Revisiting the Limit Behaviour of "El Botellón"

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***Abstract*—Emergent phenomena occur due to the pattern of non-linear and distributed local interactions between the elements of a system over time. An example of such phenomena is the spontaneous self-organisation of drinking parties in the squares of cities in Spain, also known as "El Botellón" [1]. The emergence of self-organisation in [1] was shown to depend critically on the chat-probability, i.e. the probability that a person finds someone to chat with in a square of the city. We consider a variant of "El Botellón" in which this probability is instead defined based on the socialisation level. For this variant it is possible to derive the mean field limit and perform a stability analysis of the related ODE. We also provide a process algebraic model of "El Botellón" and show that the phase plots of the ODE derived from the latter correspond very well to the mean field limit even for finite though relatively large populations.**

***Keywords*-Fluid flow; process algebra; crowd dynamics; self-organisation**

I. Introduction

This paper revisits the case of self-organisation of crowds in a city as described by Rowe and Gomez in [1]. Their work was inspired by a typical social phenomenon observed in Spanish cities, on summer nights, called "El Botellón", when crowds of youngsters wander between city squares in search of a party. Such self-organising parties sometimes lead to heavy drinking and noisy behaviour until late at night. Predicting when and where a large party would take place turned out to be suprisingly hard. The aim of the work by Rowe and Gomez was to gain insight into the general principles due to which parties self-organise, abstracting from specific details of individual cases. To that purpose they studied an agent based model of crowd movement under various assumptions about the likelihood that people remain in a square. In their work, agents follow two basic rules. The first rule defines when agents remain in a square, which depends on the "chat-probability", i.e. the likelihood to meet someone in the square to chat with. The second rule defines how agents move between squares. By developing an analytical model, they determined a threshold of the chat-probability below which people are freely moving through the city and above which large crowds start to form. Their theory has been validated by a simulation of a multi-agent model for a ring topology of 4 squares and up to 80 agents. The value of the threshold is n/N, where n is the number of squares and N the number of agents.

An alternative way to model and study the same crowd behaviour was proposed in [2]. In that work a fluid flow approximation, i.e. a deterministic reading of the average behaviour of the system, is studied starting from a formal process algebraic model in the Bio-PEPA language [3]. Models in Bio-PEPA are amenable to several forms of analyses, among which simulation and fluid flow analysis, which facilitate the comparison of the analysis results. In [2] a very good correspondence was found between the simulation results and a computationally much more viable fluid flow approximation. In particular, it was shown that the fluid approximation can be used efficiently to explore at which critical value of the chat-factor c the phase shift occurs between a uniformly distributed dynamic population to one in which the population gathers in a single square. Moreover, an analytical justification was provided for this correspondence result. Unfortunately, the limit behaviour of that model, a model in which N tends to infinity, is dependent on N due to the fact that the chat probability is kept constant. In that case the model has vanishing drift, which makes it not easy to exploit.

An alternative model can be obtained considering the chat probability as the fraction of the socialisation level (i.e. the average number of friends people have) w.r.t. the total population walking around in the city. This assumption is justified by the fact that in general people have a fixed number of friends and meeting them (and thus remaining in a square to chat with them) gets more and more unlikely when the total population that is walking around in the city increases. This is the scenario addressed in the present paper. It is shown that this model has in fact an interesting mean field limit that leads to different sets of stationary points with different stability properties when considering the socialisation level as a variable of the model. The stability analysis of the limit behaviour of this model is addressed and compared with the analysis results of a Bio-PEPA model of the same scenario for a fixed and finite population. The study provides a contribution towards addressing the question whether a fluid flow approximation, such as that underlying the process algebra Bio-PEPA, can provide an alternative, and computationally more viable, way to study emergent, non-linear, crowd behaviour such as that explored in [1] where simulation was used instead.

The outline of the paper is as follows. Section II recalls the crowd model by Rowe and Gomez [1]. Section III introduces an alternative crowd model based on the population's socialisation level. Section IV introduces a Bio-PEPA version of the crowd model for a city with three squares and Section V presents a fluid flow analysis for a chosen population level. Section VI presents a mean field analysis of the crowd model followed by a stability analysis. In Section VII the results of the mean field analysis are compared with the fluid flow results obtained from the ODE underlying the Bio-PEPA model.

II. Rowe and Gomez' Model of Crowd Dynamics

In this section we briefly recall the model of movement of crowds between squares in a city as presented by Rowe and Gomez in [1]. Assume a city with a general topology, with a finite set of squares $i \in \{1,..,n\}$. Let Q be the $n \times n$ routing matrix, i.e. $Q_{i,j}$ is the probability that a person moves to square j, given that she decided to leave square i. We assume Q is symmetric, i.e. $Q_{i,j} = Q_{j,i}$, so Q is a stochastic and symmetric matrix. It is further assumed that Q is irreducible (this is not a limitation since otherwise the city can be split into its connected components.)

