NON-EXISTENCE OF MULTIPLE-BLACK-HOLE SOLUTIONS CLOSE TO KERR-NEWMAN

WILLIE WAI-YEUNG WONG AND PIN YU

ABSTRACT. We show that a stationary asymptotically flat electro-vacuum solution of Einstein's equations that is everywhere locally "almost isometric" to a Kerr-Newman solution cannot admit more than one event horizon. Axial symmetry is not assumed. In particular this implies that the assumption of a single event horizon in Alexakis-Ionescu-Klainerman's proof of perturbative uniqueness of Kerr black holes is in fact unnecessary.

CONTENTS

2000 *Mathematics Subject Classification.* 83C22, 83C57.

Version as of October 4, 2012.

W.W.Wong's research for this paper was supported in part by the Commission of the European Communities, ERC Grant Agreement No 208007, at the Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK; and in part by the Swiss National Science Foundation.

P.Yu's research for this paper was supported in part by Princeton University, Princeton, NJ, USA, and in part by NSF-China Grant 11101235.

1. Introduction

The goal of the present paper is to provide a justification for the intuitively obvious fact that

> *A stationary electro-vacuum space-time that is everywhere* almost *isometric to Kerr-Newman can admit at most a single event horizon.*

Roughly speaking, we do not expect small perturbations of the metric structure to allow the topology (of the domain of outer communications) of the solution to change greatly. Or, slightly differently put, we expect that Weyl's observation for multiple-static-black-hole solutions remain true in the stationary case, that along the axes connecting the multiple black holes, the local geometry should be very different from what is present in a Kerr-Newman solution. In practice, however, one needs to be specific about what *almost isometric* means. This shall be described later in this introduction. As a direct consequence of the main result from this paper, we can slightly improve the main theorem of Alexakis-Ionescu-Klainerman [AIK10a] to remove from it the assumption that the space-time only has one bifurcate event horizon. A secondary consequence of the current paper is that it casts some new light on the tensorial characterisations of Kerr and Kerr-Newman space-times due to Mars [Mar99] and the first author [Won09b].

1.1. History and overview.

The greater setting in which this paper appears is in the study of the "black hole uniqueness theorem". Prosaically stated, the theorem claims that

> *The only stationary*[1] *electro-vacuum asymptotically flat space-times are described by the three-parameter Kerr-Newman family.*

The expectation that one such theorem may be available goes back at least to Carter's lecture [Car73], where a first version of a "no hair" theorem was proven; the hypotheses for this theorem assumes, in particular, that the space-time is axisymmetric in addition to being stationary. For *static*[2] solutions a general uniqueness theorem was already established without additional symmetry assumptions by Israel [Isr67, Isr68]. By appealing to Hawking's strong rigidity theorem (see next paragraph), however, one can assume (with some loss of generality) that any reasonable stationary black-hole space-time is in fact axisymmetric. This additional symmetry can be used to great effect: for the Kerr-Newman solutions the stationary Killing field is not everywhere time-like due to the presence of the ergoregions. Thus a symmetric reduction of Einstein's equations with just a stationarity assumption (as opposed to a staticity one) is insufficient to reduce the hyperbolic system of equations to an elliptic one, for which uniqueness theorems are more readily available (or widely known). With the additional axial symmetry, the equations of motions for general relativity can be shown to reduce to that of a harmonic map [Bun83, Maz82, Car85, Rob75], for which elliptic techniques (maximum principle etc.) can be used to obtain the uniqueness result. For a modern discussion one can consult Heusler's monograph [Heu96] in which various natural generalisations

[1] Admitting a Killing vector field that becomes the time-translation at spatial infinity

[2] Admitting a *hypersurface-orthogonal* Killing vector field that is the time-translation at spatial infinity

of this method are considered. For some more historical notes and critical analysis of these more classical results, a good reference is [Chr94]. More recently, Costa in his PhD dissertation [Cos10] gave a complete and modern derivation of the black hole uniqueness theorem, in the formulation which is amenable to the approach described above (namely first establishing axial symmetry and then obtaining uniqueness using elliptic methods).

One of the main shortfalls of the above approach is that Hawking's rigidity theorem, as originally envisioned, requires that the space-time be real analytic. Thus the result established for black hole uniqueness is conditional on either the space-time being *a priori* axisymmetric, or real analytic. To overcome this problem, Ionescu and Klainerman initiated a program to study the black hole uniqueness problem as a problem of "unique continuation"; namely, one considers the *ill-posed* initial value problem for the Einstein equations with data given on the event horizon and try to demonstrate a uniqueness property for the solution in the domain of outer communications (outside the black hole). Their first approach to this problem [IK09b, IK09a] (see also the generalisation by the first author [Won09a]) provided a different conditional black hole uniqueness result: instead of demanding the space-time be axisymmetric or real analytic, the extra condition is provided by, roughly speaking, prescribing the geometry of the event horizon as an embedded null hypersurface in the space-time. Through unique continuation, this boundary condition suffices to imply that the so-called Mars-Simon tensor [Mar99, Won09b] vanishes everywhere, which shows that the exterior domain of the space-time is everywhere locally isometric to a Kerr(-Newman) black hole. A second approach to this problem was later taken together with Alexakis [AIK10a, AIK10b], where under the assumption that the Mars-Simon tensor is "small" one can extend Hawking's rigidity theorem to the non-analytic case (see also the generalisation by the second author [Yu10]). By appealing to the *axisymmetric* version of the black hole uniqueness theorem, this last theorem returns us to a statement similar to Carter's original "no hair" theorem: there are no other stationary electro-vacuum asymptotically flat space-times in a small neighbourhood of the Kerr-Newman family. One of the technical assumptions made in [AIK10a] is that the space-time admits only one connected component of the event horizon; in this paper we remove that assumption.

The arguments described in the previous paragraph relied upon a tensorial *local* characterisation of the Kerr-Newman space-times due to Mars and then to the first author [Mar99, Won09b]. In those two papers, that a region in a stationary solution to Einstein's equations is locally isometric to a Kerr(-Newman) space-time is shown to be equivalent to the vanishing of certain algebraic expressions relating the Weyl curvature, the Ernst potential, the Ernst two form, and the electromagnetic field. It is clear from the algebraic nature of the expression that if the metric of a stationary solution and the electromagnetic field are C^2 close to that of a Kerr-Newman space-time in local coordinates, the algebraic expressions will also be suitably small. The converse, however, is not obviously true: the demonstration in [Mar99, Won09b] constructs a local coordinate system by first finding a holonomic frame field, and hence exact cancellations, and not just approximate ones, are necessary to guarantee integrability. As already was used in [AIK10a], and generalised further in the current paper, we show what can be interpreted as a partial converse. In particular, we show that one can reconstruct the analogue of the r coordinate of Boyer-Lindquist presentation of the Kerr-Newman metric with the expectation that it behaves similarly to said r coordinate. Critically used in [AIK10a] and [Yu10] is that the level surfaces of this "analogue-r" have good pseoduconvexity properties for a unique continuation argument; in this paper we use the property that the "analogue-r" function behaves like the distance function from a large sphere near infinity, and cannot have a critical point outside the event horizons.

That some analogue of the r coordinate plays an important role in black hole uniqueness theorems is not new. They typically appear as the inverse of the Ernst potential, and are used implicitly in Israel's proofs for the static uniqueness theorems [Isr67, Isr68] (see also [Rob77] and [uA92], the latter of which shares some motivation with the present paper).

In the present paper we show that multiple stationary black hole configurations cannot be possible were the solution to be everywhere (in the domain of outer communications) be locally close to, but not necessarily isometric to, Kerr-Newman solutions. We would be remiss not to mention the literature concerning the case where the "smallness parameter" of being close to Kerr-Newman solutions is replaced by the restriction of axisymmetry. On the one hand we have the construction (see [Wei90, Wei92, Wei96] and references therein) of solutions with multiple spinning black holes sharing the same axis of rotation, which may be singular along the axis (see also [Ngu11] for an analysis of their regularity property). This construction uses again the stationary and axial symmetries to reduce the question to the existence of certain harmonic maps with boundary conditions prescribed along the axis of symmetry and the event horizon. On the other hand we also have the approach by studying the Ernst formulation of Einstein's equations in stationary-axisymmetric case, and using the inverse scattering method to obtain a non-existence result; see [NH09, NH12] and references therein.

One last remark about the theorem proved in this paper. *A posteriori*, by combining the results of the present paper with [AIK10a] and the axisymmetric uniqueness result of [Cos10], we have that the only space-times that satisfy our hypotheses are in fact the Kerr-Newman solutions. Hence while it is *a priori* necessary to state our theorem and perform our computations in a way that admits the possibility such additional non-Kerr-Newman solutions exist, one should not try too hard to precisely imagine such additional solutions.

1.2. **Main idea of proof.** We will not state the full detail of the main theorem until Section 2.2, seeing that we need to first clarify notations and definitions. Suffice it to say for now that under some technical assumptions (a subset of that which was assumed in [AIK10a]) and a smallness condition (that the space-time is everywhere locally close to Kerr-Newman), the event horizon of a stationary asymptotically flat solution to the Einstein-Maxwell equations can have at most one connected component.

We obtain the conclusion by studying a Cauchy hypersurface of the domain of outer communications of such space-time. We show that its topology must be that of $\mathbb{R}^3$ with a single ball removed. We argue by contradiction using a "mountain pass lemma" applied to the function we denote by y, representing the real part of the inverse of the Ernst potential. We will show

- Firstly, the function y is well-defined in the domain of outer communications. Noting that y is defined by the inverse of the values of a smooth function, we need to show that the Ernst potential does not vanish. This will occupy the bulk of the paper.

- Secondly, we need to show that y satisfies the hypotheses of a mountain pass lemma. To do so we use quantitative estimates derived from the smallness conditions. On the domain of outer communications of Kerr-Newman space-time, the function y attains its minimum precisely on the event horizon, and does not admit any critical points outside the event horizon. We show that these properties remain approximately true for our solutions.
- Lastly, to conclude the theorem, we observe that were there to be more than one "hole" in the Cauchy hypersurface, the function y must be "small" along two disconnected sets (the event horizons), and "big" somewhere away from those two sets. By the mountain pass lemma y must then have a critical point, which gives rise to the contradiction.

2. Preliminaries

We begin with definitions. A space-time $(\mathcal{M}, g_{ab})$ — that is, (i) a four-dimensional, orientable, para-compact, simply-connected manifold $\mathcal{M}$ endowed with (ii) a Lorentzian metric g_{ab} with signature $(-{+}{+}{+})$ such that $(\mathcal{M}, g_{ab})$ is time-orientable — is said to be electro-vacuum if there exists a (real) two-form H_{ab} on $\mathcal{M}$ called the *Faraday tensor* such that the Einstein-Maxwell-Maxwell (to distinguish it from non-linear electromagnetic theories such as Einstein-Maxwell-Born-Infeld [Kie04a, Kie04b, Spe08]) equations are satisfied:

$$\begin{aligned} Ric_{ab} &= 2H_{ac}H_b{}^c - \frac{1}{2}g_{ab}H_{cd}H^{cd} \\ &\quad (= (H + i^*H)_{ac}(H - i^*H)_b{}^c) \\ \nabla^a(H + i^*H)_{ac} &= 0 \end{aligned}$$

where * is the Hodge-star operator: $^*H := \frac{1}{2}\varepsilon_{abcd}H^{cd}$ with ε_{abcd} the volume form for the metric g_{ab}. On a four-dimensional Lorentzian manifold, Hodge-star defines an endomorphism on the space of two-forms which squares to negative the identity. Hence we can factor over the complex numbers and call a complex-valued two-form $\mathcal{X}_{ab}$ (anti-)self-dual if $^*\mathcal{X}_{ab} = (-)i\mathcal{X}_{ab}$. (See Section 2.1 in [Won09b] for a more detailed discussion of self-duality.) Observe that $H_{ab} + i^*H_{ab}$ is anti-self-dual. So equivalently we say the space-time is electro-vacuum if there exists a complex, anti-self-dual two-form $\mathcal{H}_{ab}$ such that

$$Ric_{ab} = 4\mathcal{H}_{ac}\bar{\mathcal{H}}_b{}^c \tag{2.0.1a}$$

$$\nabla^a\mathcal{H}_{ac} = 0\ . \tag{2.0.1b}$$

One can easily convert between the two formulations by the formulae $2\mathcal{H}_{ab} = H_{ab} + i^*H_{ab}$, and $H_{ab} = \mathcal{H}_{ab} + \bar{\mathcal{H}}_{ab}$.

Throughout we will assume the electro-vacuum space-time $(\mathcal{M}, g_{ab}, \mathcal{H}_{ab})$ admits a continuous symmetry, that is, there exists a vector field t^a on $\mathcal{M}$ such that the Lie derivatives $\pounds_t g_{ab} = 0$ (t^a is Killing) and $\pounds_t\mathcal{H}_{ab} = 0$.

We will use C_{abcd} to denote the Weyl curvature, and $\mathcal{C}_{abcd} = \frac{1}{2}(C_{abcd} + i^*C_{abcd})$ its anti-self-dual part (see Section 2.2 of [Won09b]). For an arbitrary tensor field $Z^{a_1\ldots a_k}_{b_1\ldots b_j}$ we write Z^2 for its Lorentzian norm relative to the metric g_{ab}, extended linearly to complex-valued fields. Hence for real Z, Z^2 may carry either sign; for complex $\mathcal{Z}$, $\mathcal{Z}^2$ can be a complex number. We also define

$$\mathcal{I}_{abcd} := \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} + i\varepsilon_{abcd})$$

the projector to, and induced metric on, the space of anti-self-dual two-forms. We also introduce the short-hand

$$(\mathcal{X}\tilde{\otimes}\mathcal{Y})_{abcd} := \frac{1}{2}(\mathcal{X}_{ab}\mathcal{Y}_{cd} + \mathcal{Y}_{ab}\mathcal{X}_{cd}) - \frac{1}{3}\mathcal{I}_{abcd}\mathcal{X}_{ef}\mathcal{Y}^{ef} \tag{2.0.2}$$

which combines two anti-self-dual two-forms to form an anti-self-dual Weyl-type tensor.

