Avoiding feedback-linearization singularity using a quotient method – The field-controlled DC motor case

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***Abstract*— Feedback linearization requires a unique feedback law and a unique diffeomorphism to bring a system to Brunovský normal form. Unfortunately, singularities might arise both in the feedback law and in the diffeomorphism. This paper demonstrates the ability of a quotient method to avoid or mitigate the singularities that typically arise with feedback linearization. The quotient method does it by relaxing the conditions on diffeomorphism, which can be achieved since there is an additional degree of freedom at each step of the iterative procedure. This freedom in choosing quotients and the resulting advantage are demonstrated for a field-controlled DC motor. Using a Lyapunov function, the domain of attraction of the control law obtained with the quotient method is proved to be larger than the domain of attraction of a control law obtained using feedback linearization.**

I. Introduction

Feedback linearization is a widely studied method for designing control laws for nonlinear system [1]. In feedback linearization, a system is transformed to Brunovský normal form using a feedback law and a diffeomorphism. All controllable linear systems can be brought to Brunovský normal form [2]. In the nonlinear setting, the input-affine single-input system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u \qquad (1)$$

is feedback linearizable (FBL) (Theorem 4.2.3 of [3]) if and only if the following conditions are satisfied:

1) Involutivity of the distribution
$$\Delta = \text{Span}\left\{g, ad_f g, \dots, ad_f^{n-2} g\right\}.$$
2) Full rank of the accessibility matrix
$$\mathcal{L} = \left\{g, ad_f g, \dots, ad_f^{n-1} g\right\},$$

where $ad_f g$ represents the Lie bracket of $f(\mathbf{x})$ and $g(\mathbf{x})$. Achieving feedback linearization requires a feedback linearizing output $h(\mathbf{x})$ of relative degree n such that the 1-form $\omega(\mathbf{x}) = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}$ belongs to the kernel of Δ. The output $h(\mathbf{x})$ is then used to obtain the feedback law

$$\begin{aligned} v &= L_f^n h(\mathbf{x}) + L_g L_f^{n-1} h(\mathbf{x}) u, \\ \text{or} \quad u &= \frac{v - L_f^n h(\mathbf{x})}{L_g L_f^{n-1} h(\mathbf{x})}, \end{aligned}$$

and the diffeomorphism

$$\mathbf{\Phi} = \begin{pmatrix} h(\mathbf{x}) \\ L_f h(\mathbf{x}) \\ \vdots \\ L_f^{n-1} h(\mathbf{x}) \end{pmatrix},$$

where v is the input to the Brunovský normal form. The domain of attraction of the control law depends on the domain of validity of $\mathbf{\Phi}$ and $\mathbf{\Phi}^{-1}$ and the zeros of the function $L_g L_f^{n-1} h(\mathbf{x})$. Hence the domain of attraction excludes the singularity points where the determinant of $\frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = 0$ or $L_g L_f^{n-1} h(\mathbf{x}) = 0$ as well as all the points where a singularity is reached following the trajectory of the closed loop system.

There are algorithms to determine the feedback linearizing outputs for single-input systems ([4], [5]) and for multi-input systems ([6], [7]). The algorithm proposed in [4] generates quotients to obtain $h(\mathbf{x})$. Based on this algorithm, a quotient method [8] has been developed to directly obtain the control law without the need of achieving Brunovský normal form. Due to the freedom in choosing the quotient foliation in the design process of the quotient method, it was observed, in simulation, that the control law was able to overcome the singularity in feedback linearization [8]. The present paper demonstrates the application of the quotient method to a field-controlled DC motor and proves, using a Lyapunov function, that the domain of attraction of the control law obtained through the quotient method is indeed larger than the domain of attraction of the control law obtained through feedback linearization.

The paper is organized in the following manner. The next section briefly introduces the steps in the quotient method. Section III presents the mathematical model of the field-controlled DC motor and discusses feedback linearization of the DC motor model. Section IV describes the steps involved in designing the control law. Section V proves the domain of attraction of the control law and presents simulation results. Finally, section VI provides concluding remarks.

II. Quotient Method

The quotient method is an iterative design technique to obtain a control law for nonlinear input-affine single-input systems [8]. The method proceeds in two stages, namely a forward decomposition stage and a backward control design stage. Both stages require several iterative steps.

A. Forward decomposition

At every step of the forward decomposition stage, an equivalence relation ($\sim$) is defined between two vector fields. For example, for the vector fields $m_1(\mathbf{x})$ and $m_2(\mathbf{x})$, the equivalence relation is:

$$m_1(\mathbf{x}) \sim m_2(\mathbf{x}) \text{ iff } m_1(\mathbf{x}) - m_2(\mathbf{x}) = \kappa(\mathbf{x}) g(\mathbf{x}),$$

where $\kappa(\mathbf{x})$ is any function. This defines the equivalence class as

$$[m_r(\mathbf{x})] = \{m(\mathbf{x}) | m(\mathbf{x}) - m_r(\mathbf{x}) = \kappa(\mathbf{x})g(\mathbf{x}) \ \ \forall \kappa(\mathbf{x})\},$$

where $m_r(\mathbf{x})$ is the representative of the equivalence class. To define the representative of the equivalence class, we choose the exact 1-form $\omega(\mathbf{x}) = \frac{\partial \gamma(\mathbf{x})}{\partial \mathbf{x}}$, where $\gamma(\mathbf{x})$ is any chosen function such that $\omega(\mathbf{x})g(\mathbf{x}) \neq 0$. Then, we define the representative of any vector field $m(\mathbf{x})$ as

$$m_r(\mathbf{x}) = m(\mathbf{x}) - \frac{\omega(\mathbf{x})m(\mathbf{x})}{\omega(\mathbf{x})g(\mathbf{x})} g(\mathbf{x}). \tag{2}$$