Let us further assume that people's behaviour is modelled by "agents" that are following a simple set of rules. Agents are located in squares. The number of agents in square i at time t, with t representing discrete time steps, is represented by $p_i(t)$. The state of the system at t is given by the vector $\mathbf{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t))$. The total number of agents N is constant, so at any time $t > 0$ we have: $N = \sum_{i=1}^{n} p_i(t)$.

At every step each agent tries to find a partner to chat with. If this succeeds it stays where it is; else, it moves to some other square. It is assumed that the probability of the latter is $(1-c)^{p_i-1}$ when this agent is at square i, and $p_i > 0$ is the number of agents currently at square i. The parameter c (representing the *chat probability*, $0 \le c \le 1$) is the probability that an agent finds a partner to talk to and thus remains in the square. Note that when there is only one agent in the square, it decides to leave with probability 1, since there is nobody else to talk to.

Let $\mathbf{M}^N = (M_1^N(t), ..., M_n^N(t))$ be the occupancy measure of sites, assuming the population is constant. This is a discrete time Markov chain with drift given by

$$\mathbb{E}\left(M_i^N(t+1) - M_i^N(t) \,\middle|\, \mathbf{M}^N(t) = \mathbf{m}\right) = -f_i^N(t) + \sum_j f_j^N(t) Q_{j,i} \quad (1)$$

where $f_i^N(t) = m_i(1-c)^{Nm_i-1}$. Let $\mathbf{f}^N(t) = (f_1^N(t), \ldots, f_n^N(t))$. The expected distribution of agents over squares at $t+1$ can be defined as:

$$\mathbf{m}^N(t+1) = \mathbf{m}^N(t) - \mathbf{f}^N(t) + \mathbf{f}^N(t)Q \quad (2)$$

From equation (2) it follows that a stationary point is reached when $\mathbf{f}^N(t) = \mathbf{f}^N(t)Q$. In other words, when the number of people entering a square is equal to the number leaving the square. Rowe and Gomez show that there are two possibilities for such a stationary point. In one case the agents freely move between squares and their distribution is proportional to the number of streets connected to each square. In the second case agents gather in large groups in a small number of squares corresponding to emergent self-organisation of parties. Which of the two situations will occur depends critically on the value of the chat probability c. When all squares have the same number of neighbouring squares a phase shift occurs at about $c = n/N$ where n is the number of squares and N the number of agents. For $c < n/N$ people freely move between squares whereas for $c > n/N$ agents self-organise into large groups. Stochastic simulation of the model confirms in an empirical way that this estimate for the critical value $c = n/N$ is quite accurate when the population is large enough where large means about 60 agents or more in a 4-square topology.

The probability $(1-c)^{p_i-1}$ can also be interpreted as a *rate* leading to a CTMC model. In this case, each agent will move asynchronously. The rate at which an agent in square i will leave that square is $p_i(1-c)^{p_i-1}$, which, expressed in terms of occupancy measure becomes $m_i(1-c)^{Nm_i-1}$. The drift of the normalised model with N agents in this case is dependent on N. For example, the drift of the CTMC model for the occupancy measure m_1 for square 1 assuming $n=3$ is:

$$-m_1(1-c)^{N\cdot m_1-1}+0.5m_2(1-c)^{N\cdot m_2-1}$$
$$+0.5m_3(1-c)^{N\cdot m_3-1} \quad (3)$$

For fixed c, letting the state space E be the unit simplex in $\Delta_3 = \{x \in \mathbb{R}^3 : x_i \geq 0 \text{ and } \sum_i x_i = 1\}$, we have that

$$\lim_{N\to\infty} \sup_{m_1,m_2,m_3\in E} ||F_{RG^{(N)}}(m_1,m_2,m_3)|| = \mathbf{0}$$

so that the drift $F_{RG^{(N)}}$ of the Rowe and Gomez model with N agents converges uniformly to the constant function yielding 0 for each point in E. Hence, the fluid limit $\mathbf{x}(t)$ of the sequence is the not very informative constant function $\mathbf{x}(t)=\mathbf{x}_0$ where $\mathbf{x}_0$ is the initial value.

