ON TYPE I BLOW UP FORMATION FOR THE CRITICAL NLW

JOACHIM KRIEGER, WILLIE WONG

ABSTRACT. We introduce a suitable concept of weak evolution in the context of the radial quintic focussing semilinear wave equation on $\mathbb{R}^{3+1}$, that is adapted to continuation past type II singularities. We show that the weak extension leads to type I singularity formation for initial data corresponding to: (i) the Kenig-Merle blow-up solutions with initial energy below the ground state and (ii) the Krieger-Nakanishi-Schlag blow-up solutions sitting initially near and "above" the ground state static solution.

1. INTRODUCTION

We consider the critical focussing nonlinear wave equation on $\mathbb{R}^{3+1}$, given by

$$\Box u := -u_{tt} + \triangle u = -u^5 \,, \tag{1.1}$$

which has a (possibly negative) conserved energy

$$E(u) := \int_{\mathbb{R}^3} \Big(\frac{1}{2}(u_t^2 + |\nabla_x u|^2) - \frac{u^6}{6}\Big)\,\mathrm{d}x\,.$$

We restrict to radial solutions of the form $u(t,x) = v(t,|x|)$. It is well-known and easy to show that this model admits finite time blow up solutions with finite initial free energy

$$E_{\text{free}}(u)(t) := \int_{\mathbb{R}^3} \Big[\frac{1}{2}(u_t^2 + |\nabla_x u|^2)\Big]\,\mathrm{d}x\,.$$

One can start with the explicit ODE-type solutions

$$u(t,x) = \frac{(\frac{3}{4})^{\frac{1}{4}}}{(T-t)^{\frac{1}{2}}}$$

for any $T \in \mathbb{R}_+$. By truncation to a backward (or forward) light cone and invocation of Huygens' principle, one can modify these to solutions for which the initial free energy is finite. Indeed, one may consider data $u[0] = \big(u(0,\cdot), u_t(0,\cdot)\big)$ with

$$u(0,\cdot) = \chi_{|x|<3T}\frac{(\frac{3}{4})^{\frac{1}{4}}}{T^{\frac{1}{2}}}\,, \quad u_t(0,\cdot) = \chi_{|x|<3T}\frac{(\frac{3}{64})^{\frac{1}{4}}}{T^{\frac{3}{2}}}$$

2010 *Mathematics Subject Classification.* 35L05, 35B44.
Key words and phrases. critical wave equation, hyperbolic dynamics, blowup, scattering, stability, invariant manifold.
Support of the Swiss National Fund for the first author is gratefully acknowledged.

where the cut-off function $\chi_{|x|<3T}|_{|x|\leqslant 2T} = 1$ and smoothly truncates to the region $|x| < 3T$. Observe that these solutions satisfy

$$\limsup_{t\nearrow T} \int_{|x|<T} \left[\frac{1}{2}(u_t^2(t) + |\nabla_x u(t)|^2)\right] \mathrm{d}x = +\infty\,,$$

and thus cannot be continued past time T (though a singularity may form at some earlier time, depending on the choice of cut-off $\chi_{|x|<3T}$).

Motivated by these ODE type blow-ups, we say a blow-up solution u with maximum forward time of existence T is of *type I* if

$$\limsup_{t\nearrow T} E_{\text{free}}(u)(t) = +\infty$$

and *type II* otherwise, that is

$$\limsup_{t\nearrow T} E_{\text{free}}(u)(t) < +\infty\,.$$

Recent works by Duyckaerts-Kenig-Merle [3–6] have provided a complete classification near the blow-up time of type II solutions for (1.1), while existence of solutions of this type was established in [12] and [1]. Here, we would like to discuss the formation of type I blow up. To the best of the authors' knowledge, previously demonstrated type I blow up mechanisms all derive in principle from u_t having a sign pointwise. In addition to the explicit ODE solutions (and perturbations thereof as in [2]), Duyckaerts-Kenig-Merle showed in [5] that monotonicity in time of a radial solution close to the blow up time implies type I blow up, which they then used to show that the W^+ solution of [7] as well as solutions given by initial data $u[0] = (cW, 0)$ with $c > 1$ all evolve into type I blow ups.

In a recent work [10], the study of all possible dynamics which result as *perturbations of the static solution* $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$ was begun. Note that these static solutions are a special feature of the energy critical case. Also, crucially for the analysis of [10], the perturbations are close to W with respect to a norm strictly stronger than the energy. It was then shown in [10] that there exists a co-dimension one Lipschitz manifold Σ passing through W such that within a sufficiently close neighbourhood to W, data 'above' Σ result in finite time blow up while data 'below' Σ scatter to zero, all in forward time. Further, data precisely located on Σ lead to solutions in forward time scattering toward a re-scaling of W.

