_{H^m(\Omega)} \lesssim \|\partial_t^i u^\varepsilon(\cdot, t) - \pi_h(\partial_t^i u^\varepsilon(\cdot, t))\|_{H^m(\Omega)} \\ & + (H^{2+s-m} + k^{d(2-m)/2} \theta^{(2-m)k}) (\|\partial_t^{2+i} u^\varepsilon(\cdot, t) - \partial_t^i F(\cdot, t)\|_{H^s(\Omega)} + \|\partial_t^i u^\varepsilon(\cdot, t)\|_{H^1(\Omega)}). \end{aligned} \quad (40)$$

*Proof.* Error estimate under low regularity assumptions (39). We use an Aubin-Nitsche duality argument for some arbitrary  $v \in L^1(0, T; H_0^1(\Omega))$ . Let us define  $e_{H,k} := \pi_{H,k}^{\text{ms}}(v) - \pi_h(v)$ . We regard the dual problem: find  $z_h \in L^1(0, T; V_h)$  with

$$b^\varepsilon(w_h, z_h(\cdot, t)) = (e_{H,k}(\cdot, t), w_h)_{L^2(\Omega)} \quad \text{for all } w_h \in V_h, \quad \text{for a.e. } t \in (0, T) \quad (41)$$

and the dual problem in the multiscale space: find  $z_{H,k}^{\text{ms}} \in V_{H,k}^{\text{ms}}$  with

$$b^\varepsilon(w^{\text{ms}}, z_{H,k}^{\text{ms}}(\cdot, t)) = (e_{H,k}(\cdot, t), w^{\text{ms}})_{L^2(\Omega)} \quad \text{for all } w^{\text{ms}} \in V_{H,k}^{\text{ms}}, \quad \text{for a.e. } t \in (0, T). \quad (42)$$

Obviously we have  $b^\varepsilon(w^{\text{ms}}, (z_h - z_{H,k}^{\text{ms}})(\cdot, t)) = 0$  for all  $w^{\text{ms}} \in V_{H,k}^{\text{ms}}$  and for almost every  $t \in [0, T]$ . This implies that  $(z_h - z_{H,k}^{\text{ms}})(\cdot, t)$  is in the  $b^\varepsilon(\cdot, \cdot)$ -orthogonal complement of  $V_{H,k}^{\text{ms}}$  (for almost every  $t$ ), hence it is in the kernel of quasi-interpolation operator  $I_H$ . Omitting the  $t$ -dependency, we obtain

$$\begin{aligned} b^\varepsilon(z_h - z_{H,\Omega}^{\text{ms}}, z_h - z_{H,\Omega}^{\text{ms}}) &= (e_{H,k}, z_h - z_{H,\Omega}^{\text{ms}})_{L^2(\Omega)} \\ &= (e_{H,k}, (z_h - z_{H,\Omega}^{\text{ms}}) - I_H(z_h - z_{H,\Omega}^{\text{ms}}))_{L^2(\Omega)} \\ &\stackrel{(31)}{\lesssim} H \|e_{H,k}\|_{L^2(\Omega)} \|z_h - z_{H,\Omega}^{\text{ms}}\|_{H^1(\Omega)}. \end{aligned} \quad (43)$$

Next, let us define the energy

$$E(v_H) := b^\varepsilon(z_h - v_H - Q_{h,k}(v_H), z_h - v_H - Q_{h,k}(v_H)) \quad \text{for } v_H \in V_H$$

and let us write  $z_{H,\Omega}^{\text{ms}} = z_{H,\Omega} + Q_{h,\Omega}(z_{H,\Omega})$  and  $z_{H,k}^{\text{ms}} = z_{H,k} + Q_{h,k}(z_{H,k})$  with  $z_{H,\Omega}, z_{H,k} \in V_H$ . Since we have

$$b^\varepsilon(z_h - z_{H,k} - Q_{h,k}(z_{H,k}), v_H + Q_{h,k}(v_H)) = 0 \quad \text{for all } v_H \in V_H,$$

we know that this is equivalent to the fact that  $z_{H,k} \in V_H$  must minimize the energy  $E(\cdot)$  on  $V_H$ . Hence

$$\begin{aligned}
\|z_h - z_{H,k} - Q_{h,k}(z_{H,k})\|_{H^1(\Omega)} &\lesssim \|z_h - z_{H,\Omega} - Q_{h,k}(z_{H,\Omega})\|_{H^1(\Omega)} \\
&\leq \|z_h - z_{H,\Omega}^{\text{ms}}\|_{H^1(\Omega)} + \|(Q_{h,\Omega} - Q_{h,k})(z_{H,\Omega})\|_{H^1(\Omega)} \\
&\stackrel{(43),(37)}{\lesssim} H\|e_{H,k}\|_{L^2(\Omega)} + \theta^k k^{d/2} \|Q_{h,\Omega}(z_{H,\Omega})\|_{H^1(\Omega)} \\
&\stackrel{(38)}{\lesssim} (H + \theta^k k^{d/2}) \|e_{H,k}\|_{L^2(\Omega)}.
\end{aligned}$$

And as a direct consequence, using  $b^\varepsilon(e_{H,k}, z_{H,k}^{\text{ms}}) = 0$  (combining (16) and (17) for test functions in  $V_{H,k}^{\text{ms}}$ )

$$\|e_{H,k}\|_{L^2(\Omega)}^2 = b^\varepsilon(e_{H,k}, z_h) = b^\varepsilon(e_{H,k}, z_h - z_{H,k}^{\text{ms}}) \lesssim \|e_{H,k}\|_{H^1(\Omega)} (H + \theta^k k^{d/2}) \|e_{H,k}\|_{L^2(\Omega)}. \quad (44)$$

The bound  $\|e_{H,k}\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}$  and  $\pi_{H,k}^{\text{ms}}(v) = (P_H \circ \pi_{H,k}^{\text{ms}})(v) + (Q_{h,k} \circ P_H \circ \pi_{H,k}^{\text{ms}})(v)$  conclude the estimate

$$\begin{aligned}
\|(P_H \circ \pi_{H,k}^{\text{ms}})(v) - v\|_{L^2(\Omega)} &\leq \|\pi_{H,k}^{\text{ms}}(v) - v\|_{L^2(\Omega)} + \|(Q_{h,k} \circ P_H \circ \pi_{H,k}^{\text{ms}})(v)\|_{L^2(\Omega)} \\
&\stackrel{(31),(38)}{\lesssim} \|v - \pi_h(v)\|_{L^2(\Omega)} + (H + \theta^k k^{d/2}) \|v\|_{H^1(\Omega)} + H \|v\|_{H^1(\Omega)}.
\end{aligned}$$

Hence for all  $v \in H_0^1(\Omega)$

$$\|(P_H \circ \pi_{H,k}^{\text{ms}})(v) - v\|_{L^2(\Omega)} + \|\pi_{H,k}^{\text{ms}}(v) - v\|_{L^2(\Omega)} \lesssim \|v - \pi_h(v)\|_{L^2(\Omega)} + (H + \theta^k k^{d/2}) \|v\|_{H^1(\Omega)}. \quad (45)$$

The results follows with  $v = \partial_t^i u^\varepsilon(\cdot, t)$ .

*Error estimate under high regularity assumptions (40).* For the next estimate, we restrict our considerations to the solution  $u^\varepsilon$  of (2). Let the regularity assumptions of the lemma hold true and let us introduce the simplifying notation

$$v^\varepsilon := \partial_t^i u^\varepsilon \quad \text{and} \quad \bar{F} := \partial_t^i F.$$

We observe that  $v^\varepsilon$  solves the equation

$$(\partial_{tt} v^\varepsilon(\cdot, t), w)_{L^2(\Omega)} + b^\varepsilon(v^\varepsilon(\cdot, t), w) = (\bar{F}(\cdot, t), w)_{L^2(\Omega)}$$

for all  $w \in H_0^1(\Omega)$ , for almost every  $t \in (0, T)$ . By the definition of projections, we have

$$b^\varepsilon((\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon))(\cdot, t), w) = 0 \quad \text{for all } w \in V_{H,k}^{\text{ms}}, \quad \text{for almost every } t \in (0, T).$$

We conclude  $(\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon))(\cdot, t) \in W_h$  for almost every  $t$  and in particular

$$I_H((\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon))(\cdot, t)) = 0 \quad \text{for almost every } t \in (0, T). \quad (46)$$

Furthermore, with the notation  $\pi_{H,k}^{\text{ms}}(v^\varepsilon) = v_{H,k} + Q_{h,k}(v_{H,k})$  and  $\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) = v_{H,\Omega} + Q_{h,\Omega}(v_{H,\Omega})$  we have again that  $v_{H,k}(\cdot, t) \in V_H$  minimizes the energy

$$E(\Phi_H) := b^\varepsilon(\pi_h(v^\varepsilon(\cdot, t) - \Phi_H(\cdot, t) - Q_{h,k}(\Phi_H)(\cdot, t), \pi_h(v^\varepsilon(\cdot, t) - \Phi_H(\cdot, t) - Q_{h,k}(\Phi_H)(\cdot, t)))$$

for  $\Phi_H \in V_H$  and therefore

$$\begin{aligned} \|\pi_{H,k}^{\text{ms}}(v^\varepsilon(\cdot, t)) - \pi_h(v^\varepsilon(\cdot, t))\|_{H^1(\Omega)} &= \|v_{H,k}(\cdot, t) + Q_{h,k}(v_{H,k})(\cdot, t) - \pi_h(v^\varepsilon(\cdot, t))\|_{H^1(\Omega)} \\ &\lesssim \|v_{H,\Omega}(\cdot, t) + Q_{h,k}(v_{H,\Omega})(\cdot, t) - \pi_h(v^\varepsilon(\cdot, t))\|_{H^1(\Omega)}. \end{aligned}$$

For brevity, let us from now on leave out the  $t$ -dependency in the functions for the rest of the proof. Hence, we obtain in the same way as for the low regularity estimate

$$\begin{aligned} &\|\pi_{H,k}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon)\|_{H^1(\Omega)} \\ &\leq \|u_{H,\Omega} + Q_{h,\Omega}(v_{H,\Omega}) - \pi_h(v^\varepsilon)\|_{H^1(\Omega)} + \|(Q_{h,\Omega} - Q_{h,k})(v_{H,\Omega})\|_{H^1(\Omega)} \\ &\lesssim \|v_{H,\Omega} + Q_{h,\Omega}(v_{H,\Omega}) - \pi_h(v^\varepsilon)\|_{H^1(\Omega)} + k^{d/2}\theta^k \|\pi_{H,k}^{\text{ms}}(v^\varepsilon)\|_{H^1(\Omega)} \\ &\lesssim \|\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon)\|_{H^1(\Omega)} + k^{d/2}\theta^k \|v^\varepsilon\|_{H^1(\Omega)}. \end{aligned} \quad (47)$$