Two important product properties of anti-self-dual two-forms that will be used frequently in computations are

$$\mathcal{X}_{ac}\bar{\mathcal{X}}_b{}^c = \mathcal{X}_{bc}\bar{\mathcal{X}}_a{}^c \ , \tag{2.0.3}$$

$$\mathcal{X}_{ac}\mathcal{Y}_b{}^c + \mathcal{Y}_{ac}\mathcal{X}_b{}^c = \frac{1}{2}g_{ab}\mathcal{X}_{cd}\mathcal{Y}^{cd} \ . \tag{2.0.4}$$

Lastly the symbols $\Re$ and $\Im$ will mean to take the real and imaginary parts respectively.

2.1. **The "error" tensors.** Now, since $\mathcal{H}$ solves Maxwell's equations, it is closed. Cartan's formula gives

$$d\iota_t\mathcal{H} + \iota_t d\mathcal{H} = \pounds_t\mathcal{H}$$

and hence by our assumptions $\iota_t\mathcal{H}$ is a closed form. Since we assumed our space-time is simply connected (a reasonable hypothesis in view of the topological censorship theorem [FSW93] since we will only consider a neighborhood of the domain of outer communications), up to a constant there exists some complex-valued function Ξ such that $d\Xi = \iota_t\mathcal{H}$.

Observe that since t^a is Killing, $\nabla_a t_b$ is anti-symmetric. Define $\hat{\mathcal{F}}_{ab} = \nabla_a t_b + \frac{i}{2}\varepsilon_{abcd}\nabla^c t^d$. Now we define the complex Ernst two-form

$$\mathcal{F}_{ab} := \hat{\mathcal{F}}_{ab} - 4\bar{\Xi}\mathcal{H}_{ab} \ . \tag{2.1.1}$$

One easily checks that $\mathcal{F}$ also satisfies Maxwell's equations, by virtue of the Jacobi equation for the Killing vector field t^a, which implies that $\nabla^a\hat{\mathcal{F}}_{ab} = -Ric_{ab}t^a$. Thus analogous to how Ξ is defined, we can define (again up to a constant) σ to be a complex valued function such that $d\sigma = \iota_t\mathcal{F}$ called the Ernst potential.

The main objects we consider are

Definition 2.1.2. The *characterization* or *error* tensors are the following objects defined up to four normalizing constants: the two complex constants in the definition of σ and Ξ, a complex constant κ, and a real constant μ. We define the two-form $\mathcal{B}$ and the four-tensor $\mathcal{Q}$ by

$$\mathcal{B}_{ab} := \kappa\mathcal{F}_{ab} + 2\mu\mathcal{H}_{ab} \tag{2.1.3a}$$

$$\mathcal{Q}_{abcd} := \mathcal{C}_{abcd} + \frac{6\kappa\bar{\Xi} - 3\mu}{2\mu\sigma}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{abcd} \tag{2.1.3b}$$

These tensors are the natural generalization of the Mars-Simon tensor [Mar99, IK09a] which characterizes Kerr space-time among stationary solutions of the Einstein vacuum equations. More precisely, we have the following theorem due to the first author [Won09b].

Theorem 2.1.4. *Let $(\mathcal{M}, g_{ab}, \mathcal{H}_{ab})$ be an electro-vacuum space-time admitting the symmetry t^a. Let $U \subset \mathcal{M}$ be a connected open subset, and suppose there exists a normalization such that on U we have $\sigma \neq 0$, $\mathcal{B} = 0$, and $\mathcal{Q} = 0$. Then we have*

$$t^2 + 2\Re\sigma + \frac{|\kappa\sigma|^2}{\mu^2} + 1 = \text{const.} \qquad \text{and} \qquad \mu^2\mathcal{F}^2 + 4\sigma^4 = \text{const.}$$

If, furthermore, both the above expressions evaluate to 0, and t^a is time-like somewhere on U, then U is locally isometric to a domain in Kerr-Newman space-time with charge κ, mass μ, and angular momentum $\mu\sqrt{\mathfrak{A}}$, where

$$\mathfrak{A} := \left|\frac{\mu}{\sigma}\right|^2 \left(\Im\nabla\frac{1}{\sigma}\right)^2 + \left(\Im\frac{1}{\sigma}\right)^2$$

is a constant on U.

Remark 2.1.5. Algebraically the definitions given herein are normalized differently from the definitions in [Won09b]. For $\kappa \neq 0$ by rescaling one can see that the statements in the above theorem are algebraically identical to the hypotheses in the main theorem in [Won09b]. For $\kappa = 0$ it is trivial to check that the conditions given above reduces to the case given in [Mar99].

Remark 2.1.6. The condition that t^a is time-like somewhere on U can be relaxed to the condition that there is some point in U where t^a is not orthogonal to either of the principal null directions of $\mathcal{F}$. Also note that asymptotic flatness is not required for the theorem.

In view of Theorem 2.1.4, we expect to use the tensors $\mathcal{B}$ and $\mathcal{Q}$ as a measure of deviation of an arbitrary stationary electro-vacuum solution from the Kerr-Newman family. Indeed, the main assumption to be introduced in the next section is a uniform smallness condition on the two tensors. In fact, we say that

Definition 2.1.7. A tensor $\mathcal{X}_{a_1\ldots a_k}$ is said to be an *algebraic error term* if there exists smooth tensors $\mathcal{A}^{(1)}_{a_1\ldots a_k}{}^{bc}$, $\mathcal{A}^{(2)}_{a_1\ldots a_k}{}^{bcd}$, and $\mathcal{A}^{(3)}_{a_1\ldots a_k}{}^{bcde}$ such that

$$\mathcal{X}_{a_1\ldots a_k} = \mathcal{A}^{(1)}_{a_1\ldots a_k}{}^{bc}\mathcal{B}_{bc} + \mathcal{A}^{(2)}_{a_1\ldots a_k}{}^{bcd}\nabla_b\mathcal{B}_{cd} + \mathcal{A}^{(3)}_{a_1\ldots a_k}{}^{bcde}\mathcal{Q}_{bcde} \ .$$

Morally speaking, an algebraic error term is one that can be "made small" by putting suitable smallness assumptions on the error tensors. In view of the indefiniteness of the Lorentzian geometric, the smallness needs to be stronger than smallness in "Lorentzian norm"; see Assumption **(KN)** in the next section. Of course, we note that should the No-Hair Theorem (see [Cos10] for a modern discussion; also [Chr94, Chr96, Bun83, Car73, Car85, Maz82, Rob75]) be proved in the smooth category (as opposed to the state-of-the-art that only holds for real-analytic space-times), then with some reasonable conditions imposed on the space-time $\mathcal{B}$ and $\mathcal{Q}$ must vanish identically.

Following the definition by Equation (2.1.3a), we immediately have

Lemma 2.1.8. *The exterior derivative dV of the potential sum $V := \kappa\sigma + 2\mu\Xi$ is an error term.*

For conciseness, we will also use the notation $P_0 := 2\bar{\kappa}\Xi - \mu$, and define the real-valued quantities y, z such that $y + iz := -\sigma^{-1}$ when the right-hand side is finite. For motivation, we mention the main lemma used in proving Theorem 2.1.4.

Lemma 2.1.9 (Mars-type Lemma [Won09b])**.** *Under the assumptions of Theorem 2.1.4 with the requirement that the two expressions evaluate to 0, we have* $g^{ab}\nabla_a y \nabla_b z = 0$ *and*

$$(\nabla z)^2 = \frac{1}{\mu^2}\frac{\mathfrak{A} - z^2}{y^2 + z^2} \qquad (\nabla y)^2 = \frac{1}{\mu^2}\frac{\mathfrak{A} + |\kappa/\mu|^2 + y^2 - 2y}{y^2 + z^2}$$

for the constant $\mathfrak{A}$ *as given in Theorem 2.1.4.*

Compare the above lemma to Lemma 2.3.12. For the expression involving $(\nabla z)^2$, we note that $\mathfrak{A}$ is now no longer a constant, but *almost* so. For the expression involving $(\nabla y)^2$, we apply (2.3.10b) of Corollary 2.3.9 and pick up a few additional error terms. For the statement about orthogonality of ∇y and ∇z, see (2.3.10a) of Corollary 2.3.9.

2.2. **Geometric assumptions and the Main Theorem.** Now we provide the precise set-up for our main theorem.

(TOP) We assume that there is a embedded partial Cauchy hypersurface $\Sigma \subset \mathcal{M}$ which is space-like everywhere. To model the multiple black holes we assume, in view of the Topology Theorem [GS06], that Σ is diffeomorphic to $\mathbb{R}^3 \setminus \cup_{i=1}^{\mathfrak{k}} B_i$, which is the Euclidean three-space with finitely many disjoint balls removed. We denote the diffeomorphism by

$$\Phi : \mathbb{R}^3 \setminus \cup_{i=1}^{\mathfrak{k}} B_i \to \Sigma$$

and require that $\mathfrak{k}$ is the total number of black holes. Each B_i is a ball centered at b_i with radius $\frac{1}{2}$. We also required that $|b_i - b_j| > 2$ when $i \neq j$. Near infinity of $\mathbb{R}^3$ we use the usual Euclidean coordinate functions (x^1, x^2, x^3) with the convention $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2)}$. Thus for large enough R_0 the set $E(R_0) := \{p \in \mathbb{R}^3 \setminus \cup_{i=1}^{\mathfrak{k}} B_i | r > R_0\}$ is unambiguously $\mathbb{R}^3$ with a large ball removed.

Furthermore we assume that for sufficiently large R_0, the Killing vector field t^a is transversal to $E(R_0)$, and thus by integrating the symmetry we define extend a diffeomorphism

$$\tilde{\Phi} : \mathbb{R} \times E(R_0) \to \mathcal{M}^{\text{end}}$$

where $\mathcal{M}^{\text{end}}$ is an open subset in $\mathcal{M}$ which we call the *asymptotic region*. In particular this defines local coordinates (x^0, x^1, x^2, x^3) on $\mathcal{M}^{\text{end}}$ with $t = \partial_0$.

(AF) In view of the dipole expansions in [MTW73] (see also [BS81]), we assume the following asymptotic properties for the metric and Faraday tensors in the local coordinates on $\mathcal{M}^{\text{end}}$. The notation $O_k(r^m)$ stands for smooth functions f obeying $|\partial^\beta f| \lesssim r^{m-|\beta|}$ for any multi-index β with $0 \leq |\beta| \leq k$. The metric components are

$$\begin{cases} g_{(0)(0)} &= -1 + 2Mr^{-1} + O_4(r^{-2}) \\ g_{(0)(i)} &= -2\sum_{j,k=1}^{3} \varepsilon_{ijk} S^j x^k r^{-3} + O_4(r^{-3}) \\ g_{(i)(j)} &= (1 + 2Mr^{-1})\delta_{ij} + O_4(r^{-2}) \end{cases} \tag{2.2.1}$$

where (S^1, S^2, S^3) form the angular momentum vector and ε_{ijk} is the fully anti-symmetric Levi-Civita symbol with 3 indices. $M > 0$ is, of course, the ADM mass. Using the gauge symmetry of the Maxwell-Maxwell equations, we shall apply a charge conjugation and assume that the space-time carries a total electric charge $q \geq 0$ and no magnetic charge. Then components of the Faraday tensor read

$$\begin{cases} H_{(i)(0)} &= \frac{q}{r^3} x^i + O_4(r^{-3}) \\ H_{(i)(j)} &= \frac{q}{Mr^3} \sum_{k=1}^3 \varepsilon_{ijk} \left(\frac{3\sum_{l,m=1}^3 \delta_{lm} S^l x^m}{r^2} x^k - S^k \right) + O_4(r^{-4}) \end{cases} \tag{2.2.2}$$

We define the total angular momentum of the space-time to be

$$\mathfrak{a}^2 := \frac{(S^1)^2 + (S^2)^2 + (S^3)^2}{M^2} \tag{2.2.3}$$

and require the non-extremal condition

$$q^2 + \mathfrak{a}^2 < M^2 \tag{2.2.4}$$

to hold.

(SBS) Define $\mathcal{E} := \mathcal{I}^-(\mathcal{M}^{\text{end}}) \cap \mathcal{I}^+(\mathcal{M}^{\text{end}})$ to be the domain of outer communications. We assume that $\mathcal{E}$ is globally hyperbolic and

$$\Sigma \cap \mathcal{I}^-(\mathcal{M}^{\text{end}}) = \Sigma \cap \mathcal{I}^+(\mathcal{M}^{\text{end}}) = \Phi(\mathbb{R}^3 \setminus \cup_{i=1}^{\mathfrak{k}} B_i') \tag{2.2.5}$$

where B_i' are balls of radius 1 centered at b_i, i.e. they are "doubles" of the balls B_i. Furthermore, we require that

$$\Phi(\cup_{i=1}^{\mathfrak{k}} \partial B_i') = \partial\mathcal{I}^-(\mathcal{M}^{\text{end}}) \cap \partial\mathcal{I}^+(\mathcal{M}^{\text{end}})$$

in other words, that Σ passes through the bifurcate spheres of all black holes. (That this is not unreasonable can be seen by noting that were there a causal curve joining two bifurcate spheres, the corresponding horizons must intersect.) We denote by $\mathfrak{h}_i^0 = \Phi(\partial B_i')$. Write $\mathfrak{h}^+ = \partial\mathcal{I}^-(\mathcal{M}^{\text{end}})$ and $\mathfrak{h}^- = \partial\mathcal{I}^+(\mathcal{M}^{\text{end}})$; let $\mathfrak{h}^0 = \cup_{i=1}^k \mathfrak{h}_i^0$, and denote by $\mathfrak{h}_i^\pm$ the connected component of $\mathfrak{h}^\pm$ containing $\mathfrak{h}_i^0$. We shall assume each $\mathfrak{h}_i^\pm$ is a smooth, embedded, null hypersurface, and require that $\mathfrak{h}_i^+$ and $\mathfrak{h}_i^-$ intersects transversally at $\mathfrak{h}_i^0$. We remark that the existence of t^a ensures that each $\mathfrak{h}_i^\pm$ is non-expanding, i.e. has vanishing null second fundamental form, and that t^a is tangent to each $\mathfrak{h}_i^\pm$ (see [Won09a, Chapter 2] for more detailed discussion of these facts). We assume that the orbits of t^a are complete in $\mathcal{E}$ and are transversal to $\mathcal{E} \cap \Sigma$.

