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Testing additive integrality gaps

Friedrich Eisenbrand · Nicolai Hähnle · Dömötör Pálvölgyi · Gennady Shmonin

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Abstract We consider the problem of testing whether the maximum additive integrality gap of a family of integer programs in standard form is bounded by a given constant. This can be viewed as a generalization of the integer rounding property, which can be tested in polynomial time if the number of constraints is fixed. It turns out that this generalization is NP-hard even if the number of constraints is fixed. However, if, in addition, the objective is the all-one vector, then one can test in polynomial time whether the additive gap is bounded by a constant.

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1 Introduction

Linear programming is a very successful tool for solving combinatorial optimization problems that can be modeled as an integer linear program

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F. Eisenbrand (✉) · N. Hähnle · G. Shmonin
DISOPT, Ecole Polytechnique Fédérale de Lausanne, Station 8,
1015 Lausanne, Switzerland
e-mail: friedrich.eisenbrand@epfl.ch

N. Hähnle
e-mail: nicolai.haehnle@epfl.ch

G. Shmonin
e-mail: shmonin@gmail.com

D. Pálvölgyi
Eötvös University, Számítógéptudomány tanszék, Pázmány Péter sétány 1/c,
Budapest 1118, Hungary
e-mail: dom@cs.elte.hu

$$\min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0 \text{ integral}\right\}, \tag{1}$$

where $A = \left(a_{ij}\right)$ is a $d \times n$ integral matrix of full row-rank, b is a d-element and c is an n-element integral vector, respectively. Common to most techniques is solving the *linear programming relaxation*

$$\min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0\right\} \tag{2}$$

and constructing a feasible integer solution from this fractional solution.

Some combinatorial optimization problems like bipartite matching or minimum cost network flows have the property that the matrix A in the integer programming model (1) is *unimodular*, which means that each $d \times d$ submatrix of A has determinant $0, \pm 1$. In this case, the integer program (1) can be solved exactly with linear programming since an optimal extreme point of the linear programming problem is integral. This is a property of the matrix A. More precisely all extreme points of the polyhedron $\{x \in \mathbb{R}^n : Ax = b,\ x \geqslant 0\}$ are integral for each integral b if and only if the matrix A is unimodular [21]. This is also related to the study of *totally unimodular matrices* and the Hoffman–Kruskal theorem [8], see e.g. [17]. Seymour [18] showed that totally unimodular matrices, and consequently, also unimodular matrices can be recognized in polynomial time. Thus this convenient property of combinatorial optimization problems having integral linear programming relaxations can be efficiently recognized.

If the linear programming relaxation does not immediately yield an integral optimal solution, one *rounds* this fractional solution to a feasible integral solution. This paradigm has been very successfully applied in the area of approximation algorithms, see e.g. [20]. As the solution which is thereby found is compared to the solution of the linear programming relaxation, the approximation guarantee of the algorithm depends crucially on the *integrality gap* of the linear programming relaxation. In this paper we consider *additive* integrality gaps, i.e., the *difference* between the optimal solution value of the combinatorial optimization problem and its linear relaxation.

The integer program (1) is defined by three parameters: the matrix A, the objective vector c, and the right-hand side b. If the optimum value of the relaxation (2) is strictly smaller than the optimum value of the integer program, then this difference can be made arbitrarily large by scaling the objective function vector c with a constant. This shows that, while unimodularity is a sufficient condition on the matrix A yielding integer polyhedra regardless of the objective function vector and the right-hand side, one needs to take the objective function vector into account if one wants to have a similar notion for additive integrality gaps.

The tuple (A, c) has the *integer rounding property* [1] if

$$\min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0 \text{ integral}\right\} = \left\lceil \min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0\right\}\right\rceil$$

for each b such that the linear programming relaxation is feasible and bounded. We shall always assume that the system $y^{\mathrm{T}}A \leqslant c^{\mathrm{T}}$ has a solution; then both the integer program (1) and the associated linear programming relaxation (2) are not unbounded for all b.

There are many examples of tuples (A, c) stemming from combinatorial optimization problems that have the integer rounding property [1,5,19,22]. In this case the integral problem can also be solved exactly by linear programming techniques, see e.g. [16, Theorem 22.15] due to the relation of the integer rounding property with the notions of a Hilbert basis and total dual integrality, which we briefly describe.

The integral vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^d$ form a *Hilbert basis* if every integral vector in the *cone* generated by $a_1, a_2, \ldots, a_n$ can be expressed as an integral non-negative linear combination of $a_1, a_2, \ldots, a_n$. Giles and Orlin [6] showed that the tuple (A, c) has the integer rounding property if and only if the columns of the matrix

$$\begin{pmatrix} 1 & c^{\mathrm{T}} \\ 0 & A \end{pmatrix} \tag{3}$$

form a Hilbert basis. The question now arises, whether this property can also be tested in polynomial time, like it is the case for total unimodularity. Pap [13] recently proved that the recognition of Hilbert bases is coNP-complete, see also [3]. Via the result of Giles and Orlin this also means that testing whether a tuple (A, c) has the integer rounding property is co-NP complete.