We next estimate the term  $\|\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon)\|_{H^1(\Omega)}$  in this estimate. For this, we use the equality

$$(v, w_h)_{L^2(\Omega)} = (v - I_H(v), w_h - I_H(w_h))_{L^2(\Omega)} \quad \text{for all } v \in L^2(\Omega), w_h \in W_h. \quad (48)$$

This equation holds because of  $I_H(w_h) = 0$  for all  $w_h \in W_h$  and  $(v_H, w_h)_{L^2(\Omega)} = 0$  for all  $v_H \in V_H$  (because  $w_h$  is in the kernel of the  $L^2$ -projection). With that we obtain

$$\begin{aligned} &b^\varepsilon(\pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon), \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon)) \\ &= b^\varepsilon(\pi_h(v^\varepsilon), \pi_h(v^\varepsilon) - \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon)) \\ &= b^\varepsilon(v^\varepsilon, \pi_h(v^\varepsilon) - \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon)) \\ &= (\bar{F} - \partial_{tt}v^\varepsilon, \pi_h(v^\varepsilon) - \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon))_{L^2(\Omega)} \\ &\stackrel{(48)}{=} ((\bar{F} - \partial_{tt}v^\varepsilon) - I_H(\bar{F} - \partial_{tt}v^\varepsilon), (\pi_h(v^\varepsilon) - \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon)) - I_H(\pi_h(v^\varepsilon) - \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon)))_{L^2(\Omega)} \\ &\stackrel{(31)}{\lesssim} H^{s+1} \|\bar{F} - \partial_{tt}v^\varepsilon\|_{H^s(\Omega)} \|\pi_h(v^\varepsilon) - \pi_{H,\Omega}^{\text{ms}}(v^\varepsilon)\|_{H^1(\Omega)}. \end{aligned}$$

Combining this with (47) we get

$$\|\pi_{H,k}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon)\|_{H^1(\Omega)} \lesssim H^{s+1} \|\partial_{tt}v^\varepsilon - \bar{F}\|_{H^s(\Omega)} + k^{d/2}\theta^k \|v^\varepsilon\|_{H^1(\Omega)}, \quad (49)$$

which proves the estimate (40) for the case  $m = 1$ . Now, we prove the estimate for the case  $m = 0$  by applying the same Aubin-Nitsche argument as above. Defining  $e_{H,k} := \pi_{H,k}^{\text{ms}}(v^\varepsilon) - \pi_h(v^\varepsilon)$  we are looking for  $z_h \in L^2(0, T; V_h)$  and  $z_{H,k}^{\text{ms}} \in V_{H,k}^{\text{ms}}$  that are defined analogously to (41) and (42). Hence we get with same strategy as before

$$\|z_{H,k}^{\text{ms}} - z_h\|_{H^1(\Omega)} \lesssim (H + k^{d/2}\theta^k) \|e_{H,k}\|_{L^2(\Omega)}$$

and hence together with (49)

$$\begin{aligned} \|e_{H,k}\|_{L^2(\Omega)}^2 &= |b^\varepsilon(e_{H,k}, z_h - z_{H,k}^{\text{ms}})| \\ &\lesssim (H^{s+2} + k^d\theta^{2k}) (\|\partial_{tt}v^\varepsilon - \bar{F}\|_{H^s(\Omega)} + \|v^\varepsilon\|_{H^1(\Omega)}) \|e_{H,k}\|_{L^2(\Omega)}. \end{aligned}$$

In total we proved (40) for  $m = 1$ , i.e.

$$\begin{aligned} &\|\pi_{H,k}^{\text{ms}}(v^\varepsilon) - v^\varepsilon\|_{L^2(\Omega)} \\ &\lesssim \|v^\varepsilon - \pi_h(v^\varepsilon)\|_{L^2(\Omega)} + (H^{s+2} + k^d\theta^{2k}) (\|\partial_{tt}v^\varepsilon - \bar{F}\|_{H^s(\Omega)} + \|v^\varepsilon\|_{H^1(\Omega)}). \end{aligned}$$

□

In view of Lemma 6.4, it is necessary to bound the various time derivatives of  $u^\varepsilon$  independent of  $\varepsilon$ . In Theorem 5.3 this issue was shifted to making various additional assumptions on the existence of corresponding uniform bounds. In the next remark, we show a possibility of how to justify this.

**Remark 6.5** (Uniform bounds in  $\varepsilon$  for the time derivatives of  $u^\varepsilon$ ). The regularity required in Theorem 5.3 (including the  $\varepsilon$ -uniform bounds) can be guaranteed by making assumptions on the data (c.f. [16, Chapter 7, Theorem 6]). The uniform  $\varepsilon$ -bounds can be for instance justified by making additional assumptions on  $f$  and  $g$  in the following way:

Let  $m \in \mathbb{N}_{\geq 0}$ , then (under the assumption of sufficient regularity) we have

$$(\partial_t^{m+2} u^\varepsilon(\cdot, t), \partial_t^{m+1} u^\varepsilon(\cdot, t)) + b^\varepsilon(\partial_t^m u^\varepsilon(\cdot, t), \partial_t^{m+1} u^\varepsilon(\cdot, t)) = (\partial_t^m F(\cdot, t), \partial_t^{m+1} u^\varepsilon(\cdot, t)).$$

Hence

$$\begin{aligned} & \|\partial_t^{m+1} u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t^m u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \\ & \lesssim \|\partial_t^m F\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t^{m+1} u^\varepsilon(0, \cdot)\|_{L^2(\Omega)} + \|\nabla \cdot (a^\varepsilon \nabla \partial_t^m u^\varepsilon(0, \cdot))\|_{L^2(\Omega)}. \end{aligned}$$

For  $m = 1$  and by  $\partial_t u^\varepsilon(\cdot, 0) = g$ ,  $u^\varepsilon(\cdot, 0) = f$  and the compatibility condition  $\partial_{tt} u^\varepsilon(\cdot, 0) = F(\cdot, 0) + \nabla \cdot (a^\varepsilon \nabla u^\varepsilon(\cdot, 0))$  (which is required to obtain sufficient regularity) we get

$$\begin{aligned} & \|\partial_{tt} u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \\ & \lesssim \|\partial_t F\|_{L^2(0,T;L^2(\Omega))} + \|F(\cdot, 0) + \nabla \cdot (a^\varepsilon \nabla f)\|_{L^2(\Omega)} + \|\nabla \cdot (a^\varepsilon \nabla g)\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, if the initial data is such that  $\|\nabla \cdot (a^\varepsilon \nabla g)\|_{L^2(\Omega)}$  and  $\|\nabla \cdot (a^\varepsilon \nabla f)\|_{L^2(\Omega)}$  can be bounded independent of  $\varepsilon$ , we can bound  $\|\partial_{tt} u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}$  and  $\|\partial_t u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$  in the same way. This easily generalizes to higher time derivatives of  $u^\varepsilon$  by exploiting the corresponding higher order compatibility conditions.

The estimates (19) and (20) in Theorem 5.3 are a direct consequence of the following lemma. Observe that this lemma is a data-explicit (in particular  $\varepsilon$ -explicit and  $T$ -explicit) version of (19) and (20).

**Lemma 6.6.** *Let the same notation and the same assumptions as in Lemma 6.4 be fulfilled and let  $s \in \{0, 1\}$ . If assumption (H2) is fulfilled it holds*

$$\begin{aligned} & \|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(L^2)} \\ & \lesssim \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(L^2)} + \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^1(L^2)} \\ & \quad + (H^{2+s} + k^d \theta^{2k}) (\|u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_t u^\varepsilon\|_{L^1(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(H^s)} + \|\partial_{ttt} u^\varepsilon\|_{L^1(H^s)} \\ & \quad + \|F\|_{L^\infty(H^s)} + \|\partial_t F\|_{L^1(H^s)}). \end{aligned} \tag{50}$$

Recall that the  $\lesssim$ -notation only contains dependencies on  $\Omega$ ,  $d$ ,  $\alpha$ ,  $\beta$  and the shape regularity of  $\mathcal{T}_H$ , but not on  $T$  and  $\varepsilon$ .

*Proof.* To prove the result, we can follow the arguments of Baker [7]. For the numerically homogenized solution  $u_{H,k}$  of (15), we define  $u_{H,k}^{\text{ms}} := u_{H,k} + Q_{h,k}(u_{H,k})$ . For brevity, we denote  $(\cdot, \cdot) := (\cdot, \cdot)_{L^2(\Omega)}$ . Furthermore, we use the notation from Lemma 6.4 and define the errors

$$e^{\text{ms}} := u^\varepsilon - u_{H,k}^{\text{ms}}, \quad e^\pi := u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon) \quad \text{and} \quad \psi^\pi := u_{H,k}^{\text{ms}} - \pi_{H,k}^{\text{ms}}(u^\varepsilon).$$

Observe that we have for  $v \in L^2(0, T; V_{H,k}^{\text{ms}})$  and almost every  $t \geq 0$ :

$$\begin{aligned} 0 &= (\partial_{tt}\psi^\pi(\cdot, t) - \partial_{tt}e^\pi(\cdot, t), v(\cdot, t)) + b^\varepsilon(\psi^\pi(\cdot, t), v(\cdot, t)) \\ &= \partial_t(\partial_t\psi^\pi(\cdot, t) - \partial_te^\pi(\cdot, t), v(\cdot, t)) - (\partial_t\psi^\pi(\cdot, t) - \partial_te^\pi(\cdot, t), \partial_tv(\cdot, t)) + b^\varepsilon(\psi^\pi(\cdot, t), v(\cdot, t)) \\ &= -\partial_t(\partial_te^{\text{ms}}(\cdot, t), v(\cdot, t)) - (\partial_t\psi^\pi(\cdot, t) - \partial_te^\pi(\cdot, t), \partial_tv(\cdot, t)) + b^\varepsilon(\psi^\pi(\cdot, t), v(\cdot, t)). \end{aligned}$$