Note that $m_r(\mathbf{x})$ represents the entire equivalence class to which $m(\mathbf{x})$ belongs. Hence, it can be verified that $m_r(\mathbf{x})$ remains unchanged when $m(\mathbf{x})$ is replaced by $m(\mathbf{x}) + \kappa(\mathbf{x})g(\mathbf{x})$, for all $\kappa(\mathbf{x})$. Using this equivalence relationship, one can obtain the representative $f_r(\mathbf{x})$ of $f(x) + g(\mathbf{x})u$. By definition, $f_r(\mathbf{x})$ remains unchanged for all control laws $u = \kappa(\mathbf{x})$. This whole process is facilitated by designing the diffeomorphism

$$\mathbf{z} = \mathbf{\Phi}_p(\mathbf{x}) = \begin{pmatrix} \phi_1(\mathbf{x}) \\ \vdots \\ \phi_{n-1}(\mathbf{x}) \\ \gamma(\mathbf{x}) \end{pmatrix},$$

where $\phi_1(\mathbf{x})$ to $\phi_{n-1}(\mathbf{x})$ are scalar functions such that $L_g\phi_i = 0$ for $i = 1, \ldots, n-1$ and $\text{rank}\left(\frac{\partial \mathbf{\Phi}_p(\mathbf{x})}{\partial \mathbf{x}}\right) = n$, so that

$$\mathbf{\Phi}_* g(\mathbf{x}) \triangleq \left.\frac{\partial \mathbf{\Phi}_p}{\partial \mathbf{x}} g(\mathbf{x})\right|_{\mathbf{x}=\mathbf{\Phi}_p^{-1}(\mathbf{z})} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha(\mathbf{z}) \end{pmatrix}, \tag{3}$$

where $\alpha(\mathbf{z}) := L_g\gamma(\mathbf{x}) \circ \mathbf{\Phi}_p^{-1}(\mathbf{z})$. In these transformed coordinates, obtaining $\mathbf{\Phi}_* f_r(\mathbf{x})$ reduces to simply eliminating the last line of $\mathbf{\Phi}_* f(\mathbf{x})$. The eliminated coordinate can be regarded as the input of the system described by the vector field $\mathbf{\Phi}_* f_r(\mathbf{x})$, and thus results in a single-input system of dimension reduced by one. This process can be repeated $n-1$ times to obtain a single-dimensional system. The diffeomorphisms obtained at each step can be combined to put the system in feedback form:

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2) \\ \dot{y}_2 &= f_2(y_1, y_2, y_3) \\ &\vdots \\ \dot{y}_n &= f_n(y_1, \ldots, y_n, u). \end{aligned} \tag{4}$$

where y_1 to y_n represents the coordinates of the new system obtained using the combined diffeomorphism.

B. Backward control design

The control design stage starts by designing a control law for the system obtained at the end of the forward stage, i.e. the first equation of (4). For this subsystem y_2 is considered as the input. Next, assign the target value $y_{2,d}(y_1)$ to the input y_2 by solving $-k_1y_1 = f_1(y_1, y_2)$ for y_2. Here, k_1 is any positive constant. This target value, which depends on y_1, is then tracked using proportional feedback. The error

$$e_2 = y_2 - y_{2,d}$$

is defined to track the desired target.

Next consider the first two equations of (4) and assume y_3 as the input to this subsystem. The error defined above is then driven asymptotically to zero by assigning the stabilizing dynamics

$$\dot{e}_2 = -k_2 e_2,$$

where k_2 is a positive gain for the proportional feedback controller . Substituting for $\dot{e}_2$ and e_2 and solving for y_3 results in $y_{3,d}(y_1, y_2)$, which is a function of y_1 and y_2. The whole step can be repeated by defining the error $e_3 = y_3 - y_{3,d}$ to obtain $y_{4,d}(y_1, y_2, y_3)$. The backward stage continues this way until a control law is obtained for the original system. It is easy to show, using the center manifold theory (see appendix B of [3] and corollary 1 of [8]), that the resulting control law is locally asymptotically stable. Furthermore, a Lyapunov-type analysis will be used to estimate the domain of attraction of the control law.

III. MODEL OF THE FIELD-CONTROLLED DC MOTOR

The example of a field-controlled DC motor is chosen to illustrate how the possibility of choosing $\gamma(\mathbf{x})$ during the forward decomposition helps avoid the singularity that arises due to the particular choice of $\gamma(\mathbf{x})$ required for feedback linearization. The field-controlled DC motor is a FBL system. However, the quotient method is not restricted to FBL systems, and application to non-FBL system are illustrated in [9] and [10].

