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Solving Diffusion Problems on Rough Surfaces with a Hierarchical Multiscale FEM

Assyr Abdulle*
[1] University of Basel

Diffusion on rough surfaces is a basic problem for many applications in engineering and the sciences. Solving these problems with a standard finite element method is often difficult or even impossible, due to the computational work and the amount of memory needed to triangulate the whole surface with a mesh which resolves its oscillations. We discuss in this paper a hierarchical Finite Element Method of "heterogeneous multiscale" type, which only needs to resolve the surface's fine scale on small sampling domains within a macro triangulation of the underlying smooth surface. This method converges, for periodic surface roughness and sufficiently small amplitude, at a robust (i.e. scale independent) rate, to the homogenized solution.

1 Introduction

Our aim is to develop a Finite Element Method (FEM) for elliptic problems on oscillatory surfaces with a macro triangulation which can be much larger than the typical oscillation lenght of the surface and which is at the same time, able to capture the right macro-scale behavior of the solution. To present the idea we restrict ourselves to the model problem

$$-\Delta_{\Gamma^\varepsilon}\tilde{u}^\varepsilon = f \text{ in } \Gamma^\varepsilon, \quad \tilde{u}^\varepsilon = 0 \text{ on } \partial\Gamma^\varepsilon, \tag{1}$$

where $\Delta_{\Gamma^\varepsilon} = \nabla_{\Gamma^\varepsilon} \cdot \nabla_{\Gamma^\varepsilon}$ is the Laplace-Beltrami operator, $\nabla_{\Gamma^\varepsilon}$ is the tangential gradient and Γ^ε is an oscillatory surface (the general case, i.e. with a diffusion tensor, can be treated similarly). Throughout, we add a superscript on the solution u to emphasize its dependence on ε.

2 Transformation to a homogenization problem

Let Ω be a bounded subset of $\mathbb{R}^2$. We consider a family of surfaces $\Gamma^\varepsilon \subset \mathbb{R}^3$ parameterized by $F^\varepsilon(\xi) = F^0(\xi) + \varepsilon a^\varepsilon(\xi) n^0(\xi)$, where $\xi \in \Omega$, $a^\varepsilon : \Omega \to \mathbb{R}$, $n^0(\xi)$ is a unit vector orthogonal to a smooth surface denoted by $F^0(\xi)$. We assume that (A1): $F^0 \in (\mathcal{C}^2(\bar{\Omega}))^3$, $F^0|_\Omega$ is injective and its differential dF^0_ξ is injective $\forall \xi \in \bar{\Omega}$ and (A2): $\|a^\varepsilon\|_{L^\infty(\Omega)} \leq \alpha_1$, $\|\nabla a^\varepsilon\|_{L^\infty(\Omega)} \leq \alpha_2/\varepsilon$, where α_1, α_2 are independent of ε.

We first transform the weak formulation of problem (1) as an elliptic homogenization problem in the parameter domain Ω. Let $u^\varepsilon(\xi) = \tilde{u}^\varepsilon(F^\varepsilon(\xi))$ then

$$\int_\Omega \nabla_\xi u^\varepsilon (g^\varepsilon)^{-1}(g^\varepsilon)^{-T}(\nabla_\xi v)^T \sqrt{\det g^\varepsilon} d\xi \; = \int_\Omega f v \sqrt{\det g^\varepsilon} d\xi \quad \forall v \in H^1_0(\Omega), \tag{2}$$

where g^ε is the metric tensor of the first fundamental form of the surface Γ^ε and $H^1_0(\Omega)$ consists of those functions in $H^1(\Omega)$ (the standard L^2-based Sobolev space of order 1) which vanish on $\partial\Omega$.

Theorem 2.1 [2] *Suppose that the assumptions (A1) and (A2) hold, then A^ε is uniformly elliptic and bounded in ε and ξ, i.e., there exist $\gamma_1, \gamma_2, \varepsilon_0 > 0$ such that $\forall \xi \in \bar{\Omega}$, $\forall \eta \in \mathbb{R}^2$ and $\forall \varepsilon$ $0 < \varepsilon \leq \varepsilon_0$*

$$\gamma_1 |\eta|^2 \leq \eta^T A^\varepsilon \eta \leq \gamma_2 |\eta|^2, \tag{3}$$

where $A^\varepsilon = (g^\varepsilon)^{-1}(g^\varepsilon)^{-T}\sqrt{\det g^\varepsilon}$. If $a^\varepsilon(\xi) = a(\xi, \xi/\varepsilon) = a(\xi, y)$ is $1-$periodic w.r.t. y_1, y_2 then $A^\varepsilon(\xi) = A(\xi, \xi/\varepsilon) = A(\xi, y)$ is $1-$periodic w.r.t. y_1, y_2.

Observe that the right hand side of problem (2) is oscillating due to the term $\sqrt{\det g^\varepsilon}$. We replace it by its weak limit $\bar{f}(\xi) = f(\xi)\int_Y \sqrt{\det g(\xi, y)} dy$. The difference between the solution u^ε of problem (2) and the solution $\bar{u}^\varepsilon$ of problem (2) with the averaged right hand side $\bar{f}(\xi)$ can be caracterised [2]. We have now a standard elliptic homogenization problem, and it is known (see e.g. [3, Chap.1],[6, Chap.1.4] and the references therein) that $\bar{u}^\varepsilon$ converges (usually in a weak sense) to $\bar{u}^0$, solution of a "homogenized" elliptic problem

$$\int_\Omega \nabla_\xi \bar{u}^0 A^0(\xi) v^T d\xi \; = \int_\Omega \bar{f} v d\xi \quad \forall v \in H^1_0(\Omega), \tag{4}$$

with an elliptic tensor $A^0(\xi)$ where the small scales have been averaged out (formulas for $A^0(\xi)$ are available [3],[6]).