(KN) Under the asymptotic flatness, we shall fix Ξ and σ to vanish at spatial infinity, and set $\mu = M$ and $\kappa = q$ in the definition of $\mathcal{Q}$ and $\mathcal{B}$. Fix, once and for all, a coordinate system (x^0, x^1, x^2, x^3) in a tubular neighborhood of Σ such that it agrees with the coordinate system at $\mathcal{M}^{\text{end}}$ (perhaps after enlarging R_0) and such that the metric g, its inverse, its Christoffel symbols and the Faraday tensor H are uniformly bounded in the coordinates. We require the following smallness assumption along Σ: for some ϵ sufficiently small (compared to $M, q, \mathfrak{a}$, the number R_0, and the uniform bound above) we have

$$\sum_{0 \leq \alpha,\beta,\gamma,\delta \leq 3} |\mathcal{Q}_{(\alpha)(\beta)(\gamma)(\delta)}| + \sum_{0 \leq \alpha,\beta \leq 3} |\mathcal{B}_{(\alpha)(\beta)}| + \sum_{0 \leq \alpha,\beta,\gamma \leq 3} |\partial_{(\gamma)} \mathcal{B}_{(\alpha)(\beta)}| < \epsilon |P_0| \tag{2.2.6}$$

where ∂ denotes coordinate derivative, and (a) denotes coordinate evaluation of the tensor object.

Our main theorem is

Theorem 2.2.7 (Non-existence of multi-black-holes)**.** *Under the assumptions* ***(TOP)****,* ***(AF)****,* ***(SBS)****, and* ***(KN)****,* $\mathfrak{k}$ *(the number of components of the horizon) must equal 1. In other words, there can only be one black hole.*

Remark 2.2.8. Under the above definitions, we can recover the Einstein-vacuum case directly as a corollary. Note that by *a priori* setting, in the hypotheses to Theorem 2.2.7, $q = 0$ and taking the Faraday tensor $H_{ab} \equiv 0$, we restrict ourselves to stationary Einstein-vacuum solutions with only vacuum perturbations.

2.3. **Algebraic lemmas.** In this section we document some algebraic manipulations that will be useful in the sequel. Note that unless specified, none of the four assumptions **(TOP)**, **(AF)**, **(SBS)**, and **(KN)** are used. The identities we derive, of course, will only hold when both sides of the equal sign are well-defined. Part of the bootstrap in the proof of the main theorem shall be demonstrating that all the quantities in these identities remain finite and smooth.

First we note some immediate consequences of Equation (2.1.3a) that measure the differences between $\hat{\mathcal{F}}$, $\mathcal{F}$, and $\mathcal{H}$ in terms of $\mathcal{B}$:

$$2\bar{\Xi}\mathcal{B}_{ab} - \mu\hat{\mathcal{F}}_{ab} = \bar{P}_0\mathcal{F}_{ab} \tag{2.3.1a}$$

$$\kappa\hat{\mathcal{F}}_{ab} - \mathcal{B}_{ab} = 2\bar{P}_0\mathcal{H}_{ab} \tag{2.3.1b}$$

Hence

$$\begin{aligned}\bar{P}_0\nabla_c\mathcal{F}_{ab} &= 2\nabla_c(\bar{\Xi}\mathcal{B}_{ab}) - \mu\nabla_c\hat{\mathcal{F}}_{ab} - \nabla_c\bar{P}_0\mathcal{F}_{ab} \\ &= 2\nabla_c(\bar{\Xi}\mathcal{B}_{ab}) - 2\kappa\nabla_c\bar{\Xi}\mathcal{F}_{ab} - 2\mu\mathcal{C}_{dcab}t^d \\ &\qquad - 2\mu(Ric_d{}^e g_c{}^f - Ric_c{}^e g_d{}^f)\mathcal{I}_{efab}t^d\end{aligned}$$

via the Jacobi equation for the Killing vector field t^a. Thus

$$\begin{aligned}\frac{1}{2}\bar{P}_0\nabla_c\mathcal{F}^2 &= 2\mathcal{F}^{ab}\nabla_c(\bar{\Xi}\mathcal{B}_{ab}) - 2\kappa\mathcal{F}^2\nabla_c\bar{\Xi} - 2\mu\mathcal{Q}_{dcab}\mathcal{F}^{ab}t^d \\ &\qquad + \frac{3\bar{P}_0}{\sigma}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{dcab}\mathcal{F}^{ab}t^d - 2\mu(Ric_{de}g_{cf} - Ric_{ce}g_{df})\mathcal{F}^{ef}t^d \\ &= 2\mathcal{F}^{ab}\nabla_c(\bar{\Xi}\mathcal{B}_{ab}) - 2\mu\mathcal{Q}_{dcab}\mathcal{F}^{ab}t^d - 2\kappa\mathcal{F}^2\bar{\mathcal{H}}_{dc}t^d + \frac{2\bar{P}_0}{\sigma}\mathcal{F}^2\mathcal{F}_{dc}t^d \\ &\qquad - 4[\bar{\mathcal{H}}_{da}(\mathcal{B}^{ea} - \kappa\mathcal{F}^{ea})\mathcal{F}_{ec} - \bar{\mathcal{H}}_{ca}(\mathcal{B}^{ea} - \kappa\mathcal{F}^{ea})\mathcal{F}_{ed}]t^d \\ &= 2\mathcal{F}^{ab}\nabla_c(\bar{\Xi}\mathcal{B}_{ab}) - 2\mu\mathcal{Q}_{dcab}\mathcal{F}^{ab}t^d - 4(\bar{\mathcal{H}}_{da}\mathcal{F}_{ec} - \bar{\mathcal{H}}_{ca}\mathcal{F}_{ed})\mathcal{B}^{ea}t^d \\ &\qquad - 2\kappa\mathcal{F}^2\bar{\mathcal{H}}_{dc}t^d + \frac{2\bar{P}_0}{\sigma}\mathcal{F}^2\mathcal{F}_{dc}t^d + 2\kappa\bar{\mathcal{H}}_{dc}\mathcal{F}^2t^d\end{aligned}$$

From which we conclude

$$\begin{aligned}\bar{P}_0\sigma^4\nabla_c\left(\frac{\mathcal{F}^2}{4\sigma^4}\right) \\ = \mathcal{F}^{ab}\left[\nabla_c(\bar{\Xi}\mathcal{B}_{ab}) - \mu\mathcal{Q}_{dcab}t^d\right] - 2(\bar{\mathcal{H}}_{da}\mathcal{F}_{ec} - \bar{\mathcal{H}}_{ca}\mathcal{F}_{ed})\mathcal{B}^{ea}t^d\end{aligned} \tag{2.3.2}$$

In other words

Lemma 2.3.3. *The quantity* $\bar{P}_0\sigma^4\nabla_c(\mathcal{F}^2/4\sigma^4)$ *is an algebraic error term.*

Next we show that

Lemma 2.3.4. *The following identities hold:*

$$\left(\nabla \frac{1}{\sigma}\right)^2 = \frac{\mathcal{F}^2}{4\sigma^4} t^2 \tag{2.3.5}$$

$$-|\kappa|^2 t^2 = \Re(2\bar{\kappa} V) + |P_0|^2 + \text{const.} \tag{2.3.6}$$

also

$$-t^2 - 1 = \frac{1}{\mu^2} \left|V - \kappa\sigma\right|^2 + \sigma + \bar{\sigma} + \text{const.} \tag{2.3.6$'$}$$

and lastly

$$\begin{aligned} \Box \frac{1}{\sigma} &= -\frac{\mathcal{F}^2}{2\sigma^3}(1 + \text{const.} + \bar{\sigma}) \\ &\quad + \frac{\bar{\Xi}}{\mu\sigma^2}\mathcal{F}\cdot\mathcal{B} - \frac{1}{\mu^2}\frac{\mathcal{F}^2}{\sigma^3} V\overline{(V - \kappa\sigma)} \end{aligned} \tag{2.3.7}$$

where the constants in (2.3.7) *and* (2.3.6$'$) *are the same.*

Remark 2.3.8. Under the asymptotic flatness assumption **(AF)**, our normalization convention fixes Ξ and σ to vanish at spatial infinity; by definition V also tends to zero, while P_0 tends to $-\mu$. Hence under this assumption, the free constant in (2.3.6) will be $|\kappa|^2 - \mu^2$, and the constants in (2.3.6$'$) and (2.3.7) will both be 0.

Proof. The first equation (2.3.5) can be directly derived by appealing to the definitions. The second expression follows from

$$\nabla_a t^2 = 2t^b \Re\hat{\mathcal{F}}_{ab} = -2\Re\left(\frac{2}{\kappa}\bar{P}_0\nabla_a\Xi + \frac{1}{\kappa}\nabla_a V\right) = -\frac{1}{\kappa\bar{\kappa}}\nabla_a|P_0|^2 + \nabla_a\Re(\frac{2}{\kappa}V)\ .$$

The computation for (2.3.6$'$) is slightly less trivial:

$$\begin{aligned} \nabla_a t^2 = 2t^b\Re\hat{\mathcal{F}}_{ab} &= -\frac{2}{\mu}\Re\left(2\bar{\Xi}\nabla_a V - \bar{P}_0\nabla_a\sigma\right) \\ &= -\frac{2}{\mu}\Re\left(2\bar{\Xi}\nabla_a(V - \kappa\sigma) + \mu\nabla_a\sigma\right) \\ &= -2\Re\left(\frac{1}{\mu^2}\overline{(V-\kappa\sigma)}\nabla_a(V-\kappa\sigma) + \nabla_a\sigma\right) \end{aligned}$$

And lastly we observe

$$\begin{aligned} \Box\frac{1}{\sigma} = \nabla^a\nabla_a\frac{1}{\sigma} &= -\nabla^a\left(\frac{1}{\sigma^2}\mathcal{F}_{ba}t^b\right) \\ &= -\frac{1}{\sigma^2}\mathcal{F}_{ba}\nabla^a t^b + \frac{2}{\sigma^3}\mathcal{F}_{ba}\mathcal{F}^{ca}t^b t_c \\ &= \frac{1}{2\sigma^2}\mathcal{F}_{ba}\hat{\mathcal{F}}^{ba} + \frac{1}{\sigma^3}\mathcal{F}^2 t^2 \\ &= \frac{\bar{\Xi}}{\mu\sigma^2}\mathcal{F}_{ba}\mathcal{B}^{ba} + \frac{\mathcal{F}^2}{2\sigma^3}\left(t^2 - \frac{1}{\mu}\sigma\bar{P}_0\right) \\ &= \frac{\bar{\Xi}}{\mu\sigma^2}\mathcal{F}_{ba}\mathcal{B}^{ba} + \frac{\mathcal{F}^2}{2\sigma^3}\left(t^2 + \sigma - \frac{2\kappa\sigma}{\mu^2}\overline{(V-\kappa\sigma)}\right) \end{aligned}$$

Applying (2.3.6′) we see

$$t^2+\sigma-\frac{2\kappa\sigma}{\mu^2}\overline{(V-\kappa\sigma)} = t^2+\sigma+\frac{2}{\mu^2}\left|V-\kappa\sigma\right|^2-\frac{2}{\mu^2}V\overline{(V-\kappa\sigma)}$$
$$= -\left(1+\text{const.}+\bar\sigma+\frac{2}{\mu^2}V\overline{(V-\kappa\sigma)}\right)$$

Combining the expressions we obtain (2.3.7) as claimed. □

In view of Remark 2.3.8, and recalling the definition $(y+iz)^{-1}=-\sigma$ we have the following expressions

Corollary 2.3.9. *Under the asymptotic flatness assumption **(AF)**,*

$$g^{ab}\nabla_a y\nabla_b z = \frac{t^2}{2}\Im\mathfrak{e}_1 \tag{2.3.10a}$$

$$(\nabla y)^2-(\nabla z)^2 = \frac{y^2+z^2-2y+\frac{|\kappa|^2}{\mu^2}}{\mu^2(y^2+z^2)}+\frac{|V|^2-2\Re(\kappa\sigma\bar V)}{\mu^4}+t^2\Re\mathfrak{e}_1 \tag{2.3.10b}$$

$$\Box y+\frac{2}{\mu^2}\frac{1-y}{y^2+z^2} = 2\Re\left(\sigma(1+\bar\sigma)\mathfrak{e}_1+\frac{1}{\sigma^2}\mathfrak{e}_2-8\sigma\bar\Xi\mathfrak{e}_3\right) \tag{2.3.10c}$$

$$\Box z+\frac{2}{\mu^2}\frac{z}{y^2+z^2} = 2\Im\left(\sigma(1+\bar\sigma)\mathfrak{e}_1+\frac{1}{\sigma^2}\mathfrak{e}_2-8\sigma\bar\Xi\mathfrak{e}_3\right) \tag{2.3.10d}$$

Where the terms $\mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3$ are given by

$$\mathfrak{e}_1=\frac{1}{\mu^2}+\frac{\mathcal{F}^2}{4\sigma^4}\qquad \mathfrak{e}_2=\frac{1}{\mu}\bar\Xi\mathcal{F}\cdot\mathcal{B}\qquad \mathfrak{e}_3=\frac{1}{\mu}\frac{\mathcal{F}^2}{4\sigma^4}V \tag{2.3.11}$$

each has the property that its exterior derivative is an algebraic error term up to a multiplicative factor of σ^{-4}.

