SIAM/ASA J. Uncertainty Quantification
Vol. 2, pp. 223–244

Perturbation Analysis for the Darcy Problem with Log-Normal Permeability*

Francesca Bonizzoni† and Fabio Nobile‡

Abstract. We study the single-phase flow in a saturated, bounded heterogeneous porous medium. We model the permeability as a log-normal random field. We perform a perturbation analysis, expanding the solution in Taylor series. The approximation properties of the Taylor polynomial are studied, and the *local convergence* of the Taylor series is proved. With a counterexample we show that, in general, the Taylor series is *not globally convergent* to the stochastic solution as the polynomial degree goes to infinity. Nevertheless, for small variability of the permeability field and low degree of the Taylor polynomial, the perturbation approach is feasible and provides a good approximation of both the stochastic solution and the statistical moments of the stochastic solution. We derive an upper bound on the norm of the residual of the Taylor series, which predicts the optimal degree of the Taylor polynomial to consider. The upper bound is quite pessimistic. In the simple case of a permeability field described by only one random variable, we show numerically that a simple "tuning" of the upper bound, which uses estimates of the growth of the derivatives, provides sharp bounds.

Key words. perturbation technique, uncertainty quantification, elliptic PDE with random coefficient, log-normal distribution

AMS subject classifications. 65C20, 35R60, 41A58, 65N15, 35B20, 60H25

DOI. 10.1137/130949415

1. Introduction.

In many mathematical models, the input parameters are affected by uncertainty, which may be due to the incomplete knowledge or the intrinsic variability of certain phenomena. Some illustrative examples are flows in porous media, combustion problems, earthquake engineering, biomedical engineering, and finance.

Starting from a suitable partial differential equation (PDE) model, we describe the uncertain parameters as random variables or random fields. The aim of *uncertainty quantification* is to infer the solution of the stochastic PDE (SPDE) by computing statistics of the solution or of functionals of it.

The situation we are interested in is the study of single-phase flow of a fluid in a bounded heterogeneous saturated porous medium. Randomness typically arises in the forcing terms, as, for instance, pressure gradients (see, e.g., [15, 40, 41, 46, 11]), as well as in the permeability tensor, due to the impossibility of a full characterization of conductivity properties of subsurface media (see, e.g., [48, 42, 28, 29, 20, 8]). In this work, we focus on the following

*Received by the editors December 17, 2013; accepted for publication (in revised form) April 4, 2014; published electronically June 17, 2014. This work was supported by Italian grant FIRB-IDEAS (project RBID08223Z), "Advanced Numerical Techniques for Uncertainty Quantification in Engineering and Life Science Problems," and by the Center for Advanced Modeling Science (CADMOS).

http://www.siam.org/journals/juq/2/94941.html

†Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria (francesca.bonizzoni@univie.ac.at).

‡Calcul Scientifique et Quantification de l'Incertitude – MATHICSE, École Polytechnique Fédérale de Lausanne, Station 8, Lausanne CH-1015, Switzerland (fabio.nobile@epfl.ch).

linear elliptic SPDE posed in the bounded domain $D \subset \mathbb{R}^d$:

$$-\mathrm{div}(a(\omega, x)\nabla u(\omega, x)) = f(x), \quad x \in D, \tag{1.1}$$

where $u(\omega, x)$ represents the hydraulic head, the forcing term $f(x)$ is deterministic, and the permeability tensor $a(\omega, x)$ is modeled as a *log-normal random field*, i.e., $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, with $Y(\omega, x)$ a centered Gaussian random field. Here ω represents a random elementary event. Note that in (1.1) the differential operators refer to the spatial variable $x \in D$. The log-normal model is widely used in geophysical applications; see, e.g., [8, 20, 28, 29, 42]. In recent years, it has appeared also in the mathematical literature [12, 13, 23, 26]. Given complete statistical information on the permeability field $a(\omega, x)$, the aim of this work is to infer the statistical moments of the stochastic solution $u(\omega, x)$.

The Monte Carlo sampling method is the easiest way to compute the statistics of $u(\omega, x)$. It features a rate of convergence on the order of $M^{-1/2}$, M being the number of independent realizations independent of the dimension of the probability space. On the other hand, a large number of realizations has to be considered in general to reach a satisfactory accuracy. In recent years, a number of improvements have been proposed and applied to SPDEs. Among them, we recall the multilevel Monte Carlo method [7, 18, 43] and the quasi Monte Carlo method [27, 31].

The generalized polynomial chaos expansion of the stochastic solution gives rise to a second family of methods. It can be coupled with a projection strategy (stochastic Galerkin method [6, 22, 25, 26, 35, 40, 45]) or an interpolation strategy (stochastic collocation method [5, 24, 36, 37, 47]). These approaches exploit the regularity of the solution in the random variables, but suffer in handling very high dimensional probability spaces.

In this work, we address the case of small randomness and consider a *perturbation approach*, alternative to Monte Carlo sampling or polynomial chaos expansion, based on the Taylor expansion of the solution $u : L^\infty(D) \to H^1(D)$ with respect to the Gaussian random field $Y \in L^\infty(D)$:

$$T^K u(Y, x) := \sum_{k=0}^{K} \frac{D^k u(0)[Y]^k}{k!}, \quad K \geq 1,$$

where $D^k u(0)[Y]^k$ denotes the kth Gâteaux derivative of u in $Y = 0$ evaluated along the vector

$$\underbrace{(Y, \dots, Y)}_{k \text{ times}}.$$

We want to use the Taylor polynomial $T^K u(Y, x)$ to approximate the statical moments of u or of functionals of it: $\mathbb{E}\left[g(u)\right] \simeq \mathbb{E}\left[g(T^K u)\right]$, g being a smooth function.

One possible strategy to compute the expected value of the Taylor polynomial $\mathbb{E}\left[T^K u\right]$ is to develop in series the random field Y (e.g., the Karhunen–Loève expansion or Fourier expansion) and then truncate it. Then the solution u will depend only on a finite number of random variables, and its Taylor polynomial can be explicitly computed. The limitation of this approach is that it suffers, in general, the curse of dimensionality as more and more random variables are retained in the expansion of Y, although recent results that overcome this limitation in particular cases can be found in [17, 19].

As an alternative, one may adopt the so-called *moment equations* approach, which consists in deriving, analyzing, and solving the deterministic equations solved by $\mathbb{E}\left[D^k u(0)[Y]^k\right]$ for $k \geq 0$.

Perturbation approaches coupled with the moment equations have been widely used in the literature. In the context of perturbations with respect to random fields (infinite-dimensional parameter space), we mention, for instance, the work [30], which considers (1.1) in a random domain; the contributions [28, 29, 42] from the hydrology literature, where log-normal random models for the permeability field are considered; [19, 44], which address problem (1.1) with a permeability field described as a linear combination of countably many bounded random variables; and [16], which considers a first order approximation of the mth moment equation, $m \geq 1$.

In the literature, whenever an infinite-dimensional random field is considered, the majority of authors compute only a second order correction to the mean and variance of the stochastic solution. The aim of the present work, concerning the log-normal model, is rather to investigate the *approximation properties of Taylor polynomials*. We do not deal with the computational aspects related to the numerical solution of the moment equations, described in [10], which will be the subject of a forthcoming paper.

The main achievements of the present paper are the following. We prove that the Taylor series of $u : L^\infty(D) \to H^1(D)$ is *locally convergent* in a bounded open ball of $L^\infty(D)$ of sufficiently smooth radius for every $\sigma > 0$, σ being the standard deviation of the Gaussian random field $Y(\omega, x)$. With a counterexample we also show that, in general, the Taylor series is not globally convergent (on all $L^\infty(D)$, with the Gaussian measure on it), and we should not expect $\mathbb{E}\left[T^K u\right]$ to converge to $\mathbb{E}[u]$. Nevertheless, for small values of σ and K, the perturbation approach can still provide a good approximation of both the stochastic solution and the statistical moments of it. We derive an a priori upper bound on the norm of the residual of the Taylor series, which predicts the optimal degree of the Taylor polynomial to consider. The numerical tests developed show that the upper bound is quite pessimistic. However, in a simple test case where the permeability is described using only one random variable, we perform an a posteriori fitting on the H^1-norm of the Gâteaux derivatives $D^k u(0)[Y]^k$, and obtain a sharp estimate which properly predicts the behavior of the error $\mathbb{E}\left[\left\|u - T^K u\right\|_{H^1}\right]$.

The outline of the paper is as follows. Section 2 introduces the problem at hand and states some results on the statistical moments of the L^∞-norm of a sufficiently smooth Gaussian random field, extending the results in [12]. In section 3 we expand the stochastic solution of the SPDE in Taylor series and prove the local convergence of the Taylor series. In section 4 the global approximation properties of the Taylor polynomial are studied, and an estimate on the optimal degree of the Taylor polynomial is derived. Finally, section 5 is focused on some numerical tests in a one-dimensional case. Section 6 draws some conclusions.