However, if d (the number of rows of A) is fixed, then one can test in polynomial time whether a given family of integer programs has the integer rounding property, by using the algorithm of Cook et al. [2] for recognizing Hilbert bases in fixed dimension.

1.1 Contribution of this paper

In this paper we consider a generalization of the integer rounding property. We say that (A, c) has an *additive integrality gap of at most* γ if

$$\min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0 \text{ integral}\right\} \leqslant \left\lceil \min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0\right\}\right\rceil + \gamma, \tag{4}$$

for each b for which the linear programming relaxation (2) is feasible. If there is a b for which (2) is feasible but (1) is infeasible, then (A, c) has infinite integrality gap.

For $\gamma = 0$ this is exactly the integer rounding property. Our main results are as follows.

(a) It is NP-hard to test whether (A, c) has additive gap of at most γ even if d is fixed and $\gamma = 1$. This is in contrast to the integer rounding property ($\gamma = 0$), which can be tested in polynomial time if d is fixed.

(b) For fixed d and γ, there exists a polynomial time algorithm which tests whether $(A, \mathbf{1})$ has additive gap of at most γ. We use $\mathbf{1}$ to denote the all-one vector.

Many combinatorial optimization problems of set-packing or set-partitioning nature have objective function $\mathbf{1}^{\mathrm{T}}x$ which makes the result (b) a relevant extension of the test for the integer rounding property of combinatorial optimization problems. While the result (a) seems surprising at first, we describe our algorithm which solves (b) such that it runs in polynomial time under a more general assumption than all-one objective function. The case $\gamma = 0$ also falls under this restriction and thus our algorithm can be interpreted as a new Hilbert basis test in fixed dimension that may be more geometrically intuitive than the algorithm of Cook et al. [2].

Here is a brief outline on how these results are proved. First we extend the notion of a Hilbert basis to a so-called γ-relaxed Hilbert basis. We prove that the tuple (A, c) has additive integrality gap at most γ if and only if the columns of the matrix (3) form a γ-relaxed Hilbert basis. Then we consider the recognition problem for γ-relaxed Hilbert bases in fixed dimension. Via a reduction from the *Frobenius problem* we show that it is NP-hard to decide whether n positive integers form a 1-relaxed Hilbert basis. This is then extended to the result (a).

The positive result (b) is obtained from an algorithm to recognize γ-relaxed Hilbert bases. Given a cone spanned by the columns of a matrix, we search for *witnesses*, that is integral vectors in the cone that cannot be written as a non-negative integer combination of the columns of the matrix. The search is performed by constructing a hyperplane arrangement to divide the search space into polynomially many half-open polytopes, followed by solving an integer program in fixed dimension for each of those cells.

1.2 Related work

Hoşten and Sturmfels [7] considered a similar recognition problem, where the right-hand side b ranges over all integral vectors, for which the integer program (1) is feasible (in contrast, we consider integral vectors b, for which the associated linear program is feasible; thus the additive integrality gap may happen to be infinite). They designed an algorithm that tests if the additive integrality gap is bounded by γ in polynomial time if both the number of constraints d and the number of variables n are fixed. Eisenbrand and Shmonin [4] generalized this result, presenting an algorithm that tests the additive integrality gap for families of integer programs of the form

$$\max\left\{c^{\mathrm{T}}x : Ax \leqslant b,\ x \text{ integral}\right\},$$

where b ranges over all vectors, for which the integer program is feasible. This algorithm runs in polynomial time assuming only that the number of variables n is fixed. These algorithms are based on an algorithm of Kannan [9,10] to decide general $\forall\exists$-statements.

1.3 Some basic definitions and notation

Let $a_1, a_2, \ldots, a_n$ be vectors in the Euclidean space $\mathbb{R}^d$. The *cone* generated by these vectors is the set

$$\operatorname{cone}(a_1, a_2, \ldots, a_n) := \left\{\sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geqslant 0 \text{ for all } i\right\}.$$

If $b \in \operatorname{cone}(a_1, a_2, \ldots, a_n)$, then we say that b can be expressed as a *non-negative combination* of vectors $a_1, a_2, \ldots, a_n$. The *integer cone* generated by $a_1, a_2, \ldots, a_n$ is the set

$$\text{int.cone}(a_1, a_2, \ldots, a_n) := \left\{ \sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geqslant 0 \text{ integer for all } i \right\},$$

which forms a semigroup with addition. We say that $b \in \text{int.cone}(a_1, a_2, \ldots, a_n)$ can be expressed as an *integral non-negative combination* of vectors $a_1, a_2, \ldots, a_n$. The integral vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^d$ form a *Hilbert basis* if

$$\text{cone}(a_1, a_2, \ldots, a_n) \cap \mathbb{Z}^d = \text{int.cone}(a_1, a_2, \ldots, a_n).$$

A cone is *pointed* if it has an extreme point, or equivalently, there is a vector d such that $d^{\mathrm{T}} x > 0$ for all non-zero vectors x from the cone. It is known that every pointed finitely generated rational cone has a unique minimal Hilbert basis; see e.g. [16].

