Multiscale adaptive method for Stokes flow in heterogenenous media

Assyr Abdulle and Ondrej Budáč

ANMC, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, Switzerland, {assyr.abdulle,ondrej.budac}@epfl.ch*

Abstract. We present a multiscale micro-macro method for the Stokes problem in heterogeneous media. The macroscopic method discretizes a Darcy problem on a coarse mesh with permeability data recovered from solutions of Stokes problems around quadrature points. The accuracy of both the macro and the micro solvers is controlled by appropriately coupled a posteriori error indicators, while the total cost of the multiscale method is independent of the pore size. Two and three-dimensional numerical experiments illustrate the capabilities of the adaptive method.

1 Introduction

Fluid flow in porous media is a basic problem in science and engineering. It enters the modeling of geothermal and petroleum reservoirs, subsurface contamination, textile modeling or biomedical materials. Since the pore size is usually much smaller than the considered porous material, global discretization that resolves the pore geometry and standard single-scale techniques such as finite element method (FEM) are extremely expensive.

Averaging techniques such as homogenization of Stokes flow in porous media is thus required in many applications. The homogenization method has been studied by various authors in the past several decades assuming periodic porosity [7, 16, 20, 22]. The effective solution is shown to be given by a Darcy equation where the permeability tensor can be computed from so-called micro problems.

Various multiscale methods have been recently proposed for the numerical approximation of Stokes (or Navier-Stokes) equations in porous media that rely on a Darcy macro problem, recovering the effective permeability from local pore geometries by numerically solving appropriate micro problems. We mention a hierarchical multiscale FEM derived in [10], a two-scale finite element method proposed in [21], and a control volume heterogeneous multiscale method described in [8].

Most of the aforementioned works discuss a priori convergence rates and assume regularity of the micro problems that might not always hold. Indeed, complicated pore structures in typical applications and non-convexity of microscopic fluid domains result in sub-optimal a priori convergence rates. In

* This work is partially supported by the Swiss National Foundation grant 200021_134716.

this contribution, we give a concise description and illustrate numerically a new adaptive numerical homogenization methods for Stokes flow proposed in [3]. The method is built using the framework of the finite element heterogeneous multiscale method (FE-HMM) [1, 4, 13]. Adaptive FE-HMM for elliptic problems has been studied in [2, 5, 18]. Our new method relies on adaptive mesh refinement on macro and micro problems and on rigorous residual-based a posteriori error estimates derived in [3]. One challenge is to adequately couple macro and micro error indicators as to achieve optimal accuracy with minimal computational cost.

The paper is organized as follows. We first review the model problem in Section 2. We then describe the FE-HMM for Stokes flow in Section 3 and the adaptive method in Section 4. In Section 5 we provide two and three-dimensional numerical experiments to test the capabilities of the adaptive method.

2 Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded connected domain, where $d \in \mathbb{N}$ and $d > 1$. Denote Y the d-dimensional unit cube $(-1/2, 1/2)^d$. For any $x \in \Omega$ let $Y_{\mathrm{S}}^x \subset \overline{Y}$ and denote $Y_{\mathrm{F}}^x = Y \backslash Y_{\mathrm{S}}^x$. The sets Y_{F}^x and Y_{S}^x represent the local fluid and solid geometry, respectively. Given a pore size $\varepsilon > 0$, we define the locally periodic porous medium by

$$\Omega_\varepsilon = \Omega \backslash \Big(\varepsilon \bigcup_{m \in \mathbb{Z}^d} (1/2 + m + Y_{\mathrm{S}}^{\varepsilon(1/2+m)} \Big)$$

and consider the following Stokes problem

$$\begin{aligned} -\Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon &= \mathbf{f} && \text{in } \Omega_\varepsilon, \\ \operatorname{div} \mathbf{u}^\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ \mathbf{u}^\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon, \end{aligned}$$

for the velocity field $\mathbf{u}^\varepsilon$ and pressure p^ε, where $\mathbf{f}$ is a given force field.

