

A comment on the construction of the maximal globally hyperbolic Cauchy development

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Under mild assumptions, we remove all traces of the axiom of choice from the construction of the maximal globally hyperbolic Cauchy development in general relativity. The construction relies on the notion of direct union manifolds, which we review. The construction given is very general: any physical theory with a suitable geometric representation (in particular all classical fields), and such that a strong notion of “local existence and uniqueness” of solutions for the corresponding initial value problem is available, is amenable to the same treatment.

A celebrated theorem on the local Cauchy problem for Einstein’s equations is that of Choquet-Bruhat and Geroch (1969) which asserts that every initial data set leads to a unique *maximal globally hyperbolic Cauchy development*. In their original proof (as well as many subsequent treatments, see e.g. Ringström (2009)) the authors appealed to Zorn’s Lemma in their construction of the space-time manifold, which led to the common misconception that the proof is non-constructive as the argument seemingly depends on the axiom of choice.

In this paper, we will show that, insofar as the actual *construction* of the space-time manifold is concerned, the use of axiom of choice is not necessary. As it turns out, however, the manifold constructed will not, in general, be second countable, making geometry and analysis somewhat awkward on the space-time. One can circumvent this difficulty in two ways: firstly in many situations assuming the axiom of *countable choice* (or even weaker statements (Howard and Rubin, 1998) such as “every countable union of countable sets is countable”) can allow us to recover statements about countability of a basis for the topology; secondly, an option that the author hopes to emphasize here, is that sometimes adding some *additional structures* (in a manner that is natural and physical) to the definition of a space-time will allow us to sidestep the issue of choice entirely.

Recently the same question, in the context of general relativity, has been treated exhaustively in a pre-print by Sbierski (2013). Our approach here offers two minor improvements:

1. We were able to avoid the axiom of choice entirely, *including the axiom of countable choice*. In Sbierski’s construction he appealed to a theorem of Geroch (1968, Appendix) to obtain second countability of the space-time. This theorem depends on the statement “every countable union of countable sets is countable” alluded to above, which cannot be proven in ZF (that is, Zermelo-Fraenkel set theory without axiom of choice) alone.
2. We isolate the structures which allows the general construction to proceed. In particular, we shall take

as *black boxes* certain facts about the local existence and uniqueness theorems (see, e.g. Choquet-Bruhat and Geroch (1969), the monograph of Ringström (2009), as well as Sbierski (2013) for a discussion), and concentrate only at the level of the construction of the maximal development. This allows us to easily “swap out” the underlying physical model with anything else that satisfies sufficiently strong local existence and uniqueness theorems.

One may ask why bother at all about the issue of choice (and its weaker formulations): after all, much of the foundations of topology, geometry, and analysis that come up in the study of partial differential equations on manifolds (a subject within which the evolution problem of general relativity squarely sits) as commonly used depend on some (perhaps weakened) version of the axiom of choice. Here is a sampling of the statements that one may find useful but cannot be proved (in ZF) without some form of choice (the numbers in parentheses refer to the searchable form numbers from the companion website <http://consequences.emich.edu/conseq.htm> to Howard and Rubin (1998)):

- In functional analysis: Hahn-Banach theorem (#52), Krein-Milman theorem (#65), the Banach-Alaoglu theorem (#14Q), and the Arzelà-Ascoli theorem (#94Q).
- In analysis of metric spaces: the fact that on a metric space sequential continuity implies continuity (#8E), the Heine-Borel theorem for \mathbb{R}^n (#74), and that every uncountable subset of \mathbb{R} contains a condensation point (#6A).
- In topology: that a second countable topological space is separable (#8L) and Urysohn’s Lemma (#78).

One answer to that question “why” is one of aesthetics. (For some other points-of-view, the author encourages the reader to look at the MathOverflow discussion accessible at <http://mathoverflow.net/questions/22927/>.) An insistence on using some versions of axiom of choice when it can be easily avoided seems rather wasteful. Furthermore, the general tool that was actually used to construct the maximum globally hyperbolic Cauchy development

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can be easily adapted in many categories other than just smooth manifolds. One need not be in a context where some version of axiom of choice is already used or needed to find the construction useful. Lastly, should (however unlikely) the prevailing opinion on the axiom of choice change in the future, a better understanding of a physical theory's dependence on said axiom will certainly help reformulate the theory upon more culturally acceptable foundations.