The field-controlled DC motor with negligible shaft damping described in [11] is considered:

$$\begin{aligned} v_f &= R_f i_f + L_f \frac{di_f}{dt}, \\ v_a &= c_1 i_f \omega + L_a \frac{di_a}{dt} + R_a i_a, \\ J\frac{d\omega}{dt} &= c_2 i_f i_a. \end{aligned}$$

The first equation represents the field circuit, with v_f, i_f, R_f, and L_f being the voltage, current, resistance and inductance, respectively. The variables v_a, i_a, R_a, and L_a in the second equation are the corresponding variables for the armature circuit. The term $c_1 i_f \omega$ is the back electromotive force induced in the armature circuit. The third equation is the equation of motion for the shaft, with the rotor inertia J and the torque $c_2 i_f i_a$ produced by the interaction of the armature current with the field circuit flux. The voltage v_a is held constant and control is achieved by varying v_f. The system is represented by the third-order model

$$\dot{\mathbf{x}} = f(\mathbf{x}) + gu,$$

with the states $x_1 = i_f, x_2 = i_a, x_3 = \omega$, and the input $u = \frac{v_f}{L_f}$,

$$f(\mathbf{x}) = \begin{bmatrix} -ax_1 \\ -bx_2 + \rho - cx_1x_3 \\ \theta x_1 x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad (5)$$

and the positive constants $a = R_f/L_f, b = R_a/L_a, c = c_1/L_a, \theta = c_2/J, \rho = v_a/L_a$. For $\bar{u} = 0$, the motor is at equilibrium at $\bar{x}_1 = 0$, $\bar{x}_2 = \rho/b$ and any arbitrary $\bar{x}_3$ value. The aim is to design a control law that drives the system from any initial condition to the desired operating point $x^* = (0,\ \rho/b,\ \omega_0)$, where ω_0 is the desired set point for the angular velocity x_3. The diffeomorphism

$$\Phi_{FBL} = \begin{pmatrix} \theta x_2^2 + cx_3^2 \\ 2\theta x_2(\rho - bx_2) \\ -2\theta(\rho - 2bx_2)(-bx_2 + \rho - cx_1x_3) \end{pmatrix}$$

brings the system to Brunovský normal form [12].

Next, consider the determinant of $\frac{\partial \mathbf{\Phi}_{FBL}}{\partial \mathbf{x}}$ given by

$$\text{Det}(\frac{\partial \mathbf{\Phi}_{FBL}}{\partial \mathbf{x}}) = 8c^2\theta^2 x_3^2(2bx_2 - \rho)^2,$$

which shows that $\mathbf{\Phi}_{FBL}$ is singular at both $x_3 = 0$ and $x_2 = \rho/2b$. It can also be seen in the accessibility matrix,

$$\begin{aligned} \mathcal{L} &= (g, [f,g], [f,[f,g]]) \\ &= \begin{pmatrix} 1 & a & a^2 \\ 0 & cx_3 & (a+b)cx_3 \\ 0 & -\theta x_2 & -\theta\rho - a\theta x_2 + b\theta x_2 \end{pmatrix}, \end{aligned}$$

whose determinant is given by

$$\text{Det}(\mathcal{L}) = c\,\theta\, x_3(2bx_2 - \rho).$$

Note that the accessibility matrix looses rank for both $x_3 = 0$ and $x_2 = \rho/2b$. Consequently, the control law designed through feedback linearization has singularities, and the diffeomorphism is not valid there. Hence, the domain of attraction of the control law developed using feedback linearization [11] is:

$$\mathcal{D}_{FBL} = \{\ (x_1, x_2, x_3) \mid \quad x_2 > \frac{\rho}{2b} \quad \text{and} \quad x_3 > 0\}. \qquad (6)$$

Physically, the two points of singularity arise due to the presence of two cross terms. The first term is the "back e.m.f" resulting from the motion of the shaft in the magnetic flux generated by the field coils $cx_1x_3 = \frac{c_1 \omega L_f i_f}{L_a}$. The second term is $\theta x_1 x_2 = \frac{c_2 L_f i_f i_a}{J}$, which corresponds to the torque generated at the shaft due to the current flowing in the armature coil inside the magnetic flux generated by the field coils. These cross terms represent the mechanisms by which the field coils are able to influence the armature current and produce torque at the shaft. Now, for $\omega = 0$, the magnetic flux and thus, the field current looses its influence over the armature current. Similarly, if the armature current $i_a = 0$, then the field current fails to produce any torque on the shaft. However, due to the presence of a bias voltage in the armature circuit v_a, the point of singularity is shifted to $i_a = \frac{v_a}{2R_a}$.[1]

The two cases, $\omega = 0$ and $i_a = \frac{v_a}{2R_a}$, represent only a momentary loss of the influence of the input. They become singularities because the feedback linearization attempts to impose a linear affine structure (Brunovský normal form) to the system. In the quotient method, by suitably using the degree of freedom in the algorithm, we can avoid imposing a strict linear form and thus having singularities. For the field-controlled DC motor, this degree of freedom can only be used in one equation; however, we can choose which singularity to avoid. In this paper, we chose to avoid the singularity at $i_a = \frac{v_a}{2R_a}$ since we are interested in a cascade to control successively i_f, i_a and ω, which is justified by the fact that the time constants of the electrical circuits are considerably smaller than the time constant of the mechanical component. The other option to control in turn i_f, ω and i_a would avoid the singularity at $\omega = 0$.