In this note, we would like to study the finite-time blow up solutions corresponding to data slightly above Σ. Conjecturally, a generic set within these solutions ought to correspond to type I blow up solutions. At this time we cannot show this. Instead, our goal here is to introduce a suitable concept of *canonical weak solution* and show that such solutions will result eventually, in finite time, in a type I blow up scenario. This will be seen to directly result from a combination of the recent breakthrough characterization of type II blow up solutions by Duyckaerts-Kenig-Merle [6] with the techniques developed in [9]. Along the way, we will also show that the canonical weak extensions of the blow-up solutions exhibited by Kenig-Merle [8], whose initial energy is below that of the ground state, terminates in finite time with exploding free energy.

The authors would like to thank the anonymous referee for the detailed comments which improved the manuscript.

2. A canonical concept of weak evolution

Let $u(t,x)$ be a (radial) Shatah-Struwe energy class solution (see e.g. [14], [8]) of (1.1), existing on an interval $I = [0, T), T > 0$. Also, assume that I is a maximal such interval. If $T < \infty$, then the solution either has a type I singularity at T, or else a type II singularity. Assume the latter situation. According to the seminal work [6], the solution admits a decomposition (writing $W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x)$ for the $\dot{H}^1$ invariant scaling)

$$u(t,\cdot) = \sum_{i=1}^{N} \kappa_i W_{\lambda_i(t)}(\cdot) + u_1(t,\cdot) + o_{\dot{H}^1}(1), \quad \kappa_i \in \{\pm 1\},$$
$$u_t(t,\cdot) = u_{1,t}(t,\cdot) + o_{L^2}(1)$$

as $t \nearrow T$, with[1]

$$(T-t)^{-1} \ll \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_N(t)$$

and $u_1(t,\cdot)$ is an energy class solution of (1.1) in a neighborhood around time $t = T$. One easily verifies that this $u_1(t,\cdot)$ is indeed uniquely determined by u and T. It is then natural, assuming that there is a type II singularity at time $t = T$, to continue the evolution past time T by imposing data

$$u[T] = \big(u(T,\cdot), u_t(T,\cdot)\big) := \big(u_1(T,\cdot), u_{1,t}(T,\cdot)\big)$$

and then using the Shatah-Struwe evolution of $u[T]$ starting from time $T =: T_1$. Then there exists $T_2 \in (T_1, +\infty]$, such that if $T_2 < \infty$, there is either a type I or type II singularity at T_2, and then in the latter case again the Duyckaerts-Kenig-Merle profile decomposition applies at time $t = T_2$, allowing us to write

$$u(t,\cdot) = \sum_{i=1}^{N_2} \kappa_i^{(2)} W_{\lambda_i^{(2)}(t)}(\cdot) + u_2(t,\cdot) + o_{\dot{H}^1}(1),\ t \nearrow T_2,$$

where u_2 is now a solution of (1.1) in a neighborhood containing $t = T_2$. In this way, we obtain a sequence of times

$$T_1 < T_2 < T_3 < \ldots,\ T_i \in (0, +\infty]$$

with the following possibilities: (i) the sequence is finite, and the last $T_{\text{terminal}} := T_N = +\infty$, with all previous T_i being type II blow up times; (ii) the sequence is finite, and the last $T_{\text{terminal}} := T_N < \infty$ being the first type I blow up time in the evolution; (iii) the sequence is infinite and we define $T_{\text{terminal}} := \lim_{i\to\infty} T_i \in (0, \infty]$. Note that that except in case (ii) we have no *a priori* knowledge as to whether the solution blows up at T_{terminal}.

We now define the *canonical evolution* of the data $u[0]$ on $[0, T_{\text{terminal}})$ to be the function $\tilde{u}(t,\cdot)$ given by $u(t,\cdot)$ on $[0, T_1)$, by $u_1(t,\cdot)$ on $[T_1, T_2)$ etc.

[1]The notation $a(t) \ll b(t)$ here means $\lim_{t\nearrow T} \frac{b(t)}{a(t)} = +\infty$.