2. Problem setting.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where Ω is the set of outcomes, $\mathcal{F}$ the σ-algebra of events, and $\mathbb{P} : \Omega \to [0, 1]$ a probability measure. Let D be an open bounded domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with locally Lipschitz boundary. We are interested in the Darcy boundary value problem with stochastic permeability: given $f \in L^2(D)$ and $g \in H^{1/2}(\Gamma_D)$, find $u \in L^p\left(\Omega; H^1(D)\right)$ s.t. $u|_{\Gamma_D} = g$, and

$$\int_D a(\omega, x) \nabla u(\omega, x) \cdot \nabla v(x)\, dx = \int_D f(x) v(x)\, dx \quad \forall v \in H^1_{\Gamma_D}(D) \text{ a.s. in } \Omega, \tag{2.1}$$

where $\{\Gamma_D,\ \Gamma_N\}$ is a partition of the boundary of the domain ∂D, and homogeneous Neumann boundary conditions are imposed on Γ_N. We denoted with $H^1_{\Gamma_D}(D)$ the subspace of $H^1(D)$ of functions whose trace vanishes on Γ_D, and with $L^p\left(\Omega; H^1(D)\right)$ the Bochner space of functions $v(\omega,x)$ s.t. $\|v\|_{L^p(\Omega;H^1)} := \left(\int_\Omega \|v(\omega)\|^p_{H^1}\,\mathrm{d}\mathbb{P}(\omega)\right)^{1/p} < \infty$.

We describe the permeability field as a log-normal random field $a(\omega,x) = \mathrm{e}^{Y(\omega,x)}$, where $Y : \Omega \times \bar{D} \to \mathbb{R}$ is a Gaussian random field. The log-normal model is frequently used in geophysical applications; see, for example, [48, 20, 8, 28, 29, 42]. Let us define the mean-free Gaussian random field $Y'(\omega,x) = Y(\omega,x) - \mathbb{E}[Y](x)$, and assume that its covariance kernel $\mathrm{Cov}_{Y'} : \overline{D\times D} \to \mathbb{R}$ is Hölder continuous with exponent t for some $0 < t \leq 1$. In [10] the following proposition is proved, which extends the result in [12] obtained only for centered second order *stationary* random fields Y with covariance function $\mathrm{Cov}_Y(x_1,x_2) = \nu(\|x_1 - x_2\|)$ for some $\nu \in C^{0,1}(\mathbb{R}^+)$.

Proposition 2.1. *Let $Y : \Omega \times \bar{D} \to \mathbb{R}$ be a Gaussian random field with covariance function $\mathrm{Cov}_Y \in \mathcal{C}^{0,t}(\overline{D\times D})$ for some $0 < t \leq 1$. Suppose $\mathbb{E}[Y] \in \mathcal{C}^{0,t/2}(\bar{D})$. Then there exists a version of Y whose trajectories belong to $\mathcal{C}^{0,\alpha}(\bar{D})$ a.s. for $0 < \alpha < t/2$.*

In what follows, we identify the Hölder regular version of the field with $Y(\omega,x)$, so that $\|Y(\omega)\|_{L^\infty(D)}$, $a_{min}(\omega) := \min_{x\in\bar{D}} a(\omega,x)$, and $a_{max}(\omega) := \max_{x\in\bar{D}} a(\omega,x)$ are well-defined random variables. Using Fernique's theorem (see, e.g., [21]), in [12] the author shows that

$$\frac{1}{a_{min}(\omega)} \in L^p(\Omega,\mathbb{P}), \quad a_{max}(\omega) \in L^p(\Omega,\mathbb{P}) \quad \forall\, 0 < p < +\infty.$$

The well-posedness of problem (2.1) follows from the Lax–Milgram lemma applied for almost all $\omega \in \Omega$ and the L^p integrability of $\frac{1}{a_{min}(\omega)}$. See [23, 26, 12].

Remark 2.1. From the point of view of applications, it is very interesting to study also the case of a random field conditioned to available observations. Take, for example, the fluid flow in a heterogeneous porous medium: the permeability varies randomly and can be measured only in a certain number of spatial points. Assuming that N_{obs} pointwise measurements of the permeability have been collected (e.g., by exploratory wells), one can construct a conditioned random field Y whose covariance function is nonstationary but still Hölder continuous with the same exponent, so that Proposition 2.1 holds.

2.1. Upper bounds for the statistical moments of $\|Y'\|_{L^\infty(D)}$. Let us denote by $\sigma^2 := \frac{1}{|D|}\int_D \mathbb{V}ar\,[Y(\cdot,x)]\ dx$. If $Y(\omega,x)$ is a stationary field, then its variance is independent of $x \in D$ and coincides with σ^2. By a little abuse of notation, in what follows we will refer to σ^2 as the variance of Y also in the case of a nonstationary random field.

Let us start from the Karhunen–Loève expansion of the Gaussian random field $Y(\omega,x)$ (see, e.g., [33, 34]):

$$Y(\omega,x) = \mathbb{E}[Y](x) + \sigma \sum_{j=1}^{+\infty} \sqrt{\lambda_j}\ \phi_j(x)\ \xi_j(\omega), \quad (\omega,x) \in \Omega \times D, \tag{2.2}$$

where $\{\sigma^2\lambda_j\}_{j\geq 1}$ is the decreasing sequence of nonnegative eigenvalues of the operator $L^2(D) \ni v \mapsto \int_D \mathrm{Cov}_Y(x_1,x_2)v(x_2)\ dx_2 \in L^2(D)$, $\{\phi_j(x)\}_{j\geq 1}$ are the corresponding eigenfunctions, which form an orthonormal basis of $L^2(D)$, and $\{\xi_j(\omega)\}_{j\geq 1}$ are the centered independent Gaussian random variables with unit variance defined as $\xi_j(\omega) = \frac{1}{\sigma\sqrt{\lambda_j}} \int_D (Y(\omega,x) - \mathbb{E}[Y](x))\,\phi_j(x)\,dx$. Under the assumption $R_\gamma := \sum_{j=1}^{+\infty} \lambda_j \|\phi_j\|^2_{\mathcal{C}^{0,\gamma}(\bar{D})} < +\infty$, in [13] the author shows that

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq C_{Y'}\, R_\gamma^{k/2} \sigma^k\ (k-1)!! \quad \forall\ k > 0 \text{ integer}, \tag{2.3}$$

where $C_{Y'}$ is a positive constant independent of σ. Recall that, given a positive integer n, the bifactorials of $2n$ and $2n-1$ are defined as $(2n)!! = \prod_{i=1}^n 2i$ and $(2n-1)!! = \prod_{i=1}^n (2i-1)$, respectively.

An estimate of the type (2.3) can also be obtained with the *Euler characteristic heuristic* method proposed in [1] and further analyzed in [14], which, however, is valid only for smooth fields:

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq \widetilde{C}_{Y'}\ \sigma^{k-2}\ k\ (k-1)!! \quad \forall\ k > 0 \text{ integer}, \tag{2.4}$$

where $\widetilde{C}_{Y'}$ is a positive constant independent of k and σ. We refer the reader to [10] for the proof of (2.4) in the case of a field defined on a d-dimensional rectangle $D = [0,T]^d$.

The bound (2.4) is weaker than (2.3) as it predicts a scaling σ^{k-2} instead of σ^k for the kth moment of the random variable $\|Y'\|_{L^\infty(D)}$. On the other hand, the bound (2.3) involves the exponential term $R_\gamma^{k/2}$, where R_γ depends on the covariance function of the random field.

To lighten the notation, in the rest of the paper we assume the Gaussian random field $Y(\omega,x)$ to be centered.

3. Perturbation analysis: Local approximation properties of the Taylor polynomial.

Thanks to the Doob–Dynkin lemma [38], the solution u of problem (2.1) is a function of Y: $u = u(Y,x)$. We will often use the shorthand notation $u(Y)$ to denote the map $u(Y) : L^\infty(D) \to H^1(D)$, implicitly defined by (2.1). In this section, under the assumption of small standard deviation of Y, we perform a perturbation analysis based on the Taylor expansion of the solution u in a neighborhood of the mean of Y (here assumed to be zero without loss of generality (w.l.o.g.)), and we study the *local* approximation properties of the Taylor polynomial of u.

3.1. Taylor expansion.

Let $0 < \sigma < 1$ be the standard deviation of the centered Gaussian random field $Y(\omega,x)$. Given a function $u(Y) : L^\infty(D) \to H^1(D)$ which is $(K+1)$-times Gâteaux differentiable, we denote its kth $(0 \leq k \leq K+1)$ Gâteaux derivative in $\bar{Y} \in L^\infty(D)$ evaluated along the vector

$$\underbrace{(Y,\ldots,Y)}_{k \text{ times}}$$

as $D^k u(\bar{Y})[Y]^k$. The Kth order Taylor polynomial of u centered in 0 is

$$T^K u(Y,x) := \sum_{k=0}^{K} \frac{D^k u(0)[Y]^k}{k!}, \quad K \geq 1, \tag{3.1}$$

where $D^0u(0)[Y]^0 := u^0(x)$ is independent of the random field Y and solves the deterministic Laplacian problem: given $f \in L^2(D)$ and $g \in H^{1/2}(\Gamma_D)$, find $u^0 \in H^1(D)$ s.t. $u|_{\Gamma_D} = g$ and

$$\int_D \nabla u^0(x) \cdot \nabla v(x)\, dx = \int_D f(x)v(x)\, dx \quad \forall\, v \in H^1_{\Gamma_D}(D). \tag{3.2}$$

The Kth order residual of the Taylor expansion $R^K u(Y,x) := u(Y,x) - T^K u(Y,x)$ can be expressed as

$$R^K u(Y,x) := \frac{1}{K!}\int_0^1 (1-t)^K D^{K+1}u(tY)[Y]^{K+1} dt. \tag{3.3}$$

See, for example, [3, 2].