In case of periodic porous media (Y_{S}^x does not depend on x), the asymptotic behavior of p^ε, $\mathbf{u}^\varepsilon$ as $\varepsilon \to 0$ is studied in [7, 22]. An extension of p^ε and $\mathbf{u}^\varepsilon$ from Ω_ε to Ω is constructed, such that (while keeping the notation for the extensions) $\|p^\varepsilon - p^0\|_{L^2(\Omega)/\mathbb{R}} \to 0$ and $\mathbf{u}^\varepsilon/\varepsilon^2 \to \mathbf{u}^0$ weakly in $L^2(\Omega)$ for $\varepsilon \to 0$, where p^0 and $\mathbf{u}^0$ are given as follows. Find p^0 such that

$$\begin{aligned} \nabla a^0(\mathbf{f} - \nabla p^0) &= 0 && \text{in } \Omega, \\ a^0(\mathbf{f} - \nabla p^0) \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the homogenized tensor $a^0(x)$ is given by the micro problems: Solve

$$\begin{aligned} -\Delta \mathbf{u}^{i,x} + \nabla p^{i,x} &= \mathbf{e}^i && \text{in } Y_{\mathrm{F}}^x, & \mathbf{u}^{i,x} &= 0 && \text{on } \partial Y_{\mathrm{S}}^x - \partial Y, \\ \operatorname{div} \mathbf{u}^{i,x} &= 0 && \text{in } Y_{\mathrm{F}}^x, & \mathbf{u}^{i,x} \text{ and } p^{i,x} & && \text{are } Y\text{-periodic}, \end{aligned} \tag{2}$$

for $i \in \{1, \ldots, d\}$, where $\mathbf{e}^i$ is the i-th canonical basis vector in $\mathbb{R}^d$, and define

$$a^0(x) = \int_{Y_{\mathrm{F}}^x} [\mathbf{u}^{1,x}, \mathbf{u}^{2,x}, \ldots, \mathbf{u}^{d,x}] \, \mathrm{d}y. \tag{3}$$

The effective velocity is then defined as $\mathbf{u}^0 = a^0(\mathbf{f} - \nabla p^0)$.

Well-posedness of the model problem (1), (2), (3) depends on the geometric properties of the micro domains Y_{F}^x and is examined in [3].

3 FE-HMM for flow in porous media

We apply the FE-HMM framework [1,14] to the problem (1), (2), (3) following [3]. Let Ω and Ω_ε be open, connected, bounded, and polygonal subsets of $\mathbb{R}^d$ with $\Omega_\varepsilon \subset \Omega$. Let $\mathcal{T}_H$ be a family of conformal, shape-regular triangulations of Ω parametrized by the mesh size $H = \max_{K \in \mathcal{T}_H} H_K$, where $H_K = \mathrm{diam}(K)$. Define the macro FE space

$$S^l(\Omega, \mathcal{T}_H) = \{q^H \in H^1(\Omega) : q^H|_K \in \mathcal{P}^l(K), \ \forall K \in \mathcal{T}_H\},$$

where $\mathcal{P}^l(K)$ is the space of polynomials on K of degree $l \in \mathbb{N}$.

For each element $K \in \mathcal{T}_H$, consider a quadrature formula (QF) with interior quadrature nodes $\{x_{K_j}\}_{j=1}^J$ and positive weights $\{\omega_{K_j}\}_{j=1}^J$, where $J \in \mathbb{N}$. To guarantee the optimal order of accuracy (see [12, Chap. 4.1]), we assume that the QF is exact for polynomials up to order $\max(2l-2, l)$. Define $Q^K = \{x_{K_j}\}_{j=1}^J$ and $Q^H = \cup_{K \in \mathcal{T}_H} Q^K$.