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I. THE DIRECT UNION CONSTRUCTION

We are motivated by the following: let M be a smooth manifold and \mathcal{U} a collection of *open* submanifolds of M (in other words, open sets on M), their union $\bigcup \mathcal{U}$ is another open submanifold of M . With great hindsight, we see that in the case of general relativity, we can take M to be the maximal globally hyperbolic Cauchy development, and \mathcal{U} the collection of Cauchy developments. Then *morally speaking* we should be able to obtain the maximal development as the union of the elements of \mathcal{U} . The concept which allows us to consider the union of a family of objects which do not, *a priori*, exist as a subset of the same set is the notion of *direct union*. Here we give a brief review.

Recall that a directed set $(I, <)$ is a preorder such that every pair of elements have an upper-bound. In the sequel a smooth manifold refers to a topological manifold equipped with a C^∞ structure. We do *not* assume the smooth manifold to be either Hausdorff or second countable.

Definition 1. Let $(I, <)$ be a directed set. A *direct system of smooth manifolds* over $(I, <)$ is the pair $(\mathfrak{M}, \mathfrak{F})$ where $\mathfrak{M} = \{M_i\}_{i \in I}$ is a set of smooth manifolds, and

$$\mathfrak{F} = \{f_{ji}\}_{i < j, i, j \in I}, \quad \text{where } f_{ji} : M_i \rightarrow M_j$$

is smooth, satisfying the condition that whenever $i < j < k$,

$$f_{ki} = f_{kj} \circ f_{ji}.$$

Definition 2. A direct system of smooth manifolds $(\mathfrak{M}, \mathfrak{F})$ is said to be *regular* if all the maps f_{ji} are open, and are diffeomorphisms of M_i onto their image.

Now, suppose M, N are smooth manifolds, and $f : M \rightarrow N$ is an open smooth map and a diffeomorphism onto $f(M)$. If (U, ϕ) is a chart of M , then by definition $(f(U), \phi \circ f^{-1}|_{f(U)})$ is a compatible chart with the atlas of N . This is to say.

Lemma 3. Let $(\mathfrak{M}, \mathfrak{F})$ be a regular direct system of smooth manifolds. Let \mathcal{A}_i be the maximal atlas¹ for M_i , then the pushforward $f_{ji}(\mathcal{A}_i)$ is well-defined, and $f_{ji}(\mathcal{A}_i) \subseteq \mathcal{A}_j$.

Now let us be given such a regular direct system of smooth manifolds. We denote by $\varinjlim_{\text{Top}} \mathfrak{M}$ its direct limit as a topological space. That is to say, as a set we take

$$\varinjlim_{\text{Set}} \mathfrak{M} = \coprod_i M_i / \sim$$

where the equivalence relation is $x_i \sim x_j$ iff there exists $k \succ i, j$ such that $f_{ki}(x_i) = f_{kj}(x_j)$; we then give it the quotient topology induced from $\coprod_i M_i$. We let $f_{*i} : M_i \rightarrow \varinjlim_{\text{Top}} \mathfrak{M}$ the natural mapping, which we remark has the property that if $i < j$, $f_{*i} = f_{*j} \circ f_{ji}$. By the definition of the quotient topology we have that f_{*i} is continuous. From the definition of the equivalence relation, and the assumption that f_{ji} are injective, we also have that f_{*i} is injective. We claim that f_{*i} is also open. Indeed, let $U_i \subseteq M_i$ be an open set. If $j \in I$ then since it is directed there exists $k \succ i, j$. By definition $f_{*k}^{-1} \circ f_{*i}(U_i) = f_{kj}^{-1} \circ f_{ki}(U_i)$, which is open since f_{ki} is open and f_{kj} is continuous.