The following lemma is a refinement of a proposition first due to Mars in the vacuum case [Mar99] (see also Lemma 10 in [Won09b] for a version in charged space-times). In order to capture the exact contributions from the error tensors, we forgo the tetrad formalisms used by Mars and by the first author in their papers, and instead work directly and covariantly with the tensors, improving upon the approach taken by Alexakis, Ionescu, and Klainerman [AIK10a]. As a consequence, the proof is lengthy, and we defer its presentation to Appendix A.

Lemma 2.3.12 (Main lemma)**.** *Define the quantity $\mathfrak{A}:=\mu^2(y^2+z^2)(\nabla z)^2+z^2$, then $\mathfrak{A}$ is "almost constant". More precisely,*

$$\begin{aligned}\nabla_a\mathfrak{A} = {}& \frac{4\mu^2}{|\sigma|^2}\nabla^b z\Im\left(\frac{t^c}{\sigma^2\bar P_0}\nabla_a\mathcal{B}_{cb}-\frac{\mu}{\sigma^2\bar P_0}\mathcal{Q}_{dacb}t^ct^d\right)+2\nabla_a z\Im\left(\frac{\bar\kappa\bar\sigma}{\mu^2\sigma}V\right)\\ &+\mu^2t^2(z\nabla_a y-y\nabla_a z)\Im\mathfrak{e}_1-\Im\left[\frac{2\mathfrak{e}_1\mu^2}{|\sigma|^2}\nabla_a z\left(\sigma t^2+\frac{i}{\mu}\Im(\bar\sigma^2P_0)\right)\right]\\ &-\frac{z\nabla_a z}{\mu^2}(|V-\kappa\sigma|^2-|\kappa\sigma|^2)+\Im\left[\frac{4\mu^3}{|\sigma|^2\sigma^2\bar P_0}\nabla^b z(\mathfrak{e}_5)_{ab}\right]\\ &+\Im\left[\frac{4\mu}{|\sigma|^2\sigma^2}\mathcal{F}_{cb}\Re(\bar\Xi\mathcal{B}_a{}^c)\nabla^b z-\frac{\mu P_0\bar\sigma}{\sigma}\Im(\mathfrak{e}_1)\nabla_a\sigma^{-1}\right]\end{aligned} \tag{2.3.13}$$

where $\mathfrak{e}_5$ is defined in (A.4) *in the appendix. Each term on the right hand side either contains an algebraic error term, or contains a factor of V or $\mathfrak{e}_1$, whose derivatives are algebraic error terms.*

2.4. **Null decomposition.** In regions where $\mathcal{F}^2 \neq 0$, the Ernst two-form is non-degenerate and anti-self-dual, and has two distinct, future directed, principal null directions l^a and $\underline{l}^a$, which we will normalize to $g_{ab}l^a\underline{l}^b = -1$. So there exists a complex-valued scalar function f such that

$$\mathcal{F}_{ab} = f\left(\underline{l}_a l_b - l_a \underline{l}_b + i\varepsilon_{abcd}\underline{l}^c l^d\right) \ .$$

Immediately we have $\mathcal{F}^2 = -4f^2$.

We can then decompose $\nabla_a y$ and $\nabla_a z$ by noting that $\nabla_a(-\sigma^{-1}) = \sigma^{-2}\mathcal{F}_{ba}t^b$.

$$\nabla_a y = \pm\frac{1}{\mu}\left(\underline{l}\cdot t l_a - l\cdot t\underline{l}_a\right) + \Re\left[\mathfrak{e}_4\left(\underline{l}\cdot t l_a - l\cdot t\underline{l}_a + i\varepsilon_{bacd}t^b\underline{l}^c l^d\right)\right] \tag{2.4.1a}$$

$$\nabla_a z = \pm\frac{1}{\mu}\varepsilon_{bacd}t^b\underline{l}^c l^d + \Im\left[\mathfrak{e}_4\left(\underline{l}\cdot t l_a - l\cdot t\underline{l}_a + i\varepsilon_{bacd}t^b\underline{l}^c l^d\right)\right] \tag{2.4.1b}$$

$$\mathfrak{e}_4 = \left(\frac{f}{\sigma^2} \mp \frac{1}{\mu}\right) \tag{2.4.1c}$$

The $\pm$ signs in the above signal two equivalent definitions. We note that for one of the choices of signs, the term $\mathfrak{e}_4$ is "small". Indeed, notice that $(f/\sigma^2 - 1/\mu)(f/\sigma^2 + 1/\mu) = -\mathcal{F}^2/\sigma^2 - 1/\mu^2 = -\mathfrak{e}_1$. So that for one choice of sign, we have

$$|\mathfrak{e}_4|\left(2\mu^{-1} - |\mathfrak{e}_4|\right) \leq |\mathfrak{e}_1| \implies |\mathfrak{e}_4| \leq \mu|\mathfrak{e}_1| \ . \tag{2.4.2}$$

This in particular implies that up to a small error, $\nabla_a z$ is space-like, which will imply, via Lemma 2.3.12, that z is almost bounded.

3. Domain of definition of the function y

The first step in the proof of Theorem 2.2.7 is to establish that the function y is well-defined and smooth to the exterior of the black hole. More precisely, we claim that

Proposition 3.0.3. *Under the hypotheses of Theorem 2.2.7, where the constant ϵ in assumption **(KN)** is taken to be appropriately small, the function σ does not vanish on $\bar{\mathcal{E}}$, the closure of the domain of outer communication. In particular, this implies that y is smooth on $\mathcal{E}$ and extends continuously to $\bar{\mathcal{E}}$.*

We devote the current section to the proof of the above proposition. As will be indicated in (3.1.1) we have an asymptotic expansion of $|\sigma| \approx M/r$, hence there is some large radius R^* (which we fix once and for all) such that the following are true:

1. σ does not vanish on $\Sigma \setminus \Phi\circ B(R^*)$;
2. for every $R > R^*$, on the boundary $\Phi\circ\partial B(R)$, we have that $|\sigma| \approx M/R \geq R^{-2}$.

For $R > R^*$, define

$$r_0(R) := \inf\left\{r \in [0,R] \ : \ |\sigma| \geq R^{-2} \text{ on } \Phi\left[B(R)\setminus B(r)\right]\right\} \ . \tag{3.0.4}$$

Note that by construction $r_0(R) < R^*$ for all $R > R^*$. It suffices to show that there exists $\tilde{R} > R^*$ such that $r_0(\tilde{R}) = 0$. We do so by bootstrap: for $\tilde{R} > \sqrt{2}R^*$ sufficiently large, we show that on $\Phi\left[B(\tilde{R}^*)\setminus B(r_0(\tilde{R}))\right]$ we have in fact the *improved* estimate

$$|\sigma| \geq 2\tilde{R}^{-2} \ .$$

3.1. **Asymptotic identities.** To show that the bootstrap assumptions are satisfied near infinity, we observe that by our assumptions **(TOP)** (which ensures that $t = \partial_0$ in $\mathcal{M}^{\mathrm{end}}$) and **(AF)** we can compute the following asymptotic expansions. (We remark again that below, the parentheses in the indices denote coordinate evaluation in the coordinates induced by Φ introduced in assumption **(TOP)**.) The inverse metric is given by

$$\begin{aligned} g^{(0)(0)} &= -1 - \frac{2M}{r} + O_4(r^{-2}) \ , \\ g^{(0)(i)} &= -2 \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{S^j x^k}{r^3} + O_4(r^{-3}) \ , \\ g^{(i)(j)} &= \delta^{ij} - \frac{2M}{r} \delta^{ij} + O_4(r^{-2}) \ . \end{aligned}$$

The Faraday tensor has

$$\begin{aligned} H^{(0)(i)} &= \frac{q x^i}{r^3} + O_4(r^{-3}) \ , \\ H^{(i)(j)} &= \frac{q}{M r^3} \sum_{k=1}^{3} \varepsilon_{ijk} \left(\frac{3 \sum_{m,l=1}^{3} \delta_{ml} S^m x^l}{r^2} x^k - S^k \right) + O_4(r^{-4}) \ , \end{aligned}$$

which implies that the real part of the potential Ξ is $O_3(1/r)$ and the imaginary part is $O_3(1/r^2)$ (after normalising to vanish at spatial infinity). This means that asymptotically $\mathcal{F}$ is given just by the contribution of $\hat{\mathcal{F}}$, that is

$$\begin{aligned} \mathcal{F}_{(0)(j)} &= \frac{M}{r^3} x^j + O_3(r^{-3}) + i \left(\frac{1}{r^3} S^j - \frac{3 \sum_{k,l=1}^{3} \delta_{kl} S^k x^l}{r^5} x^j + O_3(r^{-4}) \right) \ , \\ \mathcal{F}_{(i)(j)} &= \frac{1}{r^3} \sum_{k=1}^{3} \varepsilon_{ijk} S^k - \frac{3 \sum_{k,l=1}^{3} \delta_{kl} S^k x^l}{r^5} \sum_{m=1}^{3} \varepsilon_{ijm} x^m + O_3(r^{-4}) \\ &\quad + i \sum_{k=1}^{3} \varepsilon_{ijk} \left(\frac{M}{r^3} x^k + O_3(r^{-3}) \right) \ . \end{aligned}$$

Now we can compute σ: integrating the expression for $\mathcal{F}_{0j}$ we have that

$$\sigma = -\frac{M}{r} + O_4(r^{-2}) + i \left(\frac{\sum_{k,l=1}^{3} \delta_{kl} S^k x^l}{r^3} + O_4(r^{-3}) \right) \ . \tag{3.1.1}$$

This means that $y + iz = -\sigma^{-1} = -\bar{\sigma}/|\sigma|^2$ has

$$y = \frac{r}{M} + O_4(1) \ , \tag{3.1.2a}$$

$$z = \frac{\sum_{k,l=1}^{3} \delta_{kl} S^k x^l}{M^2 r} + O_4(r^{-1}) \ . \tag{3.1.2b}$$

From above, we compute $\mathfrak{A} = M^2(y^2 + z^2)(\nabla z)^2 + z^2$ (see Lemma 2.3.12 and assumption **(KN)**).

$$\mathfrak{A} = \frac{|S|^2}{M^4} + O_3(r^{-1}) \ , \tag{3.1.3}$$

and we remark that $M^2 \mathfrak{A}$ converges to $\mathfrak{a}^2$, the square of total angular momentum (see assumption **(AF)**).

We also need to compute $\mathcal{F}^2$. The leading order contribution comes from

$$\sum_{j=1}^{3}(\Re\mathcal{F}_{(0)(j)})^2 g^{(0)(0)}g^{(j)(j)} - \sum_{i,j=1}^{3}(\Im\mathcal{F}_{(i)(j)})^2 g^{(i)(i)}g^{(j)(j)} \approx -\frac{4M^2}{r^4} \ .$$

This implies that

$$\frac{\mathcal{F}^2}{4\sigma^4} = -\frac{1}{M^2} + O_3(r^{-1}) \ .$$

or (see Corollary 2.3.9 and assumption **(KN)**)

$$\mathfrak{e}_1 = O_3(r^{-1}) \ . \tag{3.1.4}$$

3.2. **Controlling algebraic errors.** Given the control of various quantities at spatial infinity by the **(AF)** assumption, we can control the quantities by integrating its derivative. More precisely, we have the following lemma for *scalar* functions:

Lemma 3.2.1. *Let R_0, α be fixed positive reals, and suppose that $0 < \delta < R_0^{-(\alpha+1)}$. Let f be a function defined on $\mathbb{R}^3$ such that*

$$\sum_{j=1}^{3}|\partial_j f| \le \delta$$

everywhere and

$$|f| \le Cr^{-\alpha}$$

on $\mathbb{R}^3 \setminus B(R_0)$. Then for the same C as above,

$$|f| \le (C+1)\min(\delta^{\frac{\alpha}{\alpha+1}}, r^{-\alpha}) \ .$$

Proof. Since $R_0\delta^{\frac{1}{1+\alpha}} < 1$ by assumption, there exists $\bar{R} > R_0$ such that $\bar{R}\delta^{\frac{1}{1+\alpha}} = 1$. To the exterior of $B(\bar{R})$ we have that $|f| \le Cr^{-\alpha}$. To the interior we have by the fundamental theorem of calculus

$$|f(x)| \le \left| f\left(\frac{x\bar{R}}{|x|}\right)\right| + (\bar{R} - |x|)\cdot|\partial f| \le C\bar{R}^{-\alpha} + \bar{R}\delta = (C+1)\delta^{\frac{\alpha}{\alpha+1}} \ .$$

□

Remark 3.2.2. The $C+1$ is not sharp; the sharp estimate depends on optimising $B + CB^{-\alpha}$ for B. For the purpose of this paper, it suffices that $(C+1) - C$ is a universal constant independent of δ for δ sufficiently small.

Now we are in a situation to prove

Proposition 3.2.3 (Main error estimate)**.** *Under the assumptions of the main theorem, there exists a constant C_0 depending only on $M, q, \mathfrak{a}$ and a constant C_1 depending on the uniform bound on g, g^{-1}, the Christoffel symbols, and H (see assumption **(KN)**) such that the following estimates are true in $\Sigma \setminus \Phi \circ B(r_0(R))$*

for $R > R^*$:

$$\begin{aligned}
\mathfrak{e}_1 &\le C_0 \min(C_1\epsilon^{1/2}R^4, r^{-1})\\
\mathfrak{e}_2 &\le C_0 C_1 \epsilon\\
\mathfrak{e}_3 &\le C_0 \min(C_1\epsilon^{1/2}R^4, r^{-1})\\
\mathfrak{e}_4 &\le C_0 \min(C_1\epsilon^{1/2}R^4, r^{-1})\\
\mathfrak{e}_5 &\le C_0 C_1 \epsilon |P_0|\\
V &\le C_0 \min(C_1\epsilon^{1/2}, r^{-1})\\
\left|\mathfrak{A} - \left(\frac{\mathfrak{a}}{M}\right)^2\right| &\le C_0 \min(C_1\epsilon^{1/4}R^6, r^{-1})
\end{aligned}$$

Proof. In the following $\lesssim_0, \lesssim_1$ denote that the left hand side is bounded by the right hand side up to multiplicative constants C_0 and C_1 respectively. (The C_0, C_1 can change from line to line in the proof.)