Let $\delta \geq \varepsilon$. For each $x \in Q^H$ we define the local geometry snapshot by

$$Y_{\mathrm{S}}^{x,\delta} = ((\mathbb{R}^d - \Omega_\varepsilon) \cap (x + \delta \overline{Y}) - x)/\varepsilon, \qquad Y_{\mathrm{F}}^{x,\delta} = (\delta/\varepsilon) Y - Y_{\mathrm{S}}^{x,\delta}.$$

Let $\mathcal{T}_h^x$ be a family of conformal, shape-regular triangulations of $Y_{\mathrm{F}}^{x,\delta}$ parametrized by the mesh size $h = \max_{T \in \mathcal{T}_h^x} h_T$, where $h_T = \mathrm{diam}(T)$. We consider the Taylor-Hood $\mathbb{P}_{k+1}/\mathbb{P}_k$ FE space with $k \in \mathbb{N}$ and periodic coupling for the micro problems (for other micro FE spaces or couplings see [3]) and define

$$M(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x) = H^1_{\mathrm{per}}(Y_{\mathrm{F}}^{x,\delta})^d \cap S^k(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x),$$
$$X(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x) = \{\mathbf{v} \in H^1_{\mathrm{per}}(Y_{\mathrm{F}}^{x,\delta})^d : \mathbf{v} = 0 \text{ on } \partial Y_{\mathrm{S}}^{x,\delta}\} \cap S^{k+1}(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x)^d.$$

The FE-HMM for Stokes flow reads as follows: find $p^H \in S^l(\Omega, \mathcal{T}_H)/\mathbb{R}$ such that

$$B_H(p^H, q^H) = L_H(q^H) \qquad \forall q^H \in S^l(\Omega, \mathcal{T}_H)/\mathbb{R}, \tag{4}$$

where

$$B_H(p^H, q^H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \omega_{K_j} a^h(x_{K_j}) \nabla p^H(x_{K_j}) \cdot \nabla q^H(x_{K_j}),$$

$$L_H(q^H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \omega_{K_j} a^h(x_{K_j}) \mathbf{f}^H(x_{K_j}) \cdot \nabla q^H(x_{K_j}).$$

We observe that the precise knowledge of $\varepsilon > 0$ is not necessary to apply the above method. Here, $\mathbf{f}^H$ is a suitable interpolation of the force field $\mathbf{f} \in L^2(\Omega)^d$ and $a^h(x_{K_j})$ is a numerical approximation of the tensor $a^0(x_{K_j})$ computed by the micro Stokes problems: For any $i \in \{1, \dots, d\}$ and $x \in Q^H$ find $\mathbf{u}^{i,x,h} \in X(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x)$ and $p^{i,x,h} \in M(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x)/\mathbb{R}$ such that[1]

$$\begin{aligned} (\nabla \mathbf{u}^{i,x,h}, \nabla \mathbf{v}) - (\nabla \mathbf{v}, p^{i,x,h}) = (\mathbf{e}^i, \mathbf{v}) \quad & \forall \mathbf{v} \in X(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x) \\ (\nabla \mathbf{u}^{i,x,h}, q) = 0 \quad & \forall q \in M(Y_{\mathrm{F}}^{x,\delta}, \mathcal{T}_h^x)/\mathbb{R} \end{aligned} \tag{5}$$

and set

$$a^h(x) = \frac{\varepsilon^d}{\delta^d} \int_{Y_{\mathrm{F}}^{x,\delta}} [\mathbf{u}^{1,x,h}, \dots, \mathbf{u}^{d,x,h}] \, \mathrm{d}y.$$

A velocity field approximation can be obtained by interpolation from quadrature points. If the QF has the minimal number of nodes ($J = \binom{l+d-1}{d}$), we know [6] that any tensor $A(x) : Q^H \to \mathbb{R}^d$ uniquely defines an operator $\Pi_A : S_{\mathrm{D}}^{l-1}(\Omega, \mathcal{T}_H)^d \to S_{\mathrm{D}}^{l-1}(\Omega, \mathcal{T}_H)^d$ such that

$$\Pi_A(\mathbf{v})(x) = A(x)\mathbf{v}(x), \quad \forall x \in Q^H,$$

where the space $S_{\mathrm{D}}^{l-1}(\Omega, \mathcal{T}_H)$ is a space of functions $q^H : \Omega \to \mathbb{R}$ such that $q^H \in \mathcal{P}^{l-1}(K)$ for every $K \in \mathcal{T}_H$. We define the reconstructed velocity field by $\mathbf{u}^H = \Pi_{a^h}(\mathbf{f}^H - \nabla p^H)$.