It is possible to derive deterministic recursive equations solved by the increasing order corrections of the statistical moments of $T^K u$. See [10]. For example, for the computation of the expected value of u, one can write deterministic recursive problems for the kth order term $\mathbb{E}\left[D^k u(0)[Y]^k\right]$ and approximate $\mathbb{E}[u]$ as

$$\mathbb{E}[u] \approx \mathbb{E}\left[T^K u\right] = \sum_{k=0}^{K} \frac{1}{k!}\mathbb{E}\left[D^k u(0)[Y]^k\right].$$

This approach is known in the literature as *moment equations* (see, e.g., [41, 46, 4, 30, 39, 42, 28]). We do not detail here the derivation and algorithmic implementation of the moment equations, which can be found in [10]. Rather, we investigate the accuracy of the Taylor expansion for the problem at hand.

3.2. Local convergence of the Taylor series. The problem solved by the kth Gâteaux derivative of u, $D^k u(0)[Y]^k$ ($k \geq 1$), is as follows (see, e.g., [5, 28, 42]):

kTH DERIVATIVE PROBLEM. LOG-NORMAL RANDOM FIELD

(3.4)

Given $u^0 \in H^1(D)$ and all lower order derivatives

$$D^l u(0)[Y]^l \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right),\ l < k,$$

find $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ s.t.

$$\int_D \nabla D^k u(0)[Y]^k \cdot \nabla v\, dx = -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla v\, dx$$

$$\forall\, v \in H^1_{\Gamma_D}(D) \quad a.s.\ in\ \Omega.$$

By the Lax–Milgram lemma, the boundedness of $\|Y\|_{L^\infty(D)}$, and a recursion argument, we can state the following result.

Theorem 3.1. *Problem (3.4) is well-posed, that is, it admits a unique solution $D^k u(0)[Y]^k \in L^p\big(\Omega; H^1_{\Gamma_D}(D)\big)$ for any $0 < p < +\infty$ that depends continuously on the data. Moreover, it* *holds that*

$$\|D^k u(0)[Y]^k\|_{H^1(D)} \le C \left(\frac{\|Y\|_{L^\infty(D)}}{\log 2} \right)^k k! < +\infty \quad \forall k \ge 1 \quad a.s.\ in\ \Omega, \tag{3.5}$$

where $C = C\big(C_P, \|u^0\|_{H^1(D)}\big)$, C_P *being the Poincaré constant.*

Proof. For every fixed $\omega \in \Omega$, problem (3.4) is of the following form: find $w \in H^1_{\Gamma_D}(D)$ such that

$$\mathscr{A}(w,v) = \mathscr{L}(v) \quad \forall v \in H^1_{\Gamma_D}(D),$$

where the bilinear form $\mathscr{A}$ and the linear form $\mathscr{L}$ are defined, respectively, as

$$\mathscr{A} : H^1_{\Gamma_D}(D) \times H^1_{\Gamma_D}(D) \to \mathbb{R}, \quad \mathscr{A}(w,v) = \int_D \nabla w(x) \cdot \nabla v(x)\, dx,$$

$$\mathscr{L} : H^1_{\Gamma_D}(D) \to \mathbb{R}, \quad \mathscr{L}(v) = -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla v\, dx.$$

It is easy to verify that $\mathscr{A}$ is continuous and coercive. Moreover, $\mathscr{L}$ is continuous:

$$\begin{aligned} |\mathscr{L}(v)| &\le \sum_{l=1}^{k} \binom{k}{l} \left| \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla v\, dx \right| \\ &\le \sum_{l=1}^{k} \binom{k}{l} \|Y\|^l_{L^\infty} \left\| D^{k-l} u(0)[Y]^{k-l} \right\|_{H^1} \|v\|_{H^1}. \end{aligned}$$

Thanks to the Lax–Milgram lemma we conclude the well-posedness of problem (3.4) a.s. in Ω. To prove (3.5), let us take $v = D^k u(0)[Y]^k$ in (3.4). By the Cauchy–Schwarz inequality

$$\begin{aligned} \int_D \left| \nabla D^k u(0)[Y]^k \right|^2 \mathrm{dx} &\le \sum_{l=1}^{k} \binom{k}{l} \left| \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla D^k u(0)[Y]^k \mathrm{dx} \right| \\ &\le \sum_{l=1}^{k} \binom{k}{l} \|Y\|^l_{L^\infty} \left\| \nabla D^{k-l} u(0)[Y]^{k-l} \right\|_{L^2} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2}. \end{aligned}$$

By defining $S_k := \frac{1}{k!} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2}$, we have

$$S_k \le \sum_{l=1}^{k} \frac{\|Y\|^l_{L^\infty}}{l!} S_{k-l}. \tag{3.6}$$

We prove by induction that

$$S_k \le C_k \|Y\|^k_{L^\infty} S_0, \tag{3.7}$$

where $\{C_k\}_{k\ge1}$ are defined by recursion as

$$\begin{cases} C_0 = 1, \\ C_k = \displaystyle\sum_{l=1}^{k} \frac{1}{l!} C_{k-l}. \end{cases} \tag{3.8}$$

If $k=1$, (3.7) easily follows from (3.6). Now, let us suppose that (3.7) is verified for every S_j with $j=1,\dots,k-1$. Then, using (3.6), the inductive hypothesis, and the definition of the coefficients C_k in (3.8),

$$\begin{aligned} S_k &\le \sum_{l=1}^{k} \frac{\|Y\|_{L^\infty}^l}{l!} S_{k-l} = \sum_{l=1}^{k-1} \frac{\|Y\|_{L^\infty}^l}{l!} S_{k-l} + \frac{\|Y\|_{L^\infty}^k}{k!} S_0 \\ &\le \sum_{l=1}^{k-1} \frac{\|Y\|_{L^\infty}^l}{l!} C_{k-l} \|Y\|_{L^\infty}^{k-l} S_0 + \frac{\|Y\|_{L^\infty}^k}{k!} S_0 \\ &= \|Y\|_{L^\infty}^k \left(\sum_{l=1}^{k-1} \frac{C_{k-l}}{l!} + \frac{1}{k!} \right) S_0 = \|Y\|_{L^\infty}^k \, C_k \; S_0, \end{aligned}$$

so that (3.7) is verified. In [9], the authors show by induction that $C_k \le \left(\frac{1}{\log 2}\right)^k \ \forall\ k \ge 0$. Hence,

$$S_k \le \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k S_0.$$

In conclusion,

$$\begin{aligned} \left\| D^k u(0)[Y]^k \right\|_{H^1} &\le \sqrt{C_P^2+1} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2} \\ &\le \sqrt{C_P^2+1}\; S_0 \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k k! \\ &\le \left(\sqrt{C_P^2+1} \left\| u^0 \right\|_{H^1} \right) \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k k!, \end{aligned}$$

so that (3.5) is proved with $C = \sqrt{C_P^2+1} \left\| u^0 \right\|_{H^1}$. Moreover, since $\|Y\|_{L^\infty} \in L^q(\Omega, \mathbb{P})$ for any $0<q<+\infty$, we conclude that $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}\right)$ for any $0<p<+\infty$. ■

Thanks to Theorem 3.1 we prove the following result on the local convergence of the Taylor series.

Theorem 3.2 (local convergence of the Taylor series). *Let $Y(\omega,x)$ be a centered (w.l.o.g.) Gaussian random field with standard deviation σ and covariance function $\mathrm{Cov}_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$. Moreover, let $u(Y): L^\infty(D) \to H^1(D)$ be the infinite many times Gâteaux differentiable map defined in (2.1). Then its Taylor series is absolutely convergent in the open ball $B_{\log 2}(0) := \{Y \in L^\infty(D) : \ \|Y\|_{L^\infty} < \log 2\}$ for any positive σ.*

Proof. Using the triangular inequality and (3.5), we have

$$\left\| T^K u \right\|_{H^1} \le \sum_{k=0}^{K} \frac{\left\| D^k u(0)[Y]^k \right\|_{H^1}}{k!} \le C \sum_{k=0}^{K} \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k,$$

where $C = C\left(C_P, \left\| u^0 \right\|_{H^1}\right)$. Taking the limit for $K \to +\infty$ we immediately conclude the proof. ■

Remark 3.1. The map $u(Y): L^\infty(D) \to H^1(D)$ is analytic on $L^\infty(D)$. Indeed, Theorem 3.2 applies to the Taylor series centered in $\bar{Y} \neq 0$ and predicts local convergence for any $\bar{Y} \in L^\infty(D)$.