For $\mathfrak{e}_1$, by the bootstrap assumption (3.0.4), Lemma 2.3.3, and assumption **(KN)**, we have

$$|\partial \mathfrak{e}_1| \lesssim_1 \epsilon R^8$$

and the decay condition

$$|\mathfrak{e}_1| \lesssim_0 r^{-1}$$

which implies by Lemma 3.2.1

$$|\mathfrak{e}_1| \lesssim_0 \min(C_1\epsilon^{1/2}R^4, r^{-1}) \ .$$

This immediately implies the same bound for $\mathfrak{e}_4$.

For $\mathfrak{e}_2$, it follows directly from the definition that

$$|\mathfrak{e}_1| \lesssim_1 \frac{\epsilon}{M} \ .$$

Similarly, $\mathfrak{e}_5$ can be directly bounded by $\frac{|P_0|}{M^2}C_1\epsilon$.

For V, its derivative is a direct error term, hence $|\partial V| \le C_0 C_1 \epsilon$. Its decay rate is C_0/r, which implies by Lemma 3.2.1 that

$$|V| \lesssim_0 \min(C_0 C_1 \epsilon^{1/2}, r^{-1}) \ .$$

An estimate for $\mathfrak{e}_3$ can be directly obtained from the estimate for V, if we use the bootstrap assumption (3.0.4). However, this will lead to a term where R is not paired against ϵ, which will cause difficulties for closing the bootstrap argument. Instead, we estimate it directly from its derivatives: from the product rule we have that

$$|\partial \mathfrak{e}_3| \le C_0 C_1 R^8 \epsilon \ .$$

On the other hand, we know that the asymptotic behaviour of $\mathfrak{e}_3$ can be read-off from (3.1.4) and that of V, that is asymptotically $|\mathfrak{e}_3| \lesssim_0 r^{-1}$. This implies via our technical lemma again

$$|\mathfrak{e}_3| \lesssim_0 \min(C_0 C_1 R^4 \epsilon^{1/2}, r^{-1}) \ .$$

Lastly we estimate $\mathfrak{A}$. From the asymptotic behaviour computed in the previous section, we have that at the asymptotic end $\mathfrak{A} - (\mathfrak{a}/M)^2 \lesssim_0 r^{-1}$. Its derivative we estimate using Lemma 2.3.12, where the following points are observed:

- The terms y, z are size σ^{-1} or R^2.
- The terms ∇y and ∇z are size $\frac{1}{|\sigma|^2}\nabla\bar{\sigma}$ or $C_1 R^4$.

- The term V we (roughly) estimate by $C_0C_1\epsilon^{1/2}$.
- The term $\mathfrak{e}_1$ we (roughly) estimate by $C_0C_1\epsilon^{1/2}R^4$.

This gives us

$$\begin{aligned}
|\nabla_a\mathfrak{A}| \le \Bigg| & \frac{4\mu^2}{|\sigma|^2}\nabla^b z\Im\left(\frac{t^c}{\sigma^2\bar{P}_0}\nabla_a\mathcal{B}_{cb} - \frac{\mu}{\sigma^2\bar{P}_0}\mathcal{Q}_{dacb}t^ct^d\right) + 2\nabla_a z\Im\left(\frac{\bar{\kappa}\bar{\sigma}}{\mu^2\sigma}V\right) \\
& + \mu^2t^2(z\nabla_a y - y\nabla_a z)\Im\mathfrak{e}_1 - \Im\left[\frac{2\mathfrak{e}_1\mu^2}{|\sigma|^2}\nabla_a z\left(\sigma t^2 + \frac{i}{\mu}\Im(\bar{\sigma}^2P_0)\right)\right] \\
& - \frac{z\nabla_a z}{\mu^2}(|V-\kappa\sigma|^2 - |\kappa\sigma|^2) + \Im\left[\frac{4\mu^3}{|\sigma|^2\sigma^2\bar{P}_0}\nabla^b z(\mathfrak{e}_5)_{ab}\right] \\
& + \Im\left[\frac{4\mu}{|\sigma|^2\sigma^2}\mathcal{F}_{cb}\Re(\bar{\Xi}\mathcal{B}_a{}^c)\nabla^b z - \frac{\mu P_0\bar{\sigma}}{\sigma}\Im(\mathfrak{e}_1)\nabla_a\sigma^{-1}\right]\Bigg| \\
\le{} & C_0C_1\left[R^{12}\epsilon + R^4\epsilon^{1/2} + R^{10}\epsilon^{1/2} + R^{10}\epsilon^{1/2} + R^6\epsilon^{1/2} + R^{12}\epsilon + R^{12}\epsilon + R^8\epsilon^{1/2}\right] \\
\le{} & C_0C_1R^{12}\epsilon^{1/2}
\end{aligned}$$

where we used that ϵ will be small and R large. Integrating using Lemma 3.2.1 we get

$$\left|\mathfrak{A} - \left(\frac{\mathfrak{a}}{M}\right)^2\right| \le C_0\min(C_1R^6\epsilon^{1/4}, r^{-1}) \ .$$

□

Applying the above estimates to Corollary 2.3.9, we obtain immediately the following

Corollary 3.2.5. *The following almost identities are true:*

$$\left|\Box y + \frac{2}{M^2}\frac{1-y}{y^2+z^2}\right| \le C_0C_1R^4\epsilon^{1/2} \tag{3.2.6a}$$

$$\left|(\nabla z)^2 - \frac{\frac{\mathfrak{a}^2}{M^2} - z^2}{M^2(y^2+z^2)}\right| \le C_0C_1R^6\epsilon^{1/4} \tag{3.2.6b}$$

$$\left|(\nabla y)^2 - \frac{\frac{\mathfrak{a}^2}{M^2} + \frac{q^2}{M^2} + y^2 - 2y}{M^2(y^2+z^2)}\right| \le C_0C_1R^6\epsilon^{1/4} \tag{3.2.6c}$$

3.3. **Closing the bootstrap.** To close the bootstrap, that is, to obtain the improved decay estimate $|\sigma| \ge 2\tilde{R}^{-2}$ for sufficiently small ϵ and sufficiently large $\tilde{R}$ on the domain $E_{\tilde{R}} := \Phi\left[B(R^*) \setminus B(r_0(\tilde{R}))\right]$, it suffices to consider the domain $W_{\tilde{R}} := E_{\tilde{R}} \cap \{|\sigma| \le 4\tilde{R}^{-2}\}$. Consider first (2.3.6′). By studying the asymptotic limit, we have that the constant term is 0. On $W_{\tilde{R}}$ then we have

$$\left|t^2+1\right| \le C_0\tilde{R}^{-2} + C_0C_1\epsilon^{1/2} \ .$$

So for sufficiently large $\tilde{R} > 3R^*$ (now depending on C_0) and sufficiently small ϵ (now depending on C_0 and C_1) we have that $t^2 < -1/2$. In particular the Killing vector field is time-like. Now using that $t(y) = t(z) = 0$, we have that ∇y and ∇z are *space-like* in $W_{\tilde{R}}$.

Since $E_{\tilde{R}}$ has compact closure, we have that $W_{\tilde{R}}$ has compact closure. Using that $t^2 \le -1/2$ on this set, we have that $\sum_{i=1}^3 |\partial_i\sigma^{-1}| \le C_1\left[|(\nabla z)^2| + |(\nabla y)^2|\right]$.

The right hand side we bound by Corollary 3.2.5, and the fact that in $W_{\tilde{R}}$ we have the upper bound $(y^2+z^2)^{-1} = |\sigma|^2 \leq 16\tilde{R}^{-4}$. This leads to

$$\sum_{i=1}^{3} |\partial_i \sigma^{-1}| \leq C_0 C_1 (1 + \tilde{R}^{-4} + \tilde{R}^6 \epsilon^{1/4}) \tag{3.3.1}$$

so the fundamental theorem of calculus, integrating from the boundary of $W_{\tilde{R}}$ where $|\sigma| \geq 4\tilde{R}^{-2}$, gives that

$$\begin{aligned} |\sigma^{-1}| &\leq \frac{1}{4}\tilde{R}^2 + C_0 C_1 (1 + \tilde{R}^{-4} + \tilde{R}^6 \epsilon^{1/4}) R^* \\ &\leq \frac{1}{4}\tilde{R}^2 + C_0 C_1 \tilde{R} + C_0 C_1 \tilde{R}^{-3} + C_0 C_1 \tilde{R}^7 \epsilon^{1/4} \end{aligned}$$

where the R^* denotes the maximum coordinate distance one has to integrate (since $W_{\tilde{R}} \subseteq \Phi \circ B(R^*)$). By choosing $\tilde{R}$ sufficiently large, and

$$\epsilon^{1/4} \ll \tilde{R}^{-6} \ , \tag{3.3.2}$$

we can bound the right hand side

$$|\sigma^{-1}| \leq \frac{1}{2}\tilde{R}^2 \tag{3.3.3}$$

as desired.

Remark 3.3.4. The value $\tilde{R} > R^* > R_0$ is chosen to be sufficiently large relative to the constants C_0 and C_1 measuring the sizes of the asymptotic $M, q, \mathfrak{a}$ and uniform bounds on the metric etc. The value ϵ is now required to be sufficiently small relative to C_0, C_1, and $\tilde{R}$, which implies that ϵ only needs to be sufficiently small relative to C_0 and C_1. See also assumption **(KN)**.

Remark 3.3.5. After the bootstrap argument above, $\tilde{R}$ should be considered a fixed constant depending on C_0 and C_1. That is to say, it is understood that the right hand sides of the almost identities in Corollary 3.2.5 can be made arbitrarily small by choosing sufficiently small ϵ.

4. Proof of the Main Theorem

Now that we know the function y can be smoothly defined on the entirety of our partial Cauchy surface Σ and extended smoothly past the horizons $\mathfrak{h}^0$, we can study the local behaviour of y near a bifurcate sphere $\mathfrak{h}^0_i$. We will, in fact, demonstrate that

- y is almost constant on the bifurcate sphere, and
- y increases as we move off the horizon.

As one may expect, given that the local deviation of our space-time from the Kerr-Newman solutions is not too large (as required by assumption **(KN)**; see also Theorem 2.1.4), the constant to which y approach on the bifurcate sphere is the value of y on the corresponding Kerr-Newman black-hole, that is, $y = \frac{1}{M}\left(M + \sqrt{M^2 - \mathfrak{a}^2 - q^2}\right)$. For the Kerr-Newman solution, this value is also the largest value of y at which the function y can attain a critical point; this is captured in Lemma 2.1.9. In the case under consideration in this paper, we instead use the approximate identities of Corollary 3.2.5 to conclude that at critical points of the function y, the value of y cannot be too much greater than its value on the horizon. Together with the above two bullet points and a mountain-pass lemma, we can then conclude that y cannot have a critical point in the domain of outer communications, and hence there must only be one black hole.

In the sequel we implement the above heuristics in detail.

4.1. **Near horizon geometry.** We wish to study the behaviour of y near the bifurcate spheres; without loss of generality we consider a small neighborhood of $\mathfrak{h}_1^0$ in $\mathcal{M}$ (see Assumption **(SBS)** for definitions). We begin by establishing a double null foliation of the neighborhood and briefly recalling some implications of a non-expanding horizon (for more detailed discussion please see [AIK10a, AIK10b, Won09a]). In the sequel we will always implicitly work in a small neighborhood of $\mathfrak{h}_1^0$, whose smallness depends on $M, q, \mathfrak{a}$, and the uniform bounds on the metric, its inverse, the Christoffel symbols, and the Faraday tensor in Assumption **(KN)**, but independent of the smallness parameter ϵ.

Along $\mathfrak{h}_1^\pm$ let $L^\pm$ be future-directed geodesic generators of the respective null hypersurfaces. We choose to normalise $g(L^+, L^-) = -1$ on $\mathfrak{h}_1^0$. Along $\mathfrak{h}_1^\pm$ we define the functions $u^\mp$ by $L^\pm(u^\mp) = 1$ and $u^\mp|_{\mathfrak{h}_1^0} = 0$. The level sets of $u^\mp$ are topological spheres, and are space-like surfaces. Extend $L^\mp$ to $\mathfrak{h}_1^\pm$ to be the unique future-directed null vector orthogonal to the level sets of $u^\mp$ and satisfying $g(L^-, L^+) = -1$. Now extend $L^\mp$ off $\mathfrak{h}_1^\pm$ geodesically, and declare $L^\pm(u^\pm) = 0$. This defines a double-null foliation $u^\pm$ with associated null vector fields $L^\pm$ in the neighborhood of $\mathfrak{h}_1^0$.

Along $\mathfrak{h}_1^\pm$ the null second fundamental form $g(\nabla_X L^\pm, Y) = -g(L^\pm, \nabla_X Y)$ (for X, Y vector fields tangent to $\mathfrak{h}_1^\pm$) vanishes identically due to the horizons being non-expanding (see, e.g. [Won09a, §2.5]). This implies that $\hat{\mathcal{F}} \cdot L^\pm \propto L^\pm$ along the horizons:

$$\Re\hat{\mathcal{F}}(X, L^\pm) = g(\nabla_X t, L^\pm) = 0 \ ,$$

and the imaginary part follows once it is realised that the Hodge dual of $L^\pm \wedge X$ can be written as $L^\pm \wedge Y$ for some Y also tangent to $\mathfrak{h}_1^\pm$. Furthermore, Raychaudhuri's equation then guarantees that $\mathcal{H} \cdot L^\pm \propto L^\pm$ along the horizon, using that $Ric(L^\pm, L^\pm) = (\mathcal{H} \cdot L^\pm)_a (\bar{\mathcal{H}} \cdot L^\pm)^a$ [Won09a, §2.5]. Together these imply (via the definition (2.1.1)) that $L^\pm$ are in fact the null principal directions of $\mathcal{F}$ on $\mathfrak{h}_1^0$.