Fortunately, as shown in [2], there is a reasonable agreement between the ODE and CTMCs for sufficiently large N. Indeed, fluid flow approximation results $\mathbf{x}^{(N)}(t)$, i.e. the ODE solution for population level N, can still be used in this context for two complementary reasons: 1) $\mathbf{x}^{(N)}(t)$ is, in any case, an approximation of the average of the stochastic process (see [2]); 2) deterministic approximation theorems prove the convergence essentially by showing that $\lim_{N\to\infty}\mathbf{X}^{(N)}(t)=\mathbf{x}^{(N)}(t)$ and that $\lim_{N\to\infty}\mathbf{x}^{(N)}(t)=\mathbf{x}(t)$ where $\mathbf{X}^{(N)}$ denotes the CTMC model for population size N and $\mathbf{x}(t)$ the solution of the limit ODE (this is a consequence of the scaling hypothesis that the exit rate is bounded for all N ; hence one can always approximate the CTMC with the solution of the ODE of level N, if N is sufficiently large.)

The solutions $\mathbf{x}^{(N)}(t)$ for different population levels N form a semantic interpretation of the stochastic process algebra Bio-PEPA. In [2] this process algebra has been used to specify, in a compositional way, the crowd model by Rowe and Gomez.

III. A Socialisation Level Based Crowd Dynamics Model

The probability to meet a friend in a crowded city is in general not the same as the probability to find a friend when it is less crowded. People tend to have a fixed number of friends given a city population, and the larger the number of people walking around, the more of them will turn out not to be one of your friends. This consideration leads to an alternative crowd model in which the chat probability is defined as $c=s/N$, where s is the level of socialisation of the population, i.e. the average number of friends that people have. Using this alternative definition of c the ODE for population level N of this new model is:

$$\frac{dx_i}{dt} = -x_i(1-s/N)^{Nx_i-1} + \sum_j x_j(1-s/N)^{Nx_j-1}Q_{j,i} \quad (4)$$

The above ODE forms also the fluid flow interpretation of a Bio-PEPA model of this new scenario. The Bio-PEPA model and its analysis are shown in the following sections. Note, however, that this model has also an interesting fluid limit. For N going to ∞ the right hand side of equation (4) becomes:

$$-x_ie^{-sx_i}+\sum_j x_je^{-sx_j}Q_{j,i} \quad (5)$$

It follows from this and for example from [4] that $\mathbf{M}^N$ has a non-trivial fluid limit $\mathbf{x}(t)$, given as the solution of the ODE

$$\frac{dx_i}{dt} = -x_ie^{-sx_i}+\sum_j x_je^{-sx_j}Q_{j,i} \quad (6)$$

This is a non linear ODE, with solutions in the unit simplex $\Delta_n = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ and } \sum_i x_i = 1\}$ if the initial condition is in Δ_n. This provides an interesting occasion to compare the approximation results obtained with the Bio-PEPA model with those derived analytically. We will show that the approximation is actually rather good even for relatively small population sizes. This insight could be very helpful in the analysis of non-linear large-scale interaction systems with a much less regular structure and for which it would be extremely hard to find tractable closed-form expressions for the fluid limit or when the fluid limit is non-informative.

IV. Bio-PEPA Crowd Model

Bio-PEPA [3] is a process algebraic language that has recently been developed for the modelling and analysis of biochemical systems. We use this language to present the crowd model and explain the relevant language constructs on the fly.

Let us consider a small ring topology with 3 city squares, A, B and C, allowing bi-directional movement between squares. In Bio-PEPA the squares are modelled as locations called sqA, sqB and sqC.

The next step is the definition of the model parameters. Parameter c defines the chat-probability and the parameter d the degree or number of streets connected to a square. In the considered topology $d = 2$ for each square. The chat-probability is defined as the fraction of the socialisation factor s w.r.t. the total population N, i.e. $c = s/N$.

The actions modelling agents moving *f*rom square X *t*o square Y are denoted by $fXtY$. The associated functional rate for $fAtB$ with $P@sqA$ denoting the population in square A at time t is defined as:

$$fAtB \quad = \quad (P@sqA * (1-c)^{(P@sqA-1)})/d;$$

the other rates are defined similarly.

The behaviour of a typical agent moving between squares is modelled by sequential component P. For example, $fAtB[sqA \to sqB] \odot P$ models that an agent present in square A moves to square B according to the functional rate defined for the action $fAtB$.

$$\begin{array}{l} P = \\ fAtB[sqA \to sqB] \odot P + fBtA[sqB \to sqA] \odot P + \\ fAtC[sqA \to sqC] \odot P + fCtA[sqC \to sqA] \odot P + \\ fBtC[sqB \to sqC] \odot P + fCtB[sqC \to sqB] \odot P; \end{array}$$

The operator "+" expresses the choice between possible actions. The notation $\alpha[I \to J] \odot S$ is a shorthand for the pair $(\alpha, 1)\!\downarrow\! S@I$ and $(\alpha, 1)\!\uparrow\! S@J$ that synchronise on action α. The symbol $\downarrow$ indicates a *reactant* which will be consumed in the action, $\uparrow$ a *product* which is produced as a result of the action. The prefix term (α, κ) $\mathtt{op}$ $S@l$ is used to specify that α is performed by S in location l. The value κ captures the multiples of an entity involved in an occurring action.