2 Relaxed Hilbert bases

In this section we provide an analogous result to the one of Giles and Orlin [6] for bounded integrality gap. For this, we need to generalize the notion of a Hilbert basis.

Let $a_0, a_1, \ldots, a_n$ be integral vectors in $\mathbb{R}^d$ and let γ be a non-negative integer. We say that the vectors $a_0, a_1, \ldots, a_n$ form a *γ-relaxed Hilbert basis with respect to a_0* if every integral vector b in $\text{cone}(a_0, a_1, \ldots, a_n)$ can be written as an integral combination

$$b = \sum_{i=0}^{n} \lambda_i a_i, \ \lambda_0 + \gamma \geqslant 0 \quad \text{and} \quad \lambda_i \geqslant 0 \quad \text{for } i = 1, 2, \ldots, n.$$

In other words, we require that at least one of the vectors

$$b, \ \ b + a_0, \ \ b + 2a_0, \ldots, b + \gamma a_0$$

belongs to $\text{int.cone}(a_0, a_1, \ldots, a_n)$. Moreover, if $b + i a_0$ belongs to int.cone $(a_0, a_1, \ldots, a_n)$ for some integer $i \leqslant \gamma$, then $b + \gamma a_0$ also belongs to $\text{int.cone}(a_0, a_1, \ldots, a_n)$. Thus, vectors $a_0, a_1, \ldots, a_n$ form a γ-relaxed Hilbert basis if and only if

$$b + \gamma a_0 \in \text{int.cone}(a_0, a_1, \ldots, a_n)$$

whenever $b \in \text{cone}(a_0, a_1, \ldots, a_n) \cap \mathbb{Z}^d$.

Theorem 1 *Let A be an integral matrix and let c be an integral vector. The family of integer programs*

$$\min \left\{ c^{\mathrm{T}} x : Ax = b, \ x \geqslant 0 \text{ integral} \right\} \tag{5}$$

has an additive integrality gap of at most γ if and only if the columns of the matrix

$$\begin{pmatrix} 1 & c^{\mathrm{T}} \\ 0 & A \end{pmatrix} \tag{6}$$

form a γ-relaxed Hilbert basis with respect to the first column.

Proof Suppose that the additive integrality gap of (5) is at most γ. Let $\binom{\alpha}{b}$ be an arbitrary integral vector from the cone generated by the columns of the matrix (6). We have to show that $\binom{\alpha+\gamma}{b}$ is in the integer cone generated by the columns of the matrix (6).

One has $\alpha = \hat{\xi} + c^{\mathrm{T}}\hat{x}$ and $b = A\hat{x}$ for some number $\hat{\xi} \geqslant 0$ and some vector $\hat{x} \geqslant 0$. It follows that

$$\min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0\right\} \leqslant c^{\mathrm{T}}\hat{x} \leqslant \alpha.$$

Since the additive integrality gap of the integer program (5) is at most γ, the optimum value β of (5) is at most $\alpha + \gamma$. Thus the vector $\binom{\beta}{b}$ belongs to the integer cone generated by the columns of matrix (6). Since the vector $\binom{1}{0}$ belongs to (6), so does $\binom{\alpha+\gamma}{b}$.

Conversely, suppose that the columns of the matrix (6) form a γ-relaxed Hilbert basis and let b be an integral vector such that the system $Ax = b$ has a non-negative solution. Let α denote the optimum value of the linear program

$$\min\left\{c^{\mathrm{T}}x : Ax = b,\ x \geqslant 0\right\}.$$

Then $\alpha = c^{\mathrm{T}}\hat{x}$ and $b = A\hat{x}$ for some vector $\hat{x} \geqslant 0$. Consequently, the integral vector $\binom{\lceil\alpha\rceil}{b}$ belongs to the cone generated by the columns of matrix (6), and thus the vector $\binom{\lceil\alpha\rceil+\gamma}{b}$ belongs to the integer cone generated by the columns of matrix (6). But the latter implies that the optimum value of the integer program (5) is at most $\lceil\alpha\rceil + \gamma$. □

It is easy to see that the cone generated by the columns of the matrix (6) is pointed if and only if the system $y^{\mathrm{T}}A < c$ has a solution. We shall need the following two lemmas for our recognition algorithm in the last section of this paper. They help us to restrict the space in which we have to search for witnesses, i.e. vectors b which prove that $a_0, a_1, \ldots, a_n$ fails to be a γ-relaxed Hilbert bases. The first lemma is a generalization of a statement about classical Hilbert bases ($\gamma = 0$) by Cook et al. [2]. It is proved in a similar way.