Furthermore, observe that since t^a is tangent to $\mathfrak{h}_1^\pm$ which intersect transversely, we must have t^a is tangent to $\mathfrak{h}_1^0$. This implies that $g(L^\pm, t) = 0$ along $\mathfrak{h}_1^0$.

Proposition 4.1.1. *For ϵ sufficiently small, along $\mathfrak{h}_1^0$,*

$$\left| My - \left(M + \sqrt{M^2 - \mathfrak{a}^2 - q^2} \right) \right| \lesssim \epsilon^{1/4} \ .$$

Remark 4.1.2. The quadratic polynomial $y^2 - 2y + \frac{\mathfrak{a}^2 + q^2}{M^2}$ plays a recurring role in our argument. We note that the two roots to the polynomial are

$$y_\pm = \frac{1}{M} \left(M \pm \sqrt{M^2 - \mathfrak{a}^2 - q^2} \right) \ .$$

That we need to ensure the existence of two distinct roots, one larger than, and one smaller than 1 is why sub-extremality is assumed in **(AF)**. (Of course, the extremal Kerr-Newman black holes have very different horizon geometry, and we should not expect an analysis based on the bifurcate spheres to carry over in that case.)

Remark 4.1.3. The proposition and its proof are largely the same as Lemma 4.1 in [AIK10a]; we sketch the proof here for completeness.

Proof. Since $L^\pm$ along $\mathfrak{h}_1^0$ are the null principal directions of $\mathcal{F}$, we can apply the results of Section 2.4. In particular, we have that the orthogonality of $L^\pm$ to the Killing vector field t on the horizons implies the *exact identity* (that the following two equations do not contain error terms is very important in the sequel)

$$L^+(y) = L^+(z) = 0 \text{ on } \mathfrak{h}_1^+ \ , \tag{4.1.4a}$$

$$L^-(y) = L^-(z) = 0 \text{ on } \mathfrak{h}_1^- \ . \tag{4.1.4b}$$

These imply that on $\mathfrak{h}_1^0$

$$\nabla_a y = \Re\left[i\mathfrak{e}_4 \varepsilon_{bacd} t^b (L^-)^c (L^+)^d\right] \tag{4.1.5}$$

is of size $\epsilon^{1/2}$ by Proposition 3.2.3 and Remark 3.3.5. This implies that $(\nabla y)^2 = O(\epsilon^{1/4})$. So using Corollary 3.2.5 we obtain that along the horizon

$$\frac{\frac{\mathfrak{a}^2+q^2}{M^2} + y^2 - 2y}{M^2(y^2+z^2)} = O(\epsilon^{1/4}) \ .$$

By the bootstrap argument, we have that $(y^2+z^2)^{-1}$ is bounded above by a constant depending only on C_0, C_1 (see Remark 3.3.5 again), hence we have that on $\mathfrak{h}_1^0$

$$y^2 - 2y + \frac{\mathfrak{a}^2+q^2}{M^2} = O(\epsilon^{1/4}) \ .$$

Observe further that by (4.1.5), if X, Y are vector fields tangent to $\mathfrak{h}_1^0$, we have that

$$X(Y(y)) = \Re\left[iX(\mathfrak{e}_4)\varepsilon(t, Y, L^-, L^+) + i\mathfrak{e}_4 X(\varepsilon(t, Y, L^-, L^+))\right] \ .$$

From the definition of $\mathfrak{e}_4$ in Section 2.4, we see immediately that $\nabla_a \mathfrak{e}_4$ can be controlled by $\mathfrak{e}_1$ and $\nabla_a \mathfrak{e}_1$. That is to say, we have that the Hessian of y *along* $\mathfrak{h}_1^0$ is also of order $\epsilon^{1/4}$.

This gives two possibilities: either $|y - y_+| \lesssim \epsilon^{1/4}$ or $|y - y_-| \lesssim \epsilon^{1/4}$; it suffices to eliminate the second alternative. To do so we consider the first inequality in Corollary 3.2.5. Provided ϵ is sufficiently small (especially compared to $\sqrt{M^2 - \mathfrak{a}^2 - q^2}$), that $|y - y_-| \lesssim \epsilon^{1/4}$ along $\mathfrak{h}_1^0$ would imply $\Box y < 0$ in a small neighborhood of the bifurcate sphere. We use this fact to show that y must decrease as we move off the horizon.

Define $\tilde{y}$ by setting $\tilde{y} = y$ along $\mathfrak{h}_1^-$, and requiring that $L^+\tilde{y} = 0$. This guarantees that in a small neighborhood of $\mathfrak{h}_1^0$, $\tilde{y}$ is bounded by $\sup_{\mathfrak{h}_1^0} y$. Using that the Hessian of y tangent to $\mathfrak{h}_1^0$ is also an error term, this implies that $|\Box\tilde{y}| \lesssim \epsilon^{1/4}$; that is to say, the main contribution to $\Box y$ comes from $L^-(L^+ y)$. Using that y and $\tilde{y}$ agree on $\mathfrak{h}_1^\pm$, we can write $y = \tilde{y} + u^+u^-\hat{y}$ where $\hat{y}$ is a smooth function in a small neighborhood of $\mathfrak{h}_1^0$. Furthermore, on $\mathfrak{h}_1^0$ we have that $\Box(y - \tilde{y}) = -2\hat{y}$, hence along $\mathfrak{h}_1^0$ we have

$$\left|\hat{y} - \frac{1-y}{M^2(y^2+z^2)}\right| \lesssim \epsilon^{1/2}$$

and in particular for all ϵ sufficiently small

$$\hat{y}|_{\mathfrak{h}_1^0} \geq \frac{1-y_-}{2M^2(y^2+z^2)} > 2C_h > 0 \ .$$

By continuity, on a sufficiently small neighborhood of $\mathfrak{h}_1^0$ we have that $\hat{y} \geq C_h$. Now using that in the domain of outer communications, by construction we have $u^+u^- < 0$, this implies that

$$y \leq \tilde{y} + u^+u^-\hat{y} \leq y_- + O(\epsilon^{1/4}) - |u^+u^-|C_h$$

in the small neighborhood of $\mathfrak{h}_1^0$. Now consider all points in this neighborhood for which $-u^+u^- \geq \delta > 0$ for some fixed δ. Then for all ϵ sufficiently small, at these points we have $y < y_- - \frac{1}{2}C_h\delta$. By the asymptotic behaviour of y (growing to $+\infty$), this implies that $y|_{\Sigma\cap\mathcal{E}}$ achieves a minimum value that is at most $y_- - \frac{1}{2}C_h\delta$. But this implies (using that t^a is transverse to $\Sigma\cap\mathcal{E}$) that y attains a critical point at a value $y_- - \frac{1}{2}C_h\delta$, which is impossible for sufficiently small ϵ by Corollary 3.2.5. This concludes the proof that y must be close to y_+ on the horizon. □

Remark 4.1.6. The same argument in the contradiction step of the proof can be used to show that, given y is close to y_+ on the horizon, there exists some topological sphere in $\Sigma\cap\mathcal{E}$ that encloses $\mathfrak{h}_1^0$ and some $\delta > 0$ (δ depends on $M, q, \mathfrak{a}$, and the uniform bounds on the metric, its inverse, its Christoffel symbols, and the Faraday tensor) such that restrict to that sphere $y > y_+ + 2\delta > \sup_{\mathfrak{h}_1^0} y + \delta$ provided ϵ is sufficiently small.

4.2. **Concluding the proof.** Having established our technical results about the behaviour of y near the horizon sphere $\mathfrak{h}_1^0$ (and hence by symmetry for any $\mathfrak{h}_i^0$), we conclude our main theorem by appealing to a finite dimensional mountain pass lemma (see Appendix).

Proof of Theorem 2.2.7. Assume, for contradiction, that there are at least two black holes. By Proposition 4.1.1 and Remark 4.1.6 we know that for sufficiently small ϵ, we can find $\delta > 0$ such that $y|_{\mathfrak{h}^0} < y_+ + \delta$ and there exists a topological sphere $S \subset \Sigma\cap\mathcal{E}$ such that $\mathfrak{h}_1^0$ and $\mathfrak{h}_2^0$ are in disjoint subsets of $\Sigma\setminus S$ and such that $y|_S > y_+ + 2\delta$. By the asymptotic growth of y we know that y satisfies the Palais-Smale condition. So applying Lemma B.1 to the function y on the manifold $(\Sigma\cap\mathcal{E})\cup\mathfrak{h}^0$, y attains a critical point in $\Sigma\cap\mathcal{E}$ where the value of y is at least $y_+ + 2\delta$. Using that t^a is transverse to $\Sigma\cap\mathcal{E}$, again we have that $\nabla y = 0$ there. For sufficiently small ϵ this leads to a contradiction with Corollary 3.2.5 together with Remark 4.1.2. □

Appendix A. Proof of the Main Lemma

In this appendix we shall give the proof of Lemma 2.3.12, which claims that $\mathfrak{A} = \mu^2(y^2+z^2)(\nabla z)^2 + z^2$ is "almost constant". We start directly with the definition

$$\nabla_a\mathfrak{A} = 2\mu^2(y\nabla_a y + z\nabla_a z)(\nabla z)^2 + 2z\nabla_a z + 2\mu^2(y^2+z^2)\nabla^b z\nabla_a\nabla_b z\ . \tag{A.1}$$

The focus will be on the third term in the expansion, which contains the Hessian of z. Therefore we compute $\nabla^2_{a,b}\sigma^{-1}$.

$$\nabla_a\nabla_b\sigma^{-1} = -\nabla_a(\sigma^{-2}\nabla_b\sigma) = 2\sigma\nabla_a\sigma^{-1}\nabla_b\sigma^{-1} - \sigma^{-2}\nabla_a\nabla_b\sigma$$

Next use

$$\begin{aligned}\nabla_a\nabla_b\sigma &= \nabla_a\mathcal{F}_{cb}t^c\\ &= \mathcal{F}_{cb}\nabla_a t^c + \frac{t^c}{\bar{P}_0}\left(2\nabla_a(\bar{\Xi}\mathcal{B}_{cb}) - 2\kappa\nabla_a\bar{\Xi}\mathcal{F}_{cb}\right)\\ &\qquad - \frac{2\mu t^c t^d}{\bar{P}_0}\left(\mathcal{C}_{dacb} + \left(Ric_d{}^e g_a{}^f - Ric_a{}^e g_d{}^f\right)\mathcal{I}_{efcb}\right)\end{aligned}$$

We can expand $\mathcal{I}$ by the definition, use Einstein's equation (2.0.1a) to replace the Ricci tensor, and use the definitions (2.1.3a) and (2.1.3b) to obtain that

$$\begin{aligned}\nabla_a\nabla_b\sigma - \frac{2t^c}{\bar{P}_0}\bar{\Xi}\nabla_a\mathcal{B}_{cb} + \frac{2\mu}{\bar{P}_0}\mathcal{Q}_{dacb}t^ct^d&\\ = \frac{1}{2}\mathcal{F}_{cb}\hat{\mathcal{F}}_a{}^c + \frac{1}{2}\mathcal{F}_{cb}\bar{\hat{\mathcal{F}}}_a{}^c + \frac{4\mu}{\bar{P}_0}\nabla_a\bar{\Xi}\nabla_b\Xi + \frac{3}{\sigma}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{dacb}t^dt^c&\\ - \frac{2\mu t^ct^d}{\bar{P}_0}\left(\mathcal{H}_{dl}\bar{\mathcal{H}}_c{}^l g_{ab} - \mathcal{H}_{dl}\bar{\mathcal{H}}_b{}^l g_{ac} - \mathcal{H}_{al}\bar{\mathcal{H}}_c{}^l g_{db} + \mathcal{H}_{al}\bar{\mathcal{H}}_b{}^l g_{cd}\right)&\\ - \frac{2\mu t^ct^d}{\bar{P}_0}\left(i\bar{\mathcal{H}}^{el}\varepsilon_{eacb}\mathcal{H}_{dl} - i\mathcal{H}^{el}\varepsilon_{edcb}\mathcal{H}_{al}\right)\ .&\end{aligned}$$

For the terms in the last line, we can use the identity for self-dual two-forms

$$i\bar{\mathcal{X}}^{kh}\varepsilon_{wyzk} = g_w^h\bar{\mathcal{X}}_{yz} + g_y^h\bar{\mathcal{X}}_{zw} + g_z^h\bar{\mathcal{X}}_{wy} \tag{A.2}$$

which gives

$$\begin{aligned}-\frac{2\mu t^ct^d}{\bar{P}_0}\left(i\bar{\mathcal{H}}^{el}\varepsilon_{eacb}\mathcal{H}_{dl} - i\mathcal{H}^{el}\varepsilon_{edcb}\mathcal{H}_{al}\right)&\\ = -\frac{2\mu t^ct^d}{\bar{P}_0}\left(\mathcal{H}_{ac}\bar{\mathcal{H}}_{bd} - 2\mathcal{H}_{da}\bar{\mathcal{H}}_{cb} - \mathcal{H}_{db}\bar{\mathcal{H}}_{ac} - \mathcal{H}_{dc}\bar{\mathcal{H}}_{ba} + \mathcal{H}_{ab}\bar{\mathcal{H}}_{dc}\right)&\end{aligned}$$

where by (anti)symmetry, after the contraction against t^ct^d, the last two terms in the parenthesis evaluate to zero. Hence we can simplify

$$\begin{aligned}\nabla_a\nabla_b\sigma - \frac{2t^c}{\bar{P}_0}\bar{\Xi}\nabla_a\mathcal{B}_{cb} + \frac{2\mu}{\bar{P}_0}\mathcal{Q}_{dacb}t^ct^d&\\ = \frac{1}{2}\mathcal{F}_{cb}\hat{\mathcal{F}}_a{}^c + \frac{1}{2}\mathcal{F}_{cb}\bar{\hat{\mathcal{F}}}_a{}^c + \frac{4\mu}{\bar{P}_0}\nabla_a\bar{\Xi}\nabla_b\Xi + \frac{3}{\sigma}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{dacb}t^dt^c&\\ - \frac{2\mu}{\bar{P}_0}\left(\nabla\Xi\cdot\nabla\bar{\Xi}g_{ab} + \mathcal{H}_{al}\bar{\mathcal{H}}_b{}^l t^2 - \nabla^l\Xi\bar{\mathcal{H}}_{bl}t_a - \nabla^l\bar{\Xi}\mathcal{H}_{al}t_b\right)&\\ - \frac{2\mu}{\bar{P}_0}\left(\nabla_b\Xi\nabla_a\bar{\Xi} - \nabla_a\Xi\nabla_b\bar{\Xi}\right)\ .&\end{aligned}$$

In the following we will also group terms proportional to t_b on the left-hand-side of the expression, since in (A.1), the $\nabla_a\nabla_b z$ term is multiplied against $\nabla^b z$, and we have that $t_b\nabla^b z = 0$ by our assumption that t is a symmetry.