4. Perturbation analysis: Global approximation properties of the Taylor series. In this section we study the *global* approximation properties of the Taylor series. First, we present a counterexample in which the Taylor series is not globally convergent. Then we derive an a priori upper bound on the norm of the residual of the Taylor series, which will be used to predict the optimal degree K^{σ}_{opt} (depending on σ) of the Taylor polynomial to consider.

4.1. A preliminary example. Take $Y(\omega, x) = \xi(\omega)x$, with $\xi \sim \mathcal{N}(0, \sigma^2)$ Gaussian random variable, and consider the following SPDE:

$$\begin{cases} -\left(e^{\xi(\omega)x}u'(\xi,x)\right)' = 0, & x \in [0,1], \text{ a.s. in } \Omega, \\ u(\xi,0) = 0, \ u(\xi,1) = 1, \end{cases} \tag{4.1}$$

where the apex means the derivative with respect to x.

The exact solution of problem (4.1) can be exactly computed and reads as $u(\xi,x) = \frac{1-e^{-\xi x}}{1-e^{-\xi}}$. Observe that on the real axis (i.e., $\xi \in \mathbb{R}$) $u(\xi,x)$ is analytic as a function of ξ. On the other hand, in the complex plane (i.e., $\xi \in \mathbb{C}$) $u(\xi,x)$ is not entire since it admits countable many poles in $\xi = 2\pi i k$, $k \in \mathbb{Z}\backslash\{0\}$. As a consequence, the Taylor series centered in $\xi = 0$ converges only in the ball of radius $R < 2\pi$, and $\mathbb{E}\left[T^K u\right]$ *does not converge* to $\mathbb{E}[u]$ as $K \to +\infty$ *for any* $\sigma > 0$.

This counterexample shows that $\mathbb{E}\left[T^K u\right]$ might not converge to $\mathbb{E}[u]$ in general. Nevertheless, for small values of the standard deviation σ and degree K of the Taylor polynomial, $\mathbb{E}\left[T^K u\right]$ can still be a good approximation of $\mathbb{E}[u]$. The perturbation method we propose can be used even if the Taylor series is not globally convergent.

4.2. Upper bound on the norm of the Taylor residual. The following is the problem solved by $D^K u(tY)[Y]^K$, $t \in (0,1)$: given $u^0 \in H^1(D)$ and all lower order derivatives $D^l u(tY)[Y]^l \in L^p\big(\Omega; H^1_{\Gamma_D}(D)\big)$, $l < K$, find $D^K u(tY)[Y]^K \in L^p\big(\Omega; H^1_{\Gamma_D}(D)\big)$ s.t.

$$\begin{aligned} &\int_D \mathrm{e}^{tY} \nabla D^K u(tY)[Y]^K \cdot \nabla v \, dx \\ &\quad = -\sum_{l=1}^{K} \binom{K}{l} \int_D Y^l \mathrm{e}^{tY} \nabla D^{K-l} u(tY)[Y]^{K-l} \cdot \nabla v \, dx \end{aligned} \tag{4.2}$$

$\forall\, v \in H^1_{\Gamma_D}(D)$ a.s. in Ω. Following reasoning analogous to that in Theorem 3.1, we find that problem (4.2) is well-posed and

$$\|D^K u(tY)[Y]^K\|_{H^1(D)} \le C\ \mathrm{e}^{t\|Y\|_{L^\infty(D)}} \left(\frac{\|Y\|_{L^\infty(D)}}{\log 2}\right)^K K! < +\infty \tag{4.3}$$

$\forall\, K \ge 1$ a.s. in Ω, where $C = \sqrt{C_P^2 + 1}\,\left\|u^0\right\|_{H^1(D)}$.

Theorem 4.1 (a priori upper bound on the Taylor residual). *For a smooth random field Y that satisfies the bound (2.4), it holds that, for every $p \ge 1$ integer,*

$$\left\|R^K u\right\|_{L^p(\Omega;H^1(D))} \le C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \left\| \sum_{j=K+1}^{+\infty} \frac{\|Y\|^j_{L^\infty}}{j!} \right\|_{L^p(\Omega;H^1(D))} < +\infty, \tag{4.4}$$

where $C = C\left(C_P, \left\|u^0\right\|_{H^1(D)}, \widetilde{C}_{Y'}\right)$. *In particular, for* $p = 1$,

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!}. \tag{4.5}$$

Proof. Using (4.3), we find

$$\begin{aligned}\left\|R^K u\right\|_{H^1} &\leq \frac{1}{K!}\int_0^1 (1-t)^K \left\|D^{K+1}u(tY)[Y]^{K+1}\right\|_{H^1} dt \\ &\leq C\ (K+1)\left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^{K+1} \int_0^1 (1-t)^K \mathrm{e}^{t\|Y\|_{L^\infty}} dt,\end{aligned}$$

where $C = \sqrt{C_P^2+1}\left\|u^0\right\|_{H^1(D)}$. Let

$$I_K := \int_0^1 (1-t)^K \mathrm{e}^{t\|Y\|_{L^\infty}} dt. \tag{4.6}$$

By induction, we show that

$$I_K = \frac{K!}{\|Y\|_{L^\infty}^{K+1}} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}. \tag{4.7}$$

Indeed, for $K = 0$ we find

$$I_0 = \int_0^1 \mathrm{e}^{t\|Y\|_{L^\infty}} dt = \frac{\left(\mathrm{e}^{\|Y\|_{L^\infty}} - 1\right)}{\|Y\|_{L^\infty}} = \frac{1}{\|Y\|_{L^\infty}} \sum_{j=1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}.$$

Suppose now that relation (4.7) holds for $K-1$. Then, integrating by parts, we obtain

$$\begin{aligned}I_K &= \left[(1-t)^K \frac{\mathrm{e}^{t\|Y\|_{L^\infty}}}{\|Y\|_{L^\infty}}\right]_0^1 + \frac{K}{\|Y\|_{L^\infty}} \int_0^1 (1-t)^{K-1}\mathrm{e}^{t\|Y\|_{L^\infty}} dt \\ &= -\frac{1}{\|Y\|_{L^\infty}} + \frac{K}{\|Y\|_{L^\infty}} I_{K-1} \\ &= -\frac{1}{\|Y\|_{L^\infty}} + \frac{K}{\|Y\|_{L^\infty}} \frac{(K-1)!}{\|Y\|_{L^\infty}^K} \sum_{j=K}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!} \\ &= \frac{K!}{\|Y\|_{L^\infty}^{K+1}} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}.\end{aligned}$$

Hence,

$$\left\|R^K u(Y,x)\right\|_{H^1} \leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}.$$

Observe that since $\sum_{j=K+1}^{N}\frac{\|Y\|_{L^\infty}^j}{j!}\leq \mathrm{e}^{\|Y\|_{L^\infty}}\ \forall\ N$ and $\mathrm{e}^{\|Y\|_{L^\infty}}$ is $L^p(\Omega,\mathbb{P})$-integrable for each $1\leq p<\infty$, the dominated convergence theorem states that $\sum_{j=K+1}^{+\infty}\frac{\|Y\|_{L^\infty}^j}{j!}$ is $L^p(\Omega,\mathbb{P})$-integrable for each $1\leq p<\infty$, and relation (4.4) follows. Moreover, if $p=1$,

$$\mathbb{E}\left[\sum_{j=K+1}^{+\infty}\frac{\|Y\|_{L^\infty}^j}{j!}\right]=\sum_{j=K+1}^{+\infty}\frac{\mathbb{E}\left[\|Y\|_{L^\infty}^j\right]}{j!}.$$

Using (2.4), we conclude that

$$\left\|R^K u(Y,x)\right\|_{L^1(\Omega;H^1)}\leq C\ (K+1)!\left(\frac{1}{\log 2}\right)^{K+1}\sum_{j=K+1}^{+\infty}\frac{\sigma^{j-2}}{(j-2)!!},$$

with $C=\widetilde{C}_{Y'}\sqrt{C_P^2+1}\left\|u^0\right\|_{H^1(D)}$. ■

Using the upper bound (2.3) instead of (2.4), we predict that the $L^p\left(\Omega;H^1(D)\right)$-norm of $R^K u$ behaves as σ^{K+1} as a function of σ.

Remark 4.1. With similar techniques it is possible to prove the following a priori upper bound on the norm of the Taylor polynomial:

$$\left\|T^K u\right\|_{L^p(\Omega;H^1(D))}\leq\left\|u^0\right\|_{H^1(D)}+C\sum_{k=1}^{K}\left(\frac{\sigma}{\log 2}\right)^k\left(\sigma^{-2}\ kp\ (kp-1)!!\right)^{1/p}, \tag{4.8}$$

where $C=C\big(C_P,\left\|u^0\right\|_{H^1(D)},\widetilde{C}_{Y'}\big)$.

In Figure 1 we plot in semilogarithmic scale the estimate (4.5) as a function of the order of the residual K. We deduce the existence of an optimal degree K_{opt}^σ depending on σ and p such that adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it. We point out that we did not prove the divergence of the Taylor series. To do that, it is necessary to show the divergence of a lower bound for the norm of the residual $R^K u$.