On the other hand, we define $u(t,x) \in L^\infty([0,T_*),\dot{H}^1) \cap W^{1,\infty}([0,T_*),L^2)$ to be a *weak solution* of (1.1), provided for every $\phi \in C_0^\infty((-\infty,T_*)\times\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} u_t(0,\cdot)\phi(0,\cdot)\,\mathrm{d}x + \int_0^{T_*}\int_{\mathbb{R}^3}\big(u_t\phi_t - \nabla_x u\cdot\nabla_x\phi\big)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^{T_*}\int_{\mathbb{R}^3} u^5\phi\,\mathrm{d}x\,\mathrm{d}t\,. \quad (2.1)$$

Note that our concept of canonical weak evolution is in fact more regular than $L^\infty([0,T_*),\dot{H}^1)\cap W^{1,\infty}([0,T_*),L^2)$, since

$$\tilde{u}|_{[T_{i-1},T_i)} \in C^0\big([T_{i-1},T_i),\dot{H}^1\big)\cap C^1\big([T_{i-1},T_i),L^2\big).$$

In particular, for the canonical evolution $\tilde{u}$ is right-continuous at time 0, that is $\lim_{t\searrow 0}\tilde{u}(t,\cdot) = u(0,\cdot)$ with respect to $\dot{H}^1$. Then

Lemma 2.1. *Let $\tilde{u}(t,\cdot)$ be the canonical evolution of $u[0]\in\dot{H}^1\times L^2$, defined on $[0,T_{terminal})$. Then $\tilde{u}$ is a weak solution of* (1.1) *in the above sense with $T_* = T_{terminal}$.*

Proof. Let $\phi \in C_0^\infty((-\infty,T_{\text{terminal}})\times\mathbb{R}^3)$. Then recalling the construction of $\tilde{u}$, there exist finitely many T_i, $i=1,2,\ldots,k$, $T_0 := 0$, with $T_i \in \pi_t(\text{supp}(\phi))$ with $\pi_t : \mathbb{R}^{3+1}\to\mathbb{R}$ the projection onto the time coordinate. Then we have $\tilde{u}|_{[T_i,T_{i+1})} = u_i$, $u_0 = u$ being the evolution of the data $u[0]$. Now for each i, pick a function $\chi \in C_0^\infty([T_i,T_{i+1}))$ with $\chi(T_i)=1$; then integrating by parts the Shatah-Struwe energy class solution u_i we have

$$\int_{\mathbb{R}^3} u_{i,t}(T_i,\cdot)\phi(T_i,\cdot)\,\mathrm{d}x + \int_{T_i}^{T_{i+1}}\int_{\mathbb{R}^3}\big(u_{i,t}(\chi\phi)_t - \nabla_x u_i\cdot\nabla_x(\chi\phi)\big)\,\mathrm{d}x\,\mathrm{d}t = -\int_{T_i}^{T_{i+1}}\int_{\mathbb{R}^3} u_i^5\chi\phi\,\mathrm{d}x\,\mathrm{d}t \quad (2.2)$$

Pick a sequence $\chi^{(k)}\in C_0^\infty([T_i,T_{i+1}))$ with $\chi^{(k)}\to\chi_{[T_i,T_{i+1})}$ pointwise and locally uniformly and such that

$$\lim_{k\to\infty}\int_{T_i}^{T_{i+1}}(\chi^{(k)})'(t)f(t)\,dt = -f(T_{i+1})$$

for $f\in C^0([T_i,T_{i+1}])$. Since we can write (as $t\nearrow T_{i+1}$ and where $\kappa_k^{(i)}\in\{\pm1\}$)

$$u_i(t,\cdot)\xrightarrow{\dot{H}^1}\sum_{k=1}^{N_i}\kappa_k^{(i)}W_{\lambda_k^{(i)}(t)}(\cdot)+u_{i+1}(t,\cdot),\qquad u_{i,t}\xrightarrow{L^2}u_{i+1,t}(T_{i+1},\cdot),$$

we infer

$$\lim_{k\to\infty}\int_{T_i}^{T_{i+1}}\int_{\mathbb{R}^3}\big(u_{i,t}(\chi^{(k)}\phi)_t - \nabla_x u_i\cdot\nabla_x(\chi^{(k)}\phi)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{T_i}^{T_{i+1}}\int_{\mathbb{R}^3}\big(u_{i,t}\phi_t - \nabla_x u_i\cdot\nabla_x(\phi)\big)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^3} u_{i+1,t}(T_{i+1},\cdot)\phi(T_{i+1},\cdot)\,\mathrm{d}x$$

and so we obtain

$$\int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} \left(u_{i,t}\phi_t - \nabla_x u_i \cdot \nabla_x(\phi)\right) \mathrm{d}x\,\mathrm{d}t + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} u_i^5 \phi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathbb{R}^3} u_{i+1,t}(T_{i+1},\cdot)\phi(T_{i+1},\cdot)\,\mathrm{d}x - \int_{\mathbb{R}^3} u_{i,t}(T_i,\cdot)\phi(T_i,\cdot)\,\mathrm{d}x\,. \quad (2.3)$$