Directly expanding the terms

$$\begin{aligned}(\mathcal{F}\tilde{\otimes}\mathcal{F})_{dacb}t^ct^d &= \mathcal{F}_{da}\mathcal{F}_{cb}t^ct^d - \frac{1}{3}\mathcal{I}_{dacb}\mathcal{F}^2t^dt^c\\ &= \sigma^4\nabla_a\sigma^{-1}\nabla_b\sigma^{-1} - \frac{1}{12}\mathcal{F}^2t^2g_{ab} + \frac{1}{12}\mathcal{F}^2t_at_b\ ,\end{aligned}$$

we arrive at

$$\begin{aligned}\nabla_a\nabla_b\sigma^{-1} + \frac{2t^c}{\sigma^2\bar{P}_0}\nabla_a\mathcal{B}_{cb} - \frac{2\mu}{\sigma^2\bar{P}_0}\mathcal{Q}_{dacb}t^ct^d + \frac{1}{4\sigma^3}\mathcal{F}^2t_at_b + \frac{2\mu}{\sigma^2\bar{P}_0}\nabla^l\bar{\Xi}\mathcal{H}_{al}t_b&\\ = -\sigma\nabla_a\sigma^{-1}\nabla_b\sigma^{-1} + \frac{1}{4\sigma^3}\mathcal{F}^2t^2g_{ab} - \frac{1}{2\sigma^2}\mathcal{F}_{cb}\left(\hat{\mathcal{F}}_a{}^c + \bar{\hat{\mathcal{F}}}_a{}^c\right)&\\ - \frac{2\mu}{\sigma^2\bar{P}_0}\left(\nabla_a\Xi\nabla_b\bar{\Xi} + \nabla_b\Xi\nabla_a\bar{\Xi} - \nabla\Xi\cdot\nabla\bar{\Xi}g_{ab} + \nabla^l\Xi\bar{\mathcal{H}}_{bl}t_a - \mathcal{H}_{al}\bar{\mathcal{H}}_b{}^l t^2\right)\ .&\end{aligned} \tag{A.3}$$

To apply to (A.1), we next multiply (A.3) by $\nabla^b z = -\Im\nabla^b\sigma^{-1}$. We first consider the terms on the last line, where the expression inside the parenthesis is real-valued. So we can consider multiplication by $\nabla\sigma^{-1}$ instead of by ∇z. Observe that

$$\begin{aligned}
&\nabla_a\Xi\nabla_b\bar{\Xi} + \nabla_b\Xi\nabla_a\bar{\Xi} - \nabla\Xi\cdot\nabla\bar{\Xi}g_{ab} + \nabla^l\Xi\bar{\mathcal{H}}_{bl}t_a - \mathcal{H}_{al}\bar{\mathcal{H}}_b{}^l t^2\\
&\qquad = \mathcal{H}^{pr}\bar{\mathcal{H}}^{qs}t^m t^n\cdot\big(g_{ap}g_{bq}g_{rm}g_{sn} + g_{bp}g_{aq}g_{rm}g_{sn}\\
&\qquad\qquad - g_{pq}g_{ab}g_{rm}g_{sn} - g_{ap}g_{bq}g_{mn}g_{rs} - g_{bq}g_{rs}g_{an}g_{pm}\big)
\end{aligned}$$

since the last two terms in the parenthesis has a g_{rs} product, we can apply (2.0.3) to swap the p and q indices

$$\begin{aligned}
&= \mathcal{H}^{pr}\bar{\mathcal{H}}^{qs}t^m t^n\cdot\big(g_{ap}g_{bq}g_{rm}g_{sn} + g_{bp}g_{aq}g_{rm}g_{sn}\\
&\qquad - g_{pq}g_{ab}g_{rm}g_{sn} - g_{aq}g_{bp}g_{mn}g_{rs} - g_{bp}g_{rs}g_{an}g_{qm}\big)\\
&= \left(\frac{\kappa\bar{\kappa}}{4\mu^2}\mathcal{F}^{pr}\bar{\mathcal{F}}^{qs} - \frac{\kappa\bar{\kappa}}{4\mu^2}\mathcal{F}^{pr}\bar{\mathcal{F}}^{qs} + \mathcal{H}^{pr}\bar{\mathcal{H}}^{qs}\right)t^m t^n\cdot\big(g_{ap}g_{bq}g_{rm}g_{sn}\\
&\qquad + g_{bp}g_{aq}g_{rm}g_{sn} - g_{pq}g_{ab}g_{rm}g_{sn} - g_{aq}g_{bp}g_{mn}g_{rs} - g_{bp}g_{rs}g_{an}g_{qm}\big)
\end{aligned}$$

Inside the first parenthesis, we have that $-\frac{\kappa\bar{\kappa}}{4\mu^2}\mathcal{F}^{pr}\bar{\mathcal{F}}^{qs} + \mathcal{H}^{pr}\bar{\mathcal{H}}^{qs}$ is an error term by using (2.1.3a). So we define the *algebraic error term*

$$\begin{aligned}
\text{(A.4)}\quad (\mathfrak{e}_5)_{ab} &= \left(\mathcal{H}^{pr}\bar{\mathcal{H}}^{qs} - \frac{\kappa\bar{\kappa}}{4\mu^2}\mathcal{F}^{pr}\bar{\mathcal{F}}^{qs}\right)t^m t^n\cdot\big(g_{ap}g_{bq}g_{rm}g_{sn}\\
&\qquad + g_{bp}g_{aq}g_{rm}g_{sn} - g_{pq}g_{ab}g_{rm}g_{sn} - g_{aq}g_{bp}g_{mn}g_{rs} - g_{bp}g_{rs}g_{an}g_{qm}\big)
\end{aligned}$$

We next consider the left-over term given by $\mathcal{F}^{pr}\bar{\mathcal{F}}^{qs}$. Using that $\nabla_b\sigma^{-1} = \sigma^{-2}\mathcal{F}_{ub}t^u$, we consider

$$\begin{aligned}
&\mathcal{F}^{pr}\bar{\mathcal{F}}^{qs}\mathcal{F}^{bu}t_u t^m t^n\big(g_{ap}g_{bq}g_{rm}g_{sn}\\
&\qquad + g_{bp}g_{aq}g_{rm}g_{sn} - g_{pq}g_{ab}g_{rm}g_{sn} - g_{aq}g_{bp}g_{mn}g_{rs} - g_{bp}g_{rs}g_{an}g_{qm}\big)
\end{aligned}$$

The first and the third terms inside the parenthesis cancel each other. We can use product property (2.0.4) with g_{bp} to obtain

$$\frac{1}{4}\mathcal{F}^2 t^r t^m t^n\bar{\mathcal{F}}^{qs}\big(g_{aq}g_{rm}g_{sn} - g_{aq}g_{mn}g_{rs} - g_{rs}g_{an}g_{qm}\big)\,.$$

The first two terms cancel each other, and the third vanishes as $\bar{\mathcal{F}}$ is antisymmetric. From this we conclude that

$$\nabla^b z\left(\nabla_a\Xi\nabla_b\bar{\Xi} + \nabla_b\Xi\nabla_a\bar{\Xi} - \nabla\Xi\cdot\nabla\bar{\Xi}g_{ab} + \nabla^l\Xi\bar{\mathcal{H}}_{bl}t_a - \mathcal{H}_{al}\bar{\mathcal{H}}_b{}^l t^2\right) = \nabla^b z(\mathfrak{e}_5)_{ab}$$

is essentially an algebraic error term.

Next we consider the third term on the right hand side of (A.3). We can replace $\hat{\mathcal{F}}$ by $\mathcal{F}$ using (2.3.1a), and have

$$\mathcal{F}_{cb}\Re\hat{\mathcal{F}}_a{}^c = \frac{1}{\mu}\mathcal{F}_{bc}\Re(\bar{P}_0\mathcal{F}_a{}^c) + (\mathfrak{e}_6)_{ab}$$

where

$$(\mathfrak{e}_6)_{ab} = \frac{2}{\mu}\mathcal{F}_{cb}\Re(\bar{\Xi}\mathcal{B}_a{}^c)$$

Now

$$\mathcal{F}_{bc}\mathcal{F}_a{}^c = \frac{1}{4}\mathcal{F}^2 g_{ab}$$

and using that $\mathcal{F}_{bc}\bar{\mathcal{F}}_a{}^c$ is real valued, we have

$$\begin{aligned}
\mathcal{F}_{bc}\bar{\mathcal{F}}_a{}^c\nabla^b z &= \Im\sigma^{-2}\mathcal{F}_{bc}\bar{\mathcal{F}}_a{}^c\mathcal{F}^{db}t_d \\
&= -\frac{1}{4}\Im\sigma^{-2}\mathcal{F}^2\bar{\mathcal{F}}_{ac}t^c \\
&= -\frac{1}{4}|\sigma|^4\Im\left(\sigma^{-4}\mathcal{F}^2\nabla_a\bar{\sigma}^{-1}\right) \\
&= \frac{1}{4}|\sigma|^4\Im(\sigma^{-4}\mathcal{F}^2)\nabla_a y - \frac{1}{4}|\sigma|^4\Re(\sigma^{-4}\mathcal{F}^2)\nabla_a z
\end{aligned}$$

here we can use (2.3.5) and get

$$= \frac{|\sigma|^4}{t^2}\Im(\nabla\sigma^{-1})^2(\nabla_a y + i\nabla_a z) - \frac{|\sigma|^4}{4}\sigma^{-4}\mathcal{F}^2\nabla_a z$$

so we get, using (2.3.10a) from Corollary 2.3.9,

$$\begin{aligned}
-\frac{1}{\sigma^2}\nabla^b z\mathcal{F}_{cb}\Re\hat{\mathcal{F}}_a{}^c &= -\frac{1}{\sigma^2}(\mathfrak{e}_6)_{ab}\nabla^b z - \frac{\bar{P}_0}{8\mu\sigma^2}\mathcal{F}^2\nabla_a z \\
&\quad + \frac{P_0\bar{\sigma}^2}{2\mu}\Im(\mathfrak{e}_1)\nabla_a\sigma^{-1} + \frac{P_0\bar{\sigma}^2}{8\mu\sigma^4}\mathcal{F}^2\nabla_a z \\
&= \frac{\mathcal{F}^2}{4\mu\sigma^4}i\Im\left(\bar{\sigma}^2 P_0\right)\nabla_a z - \frac{1}{\sigma^2}(\mathfrak{e}_6)_{ab}\nabla^b z + \frac{P_0\bar{\sigma}^2}{2\mu}\Im(\mathfrak{e}_1)\nabla_a\sigma^{-1}\ .
\end{aligned}$$

Next, we can consider adding in the second term on the right-hand side of (A.3), and expanding $P_0 = \frac{\bar{\kappa}}{\mu}(V - \kappa\sigma) - \mu$ from the definition,

$$\begin{aligned}
&\frac{1}{4\sigma^3}\mathcal{F}^2t^2\nabla_a z + \frac{1}{4\mu\sigma^4}\mathcal{F}^2 i\Im\left(\bar{\sigma}^2P_0\right)\nabla_a z \\
&\qquad = (\mathfrak{e}_1 - \mu^{-2})\nabla_a z\left[\sigma t^2 + \frac{i}{\mu}\Im\left(\frac{\bar{\kappa}}{\mu}V\bar{\sigma}^2 - \frac{|\kappa\sigma|^2}{\mu}\bar{\sigma} - \mu\bar{\sigma}^2\right)\right]
\end{aligned}$$

where $\mathfrak{e}_1$ is as defined in Corallary 2.3.9

$$= (\mathfrak{e}_1 - \mu^{-2})|\sigma|^2\nabla_a z\left[\bar{\sigma}^{-1}t^2 + \frac{i}{\mu}\Im\left(\frac{\bar{\kappa}}{\mu}\frac{\bar{\sigma}}{\sigma}V - \frac{|\kappa\sigma|^2}{\mu}\sigma^{-1} - \mu\frac{\bar{\sigma}}{\sigma}\right)\right]\ .$$

Noting that V is controllable by Lemma 2.1.8, and using (2.3.6′) to replace t^2, we have

$$\begin{aligned}
&\bar{\sigma}^{-1}t^2 + \frac{i}{\mu}\Im\left(\frac{\bar{\kappa}}{\mu}\frac{\bar{\sigma}}{\sigma}V - \frac{|\kappa\sigma|^2}{\mu}\sigma^{-1} - \mu\frac{\bar{\sigma}}{\sigma}\right) \\
&\quad = (y - iz)\left(\frac{1}{\mu^2}|V - \kappa\sigma|^2 + \sigma + \bar{\sigma} + 1\right) \\
&\qquad + i\Im\left(\frac{\bar{\kappa}\bar{\sigma}}{\mu^2\sigma}V\right) + \frac{iz}{\mu^2}|\kappa\sigma^2| - i\Im\left(\frac{(y + iz)^2}{y^2 + z^2}\right) \\
&\quad = y\left(\frac{1}{\mu^2}|V - \kappa\sigma|^2 + \sigma + \bar{\sigma} + 1\right) + i\Im\left(\frac{\bar{\kappa}\bar{\sigma}}{\mu^2\sigma}V\right) \\
&\qquad - iz - \frac{iz}{\mu^2}\left(|V - \kappa\sigma|^2 - |\kappa\sigma|^2\right) - iz\frac{(-2y)}{y^2 + z^2} - i\frac{2yz}{y^2 + z^2}\ .
\end{aligned}$$

The first term is purely real: recalling that for our purpose we are interested in the imaginary part of this expression, its contribution will appear with a factor of $\mathfrak{e}_1$.