4.3. Optimal degree of the Taylor polynomial. In the previous section we have predicted the existence of an optimal degree K_{opt}^σ of the Taylor polynomial, which can be estimated as the argmin of the right-hand side in (4.4). Let

$$b(\sigma,K)=C\ (K+1)!\left(\frac{1}{\log 2}\right)^{K+1}\left\|\sum_{j=K+1}^{+\infty}\frac{\|Y\|_{L^\infty}}{j!}\right\|_{L^p(\Omega;H^1(D))}.$$

The estimate (4.4) states that, for every $\sigma>0$ fixed, the minimal error $\mathrm{err}_{min}^\sigma$ that we can commit using a perturbation approach is bounded by

$$\mathrm{err}_{min}^\sigma\leq\min_K b(\sigma,K)=b(\sigma,K_{opt}^\sigma). \tag{4.9}$$

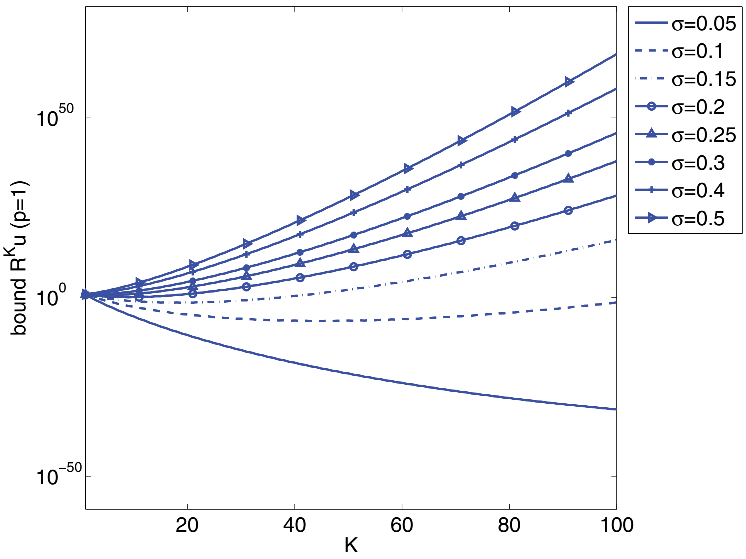

Figure 1. *Semilogarithmic plot of the estimate* (4.5) *as a function of* K *for different values of the standard deviation* σ.

Here, we provide an approximation for K^σ_{opt} and show in Remark 4.2 how $\mathrm{err}^\sigma_{min}$ behaves as a function of σ in the case $p = 1$ (estimate (4.5)).

Proposition 4.2. *Let* $0 < \sigma \le \frac{\log 2}{\sqrt{5}}$. *Then the optimal degree of the Taylor expansion can be estimated as*

$$\bar{K}^\sigma := \left\lfloor \frac{\log^2 2}{\sigma^2} \right\rfloor - 4. \tag{4.10}$$

Proof. The first step of the proof consists in showing that

$$\|R^K u\|_{L^1(\Omega;H^1(D))} \le C \frac{1}{(\log 2)^2(1-\sigma)} v(K), \tag{4.11}$$

where $v(K) = \left(\frac{\sigma}{\log 2}\right)^{K-1}(K+2)!!$ and C is independent of K. Starting from (4.5) and using that

$$\sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!} \le \frac{1}{1-\sigma}\frac{\sigma^{K-1}}{(K-1)!!}$$

we find

$$\begin{aligned}
\|R^K u\|_{L^1(\Omega;H^1(D))} &\le C \frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1} (K+1)! \,\frac{\sigma^{K-1}}{(K-1)!!} \\
&= C \frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1} \sigma^{K-1}(K+1)K(K-2)!! \\
&\le C \frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1} \sigma^{K-1}(K+2)!!,
\end{aligned}$$

Table 1

This table contains the optimal $K^\sigma_{opt} = \mathrm{argmin}_K b(\sigma, K)$ *(*$p = 1$*) and its estimate* $\bar{K}^\sigma$ *in (4.10).*

σ	K^σ_{opt}	$\bar{K}^\sigma$
0.10	45	44
0.15	19	17
0.18	11	10
0.20	9	8

so that (4.11) is proved. To find the argmin of $v(K)$, we consider $\log(v(K))$:

$$\log(v(K)) = \begin{cases} (2n-3)\log\alpha + \log(2n)!! & \text{if } K = 2n-2, \\ (2n-4)\log\alpha + \log(2n-1)!! & \text{if } K = 2n-3, \end{cases}$$

where $\alpha = \frac{\sigma}{\log 2}$. We analyze the two cases K odd or even separately, using that $(2n)!! = 2^n n!$, $(2n-1)!! = \frac{(2n)!}{2^n n!}$, and $\mathrm{e}\left(\frac{n}{\mathrm{e}}\right)^n \leq n! \leq \mathrm{e}\, n \left(\frac{n}{\mathrm{e}}\right)^n$. We conclude that

$$\log(v(n)) \leq w(n) + \bar{C},$$

where

$$w(n) := 2n\log\alpha + n\log 2 + (n+1)\log(n+1) - n \tag{4.12}$$

and $\bar{C}$ is the positive constant

$$\bar{C} = \begin{cases} -3\log\alpha + 1 & \text{if } K = 2n-2, \\ -4\log\alpha + \log 2 & \text{if } K = 2n-3. \end{cases} \tag{4.13}$$

Note that we have bounded $(n+1)\log n$ with $(n+1)\log(n+1)$ in view of having a simpler derivative $\frac{d}{dn}w(n)$. We look for the argmin$(w(n))$ by imposing $\frac{d}{dn}w(n) = 0$, that is,

$$2\log\alpha + \log 2 + \log(n+1) = 0,$$

which implies $n = \left\lfloor \frac{1}{2\alpha^2} \right\rfloor - 1$, so that we can choose $\bar{K}^\sigma = \left\lfloor \frac{1}{\alpha^2} \right\rfloor - 4$. ■

Remark 4.2. Consider the function $v(K)$ introduced in (4.11). Evaluating $v(K)$ in $\bar{K}^\sigma$ (see (4.10)), and using the Stirling formula, one can show the following exponential behavior of the minimal error as a function of σ^2:

$$\mathrm{err}^\sigma_{min} \leq Cv(\bar{K}^\sigma) \sim C(\sigma)\exp\left\{-\frac{\log^2 2}{2\sigma^2}\right\},$$

where $C(\sigma)$ is polynomial in $1/\sigma$.

In Table 1 we report the optimal $K^\sigma_{opt} = \mathrm{argmin}_K b(\sigma, K)$ and its estimate $\bar{K}^\sigma$ (4.10) for different values of σ. Figure 2 represents the upper bound $b(\sigma, K)$ of the error (see (4.9)) and the points $\left(\bar{K}^\sigma, b(\sigma, \bar{K}^\sigma)\right)$ (black circles) for different values of σ. We take the values $b(\sigma, \bar{K}^\sigma)$ as an estimate of the minimal error we can commit (maximum accuracy achievable) by performing a perturbation approach as in the previous section.

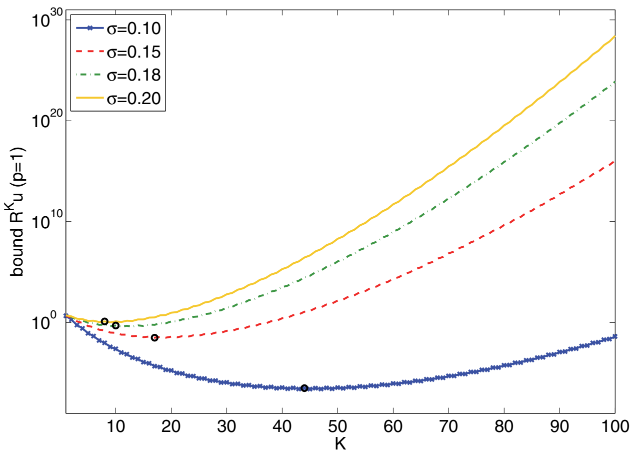

Figure 2. *Semilogarithmic plot of* $b(\sigma, K)$*, the right-hand side of* (4.5)*, and of the points* $\left(\bar{K}^{\sigma}, b(\sigma, \bar{K}^{\sigma})\right)$ *(black circles) for different values of* σ*.*

As Table 1 and Figure 2 suggest, the estimate (4.10) of the optimal K is quite sharp. Moreover, the smaller σ is, the bigger K^{σ}_{opt} is and the smaller the minimal error is that we can commit.