Summation of the relations (2.3) over $i = 1, 2, \dots, k$, we find the relation

$$\begin{aligned} &\int_{\mathbb{R}^3} \tilde{u}_t(0,\cdot)\phi(0,\cdot)\,dx + \int_0^{T_{\text{terminal}}} \int_{\mathbb{R}^3} \left(\tilde{u}_t\phi_t - \nabla_x\tilde{u}\cdot\nabla_x\phi\right) dxdt \\ &= -\int_0^{T_{\text{terminal}}} \int_{\mathbb{R}^3} \tilde{u}^5\phi\,dxdt, \end{aligned} \quad (2.4)$$

proving the lemma. □

An important consequence of the profile decomposition of Duyckaerts-Kenig-Merle [6] is that, per the asymptotic separation of profiles and energy conservation of the regular evolution, the energy of our canonical weak evolution is *strictly decreasing*. In fact, we have that

$$E(u_i) = N_i \cdot E(W) + E(u_{i+1})\,, \qquad N_i \geqslant 1 \quad (2.5)$$

since the soliton energy is scale invariant. Evaluating

$$E(W) = \frac{1}{3}\int_{\mathbb{R}^3} |\nabla W|^2\,\mathrm{d}x > 0$$

we see explicitly the energy jump as solitons get bubbled off. This also implies that our canonical weak evolution concept is time *irreversible*.

3. Formation of type I singularities: general case

Before considering the blow-up solutions of [10], we first prove some general lemmas about the eventual formation of type I singularities for our canonical weak evolution.

Lemma 3.1. *If a canonical weak solution $\tilde{u}$ satisfies $T_{terminal} < +\infty$, then it satisfies the "type I" condition*

$$\limsup_{t\nearrow T_{terminal}} E_{free}(\tilde{u})(t) = +\infty\,. \quad (3.1)$$

Proof. As discussed in Section 2, the only possibility when $T_{\text{terminal}} < +\infty$ is either $T_{\text{terminal}} = T_N < \infty$ ending in a type I blow-up, or there exists an infinite sequence $T_i \nearrow T_{\text{terminal}}$ of type II blow-up points. In the first case (3.1) follows by definition. In the second case we appeal to the energy evolution (2.5) which implies that $\lim_{i\to\infty} E(u_i) = -\infty$. Then from Sobolev's embedding we obtain, as claimed,

$$\lim_{i\to\infty} \|\nabla_{t,x}u_i(T_i,\cdot)\|_{L^2_x} = +\infty\,.$$

□

Next, we show one of the main advantages of our canonical weak solution construction: it preserves the virial type functional used in [9]. Define, for some large large $\tau > 0$ to be fixed later, and a cutoff $\chi \in C_0^\infty([0,\infty))$ with $\chi|_{[0,1]} = 1$, the functions

$$w(t,x) = \chi\Big(\frac{|x|}{t+\tau}\Big), \qquad y(t) := \langle w(t,\cdot)\tilde{u}(t,\cdot), \tilde{u}(t,\cdot)\rangle \tag{3.2}$$

Here the spatial L^2 pairing $\langle\cdot,\cdot\rangle$ is well-defined as long as $\tilde{u} \in \dot{H}^1$, due to Sobolev's embedding and the cutoff. This is definitely the case on each interval $I_j := (T_{i-1}, T_i)$. In fact, from the computations in [9, Section 5], we have that $y(t), \dot{y}(t), \ddot{y}(t)$ are continuous functions on each open interval I_j. Now suppose that $\tilde{u}$ is a canonical weak solution maximally defined on $[0, T_{\text{terminal}})$.