To finish the construction we need to give a smooth structure to $\varinjlim_{\text{Top}} \mathfrak{M}$. Consider the charts $(f_{*i}(U_i), \phi \circ f_{*i}^{-1})$ (which is well-defined since f_{*i} is injective) where $(U_i, \phi) \in \mathcal{A}_i$. Clearly the collection of all such charts cover $\varinjlim_{\text{Top}} \mathfrak{M}$. It suffices to show that they are pairwise compatible. But if $(U_i, \phi) \in \mathcal{A}_i$ and $(V_j, \psi) \in \mathcal{A}_j$ are two charts, by assumption we can find $k \succ i, j$ such that $(f_{ki}(U_i), \phi \circ f_{ki}^{-1}), (f_{kj}(V_j), \psi \circ f_{kj}^{-1}) \in \mathcal{A}_k$. Using now that f_{*k} is injective and open, we conclude that $(f_{*i}(U_i), \phi \circ f_{*i}^{-1})$ and $(f_{*j}(V_j), \psi \circ f_{*j}^{-1})$ are compatible.

Definition 4. We denote by $\varinjlim \mathfrak{M}$ the topological space $\varinjlim_{\text{Top}} \mathfrak{M}$ equipped with the atlas described above. We call it the *direct union* of our regular direct system of smooth manifolds.

Remark 5. None of the operations involved in the construction above requires any notion of choice.

From the considerations above we see that f_{*i} is continuous, open, and injective. So it is a homeomorphism onto its image. Furthermore, it is by our choice of smooth structure smooth. Therefore we have that

Proposition 6. The mappings $f_{*i} : M_i \rightarrow \varinjlim \mathfrak{M}$ are open and diffeomorphic onto their image.

¹ We remark here that it is just the union over the set of all atlases compatible to a given one, and its existence does not require Zorn's Lemma; there seems to be a lot of confusion in the literature regarding this point (see for example Schwartz (2011); Miranda (1995); compare with the simpler treatment on pg.2 of Kobayashi and Nomizu (1996)), much as the subject of the present article.

II. DIRECT LIMIT MAPS

Now suppose $(\mathfrak{M}, \mathfrak{F})$ and $(\mathfrak{N}, \mathfrak{G})$ are two regular direct systems of smooth manifolds indexed by the same direct set (I, \prec) . Suppose furthermore that there is a set $\mathfrak{H} = \{h_i\}_{i \in I}$ where $h_i : M_i \rightarrow N_i$ are smooth maps such that whenever $i \prec j$

$$h_j \circ f_{ji} = g_{ji} \circ h_i . \quad (1)$$

Proposition 7. *There exists a smooth map $h_* : \varinjlim \mathfrak{M} \rightarrow \varinjlim \mathfrak{N}$ such that for every i ,*

$$h_* \circ f_{*i} = g_{*i} \circ h_i .$$

Proof. It suffices to check that h_* is well-defined; for this we only need to check that if $x_i \sim x_j$ in $\coprod_i M_i$ that $h_i(x_i) \sim h_j(x_j)$ in $\coprod_j N_j$. But this follows from (1). That h_* is a smooth map follows from the smoothness of h_i and the fact that if (U_i, ϕ) is a chart for M_i and (V_j, ψ) for N_j

$$\psi \circ g_{*j}^{-1} \circ h_* \circ f_{*i} \circ \phi^{-1} = \psi \circ g_{*j}^{-1} \circ h_k \circ f_{kj} \circ \phi^{-1}$$

for any $k \succ i, j$ over any open set where all the operations are defined, and that an atlas for the direct union manifold is given by the collection of all pushforward charts. \square

A direct consequence of the above construction is that we can take the direct union of a regular direct system of fibred manifolds (under the obvious definition); similarly, if this system is equipped with smooth sections that obey an appropriate commutation relation of the form (1), we can extend this section to a section over the direct union of the base manifolds. In particular, noting that a pseudo-Riemannian metric on a smooth manifold M is a section² of the vector bundle $T^{0,2}M$, we have

Corollary 8. *Let $(\mathfrak{M}, \mathfrak{F})$ and $(\mathfrak{N}, \mathfrak{G})$ be two regular direct systems of smooth manifolds over the same direct set (I, \prec) , and assume that M_i and N_i are equipped with pseudo-Riemannian metrics such that the mappings f_{ji} and g_{ji} are isometries onto their image. If furthermore we have a set $\mathfrak{H} = \{h_i\}_{i \in I}$ of isometries $h_i : M_i \rightarrow N_i$, such that (1) is satisfied. Then*

1. $\varinjlim \mathfrak{M}$ and $\varinjlim \mathfrak{N}$ can be equipped with pseudo-Riemannian metrics such that the mappings f_{*i} and g_{*i} are isometries.