Assuming sufficient regularity, a priori estimates derived in [3] yield

$$|p^0 - p^H|_{H^1(\Omega)} \le CH^l + r_{\mathrm{mic}} + r_{\mathrm{mod}}, \tag{6}$$

where $|\cdot|_{H^1(\Omega)}$ denotes the standard H^1 seminorm, r_{mod} is a modeling error (vanishing if $Y_{\mathrm{F}}^{x,\delta} = Y_{\mathrm{F}}^x$ is used), and r_{mic} is a micro error. In many practical applications, we expect $r_{\mathrm{mic}} \le Ch^\theta$ with $\theta < 2$ instead of the ideal $\theta = k + 2$ (see Section 5).

4 Adaptive method

Suboptimal a priori error estimates (for non-convex microscopic fluid domain) suggest to use an adaptive methods. In [3], the residual-based FE-HMM error analysis developed in [2, 6] was coupled with an a posteriori error bound for the micro Stokes flow (5) based on [23]. This result is summarized in Theorem 1 and uses the following. Define the *macro residual* η_K by

$$\begin{aligned} \eta_K^2 = & H_K^2 \|\nabla \cdot \Pi_{a^h}(\mathbf{f}^H - \nabla p^H)\|_{L^2(K)}^2 \\ & + \textstyle\sum_{e \in \partial K} \frac{1}{2} H_e \|[\Pi_{a^h}(\mathbf{f}^H - \nabla p^H) \cdot \mathbf{n}]_e\|_{L^2(e)}^2, \end{aligned}$$

[1] We use $(\cdot, \cdot)$ for the standard scalar product in $L^2(Y_{\mathrm{F}}^{x,\delta})^m$ for any $m \in \mathbb{N}$.

the *data approximation error* $\xi_{\mathrm{data},K}$ by

$$\xi^2_{\mathrm{data},K} = \|a^0(\mathbf{f} - \nabla p^H) - \Pi_{\overline{a}}(\mathbf{f}^H - \nabla p^H)\|^2_{L^2(K)},$$

where $\overline{a}(x) = \lim_{h\to 0} a^h(x)$ and the *micro residual* $\eta_{\mathrm{mic},K}$ by

$$\begin{aligned}\eta^2_{\mathrm{mic},K} =& \|\mathbf{f}^H - \nabla p^H\|^2_{L^2(K)} \max_{x\in Q^K} \sum_{i=1}^{d} \eta^2_{\mathrm{stokes},x,i},\\ \eta^2_{\mathrm{stokes},x,i} =& \sum_{T\in\mathcal{T}_h^x} \left(\sum_{e\in\partial T\setminus\partial Y_{\mathrm{F}}^{x,\delta}} \frac{h_e}{2} \left\| \left[\frac{\partial \mathbf{u}^{i,x,h}}{\partial \mathbf{n}} - p^{i,x,h}\mathbf{n} \right]_e \right\|^2_{L^2(e)} \right.\\ &\left. + h_T^2 \|\Delta \mathbf{u}^{i,x,h} - \nabla p^{i,x,h} + \mathbf{e}^i\|^2_{L^2(T)} + \|\nabla\cdot \mathbf{u}^{i,x,h}\|^2_{L^2(T)} \right).\end{aligned}$$

Theorem 1. *There is a constant C depending only on Ω, on the continuity and coercivity constants of a^0, on the shape-regularity of $\mathcal{T}_H$ and $\mathcal{T}_h^x$, and on the Poincaré-Friedrichs and inf-sup constants related to* (5)*, such that*

$$|p^0 - p^H|^2_{H^1(\Omega)} \le C \sum_{K\in\mathcal{T}_H} (\eta_K^2 + \eta^2_{mic,K} + \xi^2_{data,K}).$$

Moreover, if $Y_F^{x,\delta} = Y_F^x$, then $\xi_{data,K} = 0$.