For some arbitrary  $0 < t_0 \leq T$  we use the function  $v(\cdot, t) = \int_t^{t_0} \psi^\pi(\cdot, s) ds$  in the above equation (and the fact that  $\partial_tv = -\psi^\pi$ ) to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\psi^\pi(\cdot, t)\|_{L^2(\Omega)}^2 - \frac{1}{2} \frac{d}{dt} b^\varepsilon \left( \int_t^{t_0} \psi^\pi(\cdot, s) ds, \int_t^{t_0} \psi^\pi(\cdot, s) ds \right) \\ &= \partial_t \left( \partial_te^{\text{ms}}(\cdot, t), \int_t^{t_0} \psi^\pi(\cdot, s) ds \right) + (\partial_te^\pi(\cdot, t), \psi^\pi(\cdot, t)). \end{aligned}$$

Integration from 0 to  $t_0$  yields

$$\begin{aligned} &\frac{1}{2} \|\psi^\pi(\cdot, t_0)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\psi^\pi(\cdot, 0)\|_{L^2(\Omega)}^2 + \frac{1}{2} b^\varepsilon \left( \int_0^{t_0} \psi^\pi(\cdot, s) ds, \int_0^{t_0} \psi^\pi(\cdot, s) ds \right) \\ &= - \left( \partial_te^{\text{ms}}(\cdot, 0), \int_0^{t_0} \psi^\pi(\cdot, s) ds \right) + \int_0^{t_0} (\partial_te^\pi(\cdot, t), \psi^\pi(\cdot, t)) dt. \end{aligned}$$

Hence

$$\begin{aligned} &\|\psi^\pi(\cdot, t_0)\|_{L^2(\Omega)}^2 \\ &\leq \|\psi^\pi(\cdot, 0)\|_{L^2(\Omega)}^2 - 2 \left( \partial_te^{\text{ms}}(\cdot, 0), \int_0^{t_0} \psi^\pi(\cdot, s) ds \right) + 2 \int_0^{t_0} (\partial_te^\pi(\cdot, t), \psi^\pi(\cdot, t)) dt \\ &\leq \|\psi^\pi(\cdot, 0)\|_{L^2(\Omega)}^2 + 2 \int_0^{t_0} (\partial_te^\pi(\cdot, t), \psi^\pi(\cdot, t)) dt \\ &\leq \|\psi^\pi(\cdot, 0)\|_{L^2(\Omega)}^2 + 2 \|\partial_te^\pi\|_{L^1(0, T; L^2(\Omega))}^2 + \frac{1}{2} \|\psi^\pi\|_{L^\infty(0, T; L^2(\Omega))}^2. \end{aligned}$$

By moving the term  $\|\psi^\pi\|_{L^\infty(0, T; L^2(\Omega))}^2$  to the left hand side, we get

$$\|\psi^\pi\|_{L^\infty(0, T; L^2(\Omega))} \lesssim \|\psi^\pi(\cdot, 0)\|_{L^2(\Omega)} + \|\partial_te^\pi\|_{L^1(0, T; L^2(\Omega))}. \quad (51)$$

However, since  $u_{H,k}^{\text{ms}}(\cdot, 0) = \pi_{H,k}^{\text{ms}}(f)$ , we get  $\psi^\pi(\cdot, 0) = \pi_{H,k}^{\text{ms}}(f) - \pi_{H,k}^{\text{ms}}(u^\varepsilon(\cdot, 0)) = 0$ . Hence, together with the triangle inequality for  $\psi^\pi = (u_{H,k}^{\text{ms}} - u^\varepsilon) + (u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon))$ , equation (51) implies

$$\|u_{H,k}^{\text{ms}} - u^\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} \lesssim \|u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon)\|_{L^\infty(0, T; L^2(\Omega))} + \|\partial_t u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_t u^\varepsilon)\|_{L^1(0, T; L^2(\Omega))}.$$

Together with Lemma 6.4 this finishes the proof of (50).  $\square$

The next lemma is an  $\varepsilon$ -explicit and  $T$ -explicit version of estimates (21) and (22) in Theorem 5.3.

**Lemma 6.7.** *Let the same notation and the same assumptions as in Lemma 6.4 be fulfilled, let  $s \in \{0, 1\}$  and assume (H3).*

*If  $s = 0$ , it holds*

$$\begin{aligned} &\|\partial_t u^\varepsilon - \partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(L^2)} + \|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(H^1)} \\ &\lesssim \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^\infty(L^2)} + \|\partial_{tt} u^\varepsilon - \pi_h(\partial_{tt} u^\varepsilon)\|_{L^1(L^2)} + \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(H^1)} \\ &\quad + (H + k^{d/2} \theta^k) \left( \sum_{i=0}^1 \|\partial_t^i u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^\infty(L^2)} + \|\partial_{tt} u^\varepsilon\|_{L^1(H^1)} + \|\partial_{ttt} u^\varepsilon\|_{L^\infty(L^2)} \right) \\ &\quad + (H + k^{d/2} \theta^k) \left( \|\partial_t^4 u^\varepsilon\|_{L^1(L^2)} + \sum_{i=0}^1 \|\partial_t^i F\|_{L^\infty(L^2)} + \|\partial_{tt} F\|_{L^1(L^2)} + \|g\|_{H^1(\Omega)} \right). \end{aligned} \quad (52)$$

If  $s = 1$  and if the initial value in (15) is picked such that  $\partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))(\cdot, 0) = \pi_{H,k}^{\text{ms}}(g)$ , then we obtain the improved estimate

$$\begin{aligned} & \|\partial_t u^\varepsilon - \partial_t(u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(L^2)} + \|u^\varepsilon - (u_{H,k} + Q_{h,k}(u_{H,k}))\|_{L^\infty(H^1)} \\ & \lesssim \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^\infty(L^2)} + \|\partial_{tt} u^\varepsilon - \pi_h(\partial_{tt} u^\varepsilon)\|_{L^1(L^2)} + \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(H^1)} \\ & \quad + (H^2 + k^{d/2}\theta^k) \left( \sum_{i=0}^2 \|\partial_t^i u^\varepsilon\|_{L^\infty(H^1)} + \|\partial_{tt} u^\varepsilon\|_{L^1(H^1)} + \|\partial_{ttt} u^\varepsilon\|_{L^\infty(L^2)} \right) \\ & \quad + (H^2 + k^{d/2}\theta^k) (\|\partial_t^4 u^\varepsilon\|_{L^1(L^2)} + \|F\|_{L^\infty(H^1)} + \|\partial_t F\|_{L^\infty(L^2)} + \|\partial_{tt} F\|_{L^1(L^2)}). \end{aligned} \quad (53)$$

Again, recall that the  $\lesssim$ -notation only contains dependencies on  $\Omega$ ,  $d$ ,  $\alpha$ ,  $\beta$  and the shape regularity of  $\mathcal{T}_H$ , but not on  $T$  and  $\varepsilon$ .

*Proof.* Again, we define the errors  $e^{\text{ms}} := u^\varepsilon - u_{H,k}^{\text{ms}}$ ,  $e^\pi := u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon)$  and  $\psi^\pi := u_{H,k}^{\text{ms}} - \pi_{H,k}^{\text{ms}}(u^\varepsilon)$ . We only consider the case  $\partial_t u_{H,k}^{\text{ms}}(\cdot, 0) = P_{H,k}^{\text{ms}}(g)$  (i.e. estimate (52)), the case  $\partial_t u_{H,k}^{\text{ms}}(\cdot, 0) = \pi_{H,k}^{\text{ms}}(g)$  (i.e. estimate (53)) follows analogously with  $\partial_t \psi^\pi(\cdot, 0) = 0$ . By Galerkin orthogonality we obtain for a.e.  $t \in [0, T]$

$$(\partial_{tt} e^{\text{ms}}(\cdot, t), v^{\text{ms}})_{L^2(\Omega)} + b^\varepsilon(e^{\text{ms}}(\cdot, t), v^{\text{ms}}) = 0 \quad \text{for all } v^{\text{ms}} \in V_{H,k}^{\text{ms}}$$

and hence

$$(\partial_{tt} \psi^\pi(\cdot, t), v^{\text{ms}})_{L^2(\Omega)} + b^\varepsilon(\psi^\pi(\cdot, t), v^{\text{ms}}) = (\partial_{tt} e^\pi(\cdot, t), v^{\text{ms}})_{L^2(\Omega)} \quad \text{for all } v^{\text{ms}} \in V_{H,k}^{\text{ms}}.$$

Testing with  $v^{\text{ms}} = \partial_t \psi^\pi$  yields for a.e.  $t \in [0, T]$

$$\frac{1}{2} \frac{d}{dt} \left( \|\partial_t \psi^\pi(\cdot, t)\|_{L^2(\Omega)}^2 + b^\varepsilon(\psi^\pi(\cdot, t), \psi^\pi(\cdot, t)) \right) = (\partial_{tt} e^\pi(\cdot, t), \partial_t \psi^\pi(\cdot, t))_{L^2(\Omega)}$$

By integration from 0 to  $t_0 \leq T$  we obtain

$$\begin{aligned} & \frac{1}{2} \|\partial_t \psi^\pi(\cdot, t_0)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\psi^\pi(\cdot, t_0)\|_{H^1(\Omega)}^2 \\ & \leq \frac{1}{2} \|\partial_t \psi^\pi(\cdot, 0)\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|\psi^\pi(\cdot, 0)\|_{H^1(\Omega)}^2 + \int_0^{t_0} |(\partial_{tt} e^\pi(\cdot, t), \partial_t \psi^\pi(\cdot, t))_{L^2(\Omega)}| dt. \end{aligned}$$

Since we have  $\psi^\pi(\cdot, 0) = \pi_{H,k}^{\text{ms}}(f) - \pi_{H,k}^{\text{ms}}(f) = 0$  we get with the Young and the Cauchy-Schwarz inequality

$$\begin{aligned} & \frac{1}{2} \|\partial_t \psi^\pi(\cdot, t_0)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\psi^\pi(\cdot, t_0)\|_{H^1(\Omega)}^2 \\ & \leq \frac{1}{2} \|\pi_{H,k}^{\text{ms}}(g) - P_{H,k}^{\text{ms}}(g)\|_{L^2(\Omega)}^2 + \|\partial_{tt} e^\pi\|_{L^1(0, T; L^2(\Omega))}^2 + \frac{1}{4} \|\partial_t \psi^\pi\|_{L^\infty(0, T; L^2(\Omega))}. \end{aligned}$$