Lemma 3.2. *The functions $y(t), \dot{y}(t)$, and $\ddot{y}(t)$ extend continuously to $(0, T_{terminal})$.*

Proof. It suffices to check that the three functions are continuous at each time T_i. Recall first the representation for $\tilde{u}$ from its definition

$$\begin{aligned} &\tilde{u}(t,\cdot) - \Big[\sum_{k=1}^{N_i} \kappa_k^{(i)} W_{\lambda_k^{(i)}(t)}(\cdot) + u_{i+1}(t,\cdot)\Big] \xrightarrow[t\nearrow T_{i+1}]{\dot{H}^1} 0\,, \\ &\tilde{u}_t(t,\cdot) - u_{i+1,t}(T_{i+1},\cdot) \xrightarrow[t\nearrow T_{i+1}]{L^2} 0\,. \end{aligned} \tag{3.3}$$

Now observe that with a finite radius cutoff

$$\int_{|x|<R} W_\lambda^2 \,\mathrm{d}x = 4\pi \int_0^R \frac{\lambda r^2}{1+\lambda^2 r^2/3}\,\mathrm{d}r \leqslant \frac{12\pi R}{\lambda}\,.$$

This implies that for each $\lambda_k^{(i)}$ we have

$$\sqrt{w} W_{\lambda_k^{(i)}(t)} \xrightarrow[t\nearrow T_{i+1}]{L^2} 0$$

and hence

$$\lim_{t\nearrow T_{i+1}} \langle w\tilde{u}, \tilde{u}\rangle = \langle w u_{i+1}, u_{i+1}\rangle|_{t=T_{i+1}}$$

showing the continuity of $y(t)$.

For the derivatives, we follow the computations in [9]. In particular, observe that

$$\dot{y}(t) = \langle \dot{w}\tilde{u} + 2w\dot{\tilde{u}}, \tilde{u}\rangle$$

provided $t \in [T_i, T_{i+1})$. Now, the same argument as above shows that using the uniformly bounded support of $\dot{w}$ and w near T_{i+1},

$$\lim_{t\nearrow T_{i+1}} \langle \dot{w}\tilde{u}, \tilde{u}\rangle = \langle \dot{w} u_{i+1}, u_{i+1}\rangle|_{t=T_{i+1}}\,.$$

Together with the L^2 convergence of $w\tilde{u} \to u_{i+1}(T_{i+1})$ and $\dot{\tilde{u}} \to u_{i+1,t}(T_{i+1})$ as $t \nearrow T_{i+1}$ we get

$$\lim_{t\nearrow T_{i+1}} \dot{y}(t) = \langle \dot{w} u_{i+1}(T_{i+1},\cdot) + 2w u_{i+1,t}(T_{i+1},\cdot),\, u_{i+1}(T_{i+1},\cdot)\rangle\,.$$

Since $\tilde{u}|_{[T_{i+1},T_{i+2})} = u_{i+1}$, the continuity of $\dot{y}(t)$ across $t = T_{i+1}$ is evident. Next, consider $\ddot{y}(t)$, which according to [9] is given by the expression

$$\ddot{y}(t) = \langle 2w,\ \dot{\tilde{u}}^2 - |\nabla\tilde{u}|^2 + \tilde{u}^6\rangle + \langle \ddot{w}\tilde{u},\ \tilde{u}\rangle + \langle 4\dot{w}\tilde{u},\ \dot{\tilde{u}}\rangle - 2\langle \tilde{u}\nabla w,\ \nabla\tilde{u}\rangle,\ t \in I_{i+1}\,. \quad (3.4)$$

The continuity of the middle two terms at times T_{i+1} is obtained exactly as shown previously. For the last term, in addition to the representation formulae (3.3) above we use also the fact that the derivative ∇w has compact spatial support uniformly (near $t = T_{i+1}$) *away* from the origin and so kills the contributions from $\nabla W_{\lambda_k^{(i)}}(t)$.

We examine the remaining term. The convergence in L^2 of $\dot{\tilde{u}}$ to $\dot{u}_{i+1}$ as $t \nearrow T_{i+1}$ implies

$$\lim_{t\nearrow T_{i+1}} \langle 2w(t,\cdot),\ \dot{\tilde{u}}^2(t,\cdot)\rangle = \langle 2w,\ u_{i+1,t}^2(T_{i+1},\cdot)\rangle\,.$$