Theorem 1 gives a foundation for an adaptive refinement algorithm on both macro and micro problems using the indicators η_K and $\eta_{\mathrm{mic},K}$. The usual refinement cycle ↱ solve → estimate → mark → refine ⅂ is implemented on both scales.

The stopping criterion in the adaptive solution of the micro problems is

$$\eta^2_{\mathrm{stokes},x,i} \le \mu d^{-1} \eta_K^2 \|\mathbf{f}^H - \nabla p^H\|^{-2}_{L^2(K)} \qquad \forall K \in \mathcal{T}_H, \tag{7}$$

where $\mu > 0$ is a problem dependent constant and can be calibrated as described in [3]. The inequality (7) implies $\eta^2_{\mathrm{mic},K} \le \mu\eta_K^2$, i.e., the micro error is dominated by the macro error.

Algorithm. Assume that the user provides Ω, Ω_ε, and δ. Then repeat:

Solve. For each quadrature point $x \in Q^H$ solve the micro problems (5) adaptively using the stopping criteria (7).[2] Then, find p^H by solving (4).
Estimate. Compute η_K and $\eta_{\mathrm{mic},K}$. Repeat the previous step until (7) is satisfied.
Mark. Using the indicator η_K and the Döfler's bulk-chasing marking strategy E [24, Chapter 4.1], mark a subset of elements of $\mathcal{T}_H$.
Refine. The marked elements are refined while maintaining conformity [9, 11].

[2] Since the right hand side of (7) is not known beforehand we use an approximation from the previous solution.

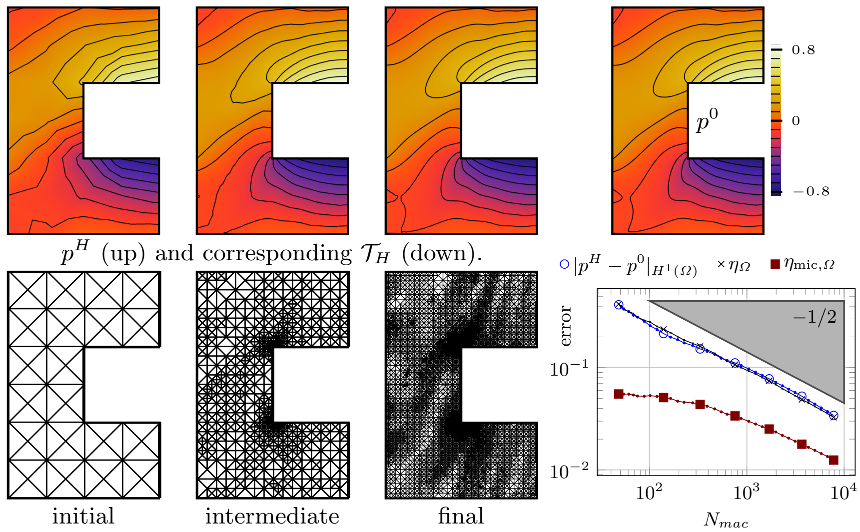

Fig. 1. FE-HMM in 2D: p^H in different stages of refinement (upper left); corresponding meshes (lower left); p^0 (upper right); error and indicators (lower right).

5 Numerical Experiments

In this section we test our adaptive algorithm by presenting two numerical experiments. The implementation is done in Matlab and makes use of AFEM [11] and gmsh [15]. Sparse saddle point linear systems arising from the micro problems were solved using the Matlab's `mldivide` in two dimensions (2D) and an Uzawa method [19] with algebraic multigrid preconditioning by AGMG [17] in three dimensions (3D).