Remark 4.3. Suppose that the permeability field is modeled using a finite number of independent standard Gaussian random variables:

$$Y(\omega, x) = \sigma \sum_{n=1}^{N} \sqrt{\lambda_n} \xi_n(\omega) \phi_n(x).$$

Define $\xi(\omega) = (\xi_1(\omega), \ldots, \xi_N(\omega))$. This situation can be achieved, for example, by approximating the Gaussian field $Y(\omega, x)$ by an N-terms Karhunen–Loève expansion (see, e.g., [25, 33, 34, 32]). The stochastic solution $u(\xi, x)$ of the Darcy problem belongs to $L^p_\rho\left(\mathbb{R}^N; H^1(D)\right)$, the Banach space of functions $v : \mathbb{R}^N \times D \to \mathbb{R}$ s.t. $\|v\|_{L^p_\rho(\mathbb{R}^N;H^1(D))} := \left(\int_{\mathbb{R}^N} \|v(\xi, \cdot)\|^p_{H^1} \rho(\xi) d\xi\right)^{1/p} < \infty$, where $\rho(\xi) = \frac{1}{(2\pi)^{N/2}} \mathrm{e}^{-\frac{\|\xi\|^2}{2}}$ is the joint probability density of the vector $\xi(\omega)$. In this setting the Kth order Gâteaux derivative simplifies to $\sum_{|\mathbf{k}|=K} \partial^{\mathbf{k}}_{\xi} u(\mathbf{0}, x) \xi^{\mathbf{k}}$, where $\mathbf{k} = (k_1, \ldots, k_N)$, $\xi^{\mathbf{k}} = \xi_1^{k_1} \cdots \xi_N^{k_N}$, and the Taylor polynomial is explicitly computable. The theoretical estimates on the norm of the Taylor polynomial and Taylor residual still hold.

Remark 4.4. In [4] (see also [19]) the authors study the Darcy problem (2.1) where the permeability is a linear combination of independent *bounded* random variables: $a(\omega, x) = \mathbb{E}\left[a\right](x) + \sum_{n=1}^{N} \phi_n(x) \xi_n(\omega)$, with $\xi_n \sim \mathcal{U}\left([-\gamma_n, \gamma_n]\right)$, $0 < \gamma_n < +\infty$ $\forall$ n, and $\phi_n \in L^\infty(D)$ $\forall$ n. In this case, under the assumption of small variability of the field, the Taylor series is proved to be convergent.

5. Single random variable: Numerical results. Here we consider the simple case where $a(\omega, x) = \mathrm{e}^{\phi(x)\xi(\omega)}$, with $\xi \simeq \mathcal{N}(0, \sigma^2)$, $0 < \sigma < 1$, and $\phi \in L^\infty(D)$. Theorem 3.1 states that the boundary value problem solved by the kth derivative of u, $\partial_\xi^k u(0,x)$, is well-posed, and

$$\left\|\partial_\xi^k u(0,x)\right\|_{H^1(D)} \le C\left(\frac{\|\phi\|_{L^\infty}}{\log 2}\right)^k k!, \tag{5.1}$$

where $C = C\big(C_P, \|u^0\|_{H^1(D)}\big)$. In the same way, (4.3) implies

$$\left\|\partial_\xi^k u(t\xi,x)\right\|_{H^1(D)} \le C\ \mathrm{e}^{t|\xi|\|\phi\|_{L^\infty}}\left(\frac{\|\phi\|_{L^\infty}}{\log 2}\right)^k k!. \tag{5.2}$$

Using the upper bound (5.1) we deduce the following upper bound on the H^1-norm of the Kth order Taylor polynomial $T^K u(\xi,x) := \sum_{k=0}^K \frac{\partial_\xi^k u(0,x)}{k!}\xi^k$:

$$\|T^K u\|_{H^1} \le C\sum_{k=0}^{K}\left(\frac{\|\phi\|_{L^\infty}|\xi|}{\log 2}\right)^k,$$

which is locally convergent in the ball $B := \left\{\xi\in\mathbb{R}:\ |\xi| < \frac{\log 2}{\|\phi\|_{L^\infty}}\right\}$ for every $\sigma > 0$. See Theorem 3.2.

Recalling the value of the statistical moments of $|\xi|$,

$$\mathbb{E}\left[|\xi|^p\right] = C\ \sigma^p(p-1)!!,\quad C = \begin{cases} 1 & \text{if } p \text{ is even,}\\ \sqrt{\frac{2}{\pi}} & \text{if } p \text{ is odd,}\end{cases} \tag{5.3}$$

and using (5.2), we derive the following estimate for the Kth order integral residual $R^K u(\xi,x) := \frac{1}{K!}\int_0^1(1-t)^K\partial_\xi^{K+1}u(t\xi,x)\xi^{K+1}dt$:

$$\|R^K u\|_{L^p_\rho(\mathbb{R};H^1(D))} \le C(K+1)!\left(\frac{1}{\log 2}\right)^{K+1}\left\|\sum_{j=K+1}^{+\infty}\frac{(|\xi|\,\|\phi\|_{L^\infty})^j}{j!}\right\|_{L^p_\rho(\mathbb{R};H^1(D))},$$

which can be particularized if $p=1$ as follows:

$$\|R^K u\|_{L^1_\rho(\mathbb{R};H^1(D))} \le C(K+1)!\left(\frac{1}{\log 2}\right)^{K+1}\sum_{j=K+1}^{+\infty}\frac{(\sigma\,\|\phi\|_{L^\infty})^j}{j!!}, \tag{5.4}$$

where $C = C\big(C_P, \|u^0\|_{H^1(D)}\big)$. See Theorem 4.1.

We develop some numerical computations in a one-dimensional case, with $D=[0,1]$, homogeneous Dirichlet boundary conditions imposed on $\Gamma_D = \{0,1\}$, $f(x)=x$, and $\phi(x) = \cos(\pi x)$. The problems solved by $u^0(x)$ and $\partial_\xi^k u(0,x)$, respectively, are

$$\int_0^1 (u^0(x))'v'(x)dx = \int_0^1 f(x)v(x)dx,\quad u^0(0)=u^0(1)=0, \tag{5.5}$$

$\forall\, v \in H_0^1([0,1])$, and

$$\int_0^1 (\partial_\xi^k u(0,x))' v'(x) dx = -\sum_{l=1}^{k} \binom{k}{l} \int_0^1 \phi(x)^l (\partial_\xi^{k-l} u(0,x))' v'(x) dx, \tag{5.6}$$

$\partial_\xi^k u(0,0) = \partial_\xi^k u(0,1) = 0$, $\forall\, v \in H_0^1([0,1])$, $\forall k \geq 1$. Note that the apex in problems (5.5) and (5.6) means the derivative with respect to x.

Let $\{\varphi_i\}_{i=1}^{N_h-1}$ be the piecewise linear finite element basis associated with a uniform partition of $[0,1]$ in N_h subintervals of length $h = 1/N_h$. Applying the finite element method (FEM) to problem (5.5), we end up with the system

$$AU^0 = F^0, \tag{5.7}$$

where the stiffness matrix is tridiagonal and symmetric and its generic element is given by $A_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx$, the right-hand side is a vector whose jth element is $F_j^0 = \int_0^1 f(x)\varphi_j(x)dx$, and U^0 is the unknown vector. Similarly, applying the linear FEM to the kth problem (5.6), we end up with the system

$$AU^k = -\sum_{l=1}^{k} \binom{k}{l} F^l U^{k-l}, \tag{5.8}$$

where the stiffness matrix is the same as in (5.7), and the right-hand side contains the solutions $U^0, \dots, U^{k-1}$ of the lth problem for $l = 0, \dots, k-1$ and the matrices $F_{ij}^l = \int_0^1 (\phi(x))^l \varphi_j'(x)\varphi_i'(x)dx$ for $l = 1, \dots, k$.

Let $u_h(\xi, x)$ denote the linear finite element solution of the Darcy problem collocated in ξ. With the notation introduced so far, we have

$$T^K u_h(\xi, x) = \sum_{k=0}^{K} \sum_{i=1}^{N_h-1} \frac{U_i^k}{k!} \varphi_i(x)\xi^k.$$

In Figure 3 we plot in semilogarithmic scale the relative error $\frac{\|u_h - T^K u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}}{\|u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}}$ ($p = 1,2$) computed by a linear FEM in space and a high order Hermite quadrature formula in the ξ variable for different values of the standard deviation $0 < \sigma < 1$. Note that we have chosen the same spatial discretization both for u_h and $T^K u_h$, so that we observe only the truncation error of the Taylor series. Consistently with the upper bound (4.4) on the residual of the Taylor series and the counterexample shown in section 4.1, these figures suggest the global divergence of the Taylor series in $L^p_\rho\left(\mathbb{R}; H^1(D)\right)$. This result does not contradict the result on local convergence of the Taylor series stated in Theorem 3.2. In agreement with the theoretical results of section 4, the existence of an optimal degree K^σ_{opt} of the Taylor polynomial (depending on σ) is numerically observed. Moreover, the higher p is, the worse the behavior of the norm of the residual is, since it starts diverging for a smaller K.

Figure 4 compares the computed absolute error $\|u_h - T^K u_h\|_{L^1_\rho(\mathbb{R};L^2(D))}$ with the theoretical estimate (5.4). It turns out that the proposed a priori estimate is quite pessimistic. This is a consequence of the estimate on $\|\partial_\xi^k u(0,x)\|_{H^1(D)}$, which is itself very pessimistic.