The second and fourth terms are controlled by Lemma 2.1.8; the last two terms cancel. So essentially we are only left with the third term, $-iz$. In other words, up to some controllable errors, the imaginary part of the sum of the second and third terms on the right-hand side of (A.3) contributes $\mu^{-2}z|\sigma|^2\nabla_a z$, which corresponds precisely to the second term on the right-hand side of (A.1).

Lastly, we deal with the first term on the right-hand side of (A.3). We directly compute that

$$\begin{aligned}-\sigma\nabla_a\sigma^{-1}\nabla_b\sigma^{-1}\nabla^b z &= |\sigma|^2(y\nabla_a y + z\nabla_a z - iz\nabla_a y + iy\nabla_a z)(\nabla_b y\nabla^b z + i(\nabla z)^2)\\ &= |\sigma|^2 i(y\nabla_a y + z\nabla_a z)(\nabla z)^2\\ &\quad - |\sigma|^2 i(z\nabla_a y - y\nabla_a z)\frac{t^2}{2}\Im\mathfrak{e}_1 + \text{real-valued terms}\ .\end{aligned}$$

The first of the terms corresponds to the first term on the right-hand side of (A.1), and the second term gives the error.

So, collecting everything into one expression, we have that

$$\begin{aligned}\nabla_a\mathfrak{A} &= \frac{4\mu^2}{|\sigma|^2}\nabla^b z\Im\left(\frac{t^c}{\sigma^2\bar{P}_0}\nabla_a\mathcal{B}_{cb} - \frac{\mu}{\sigma^2\bar{P}_0}\mathcal{Q}_{dacb}t^c t^d\right) + 2\nabla_a z\Im\left(\frac{\bar\kappa\bar\sigma}{\mu^2\sigma}V\right)\\ &\quad + \mu^2 t^2(z\nabla_a y - y\nabla_a z)\Im\mathfrak{e}_1 - \Im\left[\frac{2\mathfrak{e}_1\mu^2}{|\sigma|^2}\nabla_a z\left(\sigma t^2 + \frac{i}{\mu}\Im(\bar\sigma^2 P_0)\right)\right]\\ &\quad - \frac{z\nabla_a z}{\mu^2}(|V-\kappa\sigma|^2 - |\kappa\sigma|^2) + \Im\left[\frac{4\mu^3}{|\sigma|^2\sigma^2\bar{P}_0}\nabla^b z(\mathfrak{e}_5)_{ab}\right]\\ &\quad + \Im\left[\frac{4\mu}{|\sigma|^2\sigma^2}\mathcal{F}_{cb}\Re(\bar\Xi\mathcal{B}_a{}^c)\nabla^b z - \frac{\mu P_0\bar\sigma}{\sigma}\Im(\mathfrak{e}_1)\nabla_a\sigma^{-1}\right]\end{aligned}\tag{A.5}$$

Appendix B. A mountain pass lemma

The mountain pass theorem is perhaps most well known for its application in calculus of variations in the form given by Ambrosetti and Rabinowitz [AR73]; but a finite dimensional version goes back at least to Courant in 1950 [Cou77]. Here we give (for not being able to find the exact statement needed elsewhere) a version that is similar in statement to Katriel's topological mountain pass theorem [Kat94] but with a proof following Jabri [Jab03, Chapter 5] and Nicolaescu [Nic07, Chapter 2].

Lemma B.1. *Let $\bar S$ denote a (possibly non-compact) finite dimensional connected smooth paracompact manifold with boundary, with S its interior and ∂S the (possibly empty) boundary. Given $f \in C^\infty(S,\mathbb{R})\cap C^0(\bar S,\mathbb{R})$ such that $f^{-1}((-\infty,a])$ is compact for any $a\in\mathbb{R}$ (the Palais-Smale condition). Suppose there exists two real values $s_- < s_+$ and a closed subset $C \subsetneq S$ such that*

- *$f|_{\partial S} \le s_-$;*
- *$f|_C \ge s_+$;*
- *C separates $\bar S$ with at least two of the connected components intersecting $\{f \le s_-\}$.*

Then f attains a critical point in S where the critical value is at least s_+.

Proof. Let S_1, S_2 be two components of $\{f\le s_-\}$ separated by C (in the sense that every connected set containing both S_1 and S_2 must intersect C; the pair is guaranteed to exist by assumption). Consider the collection Γ of compact, connected subsets of $\bar{S}$ that contains $S_1 \cup S_2$. Let $m : \Gamma \to \mathbb{R}$ be defined by $m(T) = \sup_T f$. Let (T_n) be a minimising sequence for m on Γ. Observe that since each $T_n \cap C \neq \emptyset$ necessarily $m(T_n) \geq s_+$. Noting that $\overline{\cup_{j=k}^{\infty} T_j} \subset \{f \leq m(T_k)\}$ is a closed subset of a compact set, the limiting set $T_\infty = \cap_{k=1}^{\infty} \overline{\cup_{j=k}^{\infty} T_j}$ is compact as the intersection of a decreasing family of compact sets, and we have that

$$s_+ \leq m(T_\infty) \leq m(T) \quad \forall T \in \Gamma \ .$$

Let $W = \{x \in T_\infty : f(x) = m(T_\infty)\}$, we show that W contains a critical point using gradient flow: fix, once and for all, a smooth Riemannian metric g on S. Then as W is compact, $|df|_g$ attains a minimum α on W. If $\alpha = 0$ we are done. Suppose $\alpha \neq 0$, let η be a non-negative bump function supported inside $\{2m(T_\infty) > f > (s_+ + s_-)/2, |df|_g > \alpha/2\}$ with $\eta|_W = 1$. Then under the flow of $-\eta\nabla f$, T_∞ is mapped to another connected compact subset T' of $\bar{S}$. Since $-\eta\nabla f$ vanishes on S_1, S_2, the set $T' \in \Gamma$. But since $-\eta|\nabla f|_g^2 \leq 0$ and $-\eta|\nabla f|_g^2|_W \leq -\alpha^2 < 0$, we have that the flow strictly decreases m, that is $m(T') < f(W) = m(T_\infty)$, which leads to a contradiction. $\square$

References

[AIK10a] S. Alexakis, A. D. Ionescu, and S. Klainerman, *Uniqueness of smooth stationary black holes in vacuum: small perturbations of the Kerr spaces*, Comm. Math. Phys. **299** (2010), no. 1, 89–127.

[AIK10b] Spyros Alexakis, Alexandru D. Ionescu, and Sergiu Klainerman, *Hawking's local rigidity theorem without analyticity*, Geom. Funct. Anal. **20** (2010), no. 4, 845–869. MR 2729279 (2012c:53113)

[AR73] Antonio Ambrosetti and Paul H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381. MR 0370183 (51 #6412)

[BS81] R. Beig and W. Simon, *On the multipole expansion for stationary space-times*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **376** (1981), no. 1765, 333–341.

[Bun83] Gary Bunting, *Proof of the uniqueness conjecture for black holes*, Ph.D. thesis, University of New England, Australia, May 1983.

[Car73] Brandon Carter, *Black hole equilibrium states*, Black holes: les astres occlus (C. DeWitt and B. S. DeWitt, eds.), Gordon and Breach, 1973.

[Car85] B. Carter, *Bunting identity and Mazur identity for non-linear elliptic systems including the black hole equilibrium problem*, Comm. Math. Phys. **99** (1985), no. 4, 563–591.

[Chr94] Piotr T. Chruściel, *"no hair" theorems – folklore, conjectures, results*, Contemp. Math. **170** (1994), 23–49.

[Chr96] ———, *Uniqueness of stationary, electro-vacuum black holes revisited*, Helv. Phys. Acta **69** (1996), 529–552.

[Cos10] João Lopes Costa, *On black hole uniqueness theorems*, Ph.D. thesis, University of Oxford, February 2010.

[Cou77] Richard Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Springer-Verlag, New York, 1977, With an appendix by M. Schiffer, Reprint of the 1950 original. MR 0454858 (56 #13103)

[FSW93] John L. Friedman, Kristin Schleich, and Donald M. Witt, *Topological censorship*, Phys. Rev. Lett. **71** (1993), no. 10, 1486–1489.

[GS06] Gregory J. Galloway and Richard Schoen, *A generalization of Hawking's black hole topology theorem to higher dimensions*, Comm. Math. Phys. **266** (2006), no. 2, 571 – 576.

[Heu96] Markus Heusler, *Black hole uniqueness theorems*, Cambridge Lecture Notes in Physics, vol. 6, Cambridge University Press, Cambridge, 1996. MR 1446003 (98b:83057)

[IK09a] Alexandru D. Ionescu and Sergiu Klainerman, *On the uniqueness of smooth, stationary black holes in vacuum*, Invent. Math. **175** (2009), no. 1, 35–102.

[IK09b] ———, *Uniqueness results for ill-posed characteristic problems in curved space-times*, Comm. Math. Phys. **285** (2009), no. 3, 873–900.

[Isr67] Werner Israel, *Event horizons in static vacuum space-times*, Phys. Rev. (2) **164** (1967), no. 5, 1776–1779.

[Isr68] ———, *Event horizons in static electrovac space-times*, Comm. Math. Phys. **8** (1968), no. 3, 245–260.

[Jab03] Youssef Jabri, *The mountain pass theorem*, Encyclopedia of Mathematics and its Applications, vol. 95, Cambridge University Press, Cambridge, 2003, Variants, generalizations and some applications. MR 2012778 (2005k:49002)

[Kat94] Guy Katriel, *Mountain pass theorems and global homeomorphism theorems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), no. 2, 189–209. MR 1267366 (95f:58022)

[Kie04a] Michael K.-H. Kiessling, *Electromagnetic field theory without divergence problems. I. The Born legacy*, J. Stat. Phys. **116** (2004), no. 1-4, 1057–1122. MR MR2082203 (2005h:81003a)

[Kie04b] ———, *Electromagnetic field theory without divergence problems. II. A least invasively quantized theory*, J. Stat. Phys. **116** (2004), no. 1-4, 1123–1159. MR MR2082204 (2005h:81003b)

[Mar99] Marc Mars, *A spacetime characterization of the Kerr metric*, Classical Quant. Grav. **16** (1999), no. 7, 2507–2523.

[Maz82] P O Mazur, *Proof of uniqueness of the Kerr-Newman black hole solution*, J. Phys. A **15** (1982), no. 10, 3173–3180.

[MTW73] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco, Calif., 1973. MR MR0418833 (54 #6869)

[Ngu11] Luc Nguyen, *Singular harmonic maps and applications to general relativity*, Comm. Math. Phys. **301** (2011), no. 2, 411–441. MR 2764993 (2012b:58024)

[NH09] Gernot Neugebauer and Jörg Hennig, *Non-existence of stationary two-black-hole configurations*, Gen. Relativity Gravitation **41** (2009), no. 9, 2113–2130. MR 2534657 (2010i:83101)

[NH12] ———, *Stationary two-black-hole configurations: a non-existence proof*, J. Geom. Phys. **62** (2012), no. 3, 613–630. MR 2876786

[Nic07] Liviu I. Nicolaescu, *An invitation to Morse theory*, Universitext, Springer, New York, 2007. MR 2298610 (2009m:58023)

[Rob75] D. C. Robinson, *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. **34** (1975), no. 14, 905–906.

[Rob77] D. C. Robinson, *A simple proof of the generalization of Israel's theorem*, Gen. Relativity Gravitation **8** (1977), no. 8, 695–698.

[Spe08] Jared R. Speck, *On the questions of local and global well-posedness for the hyperbolic PDEs occurring in some relativistic theories of gravity and electromagnetism*, Ph.D. thesis, Rutgers, The State University of New Jersey, New Brunswick, New Jersey, May 2008.

[uA92] A K M Masood ul Alam, *Uniqueness proof of static charged black holes revisited*, Classical Quant. Grav. **9** (1992), no. 5, L53–L55.

[Wei90] Gilbert Weinstein, *On rotating black holes in equilibrium in general relativity*, Comm. Pure Appl. Math. **43** (1990), no. 7, 903–948. MR 1072397 (91h:83063)

[Wei92] ———, *The stationary axisymmetric two-body problem in general relativity*, Comm. Pure Appl. Math. **45** (1992), no. 9, 1183–1203. MR 1177481 (93k:83010)

[Wei96] ———, *N-black hole stationary and axially symmetric solutions of the Einstein/Maxwell equations*, Comm. Partial Differential Equations **21** (1996), no. 9-10, 1389–1430. MR 1410835 (97e:58058)

[Won09a] Willie Wai-Yeung Wong, *On the uniqueness of Kerr-Newman black holes*, Ph.D. thesis, Princeton University, 2009.

[Won09b] ———, *A space-time characterization of the Kerr-Newman metric*, Ann. Henri Poincaré **10** (2009), no. 3, 453 – 484.

[Yu10] Pin Yu, *On the rigidity of charged black holes*, Ph.D. thesis, Princeton University, June 2010.

École Polytechnique Fédérale de Lausane, Lausanne, Switzerland
E-mail address: willie.wong@epfl.ch

Mathematical Sciences Center, Tsinghua University, Beijing, China
E-mail address: pin@math.tsinghua.edu.cn