IV. Control design using the quotient method

Since the quotient method allows avoiding at least one singularity, the resulting domain of attraction of the control law is increased. The forward decomposition stage includes two steps, so as to achieve a single-dimensional system.

Step 1: This step brings the vector field $g(\mathbf{x})$ into the canonical form (3) and then shifts the equilibrium point from $(0,\ \rho/b,\ \omega_0)$ to $(0,0,0)$ by designing a suitable diffeomorphism. The first two functions of the diffeomorphism are [8]:

$$\begin{aligned} z_1 &= x_2 - \rho/b \qquad (7) \\ z_2 &= x_3 - \omega_0. \qquad (8) \end{aligned}$$

The third function can be chosen to maximize the size of the validity domain of the resulting diffeomorphism. One such choice is $z_3 = x_1$, which results in the globally valid diffeomorphism:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \mathbf{\Phi}_1(x_1, x_2, x_3) = \begin{pmatrix} x_2 - \rho/b \\ x_3 - \omega_0 \\ x_1 \end{pmatrix}.$$

The model in $\mathbf{z}$-coordinates becomes:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} -bz_1 - cz_3(\omega_0 + z_2) \\ \theta(\frac{\rho}{b} + z_1)z_3 \\ -az_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u.$$

The last row is removed and $\dot{z}_1$ and $\dot{z}_2$ expressed with z_3 as input to give the following two-dimensional system:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -bz_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -c(\omega_0 + z_2) \\ \theta(\frac{\rho}{b} + z_1) \end{pmatrix} z_3. \qquad (9)$$

Step 2: This step brings the input vector field of (9) into the canonical form (3) by designing a suitable diffeomorphism. The first function of this diffeomorphism is:

$$y_1 = \frac{\theta}{2}z_1^2 + \frac{\theta\rho}{b}z_1 + \frac{c}{2}z_2^2 + c\omega_0 z_2.$$

[1] This shift to $i_a = \frac{v_a}{2R_a}$ is not obvious from the equations; however, the absence of a singularity for $i_a = 0$ clearly indicates the effect of the bias voltage v_a.

The second function y_2 can be chosen to maximize the size of the validity domain of the resulting diffeomorphism. The choice $y_2 = z_1$ results in the diffeomorphism:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{\Phi}_2(z_1, z_2) = \begin{pmatrix} \frac{\theta}{2} z_1^2 + \frac{\theta\rho}{b} z_1 + \frac{c}{2} z_2^2 + c\omega_0 z_2 \\ z_1 \end{pmatrix},$$

which, however, has a singularity at $z_2 = -\omega_0$, corresponding to $x_3 = 0$ (see eq. 8). Other choices for y_2 are possible, for example:

- $y_2 = z_2$ results in a singularity at
$$z_1 = -\rho/b, \quad \text{corresponding to } x_2 = 0.$$
- $y_2 = z_1 + z_2$ results in a singularity at
$$\theta z_1 - c z_2 + \frac{\theta\rho}{b} - c\omega_0 = 0, \quad \text{corresponding to } \theta x_2 - c x_3 = 0.$$
- $y_2 = -b\theta z_1^2 - \theta\rho z_1$ results in a singularity at
$$c\theta\rho\omega_0 + 2bc\theta\omega_0 z_1 + c\theta\rho z_2 + 2bc\theta z_1 z_2 = 0,$$
corresponding to $(\rho - 2bx_2)x_3 = 0$,
that is, either $x_3 = 0$ or $x_2 = \frac{\rho}{2b}$,

  which is the same singularity as in case of feedback linearization.

With the quotient method, the choice of the last function of the diffeomorphism at each step plays a crucial role in determining the singularity of the resulting control law. Transforming using $\mathbf{\Phi}_2$ results in:

$$\begin{aligned} \dot{y_1} &= -y_2\theta(by_2 + \rho), \\ \dot{y_2} &= -by_2 + g_1(y_1, y_2) z_3, \end{aligned}$$

where

$$g_1(y_1, y_2) = -\frac{\sqrt{b^2c^2\omega_0^2 + 2b^2cy_1 - b^2c\theta y_2^2 - 2bc\theta\rho y_2}}{b}.$$

The last row is removed and $\dot{y}_1$ is expressed with y_2 as input to give the following one-dimensional system:

$$\dot{y_1} = -y_2\theta(by_2 + \rho). \quad (10)$$

Remark 1: The ability to avoid singularity is harnessed from the fact that (10) is not affine in y_2. By choosing y_2 satisfying the lemma 4 of [8], it is possible to obtain an equation affine in y_2 in lieu of equation (10). However, this would result in the same singularities as in feedback linearisation.