By taking the supremum over all  $0 \leq t_0 \leq T$  we obtain

$$\|\partial_t \psi^\pi\|_{L^\infty(L^2)}^2 + 2\alpha \|\psi^\pi\|_{L^\infty(H^1)}^2 \leq 2 \|\pi_{H,k}^{\text{ms}}(g) - P_{H,k}^{\text{ms}}(g)\|_{L^2(\Omega)}^2 + 4 \|\partial_{tt} e^\pi\|_{L^1(L^2)}^2.$$

The term  $\|\partial_{tt} e^\pi\|_{L^1(L^2)}$  can be treated with Lemma 6.4, equation (40). Hence it only remains to estimate the term  $\|\pi_{H,k}^{\text{ms}}(g) - P_{H,k}^{\text{ms}}(g)\|_{L^2(\Omega)}$  for which we get

$$\begin{aligned} \|\pi_{H,k}^{\text{ms}}(g) - P_{H,k}^{\text{ms}}(g)\|_{L^2(\Omega)} & \leq \|\pi_{H,k}^{\text{ms}}(g) - g\|_{L^2(\Omega)} + \|P_{H,k}^{\text{ms}}(g) - g\|_{L^2(\Omega)} \\ & \leq 2 \|\pi_{H,k}^{\text{ms}}(g) - g\|_{L^2(\Omega)} = 2 \|\pi_{H,k}^{\text{ms}}(g) - g - I_H(\pi_{H,k}^{\text{ms}}(g) - g)\|_{L^2(\Omega)} \\ & \lesssim H \|\pi_{H,k}^{\text{ms}}(g) - g\|_{H^1(\Omega)} \lesssim H \|g\|_{H^1(\Omega)}. \end{aligned}$$

Hence, the triangle inequality yields

$$\begin{aligned} & \|\partial_t e^{\text{ms}}\|_{L^\infty(L^2)} + \|e^{\text{ms}}\|_{L^\infty(H^1)} \\ & \lesssim \|\partial_t u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_t u^\varepsilon)\|_{L^\infty(L^2)} + \|u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon)\|_{L^\infty(H^1)} \\ & \quad + \|\partial_{tt} u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_{tt} u^\varepsilon)\|_{L^1(L^2)} + H\|g\|_{H^1(\Omega)}. \end{aligned}$$

Lemma 6.4 finishes the proof.  $\square$

Next, we sketch the proof of Theorem 5.8.

*Proof of Theorem 5.8.* The first part of the proof is completely analogous to the one presented by Baker [7, Section 4] for the classical finite element method. With the same arguments, we can show that

$$\begin{aligned} & \max_{0 \leq n \leq J} \|(u^\varepsilon - u_{H,\Delta t,k} - Q_{h,k}(u_{H,\Delta t,k}))(\cdot, t^n)\|_{L^2(\Omega)} \\ & \lesssim \max_{0 \leq n \leq J} \|(u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon))(\cdot, t^n)\|_{L^2(\Omega)} \\ & \quad + \|(u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k}))(\cdot, 0) - \pi_{H,k}^{\text{ms}}(u^\varepsilon(\cdot, 0))\|_{L^2(\Omega)} \\ & \quad + \|(\partial_t u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_t u^\varepsilon))\|_{L^2(0,T;L^2(\Omega))} + \Delta t^2 (\|\partial_t^3 u^\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t^4 u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}), \end{aligned}$$

where we note that the term  $\|(u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k}))(\cdot, 0) - \pi_{H,k}^{\text{ms}}(u^\varepsilon(\cdot, 0))\|_{L^2(\Omega)}$  is equal to zero, since obviously  $(u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k}))(\cdot, 0) = \pi_{H,k}^{\text{ms}}(f)$  by definition of the method. The last two terms are already readily estimated. It only remains to bound the two terms  $\|(u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon))(\cdot, t^n)\|_{L^2(\Omega)}$  and  $\|(\partial_t u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_t u^\varepsilon))\|_{L^2(L^2)}$  using (40). That finishes the proof.  $\square$

We next prove the homogenization result stated in Theorem 5.12. This is a direct consequence of the following lemma.

**Lemma 6.8.** *Let the same notation and the same assumptions as in Lemma 6.4 be fulfilled, let  $u^0$  be the homogenized solution of (28) and let  $\partial_t u^\varepsilon \in L^1(0, T; H^1(\Omega))$ . Then it holds*

$$\begin{aligned} & \|u^0 - u_{H,k}\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \|u^\varepsilon - u^0\|_{L^\infty(0,T;L^2(\Omega))} \\ & \quad + \|f - \pi_h(f)\|_{L^2(\Omega)} + \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^1(0,T;L^2(\Omega))} \\ & \quad + (H + \theta^k k^{d/2}) (\|F\|_{L^2(0,T;L^2(\Omega))} + \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)} + \|\partial_t u^\varepsilon\|_{L^1(0,T;H^1(\Omega))}). \end{aligned} \tag{54}$$

Recall that the  $\lesssim$ -notation does not allow dependencies on  $T$  and  $\varepsilon$ .

*Proof.* First, recall that  $u_{H,k} \in V_H$  denotes the solution of (15) and define  $u_{H,k}^{\text{ms}} := u_{H,k} + Q_{h,k}(u_{H,k})$ . We aim to split the error into the contributions

$$\|u^0 - u_{H,k}\|_{L^\infty(L^2)} \leq \underbrace{\|u^0 - u^\varepsilon\|_{L^\infty(L^2)}}_{=: \text{I}} + \underbrace{\|u^\varepsilon - u_{H,k}^{\text{ms}}\|_{L^\infty(L^2)}}_{=: \text{II}} + \underbrace{\|Q_{h,k}(u_{H,k})\|_{L^\infty(L^2)}}_{=: \text{III}}.$$

The term I does not have to be estimated any further.

For the term II, we start with the triangle inequality to obtain  $\|u^\varepsilon - u_{H,k}^{\text{ms}}\|_{L^\infty(L^2)} \leq \|u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon)\|_{L^\infty(L^2)} + \|\psi^\pi\|_{L^\infty(L^2)}$ , where  $\psi^\pi := u_{H,k}^{\text{ms}} - \pi_{H,k}^{\text{ms}}(u^\varepsilon)$ . For  $\|\psi^\pi\|_{L^\infty(L^2)}$  we can use estimate (51) that we obtained in the proof of Lemma 6.6, i.e. we have

$$\|\psi^\pi\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \|\psi^\pi(\cdot, 0)\|_{L^2(\Omega)} + \|\partial_t u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_t u^\varepsilon)\|_{L^1(0,T;L^2(\Omega))}. \tag{55}$$

In Theorem 5.12 we claimed that (15) remains valid independently of whether we use  $u_{H,k}^{\text{ms}}(\cdot, 0) = \pi_{H,k}^{\text{ms}}(f)$  or  $u_{H,k}^{\text{ms}}(\cdot, 0) = P_{H,k}^{\text{ms}}(f)$  for the initial value. For the first case we have  $\psi^\pi(\cdot, 0) = 0$

and the term in (55) vanishes. Subsequently, we therefore only consider the non-trivial  $L^2$ -projection case, i.e.  $u_{H,k}^{\text{ms}}(\cdot, 0) = P_{H,k}^{\text{ms}}(f)$ . Now we use the triangle inequality in (55) for  $\psi^\pi = (u_{H,k}^{\text{ms}} - u^\varepsilon) + (u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon))$  and obtain

$$\begin{aligned} \text{II} &= \|u_{H,k}^{\text{ms}} - u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \underbrace{\|(u_{H,k}^{\text{ms}} - u^\varepsilon)(\cdot, 0)\|_{L^2(\Omega)} + \|(u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon))(\cdot, 0)\|_{L^2(\Omega)}}_{=: \text{II}_1} \\ &+ \|u^\varepsilon - \pi_{H,k}^{\text{ms}}(u^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^\varepsilon - \pi_{H,k}^{\text{ms}}(\partial_t u^\varepsilon)\|_{L^1(0,T;L^2(\Omega))}. \end{aligned} \quad (56)$$

Recalling that  $P_{H,k}^{\text{ms}}$  denotes the  $L^2$ -projection on  $V_{H,k}^{\text{ms}}$  and  $\pi_{H,k}^{\text{ms}}$  the elliptic projection, the first term on the right hand side can be estimated as follows

$$\begin{aligned} \text{II}_1 &= \|f - P_{H,k}^{\text{ms}}(f)\|_{L^2(\Omega)} + \|f - \pi_{H,k}^{\text{ms}}(f)\|_{L^2(\Omega)} \\ &\leq \inf_{v \in V_{H,k}^{\text{ms}}} \|f - v\|_{L^2(\Omega)} + \|f - \pi_{H,k}^{\text{ms}}(f)\|_{L^2(\Omega)} \\ &\leq 2\|f - \pi_{H,k}^{\text{ms}}(f)\|_{L^2(\Omega)} \\ &\stackrel{(44)}{\lesssim} \|f - \pi_h(f)\|_{L^2(\Omega)} + (H + \theta^k k^{d/2})\|f\|_{H^1(\Omega)}. \end{aligned} \quad (57)$$

Consequently with (33), we have

$$\begin{aligned} \text{II} &\lesssim \|f - \pi_h(f)\|_{L^2(\Omega)} + (H + \theta^k k^{d/2})\|f\|_{H^1(\Omega)} \\ &+ \|u^\varepsilon - P_H(\pi_{H,k}^{\text{ms}}(u^\varepsilon))\|_{L^\infty(L^2)} + \|\partial_t(u^\varepsilon - P_H(\pi_{H,k}^{\text{ms}}(u^\varepsilon)))\|_{L^1(L^2)} \\ &+ \|(Q_{H,k}^{\text{ms}} \circ P_H \circ \pi_{H,k}^{\text{ms}})(u^\varepsilon)\|_{L^\infty(L^2)} + \|(Q_{H,k}^{\text{ms}} \circ P_H \circ \pi_{H,k}^{\text{ms}})(\partial_t u^\varepsilon)\|_{L^1(L^2)}. \end{aligned} \quad (58)$$

The terms  $\|u^\varepsilon - P_H(\pi_{H,k}^{\text{ms}}(u^\varepsilon))\|_{L^\infty(L^2)}$  and  $\|\partial_t(u^\varepsilon - P_H(\pi_{H,k}^{\text{ms}}(u^\varepsilon)))\|_{L^1(L^2)}$  can be estimated with Lemma 6.4, inequality (39). For the last two terms involving  $Q_{H,k}(\cdot)$ , we can subtract the Clément-type interpolation  $I_H(Q_{H,k}(v)) = 0$  and use the interpolation error estimate (31). This gives us  $\mathcal{O}(H)$ -terms. Then, we can then use the  $H^1$ -stability estimates in (38) to obtain

$$\begin{aligned} \text{II} &\lesssim \|f - \pi_h(f)\|_{L^2(\Omega)} + \|u^\varepsilon - \pi_h(u^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^\varepsilon - \pi_h(\partial_t u^\varepsilon)\|_{L^1(0,T;L^2(\Omega))} \\ &+ (H + \theta^k k^{d/2}) (\|F\|_{L^2(0,T;L^2(\Omega))} + \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)} + \|\partial_t u^\varepsilon\|_{L^1(0,T;H^1(\Omega))}). \end{aligned}$$

The term  $\text{III} = \|Q_{h,k}(u_{H,k})\|_{L^\infty(L^2)}$  can be also estimated in the same way, using (31) and (38). Finally, the energy estimate  $\|u_{H,k}^{\text{ms}}\|_{L^\infty(H^1)} \lesssim \|F\|_{L^2(L^2)} + \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$  is required to bound  $\text{III}$  independent of  $H$  and  $h$ . We obtain the same type of estimate as for  $\text{II}$ .