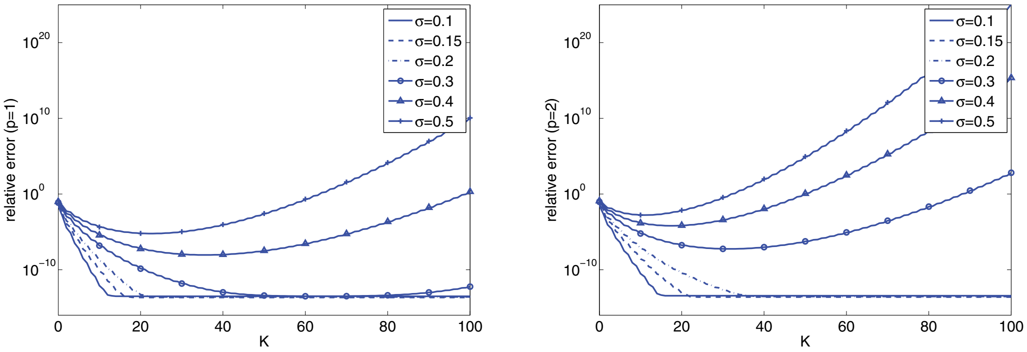

Figure 3. *Relative error* $\frac{\|u_h - T^K u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}}{\|u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}}$ *computed by linear FEM in space and a high order Hermite quadrature formula in probability for* $p = 1$ *(left) and* $p = 2$ *(right).*

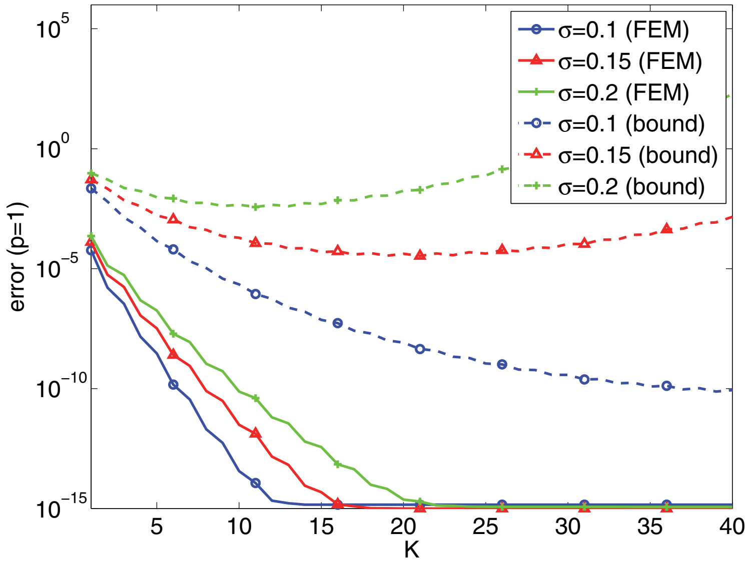

Figure 4. *Comparison between the computed error* $\|u_h - T^K u_h\|_{L^1_\rho(\mathbb{R};L^2(D))}$ *and the theoretical estimate* (5.4).

With the aim of improving the theoretical bounds on the norm of the Taylor residual, we assume that the growth of the derivatives follows the ansatz

$$\left\|\partial_\xi^k u(0,x)\right\|_{L^2(D)} \sim \gamma^k k! \tag{5.9}$$

for a suitable value of γ. Then we try to fit the value of γ starting from the numerical results obtained. In this specific example, the fitting procedure gives $\gamma = \frac{\|\phi\|_{L^\infty}}{3.5 \log 2}$. Nevertheless, we highlight that the choice of γ strongly depends on $\phi(x)$, whereas it seems to be rather

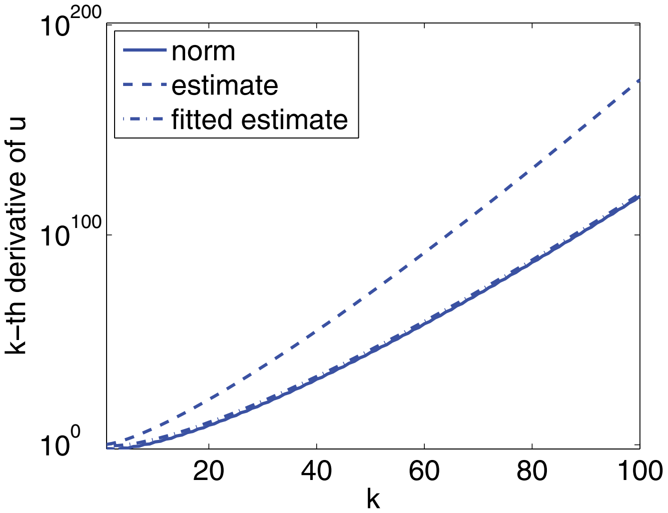

Figure 5. *Comparison between the quantity* $\left\|\partial_\xi^k u(0,x)\right\|_{L^2(D)}$ *computed by a linear FEM, its theoretical estimate* (5.1)*, and the fitted one* (5.9) *with* $\gamma = \frac{\|\phi\|_{L^\infty}}{3.5\log 2}$.

insensitive to other quantities such as the loading term $f(x)$, the boundary conditions, or the number of intervals in the mesh N_h. Moreover, this fitting procedure cannot straightforwardly be generalized to the infinite-dimensional setting. In Figure 5 we plot in semilogarithmic scale the quantity $\left\|\partial_\xi^k u(0,x)\right\|_{L^2(D)}$ computed by a linear FEM, compared with the theoretical estimate (5.1) and the fitted one (5.9) with $\gamma = \frac{\|\phi\|_{L^\infty}}{3.5\log 2}$. The agreement of the computed norm $\left\|\partial_\xi^k u(0,x)\right\|_{L^2(D)}$ with the fitted estimate (5.9) is remarkable, which strongly indicates that the ansatz (5.9) is appropriate. We then use the fitted value $\gamma = \frac{\|\phi\|_{L^\infty}}{3.5\log 2}$ in the estimate on the norm of the residual (5.4):

$$\left\|R^K u\right\|_{L^1_\rho(\mathbb{R};H^1(D))} \leq C\ (K+1)! \left(\frac{1}{3.5\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\left(\|\phi\|_{L^\infty}\sigma\right)^j}{j!!}. \tag{5.10}$$

Figure 6 compares the computed quantity $\left\|R^K u_h\right\|_{L^1_\rho(\mathbb{R};H^1(D))}$ with the fitted bound (5.10). We underline that, with the ansatz (5.9) on the growth of the derivatives, we are able to sharply predict the optimal degree of the Taylor polynomial K^σ_{opt}.

Finally, we analyze the behavior of the error $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ as a function of σ. Figure 7 shows this error in logarithmic scale. Observe that the exponential behavior σ^{K+1} predicted in (5.4) is confirmed.

6. Conclusions. The present work addresses the Darcy problem describing the single-phase flow in a bounded heterogeneous porous medium occupying the domain $D \subset \mathbb{R}^d$, $d = 1,2,3$, where the permeability tensor is modeled as a log-normal random field: $a(\omega,x) = \mathrm{e}^{Y(\omega,x)}$. Under the assumption of small variability of the field Y, we perform a perturbation analysis and study the approximation properties of the Taylor polynomial of order K. In

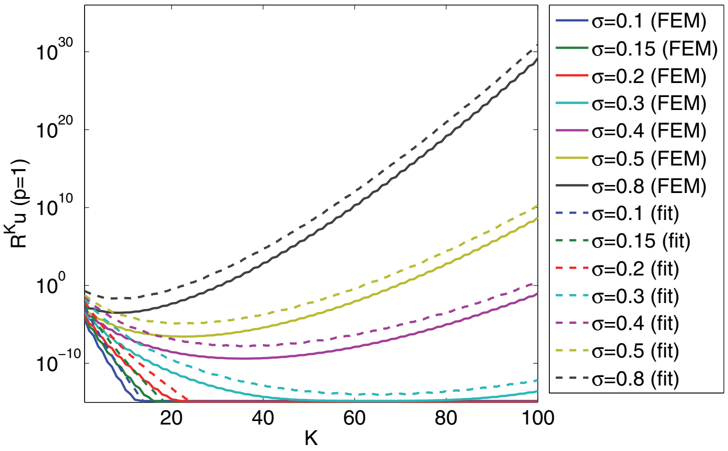

Figure 6. *Comparison between the computed quantity* $\left\|R^K u_h\right\|_{L^1(\Omega;L^2(D))}$ *and its theoretical estimate* (5.10) *with the fitted value* $\gamma = \frac{\|\phi\|_{L^\infty}}{3.5\log 2}$.

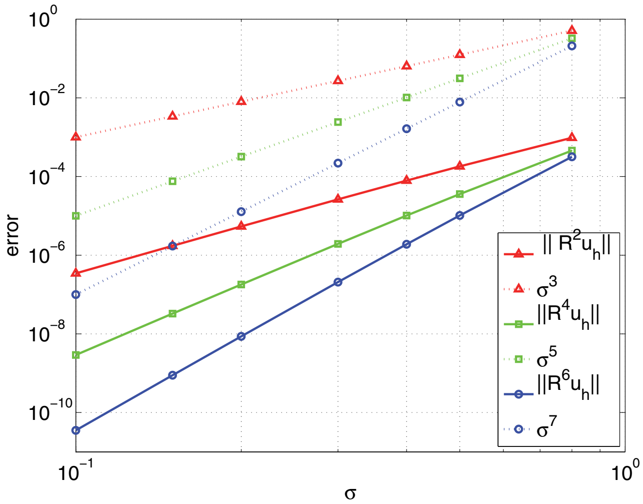

Figure 7. *Error* $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ *as a function of* σ.

particular, we prove the local convergence of the Taylor series to the solution of the Darcy problem. On a counterexample, we show that, in general, the Taylor series is not globally convergent. We derive an a priori upper bound on the norm of the Taylor residual, which predicts the existence of an optimal degree K^σ_{opt} of the Taylor polynomial to consider. We provide a formula to compute K^σ_{opt} in the case where the $L^1\left(\Omega; H^1(D)\right)$-norm is considered.