* e-mail: abdulle@math.unibas.ch, Phone: +41 61 267 3997, Fax: +41 61 267 3995, Rheinsprung 21, CH-4051 Basel, Switzerland.

3 Numerical methods

Based on the heterogeneous multiscale methods (HMM) [4], we define the following FEM for the numerical solution of problem (2) with the weak limit $\bar{f}(\xi)$ for right hand side: find $u^H \in S_0^1(\Omega, \mathcal{T}_H)$ such that [2],[4]

$$B(u^H, v^H) := \sum_{K \in \mathcal{T}_H} \frac{|K|}{|K_\varepsilon|} \int_{K_\varepsilon} \nabla u \; A(\xi_k, \xi/\varepsilon)(\nabla v)^T d\xi = \int_\Omega \bar{f} v^H d\xi \quad \forall v^H \in S_0^1(\Omega, \mathcal{T}_H), \tag{5}$$

where $S_0^1(\Omega, \mathcal{T}_H) = \{w^H \in H_0^1(\Omega);\; w^H|_K \text{ linear polynomials }, \forall K \in \mathcal{T}_H\}$, $\mathcal{T}_H$ is a triangulation of Ω, $|K|, |K_\varepsilon|$ denote the measure of K and K_ε, respectively, and where u is the solution of the following *micro problem*: find u such that $u - u^H \in W_{per}^1(K_\varepsilon)$ and

$$\int_{K_\varepsilon} \nabla u \; A(\xi_k, \xi/\varepsilon)(\nabla v)^T d\xi = 0 \quad \forall v \in W_{per}^1(K_\varepsilon), \tag{6}$$

where $\xi_k \in K_\varepsilon$ denotes the barycenter of K and $W_{per}^1(Y) = \{v \in H_{per}^1(Y);\; \int_Y v dx = 0\}$ ($H_{per}^1(Y)$ is defined as the closure of $\mathcal{C}_{per}^\infty(Y)$, $Y = (0,1)^2$ is the unite cube). We obtain v by a similar problem, replacing u^H by v^H. Observe that $\bar{f}(\xi)$ is not available in closed form and we have thus to sample it on the micro domain K_ε during the integration process. This can be done without decreasing the order of convergence of the FEM [2]. To avoid having to impose periodic essential boundary conditions on the micro FE space, we enforce weakly the periodicity through Lagrange multipliers. This allows for general non periodic meshes in the micro FE space [2].

Polyhedral mean surfaces. The above heterogeneous FEM is also applicable when the mean surface is given just by data points which define a polyhedral mean surface. Since the macro bilinear form is estimated from scale resolved computations on the sampling domains, we only need, for the application of the algorithm, the mean surface to be smooth in these subdomains (however, the theory of homogenization ceases to be valid near edges and vertices of the polyhedral midsurface).

Micro-scale reconstruction. We next discuss how the micro-scale information (the small scale solution) can be recovered. Following the procedure described in [1],[5] and also in [7], we extend periodically the known micro-scale solution u given by (6) and define u^H_{rec} as $u^H_{\text{rec}}|_K = u^H + (u - u^H)|_K^\sharp$. Here, for a function $v \in H^1(K_\varepsilon)$, $v^\sharp$ denotes its periodic extension over K defined by $v^\sharp(\xi + \varepsilon(l_1, l_2)) = v(\xi) \quad \forall l_l, l_2 \in \mathbb{Z},\; \forall \xi \in K_\varepsilon \; s.t. \; \xi + \varepsilon(l_1, l_2) \in K$. Since u^H_{rec} can be discontinuous across the macro element K, we define a broken H^1 norm $\|u\|_{\bar{H}^1(\Omega)} := (\sum_{K \in \mathcal{T}_H} \|\nabla u\|^2_{L^2(K)})^{1/2}$ for the error estimates of the reconstructed solution.

4 Error estimates

We present finally some convergence results (we refer to [2] for further estimates and proofs). In the following theorem, we assume that Ω is a convex polygon, that the solution $\bar{u}^0$ of problem (4) is H^2-regular, that $A(\xi, \cdot) \in W^{1,p}(Y),\; (p > 2)$ and that $\xi \to A(\xi, \cdot)$ is smooth.

Theorem 4.1 *Let u^H be the solution of problem (5). Let $u^\varepsilon, \bar{u}^0$ be the solutions of Problem (2) and (4), respectively. Then*

$$\|\bar{u}^0 - u^H\|_{H^1(\Omega)} \le CH\|\bar{f}\|_{L^2(\Omega)}, \tag{7}$$

$$\|u^\varepsilon - u^H\|_{L^2(\Omega)} \le C(\varepsilon + H^2)\|\bar{f}\|_{L^2(\Omega)} + \delta(\varepsilon), \tag{8}$$

$$\|u^\varepsilon - u^H_{\text{rec}}\|_{\bar{H}^1(\Omega)} \le C(\sqrt{\varepsilon} + H)\|\bar{f}\|_{L^2(\Omega)} + \delta(\varepsilon), \tag{9}$$

(10)

where $\delta(\varepsilon) := (\sup_{v \in L^2(\Omega)} |\int_\Omega (f\sqrt{\det g^\varepsilon} - F) v d\xi|)^{1/2} \to 0$ for $\varepsilon \to 0$.

A pullback lemma [2] shows that the above theorem with obvious changes is also valid on the surface Γ^ε.

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