2. The smooth map h_* of Proposition 7 is an isometry.

² In terms of applications to physics (see Section V), We only consider the case of *smooth* solutions to the Cauchy problem. Most of the statements here carry over exactly when the relevant structures are of class C^k which are suitably compatible under compositions. For some of the difficulties involved when considering Sobolev-class structures and in developing the *black box* local existence and uniqueness results (cf. Section V) in lower (but still classical) regularity, see Chruściel (2011).

III. SEPARATION AND COUNTABILITY

For reasons of analysis, it is usually convenient to work with smooth manifolds that are Hausdorff and second countable. Let us first consider the separation axioms. We have the following

Lemma 9. *Let $(\mathfrak{M}, \mathfrak{F})$ be a regular direct system of smooth manifolds. Let $x_i, y_i \in M_i$ and let $U_i \subseteq M_i$ be open, such that $x_i \in U_i$ and $y_i \notin U_i$. Then $f_{*i}(x_i) \in f_{*i}(U_i)$, $f_{*i}(y_i) \notin f_{*i}(U_i)$, and $f_{*i}(U_i)$ is open.*

Proof. By the construction in Section I, f_{*i} is open and injective; the lemma follows. \square

Now recall some of the separation axioms. A topological space is said to be

- T_0 (Kolmogorov) if given any $x \neq y$, there exists an open set U such that exactly one of x, y belongs to U .
- T_1 (Fréchet) if given any $x \neq y$, there exists open sets U, V such that $x \in U, y \in V$ and $y \notin U, x \notin V$.
- T_2 (Hausdorff) if given any $x \neq y$, there exists open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Corollary 10. *Let $(\mathfrak{M}, \mathfrak{F})$ be a regular direct system of smooth manifolds. Assume the elements M_i are all T_0 (resp. T_1 or T_2) as topological spaces. Then $\varinjlim_{Top} \mathfrak{M}$ is T_0 (resp. T_1 or T_2).*

Countability, on the other hand, is problematic.

Example 11. Let \mathfrak{M} consists of *finite* subsets of $(0, 1)$, equipped with the discrete topology; the elements are trivially 0-dimensional smooth manifolds. By definition each M_i is second countable, separable, and Lindelöf. But the (direct) union $\varinjlim \mathfrak{M}$, which is again $(0, 1)$ with the discrete topology, is none of the three.

One may be tempted into thinking that the issue can be fixed by working with spaces that are connected. But then one runs into the problem with the long line (which can be defined as a direct limit of uncountably many copies of \mathbb{R}).

We return to the resolution of this problem when we discuss physical applications in Section V.

IV. A NOTE ON THE INVERSE LIMIT

Just as the direct limit generalises the notion of *unions* of sets, we can use the notion of the *inverse limit* to generalise the notion of *intersections*. More precisely, given a direct system of smooth manifolds $(\mathfrak{M}, \mathfrak{F})$, we consider the set

$$\varprojlim_{\text{Set}} \mathfrak{M} = \left\{ \vec{x} \in \prod M_i \mid x_j = f_{ji}(x_i) \text{ whenever } i \prec j \right\} .$$

A priori the inverse limit can be empty, since the intersection of an arbitrary family of sets can be the empty set. But moreover, to even assert that $\prod M_i$ is non-empty in general is precisely the *axiom of choice*.

Ignoring this problem with choice, we also see that in many regards the inverse limit does not behave as nicely as the direct limit (union). For example, the inverse limit of a directed system of open sets is not necessarily open: consider $\mathfrak{M} = \{(-q, q) \subset \mathbb{R} \mid q \in \mathbb{Q} \cap (-1, 1)\}$ ordered by inclusion, with \mathfrak{F} the inclusion maps. Their direct union is their union which is $(-1, 1)$, but their inverse limit is the single point $\{0\}$. Given a regular direct system of smooth manifolds, supposing that the inverse limit exists, the most we can say is that the projection map $f_{i*} : \varprojlim_{\text{Set}} \mathfrak{M} \rightarrow M_i$ is injective. The projection maps are not guaranteed to be open, nor can we easily define a smooth structure.