The diffeomorphism $\mathbf{\Phi} = \mathbf{\Phi}_2 \circ \mathbf{\Phi}_1$ can be obtained by augmenting $\mathbf{\Phi}_2$ with $y_3 = z_3$:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \mathbf{\Phi}(x_1, x_2, x_3) = \begin{pmatrix} \frac{-\theta\rho^2 - b^2c\omega_0^2 + b^2\theta x_2^2 + b^2 c x_3^2}{2b^2} \\ x_2 - \rho/b \\ x_1 \end{pmatrix}, \quad (11)$$

with the inverse transformation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{\Phi}^{-1}(y_1, y_2, y_3) = \begin{pmatrix} y_3 \\ \frac{\rho}{b} + y_2 \\ \frac{\sqrt{bc(bc\omega_0^2 + 2by_1 - 2\theta\rho y_2 - b\theta y_2^2)}}{bc} \end{pmatrix}.$$

This diffeomorphism leads to the following model in $\mathbf{y}$-coordinates:

$$\begin{aligned} \dot{y_1} &= -y_2\theta(by_2 + \rho), \\ \dot{y_2} &= -by_2 + g_1(y_1, y_2) y_3, \\ \dot{y_3} &= -ay_3 + u. \end{aligned} \quad (12)$$

The first function of $\mathbf{\Phi}$ is always the static feedback linearizing output function (Propositions 3 and 4 of [4]). However, due to the choice of $y_2 = z_1$, this diffeomorphism has a singularity at $x_3 = 0$. Hence, any globally asymptotically stabilizing (GAS) control law designed for (12) through the quotient method will have the following domain of attraction:

$$\mathcal{D}_{QM} = \{ \ (x_1, x_2, x_3) \ | \quad x_3 < 0 \quad \text{or} \quad x_3 > 0\}. \quad (13)$$

The backward stage computes a control law for (12), which is given by

$$u = -k_3(y_3 - y_{3,d}) + \frac{\partial y_{3,d}}{\partial y_1}(-y_2\theta(by_2 + \rho)) + \frac{\partial y_{3,d}}{\partial y_2}(-by_2 + g_1(y_1, y_2) z_3) + a z_3, \quad (14)$$

where

$$\begin{aligned} y_{3,d} &= \frac{-k_2(y_2 - y_{2,d}) + by_2 + \frac{\partial y_{2,d}}{\partial y_1}(-y_2\theta(by_2 + \rho))}{g_1(y_1, y_2)}, \\ y_{2,d} &= \frac{-\theta\rho + \sqrt{\theta^2\rho^2 + 4\theta b k_1 y_1}}{2\theta b}. \end{aligned}$$

During the control design stage, the errors e_1, e_2 and e_3 are defined in order to obtain $y_{2,d}$, $y_{3,d}$ and u. The diffeomorphism Φ_e is obtained using these definitions:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \mathbf{\Phi}_e(y_1, y_2, y_3) = \begin{pmatrix} y_1 \\ y_2 - y_{2,d} \\ y_3 - y_{3,d} \end{pmatrix},$$

which transforms the closed-loop system composed of (12) and u given by (14) into the following form

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} -k_1e_1 - be_2^2\theta - e_2\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2} \\ -k_2e_2 - g_e(e_1, e_2)e_3 \\ -k_3e_3 \end{pmatrix}, \quad (15)$$

where

$$g_e(e_1, e_2) = \left(\frac{c\rho\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2}}{2b^2} - \frac{c\theta\rho^2}{2b^2} + \frac{ce_2\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2}}{b} + \frac{ce_1k_1}{b} + \frac{ce_2\theta\rho}{b} - c^2\omega_0^2 - 2ce_1 + ce_2^2\theta \right)^{1/2}.$$

If $\mathbf{\Phi}_e$ is globally valid and (15) is GAS, then the closed-loop system is also GAS. As a consequence, the domain of attraction for the original system is $\mathcal{D}_{QM}$. The following section proves the GAS property of (15) and presents the condition for the global validity of $\mathbf{\Phi}_e$.

V. Domain of Attraction

This section proves the global asymptotically stability (GAS) of (15). The proof is divided in two parts. Firstly, the GAS of a subsystem obtained for $e_3 = 0$ is established using a Lyapunov function. Based on this Lyapunov function, a new Lyapunov function is proposed in order to prove the GAS of the full system (15).

A. Subsystem

Let us first consider the following sub-system resulting by setting $e_3 = 0$ in (15):

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -k_1e_1 - be_2^2\theta - e_2\sqrt{4b\theta e_1k_1 + \theta^2\rho^2} \\ -k_2e_2 \end{pmatrix}, \quad (16)$$

Note that the constants k_1 and k_2 are tunable positive gains.

Lemma 1: System (16) is globally asymptotically stable.

Proof: Consider the Lyapunov function

$$V_1 = \frac{1}{2}e_1^2 + \frac{C_1}{2}e_2^2,$$

where $C1$ is a positive constant with $C_1 > \frac{\theta^2\rho^2}{4k_1k_2}$.