Finally, the model component defines the initial conditions of the model, i.e. in which squares the agents are located initially, and the relative synchronisation pattern. If, initially, there are 100 agents in square A this is expressed by $P@sqA[100]$. The fact that moving agents need to synchronise follows from the definition of the shorthand operator $\to$.

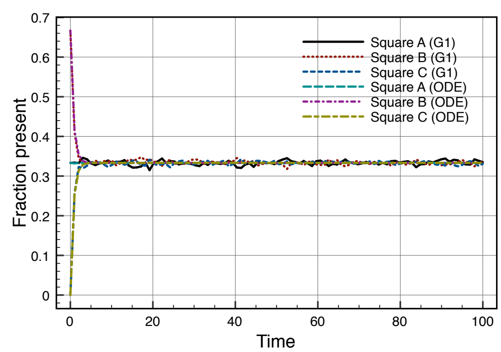
$$(P@sqA[100] \underset{*}{\bowtie} P@sqB[0]) \underset{*}{\bowtie} (P@sqC[0])$$

The total number of agents $P@sqA + P@sqB + P@sqC$ is invariant and amounts to 100 in this specific case. The occupancy measure of the population in square A can be defined as $PsqA = P@sqA/N$ and similarly for the other squares.

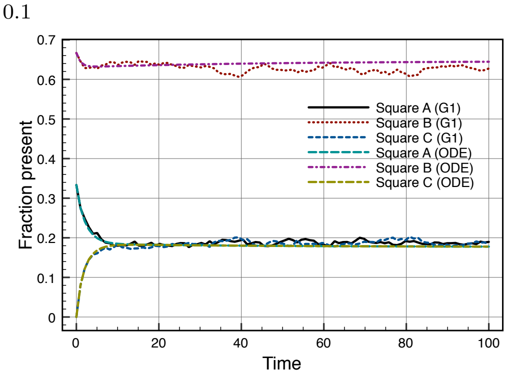
V. Bio-PEPA Fluid Approximation

This section presents a selection of the analysis results for the Bio-PEPA model of the previous section for $N = 9000$[1]. The figures below report both analysis via Gillespie stochastic simulation [6], and fluid flow analysis based on the adaptive step-size 5th order Dormand-Prince ODE solver [7][2].

Fig 1(a) shows a single simulation trajectory (G1) and a fluid flow approximation (ODE) of the fraction of agents in each square for $s = 0.1$ with initially $1/3$ of the population in square A, $2/3$ in square B and none in square C. Fig. 1(b) shows similar analysis for $s = 2.76$. Note that the



(a) Stochastic Simulation and fluid flow trajectory for $s = 0.1$



(b) Stochastic Simulation and fluid flow trajectory for $s = 2.76$

Fig. 1. Results for $n = 3$, $N = 9000$, $A = \frac{1}{3}$, $B = \frac{2}{3}$ and $C = 0$ initially.

simulation results and the fluid approximation in

[1]This number is just arbitrary; fluid approximation can be applied for any desired large number of agents since the performance of this technique is insensitive to the number of copies of agents involved. For a discussion see for example [5].

[2]All analyses have been performed with the Bio-PEPA Eclipse Plug-in tool [8] on a Macintosh PowerPC G5.

Fig. 1 (a) and (b) show very good correspondence for this non-linear model. This correspondence becomes even better when the average of multiple stochastic simulation runs is considered instead of single runs.

A phase plot of the system for different values of s gives a more complete overview. In Fig. 2(a) the ODE trajectories of the occupancy measures of square A (x_1-axes) w.r.t. square B (x_2-axes) are shown for a grid of different initial values of the occupancy measure in these squares and for $s = 0.1$ and $N = 9000$. The trajectories are the solution of the following ODE (for $i \in \{1, 2, 3\}$):

$$\frac{dx_i}{dt} = -x_i(1 - s/N)^{Nx_i - 1} + \sum_j x_j(1 - s/N)^{Nx_j - 1} Q_{j,i} \tag{7}$$