Combining the estimates for I, II and III finishes the proof.  $\square$

## 7 Numerical experiments

In this section we present the results for three different model problems. The first model problem is taken from [29] and involves a microstructure without scale separation, which however can be described by a smooth coefficient. In the second model problem we abandon the smoothness and consider a problem which involves a highly heterogenous discontinuous coefficient. The third model problem is also inspired by a problem presented in [29]. Here, we add an additional conductivity channel to the heterogenous structure of model problem 2, which results in a high contrast of order  $10^4$ . Finally, in the last experiment, we investigate briefly the accuracy of our method for increasing time.

In the first three computations, we fix the considered time interval to be  $[0, T] := [0, 1]$  and the time step size to be  $\Delta t := 0.05$ . In order to compute the errors for the obtained



multiscale approximations, we use a discrete reference solution  $u_{h,\Delta t}$  as an approximation to the exact solution of problem (1). This reference solution is determined with the Crank-Nicolson scheme for the time discretization (using equidistant time steps with time step size  $\Delta t = 0.05$ ) and a Finite Element method on the fine mesh  $\mathcal{T}_h$  for the space discretization. We use a linear interpolation between the solutions obtained for each time step. Hence,  $\partial_t u_{h,\Delta t}$  is well defined on each time interval  $[t^n, t^{n+1}]$ . By  $u_{H,\Delta t,k}$  we denote the multiscale approximation defined according to (25).

In this section, we use the following notation for the errors:

$$\begin{aligned} e^{0,n} &:= u_{H,\Delta t,k}(\cdot, t^n) - u_{h,\Delta t}(\cdot, t^n) \\ e^{\text{ms},n} &:= (u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k}))(\cdot, t^n) - u_{h,\Delta t}(\cdot, t^n), \\ \partial_t e^{\text{ms},n} &:= \lim_{t \nearrow t^n} \partial_t (u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k}))(\cdot, t) - \partial_t u_{h,\Delta t}(\cdot, t). \end{aligned} \quad (59)$$

By  $\|\cdot\|_{L^2(\Omega)}^{\text{rel}}$  (respectively  $\|\cdot\|_{H^1(\Omega)}^{\text{rel}}$ ) we denote the relative error norms, i.e. the absolute errors divided by the associated norm of the reference solution  $u_{H,\Delta t,k}$ .

## 7.1 Model problem 1

The first model problem is extracted from [29]. As pointed out in [29], a sufficiently accurate reference solution  $u_{h,\Delta t}$  is obtained for a uniform fine grid with resolution  $h = 2^{-7}$ . Hence, we fix  $\mathcal{T}_h$  to be a uniformly refined triangulation of  $\Omega$  with 66.049 DOFs.

**Problem 7.1.** *Let  $\Omega := ]-1, 1[^2$  and  $T := 1$ . Find  $u^\varepsilon \in L^\infty(0, T; H_0^1(\Omega))$  such that*

$$\begin{aligned} \partial_{tt} u^\varepsilon(x, t) - \nabla \cdot (a^\varepsilon(x) \nabla u^\varepsilon(x, t)) &= F(x) && \text{in } \Omega \times (0, T], \\ u^\varepsilon(x, t) &= 0 && \text{on } \partial\Omega \times [0, T], \\ u^\varepsilon(x, 0) &= 0 \quad \text{and} \quad \partial_t u^\varepsilon(x, 0) = 0 && \text{in } \Omega, \end{aligned} \quad (60)$$

where  $F$  is a Gaussian source term given by  $F(x_1, x_2) = (2\pi\sigma^2)^{-1/2} e^{-(x_1^2 + (x_2 - 0.15)^2)/(2\sigma^2)}$  for  $\sigma = 0.05$  and

$$\begin{aligned} a^\varepsilon(x_1, x_2) &:= \frac{1}{6} \left( 1 + \sin(4x_1^2 x_2^2) + \frac{1.1 + \sin(2\pi x_1/\varepsilon_1)}{1.1 + \sin(2\pi x_2/\varepsilon_1)} + \frac{1.1 + \sin(2\pi x_1/\varepsilon_2)}{1.1 + \cos(2\pi x_2/\varepsilon_2)} \right. \\ &\quad \left. + \frac{1.1 + \cos(2\pi x_1/\varepsilon_3)}{1.1 + \sin(2\pi x_2/\varepsilon_3)} + \frac{1.1 + \sin(2\pi x_1/\varepsilon_4)}{1.1 + \cos(2\pi x_2/\varepsilon_4)} + \frac{1.1 + \cos(2\pi x_1/\varepsilon_5)}{1.1 + \sin(2\pi x_2/\varepsilon_5)} \right). \end{aligned} \quad (61)$$

with  $\varepsilon_1 = 1/5$ ,  $\varepsilon_2 = 1/13$ ,  $\varepsilon_3 = 1/17$ ,  $\varepsilon_4 = 1/31$  and  $\varepsilon_5 = 1/65$ . The coefficient  $a^\varepsilon$  is plotted in Figure 1, together with the reference solution  $u_{h,\Delta t}$  for  $t = 1$ .

Note that the Gaussian source term will become singular for  $\sigma \rightarrow 0$ . Hence it influences the regularity of the solution and we expect the multiscale approximation to be less accurate than for a more regular source term. In particular,  $F$  has already a very large  $H^1$ -norm, which is why we cannot expect to see the third order convergence  $O(H^3)$  in (27), unless  $H/\|F\|_{H^1(\Omega)} \ll 1$ .

In Table 1 the relative errors are depicted for various combinations of  $H$  and  $k$  (recall that  $k$  denotes the truncation parameter defined in (11)). The errors are qualitatively comparable to the errors obtained in [29] for similar computations. Furthermore, we observe that the error evolution is consistent with the theoretically predicated rates. EOCs are given in Table 2.

For  $k \approx |\ln(H)| + 1$ , we observe roughly a convergence rate of 1.3 in  $H$  for the  $L^2$ -error of the numerically homogenized solution  $u_{H,\Delta t,k}$ . Adding the corresponding corrector

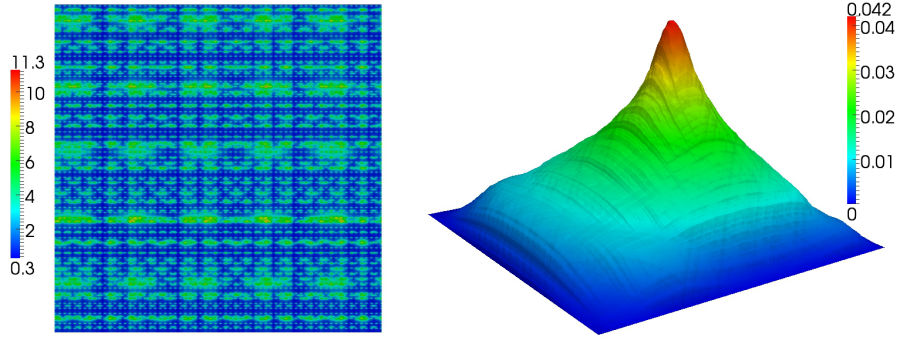


Figure 1: *Model Problem 1. Left Picture: Plot of the coefficient  $a^\varepsilon$  given by (61). Right Picture: reference solution  $u_{h,\Delta t}$  at  $t = 1$  for  $h = 2^{-7}$ .*

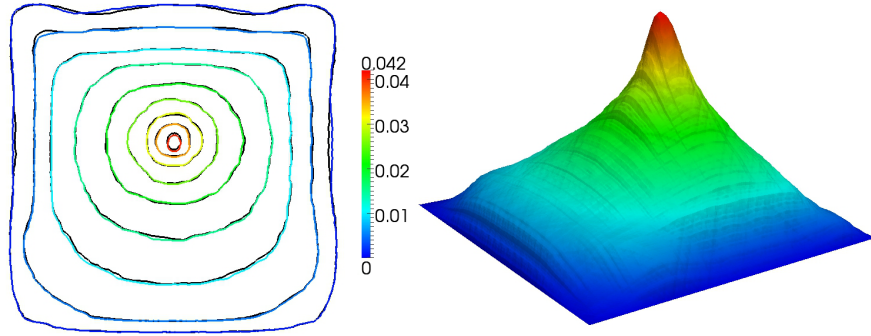


Figure 2: *Model Problem 1, results for  $t^n = 1$ . Left Picture: Comparison of the isolines of the reference solution  $u_{h,\Delta t}$  for  $h = 2^{-7}$  (black isolines) with the multiscale approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  for  $(H,h,k) = (2^{-3}, 2^{-7}, 2)$  (colored isolines). Right Picture: Plot of the multiscale approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  for  $(H,h,k) = (2^{-3}, 2^{-7}, 2)$ .*

Table 1: *Model Problem 1, results for  $t^n = 1$ . The table depicts relative  $L^2$ - and  $H^1$ -errors for the obtained multiscale approximations with respect to the reference solution. The errors are defined in (59).*

$H$	$k$	$\ e^{0,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-1}$	1	0.1448	0.1341	0.4532	0.8718	0.9957
$2^{-1}$	2	0.1394	0.1334	0.4627	0.8312	0.9822
$2^{-2}$	1	0.0780	0.0688	0.3517	0.6464	0.9424
$2^{-2}$	2	0.0687	0.0521	0.2919	0.5439	0.8949
$2^{-2}$	3	0.0675	0.0499	0.2835	0.5362	0.8929
$2^{-3}$	1	0.0368	0.0328	0.2279	0.5824	1.1262
$2^{-3}$	2	0.0242	0.0130	0.1212	0.3285	0.7769
$2^{-3}$	3	0.0234	0.0105	0.1036	0.2846	0.6998

$Q_{h,k}(u_{H,\Delta t,k})$ , the rate is close to 2 in average. In Figure 2, a visual comparison between the reference solution and the multiscale approximation is shown. We observe that for  $(H,h,k) = (2^{-3}, 2^{-7}, 2)$ , the solution  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  looks the same as the reference solution  $u_{h,\Delta t}$  depicted in (1). This is also stressed by the comparison of isolines in Figure 2.