In both experiments we took $\mathbb{P}_2/\mathbb{P}_1$ Taylor-Hood micro FE and $\mathbb{P}_1$ macro FE. We set $\delta = \varepsilon$ (eliminating the modeling error) for the micro domains $Y_{\mathrm{F}}^{x,\delta}$. Variation of Y_{F}^{x} for both examples is depicted in Figure 2.

2D experiment. Let $\Omega = ((0,2)\times(0,3))\backslash([1,2]\times[1,2])$ with periodic boundary conditions between the edges $(0,2)\times\{0\}$ and $(0,2)\times\{3\}$ and let $\mathbf{f} \equiv \mathbf{f}^H \equiv (0,-1)$. Setting $\delta = \varepsilon = 10^{-4}$, we performed the adaptive FE-HMM method. Convergence rates and examples of solutions and meshes are displayed in Figure 1. The global error estimator $\eta_\Omega = (\sum_{K\in\mathcal{T}_H} \eta_K^2)^{-1/2}$ and the error $|p^H - p^0|_{H^1(\Omega)}$ are both following the expected rate $O(N_{\mathrm{mac}}^{-1/2})$, where N_{mac} is the number of degrees of freedom of the macro problem (4). The micro error estimator $\eta_{\mathrm{mic},\Omega}^2 = \sum_{K\in\mathcal{T}_H} \eta_{\mathrm{mic},K}^2$ is dominated by η_Ω^2.

3D experiment. Let Ω be a subset of $(0,2)\times(0,2)\times(0,3)$ for which $(x_3-2)(x_3-1) > 0$ or $\max(x_1,x_2) < 1$ and let $\mathbf{f} \equiv \mathbf{f}^H \equiv (0,0,-1)$. Consider

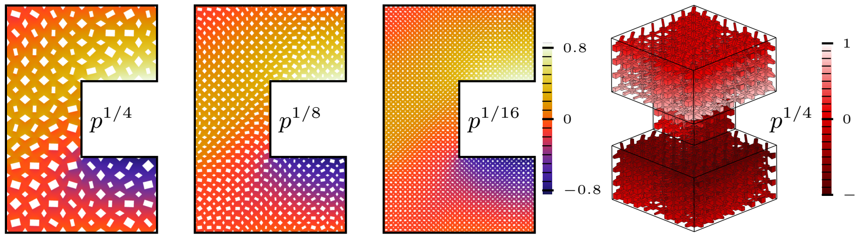

Fig. 2. Plots of p^ε for the 2D (left) and 3D (right) experiment.

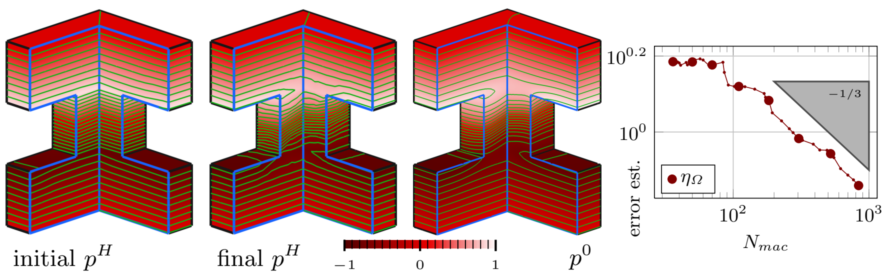

Fig. 3. FE-HMM in 3D: initial and final solution p^H (left) and p^0 (middle) displayed on a cut domain $\Omega\backslash((0.5,2)\times(0.5,2)\times(0,3))$; the error indicator (right).

periodic boundary conditions on Ω that connect the faces $(0,2)\times(0,2)\times\{0\}$ and $(0,2)\times(0,2)\times\{3\}$. The adaptive FE-HMM with $\delta=\varepsilon=10^{-2}$ yields a global error estimate η_Ω that seems to follow the right convergence rate $O(N_{\text{mac}}^{-1/3})$, which is displayed in Figure 3 along with plots of p^H and p^0.

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