where $x_1 = PsqA$, $x_2 = PsqB$ and $x_3 = PsqC$. This is the same ODE as that obtained via the Bio-PEPA semantics for this crowd model. The phase plots show a number of trajectories of the ODE for a particular value of s with initial values of the population fractions indicated by a small circle. The phase plot in Fig. 2(a) shows clearly that all trajectories lead to a single point in which $x_1 = \frac{1}{3}$ and $x_2 = \frac{1}{3}$ and consequently, since the population is fixed, $x_3 = \frac{1}{3}$. It is easy to observe that for this value of s the number of people in the three squares is equal leading to a dynamic, stationary situation, independently from the initial distribution. The phase plot for $s = 5.0$ in Fig. 2(b) shows a completely different scenario. In this case the area of possible initial values for x_1 and x_2 turns out to be divided into three different sub-areas, so called basins of attraction. The population in this case will gather in one of the three squares after some time and the particular square in which they will gather depends on the initial population distribution[3].

In the next section an analytical stability analysis is provided of the *limit* ODE of the model, i.e. for N to infinity. The results are compared with those obtained with the ODE of Eq.(7) underlying the Bio-PEPA model.

VI. Stability analysis of the limit ODE

The ODE in Eq.(6) in Section III is non-linear. It is possible to establish its set of stationary points and study their nature in terms of local stability. To be more precise, we will study a family of differential equations considering the socialisation factor s as a parameter of the ODE. We will first address the computation of stationary points and the phase space of the limit ODE. We then analyse the nature of the stationary points and show how they change depending on the value of s presenting a bifurcation diagram. Finally, we will compare the results of the limit ODE with $\mathbf{x}^N(t)$, the solution of Eq.(7), i.e. the ODE for a particular size N of the population.

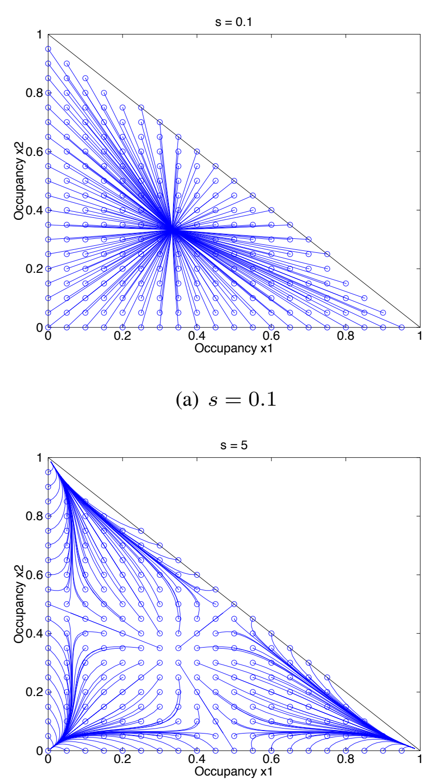
A. Stationary Points

Let $\mathcal{F}(s)$ be the set of stationary points of the ODE that are in the simplex Δ_n for socialisation factor s. Such points are defined by

$$-\mathbf{z} + \mathbf{z}Q = 0 \tag{8}$$

with $z_i \stackrel{\text{def}}{=} x_i e^{-sx_i}$ and $\mathbf{z} = (z_1, ..., z_n)$. Since Q is irreducible and stochastic, this implies that $\mathbf{z}$ is proportional to $(1, ..., 1)$, i.e. there exists some $\lambda \geq 0$ such that $z_i = \lambda$ for all i, which is equivalent to

$$x_i e^{-sx_i} = x_j e^{-sx_j} \quad \forall i, j \tag{9}$$



(a) $s = 0.1$

(b) $s = 5.0$

Fig. 2. Phase plots Bio-PEPA model

[3]The phase plots have been obtained via the Bio-PEPA plugin export of the model to SBML and from there translated into Octave code via the SBFC converter package [9].

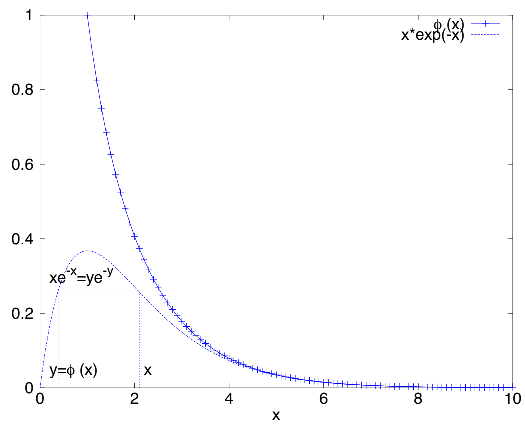


Fig. 3. Definition and graph of function φ.