Table 2: *Model Problem 1, results for  $t^n = 1$ . Overview on the EOCs associated with errors from Table 1. We couple  $k$  and  $H$  by  $k = k(H) := \lfloor |\ln(H)| + 1 \rfloor$ . For each of the errors  $\|e_H\|$  below (for  $H = 2^{-i}$ ), we define the average EOC by  $\text{EOC} := \frac{1}{2} \sum_{i=1}^2 \log_2(\|e_{2^{-i}}\| / \|e_{2^{-(i+1)}}\|) / \log_2(2)$ .*

$H$	$k(H)$	$\ e^{0,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-1}$	1	0.1448	0.1341	0.4532	0.8718	0.9957
$2^{-2}$	2	0.0687	0.0521	0.2919	0.5439	0.8949
$2^{-3}$	3	0.0234	0.0105	0.1036	0.2846	0.6998
EOC		1.31	1.84	1.06	0.81	0.25

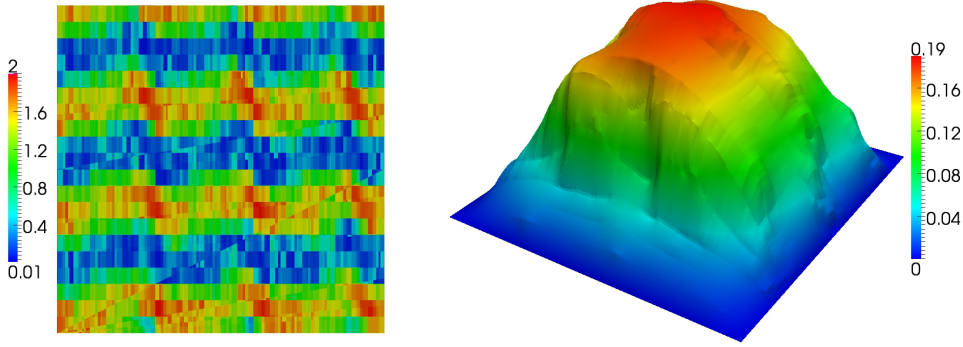


Figure 3: *Model Problem 2, plots for  $t^n = 1$ . Left Picture: Plot of the coefficient  $a^\epsilon$  given by (63). Right Picture: reference solution  $u_{h,\Delta t}$  at  $t = 1$  for  $h = 2^{-8}$ .*

## 7.2 Model problem 2

In Model Problem 2 we investigate the influence of a discontinuous coefficient  $a^\epsilon$  in our multiscale method. According to the theoretical results, it should not influence the convergence rates. The fine grid  $\mathcal{T}_h$  is a uniformly refined triangulation with resolution  $h = 2^{-8}$ .

**Problem 7.2.** *Let  $\Omega := ]0, 1[^2$  and  $T := 1$ . Find  $u^\epsilon \in L^\infty(0, T; H_0^1(\Omega))$  such that*

$$\begin{aligned}
 \partial_{tt} u^\epsilon(x, t) - \nabla \cdot (a^\epsilon(x) \nabla u^\epsilon(x, t)) &= 1 && \text{in } \Omega \times (0, T], \\
 u^\epsilon(x, t) &= 0 && \text{on } \partial\Omega \times [0, T], \\
 u^\epsilon(x, 0) = 0 \quad \text{and} \quad \partial_t u^\epsilon(x, 0) &= 0 && \text{in } \Omega.
 \end{aligned} \tag{62}$$

Here, we have

$$a^\epsilon(x) := (h \circ c_\epsilon)(x) \quad \text{with } h(t) := \begin{cases} t^4 & \text{for } \frac{1}{2} < t < 1 \\ t^{\frac{3}{2}} & \text{for } 1 < t < \frac{3}{2} \\ t & \text{else} \end{cases} \tag{63}$$

and where

$$c_\epsilon(x_1, x_2) := 1 + \frac{1}{10} \sum_{j=0}^4 \sum_{i=0}^j \left( \frac{2}{j+1} \cos \left( \lfloor ix_2 - \frac{x_1}{1+i} \rfloor + \lfloor \frac{ix_1}{\epsilon} \rfloor + \lfloor \frac{x_2}{\epsilon} \rfloor \right) \right).$$

The coefficient  $a^\epsilon$  is plotted in Figure 3 together with the reference solution on  $\mathcal{T}_h$  for  $t = 1$ .

Table 3: *Model Problem 2. Overview on relative  $L^2$ - and  $H^1$ -errors for Model Problem 2 for  $t^n = 1$ . The errors are defined in (59).*

$H$	$k$	$\ e^{0,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-2}$	1	0.1299	0.0613	0.1802	0.1762	0.6615
$2^{-2}$	2	0.1223	0.0245	0.0800	0.1298	0.6323
$2^{-3}$	1	0.0914	0.0616	0.1926	0.2194	0.7255
$2^{-3}$	2	0.0753	0.0191	0.0841	0.1049	0.5902
$2^{-3}$	3	0.0741	0.0085	0.0563	0.0870	0.5688
$2^{-4}$	1	0.0327	0.0243	0.1401	0.1197	0.6710
$2^{-4}$	2	0.0240	0.0047	0.0505	0.0600	0.5109
$2^{-4}$	3	0.0239	0.0029	0.0347	0.0562	0.5004

Table 4: *Model Problem 2, results for  $t^n = 1$ . Overview on the EOCs associated with errors from Table 3. We couple  $k$  and  $H$  by  $k = k(H) := \lfloor |\ln(H)| + 0.5 \rfloor$ . The average EOCs are computed according to (64).*

$H$	$k(H)$	$\ e^{0,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-2}$	1	0.1299	0.0613	0.1802	0.1762	0.6615
$2^{-3}$	2	0.0753	0.0191	0.0841	0.1049	0.5902
$2^{-4}$	3	0.0239	0.0029	0.0347	0.0562	0.5004
EOC		1.22	2.20	1.19	0.82	0.20

In Table 3 we depict various relative  $L^2$ - and  $H^1$ -errors for  $t^n = 1$ . We observe that the numerically homogenized solution  $u_{H,\Delta t,k}$  already yields good  $L^2$ -approximation properties with respect to the fine scale reference solution. These approximation properties can still be significantly improved by adding the corrector  $Q_{h,k}(u_{H,\Delta t,k})$ . We can see that the total approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  is also accurate in the norms  $\|\nabla \cdot\|_{L^2(\Omega)}$  and  $\|\partial_t \cdot\|_{L^2(\Omega)}$ . The errors in the norm  $\|\partial_t \cdot\|_{H^1(\Omega)}$  are noticeably larger than the errors in all other norms. Hence, the multiscale solution does not necessarily yield a highly accurate  $H^1(0, T, H^1(\Omega))$  approximation, even though the discrepancy is still tolerable.

All errors for Model Problem 2 are of the same order as the errors for Model Problem 1 depicted in Table 1. In particular, we do not see any error deterioration caused by the discontinuity of the coefficient  $a^\varepsilon$ . The method behaves nicely in both cases. This is stressed by the experimental orders of convergence (EOCs) shown in Table 4. For an error  $\|e_H\|$  on a coarse grid  $\mathcal{T}_H$  and an error  $\|e_{H/2}\|$  on a coarse grid  $\mathcal{T}_{H/2}$ , the EOC (experimental order of convergence) is given by  $\text{EOC}_H := \log_2(\|e_H\|/\|e_{H/2}\|)/\log_2(2)$ . For the EOCs in Table 4, we use the average

$$\text{EOC} := (\text{EOC}_{2^{-2}} + \text{EOC}_{2^{-3}})/2. \quad (64)$$

Motivated by Theorem 5.8, we couple the coarse mesh  $H$  and the truncation parameter  $k$  to be the closest integer to  $|\ln(H)|$ , i.e. we pick  $k = k(H) := \lfloor |\ln(H)| + 0.5 \rfloor$  for the computation of the EOCs. This gives us  $k = 1$  for  $H = 2^{-2}$ ,  $k = 2$  for  $H = 2^{-3}$  and  $k = 3$  for  $H = 2^{-4}$ . The corresponding results are stated in Table 4. We observe a close to linear convergence for the  $L^2$ -error for the numerically homogenized solution  $u_{H,\Delta t,k}$  (as predicted by the theory).

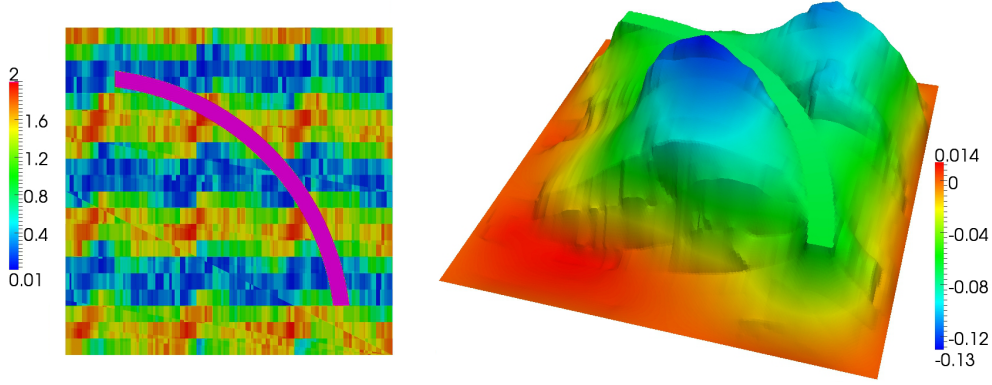


Figure 4: *Model Problem 3. Left Picture: Plot of the coefficient  $a^\varepsilon$ . The basis structure of  $a^\varepsilon$  is given by (63), but this structure is perturbed by an arc-like conductivity channel (pink). In this arc  $a^\varepsilon$  takes the value 100. Right Picture: reference solution  $u_{h,\Delta t}$  at  $t = 1$  for  $h = 2^{-8}$ .*

Adding the corrector  $Q_{h,k}(u_{H,\Delta t,k})$ , the convergence rate increases to 2.2 (slightly worse than the optimal rate of 3). For the  $H^1$ -error, we observe linear convergence. The convergence rate of the  $L^2$ -error for time derivatives is slightly below linear convergence, but still very satisfying. Only the convergence rate for the  $H^1$ -error for the time derivatives is close to stagnation.