Lemma 1 *Let $a_0, a_1, \ldots, a_n$ be integral vectors in $\mathbb{R}^d$ such that the cone* $\mathrm{cone}(a_0, a_1, \ldots, a_n)$ *is pointed. Then $a_0, a_1, \ldots, a_n$ do not form a γ-relaxed Hilbert basis with respect to a_0 if and only if there is an integral vector $b \in$* $\mathrm{cone}(a_0, a_1, \ldots, a_n)$ *such that*

$$b - a_j \notin \mathrm{cone}(a_0, a_1, \ldots, a_n) \quad \textit{for each } j = 0, 1, \ldots, n,$$

and

$$b + \gamma a_0 \notin \text{int.cone}(a_0, a_1, \dots, a_n).$$

Proof Sufficiency is clear. Suppose that $a_0, a_1, \dots, a_n$ do not form a γ-relaxed Hilbert basis. Since $\text{cone}(a_0, a_1, \dots, a_n)$ is pointed, there is a vector d with $d^{\mathrm{T}} a_j > 0$ for $j = 0, 1, \dots, n$. Let b be an integral vector in $\text{cone}(a_0, a_1, \dots, a_n)$ such that $b+\gamma a_0 \notin \text{int.cone}(a_0, a_1, \dots, a_n)$. Among all possible b's we choose one that minimizes $d^{\mathrm{T}} b$. Since $d^{\mathrm{T}}(b-a_j) < d^{\mathrm{T}} b$ one has, by the minimality of b, $b-a_j \notin \text{cone}(a_0, a_1, \dots, a_n)$ for each $j = 0, 1, \dots, n$. □

Corollary 1 *Let* $a_0,a_1,\dots,a_n$ *be integral vectors in* $\mathbb{R}^d$ *such that* $\text{cone}(a_0,a_1,\dots,a_n)$ *is a pointed cone. Then* $a_0, a_1, \dots, a_n$ *do not form a* γ*-relaxed Hilbert basis with respect to* a_0 *if and only if there is an integral vector* b *in* $\text{cone}(a_0, a_1, \dots, a_n)$ *such that*

$$b + \gamma a_0 \notin \text{int.cone}(a_0, a_1, \dots, a_n).$$

and

$$\max\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\right\} < d,$$

where $A = (a_0\ a_1\ \dots\ a_n)$ *is the matrix composed of vectors* $a_0, a_1, \dots, a_n$ *as columns.*

Proof Sufficiency is again clear. Let b be a vector satisfying the conditions of Lemma 1. Consider the linear program

$$\max\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\right\}.$$

This LP is bounded because the cone generated by A is pointed. From the theory of linear programming, we know that there is a *basic optimal solution* of the above linear program having at most d non-zero components, i.e.,

$$\sum_{j=1}^{d} x_{i_j} a_{i_j} = b, \qquad x_{i_j} \geqslant 0,$$

for some indices $i_1, i_2, \dots, i_d$. Then $x_{i_j} < 1$ for all $j = 1, 2, \dots, d$, as otherwise $b - a_{i_j}$ would belong to $\text{cone}(a_0, a_1, \dots, a_n)$, which contradicts the choice of b. But then

$$\mathbf{1}^{\mathrm{T}} x = \sum_{j=1}^{d} x_{i_j} < d.$$

□

3 Hardness of gap testing

In this section we show that it is NP-hard to test whether (A, c) has integrality gap at most 1, even if A has only one row. Since this is equivalent to deciding if the columns of the matrix (6) form a 1-relaxed Hilbert basis with respect to the first column we first consider a general problem of recognizing 1-relaxed Hilbert bases.

We proceed by a reduction from the *Frobenius problem* (also known as the *coin problem*). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive integers such that their greatest common divisor is 1. The smallest integer t, such that every integer $z > t$ is an element of $\text{int.cone}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, is called the *Frobenius number* and is denoted by $F(\alpha_1, \alpha_2, \ldots, \alpha_n)$. The condition that $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_n) = 1$ implies that the Frobenius number is finite. The *Frobenius problem* is to decide whether

$$F(\alpha_1, \alpha_2, \ldots, \alpha_n) < t,$$

for given positive integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and t. Kannan [10] showed that the Frobenius problem can be solved in polynomial time, if n is fixed. For general n it is known to be NP-hard under Turing reductions [15].

Theorem 2 *Testing whether positive integers $a_0, a_1, \ldots, a_n$ form a 1-relaxed Hilbert basis with respect to a_0 is NP-hard.*

Proof We demonstrate a Karp-reduction from the Frobenius problem. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and t be positive integers defining an instance of the Frobenius problem and consider the integers

$$a_0 = 4t,\ a_1 = 2\alpha_1, \ldots,\ a_n = 2\alpha_n,\ a_{n+1} = 2t + 1. \tag{7}$$

We claim that the integers (7) form a 1-relaxed Hilbert basis with respect to $a_0 = 4t$ if and only if $F(\alpha_1, \alpha_2, \ldots, \alpha_n) < t$. Observe that the cone generated by (7) is exactly the set of all non-negative numbers.