Luckily, in the context of the evolution problem in physics, we need not consider the inverse limit, as the appropriate object is already given to us *as the initial data*.

V. APPLICATION TO PHYSICS

In physics, as motivated in the introduction, we want to consider the initial value problem to some systems of equations. The simplest initial value problem is that of an ordinary differential equation. Here we immediately see that the issue of the long line cannot arise. This is due to the demand that we have an increasing time function on our “solution manifold”, which prevents the interval on which the solution exists from getting too long. We can codify this intuition by requiring that there be a well-defined time-function for test particles.

Definition 12. A regular directed system of smooth manifolds $(\mathfrak{M}, \mathfrak{F})$ is said to be *physical* if there exists a smooth manifold Σ and a family $\mathfrak{H} = \{h_i\}_{i \in I}$ of injective continuous open maps $h_i : M_i \rightarrow \Sigma \times \mathbb{R}$ such that $h_j \circ f_{ji} = h_i$.

As an immediate consequence of the definition, there exists an injective continuous open map $h_* : \varinjlim \mathfrak{M} \rightarrow \Sigma \times \mathbb{R}$, which implies that h_* is an homeomorphism onto its images. Therefore if Σ is second-countable, we will also have that $\varinjlim \mathfrak{M}$ is second countable.

The definition above captures crucially a notion of *global hyperbolicity* of initial data problems. Recall that a consequence of global hyperbolicity for smooth Lorentzian manifolds is that *every inextendible time-like geodesic must intersect the Cauchy hypersurface exactly once*. This in particular means (see also Ringström (2009, p.180) for a similar argument)

Lemma 13. *Let (\tilde{M}, g) be a globally hyperbolic Lorentzian manifold, and let $\tilde{\Sigma}$ be a Cauchy hypersurface. Then $\tilde{\Sigma}$ being second-countable implies \tilde{M} is second-countable.*

Proof. Define $M = \left\{ x = (p, v) \in T\tilde{M} \mid g(v, v) = -1 \right\}$ and $\Sigma = T\tilde{\Sigma}$. If $\tilde{\Sigma}$ is second countable, so is Σ . By global hyperbolicity, corresponding to each $x \in M$ there is exactly one $y = (\sigma, w) \in \Sigma$ and $s \in \mathbb{R}$ such that the geodesic γ in \tilde{M} with initial value x has $\gamma(s) = \sigma$ and $\dot{\gamma}(s)$ projects orthogonally to w . The mapping $x \mapsto y$ is continuous and injective by the wellposedness theory of ODEs. Reversing the flow we also have that the mapping is open. Thus M is homeomorphic to an open subset of $\Sigma \times \mathbb{R}$. This implies that M is second countable, and since the bundle projection $M \rightarrow \tilde{M}$ is open, so is \tilde{M} . \square

Remark 14. The usual proof that $\tilde{\Sigma}$ is second countable implies Σ is second countable uses the statement “countable union of countable sets is countable”, which is a weak form of choice. If we assume however the initial data is specified such that $\Sigma = T\tilde{\Sigma}$ is a second countable manifold, this deficiency can be circumvented. Note that the proof that finite Cartesian products of second countable manifolds is second countable follows from the fact that Cartesian products of countable sets are countable, which does *not* require any form of choice by the Cantor argument.

In general, one can think of Definition 12 as requiring there be

1. a well defined evolution for “test particles”;
2. and a notion of hyperbolicity which states that every test particle in the space-times can be traced back to one that arose from some initial data.

The Σ factor of the mapping h_i gives the initial configuration in phase space of the test particle, and the \mathbb{R} factor gives the elapsed (proper) time.

To apply the general machinery we have developed above to an *initial value problem*, we require that the solutions to the initial value problem satisfy certain nice properties. Below is the general prescription:

We represent the initial data of the problem by some fibred manifold Σ . By a set of solutions we refer to a set \mathfrak{M} of fibred manifolds such that for each $M \in \mathfrak{M}$, there is an embedding $\phi : \Sigma \rightarrow M$.