The results obtained in this work are very important in view of deriving an approximation of the statical moments of u. For example, if we look for an approximation of the expected value $\mathbb{E}\left[u\right]$, the underlying idea consists in deriving and numerically solving the recursive deterministic problem for the expected value of the kth order derivative $D^k u(0)[Y]^k$, $k = 0, \ldots, K^\sigma_{opt}$, and then linearly combining them: $\mathbb{E}[u] \approx \sum_{k=0}^{K^\sigma_{opt}} \frac{1}{k!}\mathbb{E}\left[D^k u(0)[Y]^k\right]$. The kth order derivative equation requires in turn the study of the problems solved by the correlations between $D^k u(0)[Y]^k$ and Y. These quantities belong to tensor product spaces and, when discretized, are represented by high dimensional tensors, so that suitable numerical techniques have to be adopted. This discussion can be found in [10] and is the topic of a forthcoming paper.

REFERENCES

[1] R. J. Adler and J. E. Taylor, *Random Fields and Geometry*, Springer Monographs in Mathematics, Springer, New York, 2007.

[2] H. Amann and J. Escher, *Analysis.* II, Birkhäuser Verlag, Basel, 2008; translated from the 1999 German original by S. Levy and M. Cargo.

[3] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge Stud. Adv. Math. 34, Cambridge University Press, Cambridge, UK, 1995; corrected reprint of the 1993 original.

[4] I. Babuška and P. Chatzipantelidis, *On solving elliptic stochastic partial differential equations*, Comput. Methods Appl. Mech. Engrg., 191 (2002), pp. 4093–4122.

[5] I. Babuška, F. Nobile, and R. Tempone, *A stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Numer. Anal., 45 (2007), pp. 1005–1034.

[6] I. Babuška, R. Tempone, and G. E. Zouraris, *Galerkin finite element approximations of stochastic elliptic partial differential equations*, SIAM J. Numer. Anal., 42 (2004), pp. 800–825.

[7] A. Barth, C. Schwab, and N. Zollinger, *Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients*, Numer. Math., 119 (2011), pp. 123–161.

[8] J. Bear and A. H.-D. Cheng, *Modeling Groundwater Flow and Contaminant Transport*, Theory Appl. Transp. Porous Media 23, Springer, The Netherlands, 2010.

[9] J. Beck, R. Tempone, F. Nobile, and L. Tamellini, *On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods*, Math. Models Methods Appl. Sci., 22 (2012), 1250023.

[10] F. Bonizzoni, *Analysis and Approximation of Moment Equations for PDEs with Stochastic Data*, Ph.D. thesis, Department of Mathematics, Politecnico di Milano, Italy, 2013.

[11] F. Bonizzoni, A. Buffa, and F. Nobile, *Moment equations for the mixed formulation of the Hodge Laplacian with stochastic loading term*, IMA J. Numer. Anal. (2013).

[12] J. Charrier, *Strong and weak error estimates for elliptic partial differential equations with random coefficients*, SIAM J. Numer. Anal., 50 (2012), pp. 216–246.

[13] J. Charrier and A. Debussche, *Weak truncation error estimates for elliptic PDEs with lognormal coefficients*, Stochastic Partial Differential Equations Anal. Comput., 1 (2013), pp. 63–93.

[14] D. Cheng and Y. Xiao, *The Mean Euler Characteristic and Excursion Probability of Gaussian Random Fields with Stationary Increments*, preprint, arXiv:1211.6693v2, 2012.

[15] A. Chernov, *Sparse polynomial approximation in positive order Sobolev spaces with bounded mixed derivatives and applications to elliptic problems with random loading*, Appl. Numer. Math., 62 (2012), pp. 360–377.

[16] A. Chernov and C. Schwab, *First order k-th moment finite element analysis of nonlinear operator equations with stochastic data*, Math. Comp., 82 (2013), pp. 1859–1888.

[17] A. Chkifa, A. Cohen, R. DeVore, and C. Schwab, *Sparse adaptive Taylor approximation algorithms for parametric and stochastic elliptic PDEs*, ESAIM Math. Model. Numer. Anal., 47 (2013), pp. 253–280.

[18] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup, *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients*, Comput. Vis. Sci., 14 (2011), pp. 3–15.

[19] A. Cohen, R. Devore, and C. Schwab, *Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's*, Anal. Appl. (Singap.), 9 (2011), pp. 11–47.

[20] G. Dagan, *Flow and Transport in Porous Formations*, Springer-Verlag, Heidelberg, Berlin, New York, 1989.

[21] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl. 44, Cambridge University Press, Cambridge, UK, 1992.

[22] P. Frauenfelder, C. Schwab, and R. A. Todor, *Finite elements for elliptic problems with stochastic coefficients*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 205–228.

[23] J. Galvis and M. Sarkis, *Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity*, SIAM J. Numer. Anal., 47 (2009), pp. 3624–3651.

[24] B. Ganapathysubramanian and N. Zabaras, *Sparse grid collocation schemes for stochastic natural convection problems*, J. Comput. Phys., 225 (2007), pp. 652–685.

[25] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, New York, 1991.

[26] C. J. Gittelson, *Stochastic Galerkin discretization of the log-normal isotropic diffusion problem*, Math. Models Methods Appl. Sci., 20 (2010), pp. 237–263.

[27] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan, *Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications*, J. Comput. Phys., 230 (2011), pp. 3668–3694.

[28] A. Guadagnini and S. P. Neuman, *Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains:* 1. *Theory and computational approach*, Water Resour. Res., 35 (1999), pp. 2999–3018.

[29] A. Guadagnini and S. P. Neuman, *Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains:* 2. *Computational examples*, Water Resour. Res., 35 (1999), pp. 3019–3039.

[30] H. Harbrecht, R. Schneider, and C. Schwab, *Sparse second moment analysis for elliptic problems in stochastic domains*, Numer. Math., 109 (2008), pp. 385–414.

[31] F. Y. Kuo, C. Schwab, and I. H. Sloan, *Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients*, SIAM J. Numer. Anal., 50 (2012), pp. 3351–3374.

[32] P. Lévy, *Processus stochastiques et mouvement brownien*, Grands Class. Gauthier-Villars, Éditions Jacques Gabay, Sceaux, 1992; reprint of the second (1965) edition.

[33] M. Loève, *Probability Theory.* I, 4th ed., Grad. Texts in Math. 45, Springer-Verlag, New York, 1977.

[34] M. Loève, *Probability Theory.* II, 4th ed., Grad. Texts in Math. 46, Springer-Verlag, New York, 1978.

[35] H. G. Matthies and A. Keese, *Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 1295–1331.

[36] F. Nobile, R. Tempone, and C. G. Webster, *An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal., 46 (2008), pp. 2411–2442.

[37] F. Nobile, R. Tempone, and C. G. Webster, *A sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal., 46 (2008), pp. 2309–2345.

[38] M. M. Rao and R. J. Swift, *Probability Theory with Applications*, 2nd ed., Math. Appl. (Springer) 582, Springer, New York, 2006.

[39] M. Riva, A. Guadagnini, and M. De Simoni, *Assessment of uncertainty associated with the estimation of well catchments by moment equations*, Adv. Water Resources, 29 (2006), pp. 676–691.

[40] C. Schwab and C. J. Gittelson, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer., 20 (2011), pp. 291–467.

[41] C. Schwab and R. A. Todor, *Sparse finite elements for elliptic problems with stochastic loading*, Numer. Math., 95 (2003), pp. 707–734.

[42] D. M. Tartakovsky and S. P. Neuman, *Transient flow in bounded randomly heterogeneous domains:* 1. *Exact conditional moment equations and recursive approximations*, Water Resour. Res., 34 (1998), pp. 1–12.

[43] A. L. Teckentrup, R. Scheichl, M. B. Giles, and E. Ullmann, *Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients*, Numer. Math., 125 (2013), pp. 569–600.

[44] R. A. Todor, *Sparse Perturbation Algorithms for Elliptic PDE's with Stochastic Data*, Ph.D. thesis, ETH Zurich, Switzerland, 2005.

[45] R. A. Todor and C. Schwab, *Convergence rates for sparse chaos approximations of elliptic problems*

with stochastic coefficients, IMA J. Numer. Anal., 27 (2007), pp. 232–261.

[46] T. von Petersdorff and C. Schwab, *Sparse finite element methods for operator equations with stochastic data*, Appl. Math., 51 (2006), pp. 145–180.

[47] D. Xiu and J. S. Hesthaven, *High-order collocation methods for differential equations with random inputs*, SIAM J. Sci. Comput., 27 (2005), pp. 1118–1139.

[48] D. Zhang, *Stochastic Methods for Flow in Porous Media: Coping with Uncertainties*, Academic Press, 2002.