Suppose that $F(\alpha_1, \alpha_2, \ldots, \alpha_n) < t$. Then every even integer $z \geqslant 2t$ can be expressed as an integral non-negative combination of $2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n$,

$$z \in \text{int.cone}(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n) \quad \text{for all } z \geqslant 2t \text{ even.} \tag{8}$$

If $0 \leqslant z < 2t$ is an even integer, then $z + 4t > 2t$ is also even, and (8) implies

$$z + 4t \in \text{int.cone}(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n) \quad \text{for all } 0 \leqslant z < 2t \text{ even.}$$

If $z \geqslant 0$ is odd, then $z + 4t - (2t + 1) \geqslant 2t$ is even and thus, again by (8), an element of $\text{int.cone}(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n)$. Thus we have

$$z + 4t \in \text{int.cone}(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n, 2t + 1) \quad \text{for all } z \geqslant 0 \text{ odd.}$$

This shows that (7) is a 1-relaxed Hilbert basis.

For the converse we need to show that every $z \geqslant t$ is contained in int.cone$(\alpha_1, \alpha_2, \ldots, \alpha_n)$ if (7) is a 1-relaxed Hilbert basis. It is enough to show this for each z with $t \leqslant z < 2t$ since each integer $x \geqslant 2t$ can be written as $x = qt + \ell$ with $q \geqslant 0$ and $t \leqslant \ell < 2t$.

We can write z as $z = t + y$ with $0 \leqslant y \leqslant t - 1$. Since $2y + 1$ is odd, one has

$$2y + 1 \notin \text{int.cone}(4t, 2\alpha_1, 2\alpha_2 \ldots, 2\alpha_n).$$

However, we have

$$(2y + 1) + 4t \in \text{int.cone}(4t, 2\alpha_1, 2\alpha_2 \ldots, 2\alpha_n, 2t + 1)$$

since $4t, 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n, 2t + 1$ is a 1-relaxed Hilbert basis. But $4t$ cannot have a positive coefficient in a non-negative integral combination of $4t, 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n, 2t + 1$ yielding $(2y + 1) + 4t$. This implies $(2y + 1) + 4t \in$ int.cone $(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n, 2t + 1)$. The coefficient of $2t + 1$ must be one. It cannot be two, since the result would be an even number and it cannot be larger than two since the outcome would be more than $6t$. Thus $(2y + 1) + 4t - (2t + 1) \in$ int.cone$(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n)$. But $2y + 2t = 2z$ which shows that $z \in$ int.cone$(\alpha_1, \alpha_2, \ldots, \alpha_n)$. □

The result of Theorem 2 is not directly applicable to the problem of recognizing additive integrality gaps. Now, we modify the reduction to settle the complexity of recognizing relaxed Hilbert bases, provided that the input vectors are of the form (6).

Theorem 3 *Testing whether a family of integer programs* (5) *has an additive integrality gap of at most 1 is NP-hard, even if there is only one non-trivial constraint.*

Proof Due to Theorem 1, we need to show that it remains NP-hard to test whether a set of vectors is a 1-relaxed Hilbert basis, where a_0 is a unit vector and the cone that the set of vectors generate is pointed.

We observe that for any unimodular matrix U, the vectors $a_0, a_1, \ldots, a_n$ form a γ-relaxed Hilbert basis if and only if $Ua_0, Ua_1, \ldots, Ua_n$ do. Consequently, it suffices to show hardness when a_0 and a_1 are required to form a unimodular matrix $(a_0\ \ a_1)$. By applying an appropriate unimodular transformation, we can then transform a_0 and a_1 into the unit vectors $e_1 = Ua_0$ and $e_2 = Ua_1$.

The reduction is again from the Frobenius problem and, in fact, very similar to that in the proof of Theorem 2. Given positive integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and t defining an instance of the Frobenius problem, we construct an instance of the relaxed Hilbert basis recognition problem as follows:

$$\begin{pmatrix}1\\4t\end{pmatrix}, \begin{pmatrix}2\\8t-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\alpha_1\end{pmatrix}, \begin{pmatrix}0\\2\alpha_2\end{pmatrix}, \ldots, \begin{pmatrix}0\\2\alpha_n\end{pmatrix}, \begin{pmatrix}0\\2t+1\end{pmatrix}. \tag{9}$$

We claim that $F(\alpha_1, \alpha_2, \ldots, \alpha_n) < t$ if and only if the vectors (9) form a 1-relaxed Hilbert basis with respect to a_0. It is easy to see that the cone generated by (9) is exactly the set of all non-negative vectors.

Suppose that $F(\alpha_1, \alpha_2, \ldots, \alpha_n) < t$. Similar to the proof of Theorem 2, we can show that for any integral vector $v = \binom{0}{z}$, at least one of the following holds:

$$v = \begin{pmatrix} 0 \\ z \end{pmatrix} \in \text{int.cone}\left(\begin{pmatrix} 0 \\ 2\alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 2\alpha_n \end{pmatrix}, \begin{pmatrix} 0 \\ 2t+1 \end{pmatrix}\right)$$

or

$$v + a_0 = \begin{pmatrix} 1 \\ z + 4t \end{pmatrix} \in \text{int.cone}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2\alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 2\alpha_n \end{pmatrix}, \begin{pmatrix} 0 \\ 2t+1 \end{pmatrix}\right).$$

Since the unit vector $\binom{1}{0}$ belongs to the set (9), this implies the claim for all integral non-negative vectors.