Remark 15. We have to be very careful here when speaking of the “set” of solutions. Recall that Einstein’s equation is diffeomorphism invariant. Now given a set of diffeomorphic manifolds \mathfrak{M} , we can apply the direct union construction to get a new manifold $\varinjlim \mathfrak{M}$ which is also diffeomorphic to any element of \mathfrak{M} . By the construction of $\varinjlim_{\text{Set}} \mathfrak{M}$ as an equivalence class, we must have that $\varinjlim_{\text{Set}} \mathfrak{M} \notin \mathfrak{M}$. To put it in another way, by the axiom of regularity it does not make sense to speak of “the set of all manifolds diffeomorphic to M ”, and so it also does not make sense to speak of the set of all manifolds solving Einstein’s equations with a given initial value. It is for this reason that in the *locality* part of Definition 16 below we cannot directly require that $M \in \mathfrak{M}$.

Given $M, M' \in \mathfrak{M}$, and ϕ, ϕ' their corresponding embeddings of Σ , we say that M is an *extension* of M' if there exists an open embedding $f : M' \rightarrow M$ such that $f \circ \phi' = \phi$. Denote by \mathfrak{F} a set of extension maps.

Definition 16. Given the pair $(\mathfrak{M}, \mathfrak{F})$ of a set of solutions to the initial value problem and a set of extension maps, we say that it satisfies

- *existence* if \mathfrak{M} is non-empty;
- *unique extension* if $f, f' \in \mathfrak{F}$ are both extension maps sending $M' \rightarrow M$, then $f = f'$;
- *locality* if “unions of solutions is a solution”; by this we mean if M is a smooth fibred manifold such that there exists open embeddings $f_i : M_i \rightarrow M$ where $M_i \in \mathfrak{M}$, such that
 1. $M = \cup_i f_i(M_i)$
 2. f_i commute with the embedding of the initial data Σ
 3. if for some i, j there exists $M' \in \mathfrak{M}$ with $f'_i : M' \rightarrow M_i$ and $f'_j : M' \rightarrow M_j$, then $f_i \circ f'_i = f_j \circ f'_j$ (preserves unique extension)

then there exists $N \in \mathfrak{M}$ and a diffeomorphism $g : M \rightarrow N$ such that $g \circ f_i \in \mathfrak{F}$ is the extension map from M_i to N .

- *uniqueness* if $M, M' \in \mathfrak{M}$, there exists $N \in \mathfrak{M}$ such that M, M' are both extensions of N ;

Remark 17. For Einstein’s equations, existence here is the same as Choquet-Bruhat and Geroch (1969, Theorem 1), and uniqueness is the same as Choquet-Bruhat and Geroch (1969, Theorem 2). The locality property holds for any initial value problem expressible as finite order partial differential equations (and in particular Einstein’s equations); but additional properties like Hausdorff separation may need checking. The unique extension property is implicit in Choquet-Bruhat and Geroch (1969), even though it is not explicitly mentioned.

Remark 18. The author should point out that, for Einstein’s equation, the fact that there exists a set (and not a proper class) \mathfrak{M} such that for every globally hyperbolic Cauchy development N there exists an elements $M \in \mathfrak{M}$ such that N is diffeomorphic to M , depends strongly on the requirements that the solutions are Hausdorff and second countable. Dropping these requirements, given a local solution M of Einstein’s equation with the given data, for every ordinal λ , we can form the Cartesian product $\lambda \times M$. This shows that in general the collection of *all* solutions of Einstein’s equation must form a proper class. (This example is due to Tobias Fritz (2013).) This implicit assumption where one is working with a set of solutions and not a larger collection is not justified by Choquet-Bruhat and Geroch (1969), and is used when it

is asserted that there is a poset structure on the collection of solutions. Here we give a sketch³:

Assuming solutions are Hausdorff and second countable and connected (a consequence of global hyperbolicity), a solution is representable by sections of some finite dimensional fibred manifold. Appealing to the strong Whitney embedding theorem (the author makes no claim on its dependence on some version of axiom of choice) the solution is diffeomorphic to some smooth submanifold of \mathbb{R}^K for some K sufficiently large. Therefore there exists a subset of the power set $\mathcal{P}(\mathbb{R}^K)$ which contains the diffeomorphic image of any solution. This implies that we have an appropriate “set of solutions”.