The expressions $\langle 2w, |\nabla\tilde{u}|^2\rangle$ and $\langle 2w, \tilde{u}^6\rangle$ must be taken together as they *do not individually extend continuously across* T_{i+1}. Their difference, however, does. Indeed, it is straightforward to check that

$$\lim_{t\nearrow T_{i+1}} \langle 2w,\ |\nabla h_i|^2 - h_i^6\rangle = 0, \quad h_i(t,\cdot) := \sum_{k=1}^{N_i} \kappa_k^{(i)} W_{\lambda_k^{(i)}(t)}$$

where we exploit of course the fact that W is the ground state, i.e. $\triangle W + W^5 = 0$, as well as the fact that the solitons separate in scale, i.e. $\lambda_{k-1}^{(i)} \ll \lambda_k^{(i)}$. It follows that

$$\lim_{t\nearrow T_{i+1}} \langle 2w,\ -|\nabla\tilde{u}|^2 + \tilde{u}^6\rangle = \langle 2w,\ -|\nabla u_{i+1}|^2 + u_{i+1}^6\rangle|_{t=T_{i+1}}\,.$$

The fact that $\ddot{y}(t)$ extends continuously across T_{i+1} follows easily. □

We conclude this section with the following result, obtained as a modification of the classical blow-up theorem of Levine [13].

Lemma 3.3. *Let $\tilde{u}$ be a maximally extended canonical weak solution, and suppose that at some positive time its energy $E(\tilde{u}) < 0$. Then $T_{terminal} < +\infty$ for $\tilde{u}$.*

Proof. Following [9], we observe that, due to the cut-off function w in (3.2), we can write

$$\ddot{y}(t) = 2\big(4\|\dot{\tilde{u}}\|_{L^2}^2 + 4\|\nabla\tilde{u}\|_{L^2}^2 - 6E(\tilde{u})\big) + O(E_{\text{ext}}) \quad (3.5)$$

where

$$E_{\text{ext}}(t) := \int_{|x|>t+\tau} \big(|\dot{u}|^2 + |\nabla u|^2\big)\,dx \lesssim E_{\text{ext}}(0)$$

via a continuity argument and Huygens' principle, and the observation that the bubbling off of solitons happen "at the origin". By picking the initial cutoff $\tau > 0$ sufficiently large, we can force $E_{\text{ext}}(0)$ as small as we want (as long as $E_{\text{free}}(0)$ is finite). Suppose now (as given by the hypothesis of our lemma) that for some T_i that $E(\tilde{u})|_{(T_i,T_{\text{terminal}})} < -2\varepsilon_* < 0$, where we used the monotonicity of energy. Then a suitably large choice of τ would guarantee that

$$\ddot{y}(t) \geqslant 8\|\dot{\tilde{u}}\|_{L^2}^2 + \varepsilon_* \quad (3.6)$$

holds on $(T_i, T_{\text{terminal}})$.

Now assume, for contradiction, that $T_{\text{terminal}} = +\infty$. Note that (3.6) establishes a lower bound on $\ddot{y}$ in (T_i, ∞), which implies that after some large finite time $\dot{y} > 0$ and $y > y_0$ is bounded below. Hence if one remarks just as in [9] that by Cauchy-Schwartz

$$|\dot{y}(t)| = 2\langle w\dot{\tilde{u}}, \tilde{u}\rangle + O(E_{\text{ext}}) \leqslant 2\sqrt{y}\|\dot{\tilde{u}}\|_{L^2}^{1/2} + O(E_{\text{ext}})$$

we have that at all sufficiently late times past T_i we can upgrade (3.6) to

$$\ddot{y}(t) \geqslant \frac{3}{2}\frac{\dot{y}^2}{y} + \frac{1}{2}\varepsilon_* . \tag{3.7}$$

From this inequality, however, we can apply the exact same argument as in [9]: (3.7) implies that $\frac{\mathrm{d}^2}{\mathrm{d}t^2} y^{-\frac{1}{2}} < 0$ at all $t > T_i$ sufficiently large; that $\dot{y}, y > 0$ implies that $\frac{\mathrm{d}}{\mathrm{d}t} y^{-\frac{1}{2}} < 0$ at all $t > T_i$ sufficiently large. Together the concavity implies $y(t)$ must blow up in finite time, ruling out the possibility $T_{\text{terminal}} = +\infty$ and proving our lemma. ◻

Theorem 3.4. *The maximally extended canonical weak solution for the Kenig-Merle type [8] blow-up initial data with $E(u) < E(W)$ and $\|\nabla_x u\|_{L^2} > \|\nabla W\|_{L^2}$ terminates in finite $T_{terminal}$ with the type I condition* (3.1) *satisfied.*

Proof. If T_1 ends in a type I blow-up, we are done. If not, by the profile decomposition and (2.5) we have that $E(\tilde{u})(T_1) = E(u)(0) - N_1 \cdot E(W) < 0$. The theorem then follows from Lemmas 3.1 and 3.3. ◻