Suppose that the vectors (9) form a 1-relaxed Hilbert basis with respect to a_0. We need to show that every integer $z \geqslant t$ belongs to $\text{int.cone}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Again, we may restrict our attention to integers $t \leqslant z \leqslant 2t - 1$ only. Set $z = t + y$, then $0 \leqslant y \leqslant t - 1$. The vector $v = \binom{0}{2y+1}$ does not belong to the integer cone generated by (9), since $2y + 1$ is odd and $2y + 1 < 2t + 1$. However, $v + a_0 = \binom{1}{2y+1+4t}$ is in the integer cone of (9) by the 1-relaxed Hilbert basis property, and so we can write

$$\begin{aligned} v + a_0 &= \begin{pmatrix} 1 \\ 2y + 1 + 4t \end{pmatrix} \\ &= \lambda_0 \begin{pmatrix} 1 \\ 4t \end{pmatrix} + \sum_{j=1}^{n} \lambda_j \begin{pmatrix} 0 \\ 2\alpha_j \end{pmatrix} + \xi \begin{pmatrix} 2 \\ 8t - 1 \end{pmatrix} + \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2t + 1 \end{pmatrix} \end{aligned}$$

for some non-negative integers $\lambda_0, \lambda_1, \ldots, \lambda_n, \xi, \eta$ and μ. Clearly, $\xi = 0$. Moreover, $\lambda_0 = 0$, since v itself does not belong to the integer cone of (9). Consequently, $\eta = 1$, as $\binom{1}{0}$ is the only remaining vector with non-zero in the first component. Thus, we are left with the second component of $v + a_0$ only:

$$2y + 1 + 4t = \sum_{j=1}^{n} \lambda_j (2\alpha_j) + \mu(2t + 1).$$

As in the proof of Theorem 2 we conclude that $\mu = 1$, which implies

$$2z = 2t + 2y \in \text{int.cone}(2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n).$$

Finally, we observe that the vectors a_0 and a_1 in (9) form a unimodular matrix $(a_0 \;\; a_1)$, as required. □

4 The case of all-one objectives

We now consider as a special case the family of linear programs

$$\min\{\mathbf{1}^{\mathrm{T}} x : \tilde{A}x = b, x \geqslant 0\}$$

for a fixed matrix $\tilde{A} \in \mathbb{Z}^{d \times n}$. We will show that for fixed d and γ, there exists a polynomial time algorithm which, given $\tilde{A}$ as input, decides whether this family has an additive integrality gap of at most γ. By Theorem 1, this is equivalent to testing whether the columns of

$$A := \begin{pmatrix} 1 & \mathbf{1}^{\mathrm{T}} \\ 0 & \tilde{A} \end{pmatrix}$$

are a γ-relaxed Hilbert basis, so we only consider this equivalent problem in the remainder of the section.

Let $\mathscr{A}$ be a family of matrices. We say that $\mathscr{A}$ has *property B* if there is a constant $f(\gamma, d)$ depending only on γ and d such that for every matrix $A = (a_0, \ldots, a_n) \in \mathscr{A}$, one has that

1. $b \in \operatorname{cone}(a_0, a_1, \ldots, a_n)$,
2. $\max\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\right\} \leqslant d$, and
3. $b + \gamma a_0 \in \operatorname{int.cone}(a_0, a_1, \ldots, a_n)$

together imply $\min\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b + \gamma a_0,\ x \geqslant 0 \text{ integral}\right\} \leqslant f(\gamma, d)$. Property B says that if b lies in a certain bounded region of the cone spanned by A, and $b + \gamma a_0$ can be written as the sum of columns of A, then the number of columns required (counting multiples) is bounded by $f(\gamma, d)$. This bound is crucial for bounding the running time of the algorithm presented in this section, and the following Lemma shows that it applies to the matrices we are interested in.

Lemma 2 *The family of matrices whose first row equals the all-ones vector has property B.*

Proof We will show that property B is satisfied with $f(\gamma, d) = d + \gamma$. Let A be a matrix whose first row is the all-ones vector. The objective value of $\max\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\}$ is equal to b_1 for *every* feasible solution. Hence

$$\max\{\mathbf{1}^{\mathrm{T}} x : Ax = b + \gamma a_0,\ x \geqslant 0\} = b_1 + \gamma \leqslant d + \gamma.$$

This LP has an integral solution (by the third precondition of the claim) whose objective value is of course also bounded from above by $d + \gamma$. Thus

$$\min\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b + \gamma a_0,\ x \geqslant 0 \text{ integral}\right\} \leqslant d + \gamma,$$

which is what we needed to show. □

Using only the second half of the proof of Lemma 2, one can show that a variant of property B holds for arbitrary matrices generating a pointed cone, as long as we restrict ourselves to the case $\gamma = 0$ and set $f(0, d) = d$. This implies that our algorithm generalizes the one of Cook et al. [2] to test classical Hilbert bases.