Note that the deviation of these rates from the perfect rates comes from that fact that we do not know the generic constant  $C_\theta$  in Theorem 5.8. We picked  $k = \lfloor \ln(H) \rfloor$ , instead of  $k = C_\theta \lfloor \ln(H) \rfloor$ . Still we observe that approximating  $C_\theta$  by 1 yields highly accurate results that are close to the optimal rates. Practically, this justifies the use of small localization patches  $U_k(K)$ .

### 7.3 Model problem 3

This model problem is inspired by a model problem in [29]. The source term as in [29] is given by  $F(x_1, x_2, t) = \sin(2.4x_1 - 1.8x_2 + 2\pi t)$  and as in [29]  $a^\varepsilon$  contains a conductivity channel that perturbs the original structure. As we could not access the data for the channel given in this paper in this paper we model a new one. This last model problem is set to investigate the approximation quality of our multiscale approximations for problems with channel (which do not have to be resolved by the coarse grid) and a high contrast  $\beta/\alpha \approx 10^4$ . As in the previous model problem, we chose  $\mathcal{T}_h$  as a uniformly refined triangulation with resolution  $h = 2^{-8}$ .

**Problem 7.3.** *Let  $\Omega := ]0, 1[^2$  and  $T := 1$ . Find  $u^\varepsilon \in L^\infty(0, T; H_0^1(\Omega))$  such that*

$$\begin{aligned} \partial_{tt} u^\varepsilon(x, t) - \nabla \cdot (a^\varepsilon(x) \nabla u^\varepsilon(x, t)) &= F(x, t) && \text{in } \Omega \times (0, T], \\ u^\varepsilon(x, t) &= 0 && \text{on } \partial\Omega \times [0, T], \\ u^\varepsilon(x, 0) = 0 \quad \text{and} \quad \partial_t u^\varepsilon(x, 0) &= 0 && \text{in } \Omega. \end{aligned} \tag{65}$$

Here, we have  $F(x_1, x_2, t) = \sin(2.4x_1 - 1.8x_2 + 2\pi t)$  and  $a^\varepsilon$  is given by equation (63) but additionally it is disturbed by a high conductivity channel of thickness 0.05. The precise structure of  $a^\varepsilon$  is depicted in Figure 4, together with the reference solution on  $\mathcal{T}_h$  for  $t = 1$ .

We see in Table 5 that the additional channel in the problem does not deteriorate the convergence rates compared to Model Problem 2. Again, close to optimal convergence rates are obtained for the choice  $k = k(H) := \lfloor \lfloor \ln(H) \rfloor + 0.5 \rfloor$  (which slightly underestimates the optimal truncation parameter  $k$ ). The corresponding results are given in Table 5. The

Table 5: *Model Problem 3, results for  $t^n = 1$ . Overview on relative  $L^2$ - and  $H^1$ -errors defined as in (59).*

$H$	$k$	$\ e^{0,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-2}$	1	0.2468	0.1564	0.3321	0.2066	0.4486
$2^{-2}$	2	0.2270	0.0782	0.1992	0.1168	0.3269
$2^{-3}$	1	0.1451	0.1046	0.3305	0.1639	0.4588
$2^{-3}$	2	0.1184	0.0329	0.1535	0.0607	0.2724
$2^{-3}$	3	0.1174	0.0202	0.1024	0.0468	0.2333
$2^{-4}$	1	0.0550	0.0433	0.2186	0.0667	0.3349
$2^{-4}$	2	0.0390	0.0095	0.0803	0.0250	0.1896
$2^{-4}$	3	0.0385	0.0046	0.0464	0.0198	0.1758

Table 6: *Model Problem 3, results for  $t^n = 1$ . Overview on the EOCs associated with errors from Table 5. We couple  $k$  and  $H$  by  $k = k(H) := \lfloor |\ln(H)| + 0.5 \rfloor$ . The average EOCs are computed according to (64).*

$H$	$k(H)$	$\ e^{0,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ \partial_t e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-2}$	1	0.2468	0.1564	0.3321	0.2066	0.4486
$2^{-3}$	2	0.1184	0.0329	0.1535	0.0607	0.2724
$2^{-4}$	3	0.0385	0.0046	0.0464	0.0198	0.1758
EOC		1.34	2.54	1.42	1.69	0.68

method yields accurate results even in the case of conductivity channel and despite that the coarse grid does not resolve the channel. Furthermore the high contrast of order  $10^4$  does not significantly influence the size of the optimal truncation parameter  $k$ . In the model problem we can still work with small localization patches  $U_k(K)$ , independent of the conductivity channel. This is further stressed by Figure 5 which depicts the multiscale approximation (i.e.  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$ ) for the case  $(H, k) = (2^{-4}, 2)$ . The solution looks almost identical to the reference solution  $u_{h,\Delta t}$  and the corresponding isolines match almost perfectly.

## 7.4 Model problem 4

In the last numerical experiment, we look at accuracy of the multiscale approximation for large time intervals. In contrast to the previous three model, we consider now the time interval  $[0, T] := [0, 1000]$  and a time step size  $\Delta t := 1.0$ . It has been observed in the literature that for time intervals of size  $\mathcal{O}(1/\varepsilon^2)$  homogenisation based models fail to capture dispersive effects that appear in the true solution [5, 6, 15, 24, 31]. We are thus interested to test the accuracy of our new multiscale method over such long time intervals. For this purpose, we consider a model problem with known exact solution  $u^\varepsilon$  given by

$$u^\varepsilon(x_1, x_2, t) := \sin(t/10)^2 \left( \sin(2\pi x_1) \sin(2\pi x_2) + \frac{\varepsilon}{2} \cos(2\pi x_1) \sin(2\pi x_2) \sin(2\pi \frac{x_1}{\varepsilon}) \right) \quad (66)$$

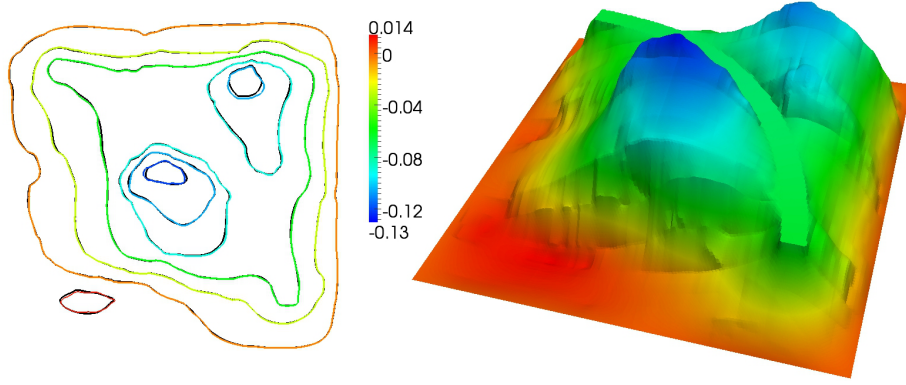


Figure 5: *Model Problem 3, results at  $t = 1$ . Left Picture: Comparison of the isolines of the reference solution  $u_{h,\Delta t}$  for  $h = 2^{-8}$  (black isolines) with the multiscale approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  for  $(H, h, k) = (2^{-4}, 2^{-8}, 2)$  (colored isolines). Right Picture: Plot of the multiscale approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  for  $(H, h, k) = (2^{-4}, 2^{-8}, 2)$ .*

Table 7: *Model Problem 4, results for  $t^n = 1000$  (i.e.  $n = 1000$ ). Overview on relative  $L^2$ - and  $H^1$ -errors defined as in (59). Furthermore, we define  $u_{H,h}^{\text{ms},n} := (u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k}))(\cdot, t^n)$ ;  $u_h^n := u_{h,\Delta t}(\cdot, t^n)(\cdot, t^n)$  and  $u^{\varepsilon,n} := u^{\varepsilon}(\cdot, t^n)$ . The relative errors are the absolute errors divided by the norm of the exact solution.*

$H$	$k$	$\ u_{H,h}^{\text{ms},n} - u^{\varepsilon,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ u_h^n - u^{\varepsilon,n}\ _{L^2(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{L^2(\Omega)}^{\text{rel}}$
$2^{-3}$	2	0.0149	0.0024	0.0143
$2^{-4}$	2	0.0079	0.0024	0.0069
$H$	$k$	$\ u_{H,h}^{\text{ms},n} - u^{\varepsilon,n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ u_h^n - u^{\varepsilon,n}\ _{H^1(\Omega)}^{\text{rel}}$	$\ e^{\text{ms},n}\ _{H^1(\Omega)}^{\text{rel}}$
$2^{-3}$	2	0.0768	0.0364	0.0679
$2^{-4}$	2	0.0652	0.0364	0.0544

for  $\varepsilon = 0.1$ . The function  $u^{\varepsilon}$  is depicted in Figure 6. To study if dispersive effects are captured by our multiscale method, we take  $T = 1000 > (1/\varepsilon^2)$  and define

$$a^{\varepsilon}(x_1, x_2) := \frac{1}{8\pi^2} \begin{pmatrix} 2(2 + \cos(2\pi \frac{x_1}{\varepsilon}))^{-1} & 0 \\ 0 & 1 + \frac{1}{2}\cos(2\pi \frac{x_1}{\varepsilon}) \end{pmatrix}$$

and

$$F^{\varepsilon}(x, t) := -\nabla \cdot (a^{\varepsilon}(x) \nabla u^{\varepsilon}(x, t)) + \partial_t t u^{\varepsilon}(x, t)$$

and regard the following model:

**Problem 7.4.** *Let  $\Omega := ]0, 1[^2$  and  $T := 1000$ . Find  $u^{\varepsilon} \in L^{\infty}(0, T; H_0^1(\Omega))$  such that*

$$\begin{aligned} \partial_{tt} u^{\varepsilon}(x, t) - \nabla \cdot (a^{\varepsilon}(x) \nabla u^{\varepsilon}(x, t)) &= F^{\varepsilon}(x, t) && \text{in } \Omega \times (0, T], \\ u^{\varepsilon}(x, t) &= 0 && \text{on } \partial\Omega \times [0, T], \\ u^{\varepsilon}(x, 0) &= 0 \quad \text{and} \quad \partial_t u^{\varepsilon}(x, 0) = 0 && \text{in } \Omega. \end{aligned} \tag{67}$$

In the following, the fine grid  $\mathcal{T}_h$  is a uniformly refined triangulation with resolution  $h = 2^{-7}$ . In Table 7 we depict relative  $L^2$ - and  $H^1$ -errors between the full multiscale approximation for  $(H, k) = (2^{-3}, 2)$  (respectively  $(H, k) = (2^{-4}, 2)$ ) and the exact solution at  $t^n = 1000$ . No



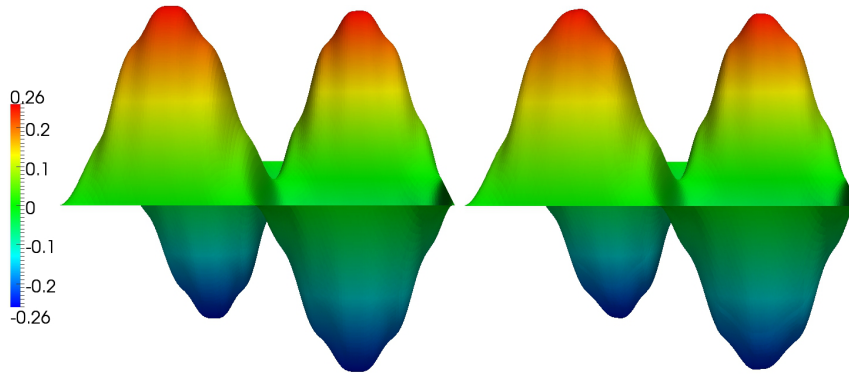


Figure 6: *Model Problem 4, plots for  $t = 1000$ . Left Picture: Plot of the exact solution  $u^\epsilon$  given by (66). Right Picture: full multiscale approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  for  $(H, h, k) = (2^{-3}, 2^{-7}, 2)$ .*

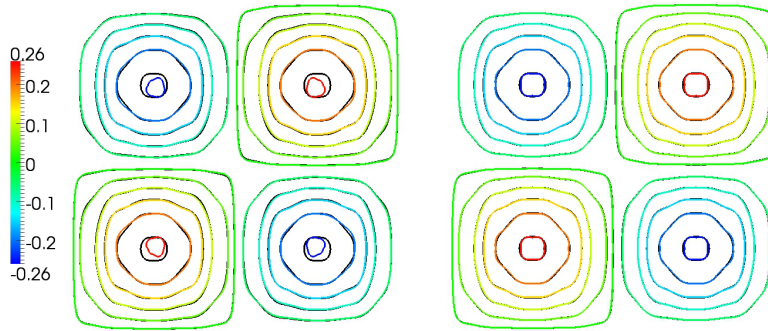


Figure 7: *Model Problem 4, results at  $t = 1000$ . Left Picture: Comparison of the isolines of the exact solution  $u^\epsilon$  (black isolines) with the multiscale approximation  $u_{H,\Delta t,k} + Q_{h,k}(u_{H,\Delta t,k})$  for  $(H, h, k) = (2^{-3}, 2^{-7}, 2)$  (colored isolines). Right Picture: Comparison of the isolines of the exact solution  $u^\epsilon$  (black isolines) with the multiscale approximation for  $(H, h, k) = (2^{-4}, 2^{-7}, 2)$  (colored isolines).*

loss accuracy is observable compared to the previous computations on short time intervals (for similar accuracies). For completeness, Table 7 also contains the errors between multiscale approximation and reference solution, as well as the errors between the reference solution and the exact solution. To underline this observation, in Figure 6 the exact solution is plotted next to the multiscale solution for  $(H, k) = (2^{-3}, 2)$  (again for  $t = 1000$ ). Clearly, all features of the exact solution are also evident in the multiscale approximation. In Figure 7 the corresponding isolines are compared. We can see that there is still a small mismatch for  $(H, k) = (2^{-3}, 2)$ , which however disappears for  $(H, k) = (2^{-4}, 2)$ . In conclusion, we cannot observe any loss in accuracy for computation over large time intervals.

**Acknowledgements.** This work is partially supported by the Swiss National Foundation, Grant: No 200021\_134716/1.

## References

- [1] A. Abdulle. On a priori error analysis of fully discrete heterogeneous multiscale FEM. *Multiscale Model. Simul.*, 4(2):447–459, 2005.



- [2] A. Abdulle. Analysis of a heterogeneous multiscale FEM for problems in elasticity. *Math. Models Methods Appl. Sci.*, 16(4):615–635, 2006.
- [3] A. Abdulle. A priori and a posteriori error analysis for numerical homogenization: a unified framework. *Ser. Contemp. Appl. Math. CAM*, 16:280–305, 2011.
- [4] A. Abdulle and M. J. Grote. Finite element heterogeneous multiscale method for the wave equation. *Multiscale Model. Simul.*, 9(2):766–792, 2011.
- [5] A. Abdulle, M. J. Grote, and C. Stohrer. FE heterogeneous multiscale method for long-time wave propagation. *C. R. Math. Acad. Sci. Paris*, 351(11-12):495–499, 2013.
- [6] A. Abdulle, M. J. Grote, and C. Stohrer. Finite element heterogeneous multiscale method for the wave equation: long-time effects. To appear in *Multiscale Model. Simul.*, 2014.
- [7] G. A. Baker. Error estimates for finite element methods for second order hyperbolic equations. *SIAM J. Numer. Anal.*, 13(4):564–576, 1976.
- [8] S. Brahim-Otsmane, G. A. Francfort, and F. Murat. Correctors for the homogenization of the wave and heat equations. *J. Math. Pures Appl.*, 71(3):197–231, 1992.
- [9] C. Carstensen. Quasi-interpolation and a posteriori error analysis in finite element methods. *M2AN Math. Model. Numer. Anal.*, 33(6):1187–1202, 1999.
- [10] D. Cioranescu and P. Donato. *An introduction to homogenization.*, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, New York, 1999.
- [11] E. De Giorgi and S. Spagnolo. Sulla convergenza degli integrali dell’energia per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital.*, 4(8):391–411, 1973.
- [12] W. E and B. Engquist. The heterogeneous multiscale methods. *Commun. Math. Sci.*, 1(1):87–132, 2003.
- [13] D. Elfverson, E. H. Georgoulis, A. Målqvist, and D. Peterseim. Convergence of a discontinuous Galerkin multiscale method. *SIAM J. Numer. Anal.*, 51(6):3351–3372, 2013.
- [14] B. Engquist, H. Holst, and O. Runborg. Multi-scale methods for wave propagation in heterogeneous media. *Commun. Math. Sci.*, 9(1), 2011.
- [15] B. Engquist, H. Holst, and O. Runborg. Multiscale methods for wave propagation in heterogeneous media over long time. In *Numerical analysis of multiscale computations*, pages 167–186. Springer, 2012.
- [16] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [17] P. Henning and A. Målqvist. Localized orthogonal decomposition techniques for boundary value problems. ArXiv e-print 1308.3379 (accepted for publication in *SISC*, 2014), 2013.
- [18] P. Henning, A. Målqvist, and D. Peterseim. A localized orthogonal decomposition method for semi-linear elliptic problems. *M2AN Math. Model. Numer. Anal.*, 2014 (eFirst).

- [19] P. Henning and M. Ohlberger. The heterogeneous multiscale finite element method for elliptic homogenization problems in perforated domains. *Numer. Math.*, 113(4):601–629, 2009.
- [20] P. Henning and D. Peterseim. Oversampling for the Multiscale Finite Element Method. *Multiscale Model. Simul.*, 11(4):1149–1175, 2013.
- [21] L. Jiang and Y. Efendiev. A priori estimates for two multiscale finite element methods using multiple global fields to wave equations. *Numer. Methods Partial Differential Equations*, 28(6):1869–1892, 2012.
- [22] L. Jiang, Y. Efendiev, and V. Ginting. Analysis of global multiscale finite element methods for wave equations with continuum spatial scales. *Appl. Numer. Math.*, 60(8):862–876, 2010.
- [23] O. Korostyshevskaya and S. E. Minkoff. A matrix analysis of operator-based upscaling for the wave equation. *SIAM J. Numer. Anal.*, 44(2):586–612 (electronic), 2006.
- [24] A. Lamacz. Dispersive effective models for waves in heterogeneous media. *Math. Models Methods Appl. Sci.*, 21(9):1871–1899, 2011.
- [25] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
- [26] A. Målqvist. Multiscale methods for elliptic problems. *Multiscale Model. Simul.*, 9(3):1064–1086, 2011.
- [27] A. Målqvist and D. Peterseim. Localization of elliptic multiscale problems. ArXiv e-print 1110.0692, accepted for publication in MathComp., 2011.
- [28] F. Murat and L. Tartar. H-convergence, topics in the mathematical modeling of composite materials. *Progr. Nonlinear Differential Equations Appl.*, 31:21–43, 1997.
- [29] H. Owhadi and L. Zhang. Numerical homogenization of the acoustic wave equations with a continuum of scales. *Comput. Methods Appl. Mech. Engrg.*, 198(3):397–406, 2008.
- [30] H. Owhadi and L. Zhang. Localized bases for finite-dimensional homogenization approximations with nonseparated scales and high contrast. *Multiscale Model. Simul.*, 9(4):1373–1398, 2011.
- [31] F. Santosa and W. W. Symes. A dispersive effective medium for wave propagation in periodic composites. *SIAM J. Appl. Math.*, 51(4):984–1005, 1991.
- [32] T. Vdovina, S. E. Minkoff, and O. Korostyshevskaya. Operator upscaling for the acoustic wave equation. *Multiscale Model. Simul.*, 4(4):1305–1338 (electronic), 2005.