Consider the time derivate of V_1,

$$\begin{aligned} \dot{V}_1 &= e_1\dot{e}_1 + C_1e_2\dot{e}_2 \\ &= -k_1e_1^2 - be_1e_2^2\theta - e_1e_2\sqrt{4b\theta e_1k_1 + \theta^2\rho^2} \\ &\quad -C_1k_2e_2^2 \\ &\leq -k_1e_1^2 - be_1e_2^2\theta + |e_1||e_2|\sqrt{4b\theta e_1k_1 + \theta^2\rho^2} \\ &\quad -C_1k_2e_2^2 \\ &= -k_1e_1^2 - be_1e_2^2\theta + |e_1|\sqrt{4b\theta e_1k_1e_2^2 + \theta^2\rho^2e_2^2} \\ &\quad -C_1k_2e_2^2. \end{aligned}$$

Using Young's inequality, $|a||b| \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$, with $a = e_1$, $b = \sqrt{4b\theta e_1k_1e_2^2 + \theta^2\rho^2e_2^2}$ and $\epsilon = 2k_1$ gives:

$$\begin{aligned} \dot{V}_1 &\leq -k_1e_1^2 - be_1e_2^2\theta \\ &\quad + \left(\frac{2k_1e_1^2}{2} + \frac{4b\theta e_1k_1e_2^2 + \theta^2\rho^2e_2^2}{4k_1}\right) - C_1k_2e_2^2 \\ &= -k_1e_1^2 - be_1e_2^2\theta \\ &\quad + \left(k_1e_1^2 + be_1e_2^2\theta + \frac{\theta^2\rho^2}{4k_1}e_2^2\right) - C_1k_2e_2^2 \\ &= -\left(C_1k_2 - \frac{\theta^2\rho^2}{4k_1}\right)e_2^2. \end{aligned} \quad (17)$$

Since $C_1k_2 > \frac{\theta^2\rho^2}{4k_1}$, $\dot{V}_1$ is negative semi-definite. Moreover, for $e_2 = 0$, one has:

$$\dot{V}_1 = -k_1e_1^2. \quad (18)$$

Next, $\dot{V}_1 = 0$ implies $e_2 = 0$ from relation (17) and $e_1 = 0$ from (18), which implies that $\dot{V}_1$ is negative definite. Hence, system (16) is globally asymptotically stable. ■

Also note that

$$e_1 = y_1 = \frac{-\theta\rho^2 - b^2c\omega_0^2 + b^2\theta x_2^2 + b^2cx_3^2}{2b^2},$$

which implies:

$$e_1 \geq \frac{-\theta\rho^2 - b^2c\omega_0^2}{2b^2}.$$

From (15), it is clear that, in order to obtain a real solution, one must ensure

$$\psi \triangleq 4b\theta e_1k_1 + \theta^2\rho^2 > 0.$$

Next, substituting the lower bound on e_1 yields an upper bound on k_1 to enforce $\psi > 0$:

$$k_1 < \frac{b\theta\rho^2}{2b^2c\omega_0^2 + 2\theta\rho^2}. \quad (19)$$

Note that $\mathbf{\Phi}_e$ involves $y_{2,d}$, which in turn has $\sqrt{\psi}$. Hence, global validity of $\mathbf{\Phi}_e$ depends on ensuring positive ψ for all y_1. By choosing k_1 such that (19) is satisfied, we will ensure global validity of $\mathbf{\Phi}_e$ for all y_1.

B. Full system

Next, consider the full system (15) and the following theorem

Theorem 1: System (15) is globally asymptotically stable.

Proof: Consider the Lyapunov function

$$V_2 = \log(1 + V_1) + \frac{C_2}{2}e_3^2,$$

where $V_1 = \frac{e_1^2}{2} + \frac{C_1e_2^2}{2}$, C_1 and C_2 are positive constants such that

$$\begin{aligned} C_1 &> \frac{\theta^2\rho^2}{4k_1k_2} + \frac{1}{2k_2} \\ C_2 &> \frac{1}{k_3}\sum_{i=1}^{8} L_i, \end{aligned}$$

and L_1 to L_8 are defined as

$$\begin{aligned} L_1 &= \max\frac{c\rho\sqrt{4b\theta e_1k_1 + \theta^2\rho^2}}{(1 + e_1^2 + e_2^2)4b^2} \\ L_2 &= \inf\frac{c\theta\rho^2}{4b^2(1 + e_1^2 + e_2^2)} = 0 \\ L_3 &= \max\left|\frac{ce_2\sqrt{4b\theta e_1k_1 + \theta^2\rho^2}}{2(1 + e_1^2 + e_2^2)b}\right| \\ L_4 &= \max\left|\frac{ce_1k_1}{2(1 + e_1^2 + e_2^2)b}\right| = \frac{ck_1}{4b} \\ L_5 &= \max\left|\frac{ce_2\theta\rho}{2(1 + e_1^2 + e_2^2)b}\right| = \frac{c\theta\rho}{4b} \\ L_6 &= \inf\frac{c^2\omega_0^2}{2(1 + e_1^2 + e_2^2)} = 0 \\ L_7 &= \max\left|\frac{ce_1}{(1 + e_1^2 + e_2^2)}\right| = \frac{c}{2} \\ L_8 &= \sup\frac{ce_2^2\theta}{(1 + e_1^2 + e_2^2)} = c\theta, \end{aligned}$$

Comparing (15) and (16) and substituting from (17) gives:

$$\dot{V}_1 \leq \left(C_1 k_2 - \frac{\theta^2 \rho^2}{4k_1} \right) e_2^2 - e_2\, g_e(e_1, e_2) e_3.$$