Thus $\mathcal{F}(s)$ is the set of $\mathbf{x} \in \Delta_n$ that satisfy Eq.(9). It contains $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$. Let us now study $\mathcal{F}(s)$ in more detail. To this end, let us introduce the function φ, defined as follows. Consider first the function $\mathbb{R}^+ \to \mathbb{R}$, $x \mapsto xe^{-x}$; it is increasing on $[0,1]$ and decreasing on $[1,+\infty)$. Thus for every $x \geq 1$, there exists a unique $y \in (0,1]$ such that $xe^{-x} = ye^{-y}$. We call φ the function that maps x to y (see Figure 3). In other words, φ is defined on $[1,+\infty)$ by

$$\varphi(x) = y \iff x \geq 1, \;\; y \leq 1 \text{ and } xe^{-x} = ye^{-y}$$

Note that the function φ can also be expressed by means of Lambert's W_0 function using the identity

$$\varphi(x) = -W_0\left(-xe^{-x}\right) \quad \forall x \geq 1$$

It can easily be seen that $\lim_{x\to\infty} \varphi(x) = 0$, $\varphi(1) = 1$. Differentiating both sides of the equation $xe^{-x} = \varphi(x)e^{-\varphi(x)}$ we obtain that $\varphi'(x) = ((1-x)e^{-x})/((1-\varphi(x))e^{-\varphi(x)})$. We observe that $\varphi(x)$ is decreasing for $x \geq 1$ and $\lim_{x\to 1^+} \varphi'(1) = -1$. for large x we use the approximation $\varphi(x) \approx xe^{-x}$, which can be derived from the value of the derivative in 0 of W_0.

With elementary but lengthy calculus, we can show that an element $\mathbf{x}$ of $\mathcal{F}(s)$ must satisfy

$$\begin{aligned} x_i &= \frac{\alpha}{s} \text{ for } i \notin \mathcal{K} && (10)\\ x_i &= \frac{1}{s}\varphi(\alpha) \text{ for } i \in \mathcal{K} && (11)\\ 1 \leq & \;\; \alpha, \;\; \alpha \in \mathbb{R} && (12)\\ \mathcal{K} &\subset \{1, ..., n\} && (13)\\ K &= \mathrm{card}(\mathcal{K}) && (14)\\ s &= (n-K)\alpha + K\varphi(\alpha) && (15) \end{aligned}$$

We are thus interested in solving Eq.(15) for α, where s is given. More precisely, with more work we can say:

Theorem 6.1: $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ is a stationary point of the ODE. Any other stationary point, if any exists, is obtained by solving the following problem:

- Find an integer $K \in \{1, ..., n-1\}$ and a real number $\alpha \geq 1$ that solve Eq.(15);

For each solution (K, α), and for any set of sites $\mathcal{K}$ of cardinality K, there is a stationary point defined by $x_i = \frac{\alpha}{s}$ for all $i \in \mathcal{K}$ and $x_i = \frac{1}{s}\varphi(\alpha)$ for $i \notin \mathcal{K}$.

Note that the function $[1,\infty) \to \mathbb{R}$, $\alpha \mapsto (n-K)\alpha + K\varphi(\alpha)$ always attains its infimum. Let

$$s^*(n,K) \overset{\text{def}}{=} \min_{\alpha\geq 1} (n-K)\alpha + K\varphi(\alpha)$$

Corollary 6.1: If $s < \min_{K=1...n-1} s^*(n,K)$, there is no stationary point other than $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$; else there are several other stationary points.

B. Example: model with three squares

In this section (and in the following ones), we focus on the model with $n = 3$ sites and with a routing matrix sending a person with equal probability to other sites:

$$Q = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

We plot the function $\alpha \mapsto (3-K)\alpha + K\varphi(\alpha)$ (right handside of Eq.(15)) in Figure 4 for $K = 1$ and 2 and we see that $s^*(3,1) = 3$ and that the minimum of $(3-K)\alpha + K\varphi(\alpha)$ is attained for $\alpha \approx 1.6055$ and $s^*(3,2) \approx 2.7456$.

Some of the solutions of equation Eq.(15) lead to stationary points of the form (v,v,v) with $v = \frac{1}{n} = \frac{1}{3}$, i.e. a stationary point with a symmetric distribution of people among the three squares. For this reason we call such a solution of equation Eq.(15) 'symmetric'. Other solutions bring to stationary points of the form (x_1, x_2, x_3) such that there exists $i, j = 1\ldots 3, i \neq j$ with $x_i = x_j$ and $x_1 + x_2 + x_3 = 1$, with $x_i \in [0,1]$, e.g. $(v, v, 1-2v)$ for appropriate v. We call any such a solution of equation Eq.(15) 'asymmetric', since they lead to stationary points with an asymmetric distribution of people in the squares. Below we provide a systemic analysis of Eq.(15) based on the value K.