We further remark that this point is also explicitly considered by Ringström (2009), who gave a different proof than the sketch below, making use of the Geroch splitting theorem (see Geroch (1970, Property 7, p.444) and Bernal and Sánchez (2003)); a similar approach is taken by Sbierski (2013).

Theorem 19. *Given the set \mathfrak{M} and \mathfrak{F} corresponding to an initial value problem that satisfies existence, uniqueness, unique extension, and locality, then there exists a maximal solution; that is to say, there exists $\bar{M} \in \mathfrak{M}$ such that for any $M \in \mathfrak{M}$ there exists an open embedding $f \in \mathfrak{F}$ sending $M \rightarrow \bar{M}$.*

Proof. We divide into several steps.

Construction of overlaps. Given $M, M' \in \mathfrak{M}$, the uniqueness property guarantees that the set

$$C^-(M, M') \stackrel{\text{def}}{=} \{ N \in \mathfrak{M} \mid \exists f_N, f'_N \in \mathfrak{F}, f_N : N \rightarrow M, f'_N : N \rightarrow M' \}$$

is non-empty. Let

$$\bar{N} = \cup_{N \in C^-(M, M')} f_N(N)$$

and

$$\bar{N}' = \cup_{N \in C^-(M, M')} f'_N(N).$$

Since f_N and f'_N are in \mathfrak{F} , unique extension is verified and so by locality we have that there exists $\tilde{N}, \tilde{N}' \in \mathfrak{M}$ diffeomorphic to \bar{N} and \bar{N}' . Now, using that $f'_N \circ f_N^{-1}|_{f_N(N)}$ is a diffeomorphism onto its image, by Proposition 7 there exists a diffeomorphism $\tilde{f} : \tilde{N} \rightarrow \tilde{N}'$, and hence $\tilde{N}, \tilde{N}' \in C^-(M, M')$. In other words, there exists some maximal element of $C^-(M, M')$ in the sense that it is an extension of any other element of $C^-(M, M')$. We denote the solution \tilde{N} constructed above by $M \wedge M'$ and call it the *maximal overlap*.

Construction of pairwise extensions. Now given $M, M' \in \mathfrak{M}$, let f, f' be the extension map from $M \wedge M'$ to M and M' respectively. Let us consider $N = M \amalg M' / \sim$ where the equivalence relation is that $x \sim x'$ iff there exists

³ This proof is inspired by comments of Omar Antolin-Camarena and Igor Belegradek on <http://mathoverflow.net/q/143947>.

$y \in M \wedge M'$ such that $f(y) = x$ and $f'(y) = x'$. By the maximality of $M \wedge M'$ we have that unique extension into N is preserved, so by locality there exists a solution $M \vee M'$ diffeomorphic to N .

Building the maximal element. By the previous steps, we have that the set \mathfrak{M} in fact forms a lattice with the partial ordering induced by the extension maps. In particular, it is in itself a directed set. Hence we can take the direct union $\varinjlim \mathfrak{M}$. The direct union construction guarantees that unique extension is preserved. Therefore we can once again appeal to locality to conclude that there must exist some element \bar{M} of \mathfrak{M} that is diffeomorphic to $\varinjlim \mathfrak{M}$. \square

Corollary 20. *For Einstein's equations in general relativity, there exists a Hausdorff, second countable, maximal globally hyperbolic Cauchy development.*

Sketch of proof. As indicated in Remark 18, assuming our initial data Σ is Hausdorff and second countable, the set of “Hausdorff, second countable, globally hyperbolic Cauchy developments” forms a set \mathfrak{M} . And as indicated in Remark 17, that this set is nonempty (*existence*), and two developments must overlap (*uniqueness*) is classical. As part of the uniqueness statement it is implicit that that *unique extension* also holds for the Einstein system. It remains to show that *locality* holds. The main difficult step occurs here, in showing that pairwise extensions remain Hausdorff. The proof occupies most of section 3.2 in Sbierski's pre-print (while being briefly sketched in the original paper of Choquet-Bruhat and Geroch), and we omit it here. For second countability a simple application of Lemma 13 suffices. \square

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