Next, consider the time derivative of V_2,

$$\dot{V}_2 = \frac{\dot{V}_1}{1+V_1} - C_2 k_3 e_3^2,$$

and substituting $\dot{V}_1$ in $\dot{V}_2$ gives:

$$\begin{aligned}\dot{V}_2 &\leq -\frac{\left(C_1 k_2 - \frac{\theta^2\rho^2}{4k_1}\right) e_2^2}{1+e_1^2+e_2^2} - \frac{e_2 g_e(e_1,e_2) e_3}{1+e_1^2+e_2^2} - C_2 k_3 e_3^2 \\ &\leq -\frac{\left(C_1 k_2 - \frac{\theta^2\rho^2}{4k_1}\right) e_2^2}{1+e_1^2+e_2^2} + \frac{|e_2||g_e(e_1,e_2)||e_3|}{1+e_1^2+e_2^2} \\ &\quad -C_2 k_3 e_3^2.\end{aligned}$$

Using Young's inequality with $a = e_2$, $b = |g_e(e_1, e_2)||e_3|$ and $\epsilon = 1$ allows writing:

$$\begin{aligned}\dot{V}_2 &\leq -\frac{\left(C_1 k_2 - \frac{\theta^2\rho^2}{4k_1}\right) e_2^2}{1+e_1^2+e_2^2} + \frac{\frac{e_2^2}{2} + \frac{g_e(e_1,e_2)^2 e_3^2}{2}}{1+e_1^2+e_2^2} \\ &\quad -C_2 k_3 e_3^2.\end{aligned}$$

Finally, substituting for $g_e(e_1, e_2)$ gives:

$$\begin{aligned}\dot{V}_2 &\leq -\frac{\left(C_1 k_2 - \frac{\theta^2\rho^2}{4k_1} - \frac{1}{2}\right) e_2^2}{(1+e_1^2+e_2^2)} \\ &\quad + \frac{(c\rho\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2} - c\theta\rho^2)}{(1+e_1^2+e_2^2)2b^2}\frac{e_3^2}{2} \\ &\quad + \frac{(ce_2\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2} + ce_1 k_1 + ce_2\theta\rho)}{(1+e_1^2+e_2^2)b}\frac{e_3^2}{2} \\ &\quad + \frac{(-c^2\omega_0^2 - 2ce_1 + ce_2^2\theta)}{(1+e_1^2+e_2^2)} - C_2 k_3 e_3^2.\end{aligned}$$

Using the definition of L_1 to L_8, $\dot{V}_2$ becomes:

$$\begin{aligned}\dot{V}_2 &\leq -\frac{\left(C_1 k_2 - \frac{\theta^2\rho^2}{4k_1} - \frac{1}{2}\right) e_2^2}{(1+e_1^2+e_2^2)} + L_1 e_3^2 + L_2 e_3^2 \\ &\quad + L_3 e_3^2 + L_4 e_3^2 + L_5 e_3^2 + L_6 e_3^2 + L_7 e_3^2 \\ &\quad + L_8 e_3^2 - C_2 k_3 e_3^2 \\ &= -\frac{\left(C_1 k_2 - \frac{\theta^2\rho^2}{4k_1} - \frac{1}{2}\right) e_2^2}{(1+e_1^2+e_2^2)} - \left(C_2 k_3 - \sum_{i=1}^{8} L_i \right) e_3^2.\end{aligned}$$

Since the explicit expressions of L_1 and L_3 are involved, there are not presented here. Nevertheless, L_1 and L_3 exist since both expressions are continuous functions that are different from zero somewhere and

$$\lim_{||(e_1,e_2)||\to\infty} \frac{c\rho\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2}}{(1+e_1^2+e_2^2)4b^2} = 0,$$
$$\lim_{||(e_1,e_2)||\to\infty} \frac{ce_2\sqrt{4b\theta e_1 k_1 + \theta^2\rho^2}}{2(1+e_1^2+e_2^2)b} = 0.$$

a	b	c	θ	ρ	ω_0
103.995	35.4034	1.45	230.769	52.7588	10

TABLE I

PARAMETER VALUES FOR THE SIMULATED DC MOTOR.

Hence, L_1 and L_3 must exist somewhere on the (e_1, e_2)-plane. Since C_1 and C_2 satisfies

$$\begin{aligned} C_1 k_2 &> \frac{\theta^2\rho^2}{4k_1} + \frac{1}{2} \\ C_2 k_3 &> \sum_{i=1}^{8} L_i, \end{aligned}$$

negative definitiveness of $\dot{V}_2$ is established. Hence, the system (15) is globally asymptotically stable. ■

Also, k_1 can be chosen according to (19) to ensure that, despite having a square root in the equation, $\mathbf{\Phi}_e$ is globally valid and the resulting control input is always real. GAS property for (15) implies that the closed-loop system is GAS with the equilibrium point:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

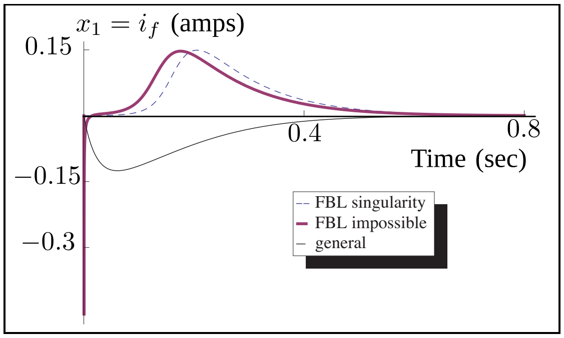
which implies:

$$\begin{pmatrix} \frac{-\theta\rho^2 - b^2 c\omega_0^2 + b^2\theta x_2^2 + b^2 c x_3^2}{2b^2} \\ x_2 - \rho/b \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