Case $K = 0$: Eq.(15) reduces to $3 \cdot \alpha = s$. Knowing that $\alpha \geq 1$ we can see that solutions only exist when $s \geq 3$. In that case there is

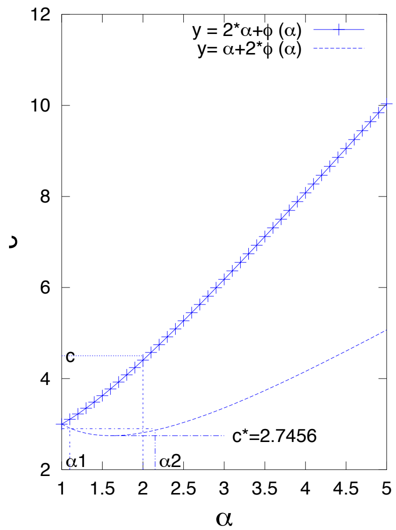


Fig. 4. The right-handsides of Eq.(15) ($n = 3$ and $K = 1, 2$).

always exactly one solution, which is symmetric. The associated stationary point is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Case $K = 1$: Inspecting function $y = 2\alpha + \varphi(\alpha)$ (see Fig. 4) we have that for $s \geq 3$ there is one asymmetric solution for α. This leads to 3 similar stationary points in which always two squares have the same occupancy measure. For $0 \leq s < 3$ there are no solutions. In the special case in which $s = 3$ the three stationary points collapse into the single point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Case $K = 2$: Inspecting function $y = \alpha + 2\varphi(\alpha)$ (see Fig. 4) we have that for $s < 2.7456$ there is no solution. For $s = 2.7456$ there is exactly one asymmetric solution leading to 3 stationary points. For $2.7456 < s \leq 3$ there are 2 asymmetric solutions for α leading to two groups of 3 similar stationary points. Note that for the special case $s = 3.0$ one of the two asymmetric solutions collapse into the single point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For $s > 3$ there is 1 asymmetric solution leading to 3 similar stationary points.

Case $K = 3$: Eq.(15) reduces to $\varphi(\alpha) = \frac{s}{3}$. We know that $\varphi(\alpha) \in (0, 1]$, so for $0 < s \leq 3$ there exists one symmetric solution, leading to stationary point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and for $s > 3$ there is no solution. Note that the solution for $s = 3.0$ coincides with the solution for Case $K = 0$.

C. Stability analysis of stationary points

In order to study the stability of the stationary points, we will investigate the sign of the real part of the eigenvalues of the Jacobian matrix, evaluated on such stationary points. We recall that, if the real part of all eigenvalues is negative, then the stationary point is asymptotically stable, while if at least one eigenvalue has a positive real part, then the point is unstable. Notice that in the other cases (i.e. in presence of eigenvalues with real part equal to zero), this approach is not informative (see e.g. [10]).

In order to simplify the following calculations, we introduce the matrix $R = (r_{ij})_{i,j \leq n}$, with $r_{ij} = Q_{ij}$ for $i \neq j$ and $r_{ii} = -1$, so that $R = Q - I$. Recall that the vector $\mathbf{z} = \mathbf{z}(x)$ is defined by $z_i = x_i e^{-sx_i}$. We can write the limit system of ODE of Eq.(6) now as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{z}R = \mathbf{F}(\mathbf{x})$$

Furthermore, we let $\frac{\partial z_i}{\partial x_i} = (1 - sx_i)e^{-sx_i} = h_i$. Clearly, $\frac{\partial z_i}{\partial x_j} = 0$ for $i \neq j$. Since $\sum_{i \leq n} x_i = 1$ we can reduce the dimension of the system by one letting $x_n = 1 - \sum_{j<n} x_i$. Let $\mathbf{x} = (x_1, \dots, x_{n-1}, 1 - \sum_{j<n} x_i)$ and $\hat{\mathbf{x}} = (x_1, \dots, x_{n-1})$ and let $\hat{\mathbf{F}}(\hat{\mathbf{x}}) = (F_1(\mathbf{x}), \dots, F_{n-1}(\mathbf{x}))$ the so obtained reduced vector field, defined in the new state space $\hat{E} = \{\hat{\mathbf{x}} \in \mathbb{R}^{n-1} \mid \sum_{j<n} \hat{x}_j \leq 1\}$.