4. Formation of type I singularities: above threshold solutions

Theorem 3.4 above settles the problem for initial data with energy below that of the ground state, in view of the dichotomy proven in [8]. We now turn our attention to whether there exist *generic sets* (not necessarily in the energy topology) of solutions which satisfy $T_{\text{terminal}} < +\infty$ with initial energy above that of the ground state. We note that by appropriately time-translating the type II blow-up solutions constructed in [12], we obtain one that satisfies $T_1 > 0$ and $T_2 = +\infty$. In order to rule out these type of behaviour, we move to a stronger topology[2]: here we review the results of [10].

First we recall that linearising (1.1) around the solution W leads us to consider the linearised operator $-\triangle - 5W^4$. On radial functions, this linearised operator has a unique negative eigenvalue $-k_d^2$ with eigenfunction g_d satisfying $g_d > 0$; this contributes to the linear instability of the ground state W. In [11], it was shown, for initial data supported in a fixed ball with the topology $H^3_{\text{rad}}(\mathbb{R}^3) \times H^2_{\text{rad}}(\mathbb{R}^3)$, that there exists a Lipschitz manifold Σ in a small neighbourhood of the ground state W, which contains the soliton curve $\mathcal{S}$ (i. e. rescalings of W), such that initial data given on Σ exists globally and scatters to $\mathcal{S}$. Moreover, this Lipschitz manifold Σ is transverse to $(g_d, 0)$: indeed, Σ is written as a Lipschitz graph over the subspace orthogonal to g_d of the tangent space at W. Therefore an ε-neighbourhood of W can

[2]This is analogous to [11], where conditional stability of the ground state W is shown for a stronger topology than energy. The same question is open in energy topology. We refer the readers to [10] for a summary.

be divided into the portion ‘above’ Σ (i. e. those that can be written as $\sigma + \delta(g_d, 0)$ for $\sigma \in \Sigma$ and $0 < \delta < \varepsilon$) and those ‘below’ Σ (with a minus sign instead). In [10], it was shown that this division provides a dichotomy: those data sitting above Σ blows up in finite time, while those data sitting below Σ has global existence in forward time and scatters to zero in energy space.

Our main theorem concerns the blow-up solutions sitting above Σ:

Theorem 4.1. *Let $u(t, \cdot)$ be one of the blow up solutions with initial data of the form $\sigma + \delta(g_d, 0)$ with $\delta > 0$ as described above. Then the canonical weak extension of this solution will satisfy $T_{terminal} < +\infty$. Furthermore, the canonical weak solutions satisfy* (3.1)*.*

Proof. By Lemma 3.1 it suffices to rule out the case $T_{\text{terminal}} = +\infty$. Now, if $T_2 < T_{\text{terminal}}$ or $T_1 < T_{\text{terminal}}$ with $N_i > 1$, by the energy jump condition (2.5) we have that the conditions of Lemma 3.3 is satisfied, since our initial energy is close to that of a single soliton, and thus $T_{\text{terminal}} < +\infty$.

It remains to rule out the case where $T_1 < T_2 = T_{\text{terminal}} = +\infty$, where *exactly one* soliton has bubbled off at T_1. For this, we will appeal to the one pass theorem of [9], which states roughly that, for initial data close to the soliton curve $\mathcal{S}$, once the solution leaves a small neighbourhood of $\mathcal{S}$ it can never return. More precisely, we can write

$$u(t, \cdot) \approx \kappa W_{\lambda(t)} + u_1(t, \cdot)\ ,\ \kappa \in \{\pm 1\},\ t \in [0, T_1)\ .$$

For $t \in [T_1, T_2)$, the energy satisfies

$$E(\tilde{u})(t) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}(\tilde{u}_t)^2 + |\nabla \tilde{u}|^2\Big) - \frac{1}{6}\tilde{u}^6\Big)\, dx < \varepsilon\ ,$$

and by using Sobolev’s inequality we get that for some constant $C_* > 0$

$$\frac{1}{2}\|\nabla \tilde{u}\|_{L^2}^2 - C_* \|\nabla \tilde{u}\|_{L^2}^6 < \varepsilon$$

which implies that if the constant ε (which we recall measures the distance from the soliton curve of our initial data) is chosen sufficiently small, by continuity we must have that throughout $t \in [T_1, T_2)$, either