Theorem 4 *There is an algorithm which decides whether a given set of vectors $a_0, \ldots, a_n \in \mathbb{Z}^d$, generating a pointed cone, is a γ-relaxed Hilbert basis. If this* *algorithm is applied for fixed γ and d on a family of inputs such that the family of matrices $A = (a_0, \dots, a_n)$ corresponding to those inputs has property B, then it runs in polynomial time in the input size.*

The remainder of this section is devoted to the proof of Theorem 4. Corollary 1 tells us that if $A = (a_0 \dots a_n)$ is *not* a γ-relaxed Hilbert basis, then there exists a *witness* to this fact in the region

$$Q = \{b \in \mathbb{R}^d : b \in \operatorname{cone}(a_0, a_1, \dots, a_n) \text{ and } \max\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\right\} \leqslant d\}.$$

In fact, a witness is a point $b \in Q \cap \mathbb{Z}^d$ such that $b + \gamma a_0 \notin \text{int.cone}(a_0, \dots, a_n)$. We say that a point $b \in Q \cap \mathbb{Z}^d$ with $b + \gamma a_0 \in \text{int.cone}(a_0, \dots, a_n)$ is a *non-witness*. Our strategy is to enumerate and eliminate all non-witnesses and then use (polynomially) many integer programs in fixed dimension to search for any remaining integer points in Q. The algorithm then concludes that A is a γ-relaxed Hilbert basis iff no remaining integer point is found.

To do all this, we need an efficient description of the set Q. We use the following fact about hyperplane arrangements; see e.g., [12, ch. 6].

Theorem 5 *The number of cells (d-faces) in an arrangement of m hyperplanes in $\mathbb{R}^d$ is bounded from above by $\Phi_d(m) = \binom{m}{0} + \cdots + \binom{m}{d}$.*

In fixed dimension, this implies that the number of cells is bounded by a polynomial in the number of hyperplanes. Note that the upper bound is achieved when the hyperplanes are in general position, i.e. no more than d planes meet in any single point.

Lemma 3 *For fixed d, there is a polynomial time algorithm that computes systems of linear inequalities that describe polyhedra $Q_1, \dots, Q_t$ such that $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_t$.*

Proof From the theory of linear programming we know that for every vector b for which the linear program

$$\max\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\right\}$$

is feasible and bounded, there is a basic optimal solution. Equivalently, there is a nonsingular $d \times d$-submatrix of A, say A', such that

$$\max\left\{\mathbf{1}^{\mathrm{T}} x : Ax = b,\ x \geqslant 0\right\} = \max\left\{\mathbf{1}^{\mathrm{T}} x : A'x = b,\ x \geqslant 0\right\}.$$

For fixed d, we can enumerate all nonsingular $d \times d$-submatrices of A in polynomial time, say $A_1, \dots, A_q$. For each submatrix A_i, the set of vectors b for which

$$\max\left\{\mathbf{1}^{\mathrm{T}} x : A_i x = b,\ x \geqslant 0\right\} \leqslant d$$

is just a simplex S_i in $\mathbb{R}^d$, and hence can be expressed as an intersection of $d + 1$ half-spaces. Furthermore, d of these half-spaces define the (simplicial) cone $C_i = \operatorname{cone}(A_i)$ of all vectors b for which the above linear program is feasible.

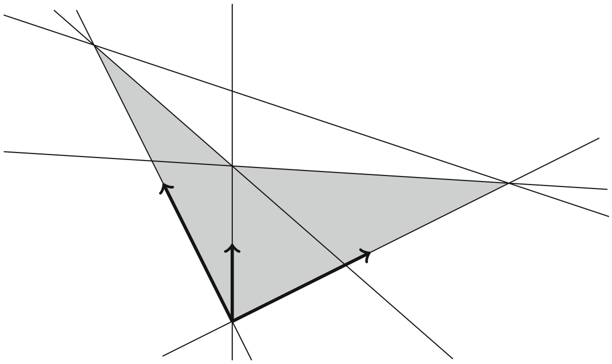

Fig. 1 The hyperplane arrangement of Lemma 3 for a cone generated by three vectors. The region Q is *shaded gray*

Now, $b \in S_i$ belongs to Q if and only if $b \in S_j$ for all indices j such that $b \in C_j$. If d is fixed, then the arrangement of hyperplanes defining the simplices S_i, $i = 1 \ldots q$, partitions the whole space $\mathbb{R}^d$ into polynomially many cells $Q_1, Q_2, \ldots, Q_l$, see Fig. 1. It is now easy to see that if $b \in Q_i$ belongs to Q, then all $b' \in Q_i$ belong to Q, and vice versa. Thus, a subset of these cells yields a required partition. □

Now that we can obtain an efficient description of Q, let us see how to enumerate all non-witnesses. Property B states that for every non-witness b, one can express $b + \gamma a_0$ as the sum of at most $f(\gamma, d)$ vectors from $a_0, \ldots, a_n$ (with repetitions allowed). We can bound the size of the set V of all such sums by $(n+1)^{f(\gamma,d)}$. This is a polynomial if d and γ are fixed, and so we can enumerate V in polynomial time. It is easy to see that an integer point in Q is a non-witness if and only if it is contained in the set

$$V^\star := \{v - \gamma a_0 : v \in V\}.$$

The search for witnesses has thus been reduced to a search for integer points in $Q \backslash V^\star$.