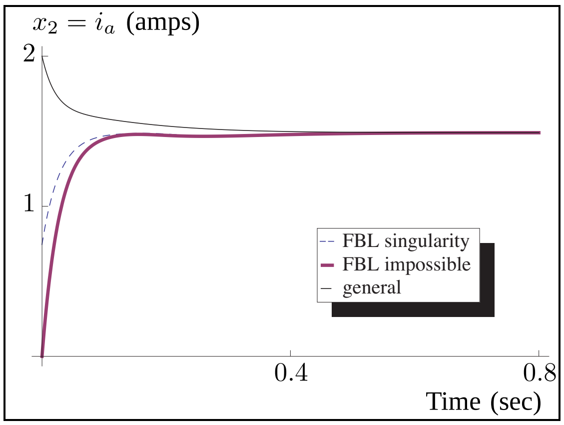
$$\text{and} \quad x_1 = 0, \quad x_2 = \rho/b \quad x_3 = \sqrt{\omega_0^2}.$$

Hence, the closed-loop system will converge to $(x_1,\, x_2,\, x_3) = (0,\, \rho/b,\, \pm\omega_0)$. Since $x_3 = 0$ is a singularity, the convergence to $\pm\omega_0$ depends on the initial condition. If $x_3|_{t=0} > 0$, then x_3 converges to $+\omega_0$, else if $x_3|_{t=0} < 0$, x_3 converges to $-\omega_0$. The domain of attraction of the control law designed using the quotient method is $\mathcal{D}_{QM}$ defined in (13). Furthermore, based on (6), it is clear that $\mathcal{D}_{FBL} \subset \mathcal{D}_{QM}$, and thus $\mathcal{D}_{QM}$ is larger than $\mathcal{D}_{FBL}$. Hence, using the quotient algorithm, we can initialise the system also from the points $x_2|_{t=0} \leq \frac{\rho}{2b}$ in addition to the points in $\mathcal{D}_{FBL}$.

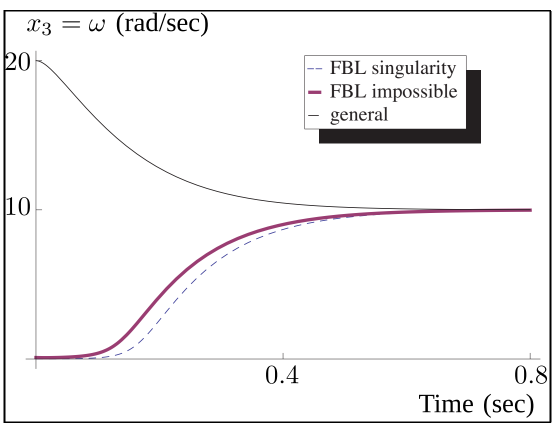
This fact is clearly seen in the simulation results presented in Figure 1. The target in these simulations is to achieve $\omega_0 = 10$. The simulations are carried out using the DC-motor parameters taken from [13] and given in Table I. Three initial conditions are chosen to simulate the control law obtained using the quotient method. The first $(0,\, \rho/(2b),\, 0.01$; *FBL singularity*) and the second $(0,\, 0,\, 0.1$; *FBL impossible*) initial conditions are outside the domain of attraction of any control law designed using feedback linearization [11] due to the presence of singularity at $x_2|_{t=0} = \rho/2b$. Only the last initial condition $(0,\, 2,\, 20$; *general*) lies in $\mathcal{D}_{FBL}$. The control law designed using feedback linearization works only for $x_2 > \rho/2b$, whereas the control law designed using the



(a) x_1



(b) x_2



(c) x_3

Fig. 1. DC motor controlled with the quotient method. The state behavior using the quotient method is depicted for three different initial conditions. The thin line represents a general case. The thick line corresponds to an initial condition that is outside $\mathcal{D}_{FBL}$. The dashed line corresponds to a case that is exactly the point of singularity for feedback linearization.

quotient method does not have this restriction. Hence, upon using the quotient method, a larger domain of attraction is achieved.

VI. Conclusion

This paper has illustrated the application of the quotient method to control a field-controlled DC motor. For this system, singularity arises in feedback linearization due to the nature of the diffeomorphism used to obtain the Brunovský normal form. The quotient method relaxes this condition by not requiring the Brunovský normal form. The advantage stems from the fact that there is an additional degree of freedom (in particular the choice of the last function in the definition of the diffeomorphism) at every step of the forward decomposition stage. This choice plays a crucial role in determining the singularity of the resulting control law. Different choices result in different domains of attraction for the resulting control law. For a particular choice, a Lyapunov-based proof of the domain of attraction has been provided. By removing singularities, the domain of attraction of the control law designed through the quotient method is shown to be larger than the domain of attraction of the control law designed through feedback linearization. To substantiate the results, simulations are provided with initial conditions that are outside the domain of attraction of any controller designed using feedback linearization.

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