$$\|\nabla \tilde{u}\|_{L^2} \lesssim \sqrt{\varepsilon} \tag{4.1a}$$

or

$$\|\nabla \tilde{u}\|_{L^2} \gtrsim 1\ . \tag{4.1b}$$

We rule out the case (4.1a): it would necessarily require a bound

$$\|\tilde{u}_t\|_{L^2} \lesssim \sqrt{\varepsilon}$$

which implies that

$$\|\nabla_{t,x} u_1(T_1, \cdot)\|_{L^2_x} \lesssim \sqrt{\varepsilon}\ .$$

This requires that there exists $\tilde{t}$ less than but arbitrarily close to T_1 such that the inequality

$$\operatorname{dist}_{\dot{H}^1 \times L^2}(u[\tilde{t}], \mathcal{S} \cup -\mathcal{S}) \lesssim \sqrt{\varepsilon} \tag{4.2a}$$

holds. But from Proposition 1.2 and the proof of Theorem 1.1 in [10] we see that, assuming $\varepsilon > 0$ is sufficiently small, for some $t \in [0, T_1)$ we must have

$$\mathrm{dist}_{\dot{H}^1 \times L^2}(u[t], \mathcal{S} \cup -\mathcal{S}) \gg \sqrt{\epsilon} \tag{4.2b}$$

due to the exponential growth of the unstable mode. The two equations (4.2a) and (4.2b) are contradictory in view of Theorem 4.1 in [9].

It follows that the alternative (4.1b) must hold. Looking at (3.5) again we see that if τ is chosen sufficiently large, and if ε is sufficiently small, the expression (3.6) would also apply in $[T_1, T_2)$ for a suitable ε_*. We can then conclude exactly as in the proof of Lemma 3.3. □

References

[1] Donninger, R., Krieger, J. (preprint 2012). Nonscattering solutions and blow up at infinity for the critical wave equation. preprint arXiv: 1201.3258v1

[2] Donninger, R., Schörkhuber, B. (preprint 2012). Stable blow up dynamics for energy supercritical wave equations. preprint, arXiv:1207.7046, to appear in Transactions of the AMS.

[3] Duyckaerts, T., Kenig, C., Merle, F. (2011). Universality of blow-up profile for small radial type II blow-up solutions of energy-critical wave equation. *J. Eur. Math. Soc.* 13:533–599.

[4] Duyckaerts, T., Kenig, C., Merle, F. (2012) Universality of the blow-up profile for small type II blow-up solutions of energy-critical wave equation: the non-radial case. *J. Eur. Math. Soc.* 14:1389–1454.

[5] Duyckaerts, T., Kenig, C., Merle, F. (preprint 2012) Profiles of bounded radial solutions of the focusing, energy-critical wave equation. preprint, arXiv:1201.4986, to appear in GAFA.

[6] Duyckaerts, T., Kenig, C., Merle F. (preprint 2012) Classification of radial solutions of the focusing, energy-critical wave equation. preprint, arXiv:1204.0031.

[7] Duyckaerts, T., Merle, F. (2008) Dynamic of threshold solutions for energy-critical wave equation. *Int. Math. Res. Pap. IMRP* 2008:rpn002

[8] Kenig, C., Merle, F. (2008) Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta Math.* 201:147–212.

[9] Krieger, J., Nakanishi, K., Schlag, W. (preprint 2010) Global dynamics away from the ground state for the energy-critical nonlinear wave equation. preprint, arXiv:1010.3799, to appear in Amer. Journal Math.

[10] Krieger, J., Nakanishi, K., Schlag, W. (preprint 2012) Threshold phenomenon for the quintic wave equation in three dimensions. preprint, arxiv:1209.0347.

[11] J. Krieger, W. Schlag (MR2325106) *On the focusing critical semi-linear wave equation.* Amer. J. Math., no. 3, **129** (2007), 843–913.

[12] Krieger, J., Schlag, W., Tataru, D. (2009) Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation. *Duke Math. J.* 147:1–53.

[13] Levine, H.A. (1974) Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.* 192:1–21.

[14] Shatah, J., Struwe, M. (1994) Well-posedness in the energy space for semi-linear wave equations with critical growth. *Internat. Math. Res. Notices* 1994:303-309.

Joachim Krieger

Bâtiment des Mathématiques, EPFL
Station 8, CH-1015 Lausanne, Switzerland
joachim.krieger@epfl.ch

Willie Wong

Bâtiment des Mathématiques, EPFL
Station 8, CH-1015 Lausanne, Switzerland

willie.wong@epfl.ch