Lemma 4 *If d and γ are fixed, then there is a polynomial time algorithm that computes open polyhedra $P_1, P_2, \ldots, P_k$ such that for any integral vector $b \in Q$ one has $b \notin V^*$ if and only if $b \in P_i$ for some index $i = 1, 2, \ldots, k$.*

Proof Since Q is a bounded set, for each $b \in V^*$ we can find a hyperplane H_b such that b is the only integral vector from Q lying on H_b, see Fig. 2. Thus, there are polynomially many hyperplanes $H_1, H_2, \ldots, H_t$ such that an integral vector $b \in Q$ belongs to V^* if and only if $b \in H_i$ for some $i = 1, 2, \ldots, t$. These hyperplanes partition the whole space $\mathbb{R}^d$ into polynomially many cells $P_1', P_2', \ldots, P_k'$, computable in polynomial time, where each cell is a polyhedron whose facets lie on some of the hyperplanes $H_1, H_2, \ldots, H_t$. The interiors of these polyhedra yield the desired open polyhedra $P_1, P_2, \ldots, P_t$. □

By Lemmas 4 and 3, it is now sufficient to check that the sets $P_i \cap Q_j$ do not contain integral vectors, for all indices i and j. So we get a polynomial number of integer

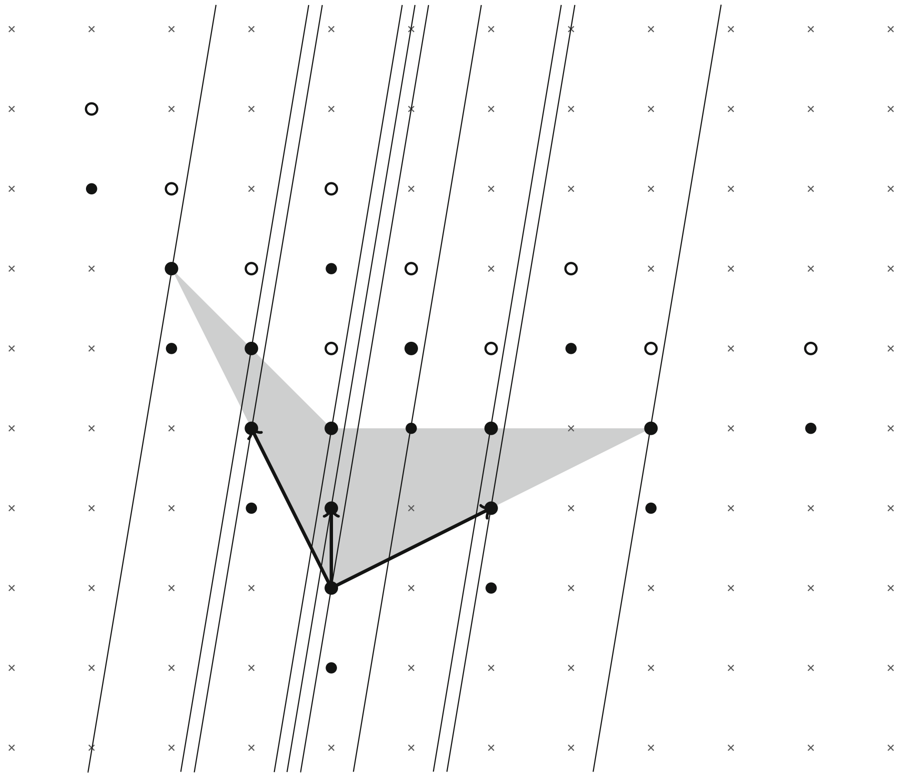

Fig. 2 The case $\gamma = 1$ of Lemma 4 for a cone generated by three vectors, $a_0^{\mathrm{T}} = (0, 1)$, $a_1^{\mathrm{T}} = (-1, 2)$, $a_2^{\mathrm{T}} = (2, 1)$. The region Q is *shaded* in *gray*, the points of V^* are represented as *filled circles*, the points of $V \setminus V^*$ are drawn as *empty circles*, and hyperplanes H_b for each $b \in V^* \cap Q$ are indicated. The integer point $(1, 1)$ lies in $Q \setminus V^*$, so the generating vectors do not form a 1-relaxed Hilbert basis for this cone

programming problems, each of which has d variables and is therefore solvable in polynomial time if d is fixed [11]. This proves Theorem 4.

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