Computing the Jacobian J of $\mathbf{F}$ and $\hat{J}$ of $\hat{\mathbf{F}}$, we obtain for $i, j \leq n$:

$$J_{ij} = \frac{\partial F_i}{\partial x_j} = r_{ij} h_j$$

and, for $i, j < n$:

$$\hat{J}_{ij} = \frac{\partial \hat{F}_i}{\partial x_j} = r_{ij}\frac{\partial z_j}{\partial x_j} + r_{in}\frac{\partial \rho(z_n)}{\partial x_j} =$$
$$r_{ij}\frac{\partial z_j}{\partial x_j} - r_{in}\rho(\frac{\partial z_n}{\partial x_n}) = r_{ij}h_j - r_{in}\rho(h_n),$$

where $\rho(f)$ denotes a substitution operator that replaces occurrences of x_n in the formula f by $1 - \sum_{j<n} x_i$. So, $\rho(z_n) = z_n[x_n/(1 - \sum_{j<n} x_j)]$ and $\rho(h_n) = h_n[x_n/(1 - \sum_{j<n} x_j)]$ are equal to z_n and h_n after replacing variable x_n by $1 - \sum_{j<n} x_i$. We will use indifferently $\rho(h_n)$ and h_n in the following, whenever this does not cause confusion.

First, we notice that the non-reduced system has always a zero eigenvalue, as $J_{ni} = -\sum_{j<n} J_{ji}$, and furthermore that any non-null eigenvalue of the Jacobian of the full system is also an eigenvalue of the reduced system. In fact, let $\lambda \neq 0$ be an eigenvalue of J and $\mathbf{v}$ an eigenvector for λ. Then $\lambda v_n = (J\mathbf{v})_n = -\sum_{j<n}(J\mathbf{v})_j = -\lambda \sum_{j<n} v_j$, so that $v_n = -\sum_{j<n} v_j$.

Now, call $\hat{\mathbf{v}}$ the *n-1*-dimensional vector obtained by removing v_n from $\mathbf{v}$. It holds for $i < n$ that

$$(J\mathbf{v})_i = \sum_{j<n} r_{ij}h_j v_j + r_{in}h_n v_n = \sum_{j<n} r_{ij}h_j v_j + r_{in}h_n(-\sum_{j<n} v_j) = (\hat{J}\hat{\mathbf{v}})_i$$

which shows that any non-null eigenvalue of J is also an eigenvalue of $\hat{J}$. We are now ready to provide a characterization of the stability of the symmetric stationary point $\mathbf{x}_{sym} = (\frac{1}{n}, \ldots, \frac{1}{n})$. We can study its stability by computing the reduced Jacobian $\hat{J}$ in $\mathbf{x}_{sym}$.

Letting $\hat{R} = (r_{ij} - r_{in})_{i,j<n}$, because in $\mathbf{x}_{sym}$ we have that $h_i = h_j = h$ for any i, j, it holds that $\hat{J} = h\hat{R}$. Now, it is easy to check that the non-null eigenvalues of $\hat{R}$ coincide with the non-null eigenvalues of R. Furthermore, if λ is a non-null eigenvalue of R, then $h\lambda$ is an eigenvalue of $\hat{J}$. As $R = Q - I$, we can observe that if λ is an eigenvalue of Q, then $\lambda - 1$ is an eigenvalue of R. By the Perron-Frobenius and the Sylvester theorems, all eigenvalues of Q are real and less than or equal to one. It follows that all non-null eigenvalues of $\hat{R}$ are negative (so that $\mathbf{x}_{sym}$ is stable) if and only if $h > 0$, where $h = (1 - \frac{s}{n})e^{-s/n}$. Therefore, we have proved the following

Proposition 6.1: $\mathbf{x}_{sym} = (\frac{1}{n}, \ldots, \frac{1}{n})$ is stable for $s < n$, and unstable for $s > n$.

For what concerns the stability of $\mathbf{x}_{asym} \in \Delta_n$ for $n > 3$ things get quite a bit more complicated and we will deal with these only in the restricted case of $n = 3$ in the following example.

D. Example: stability of stationary points

We turn now to analyse in more detail the stationary points of the model with three squares. We know via Proposition 6.1 that the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a stationary point. Furthermore, following Theorem 6.1, we have additional stationary points, of the form $(\frac{\varphi(\alpha)}{s}, \frac{\varphi(\alpha)}{s}, \frac{\alpha}{s})$, for α a solution of $2\varphi(\alpha) + \alpha = s$, and $(\frac{\alpha}{s}, \frac{\alpha}{s}, \frac{\varphi(\alpha)}{s})$, for α solution of $\varphi(\alpha) + 2\alpha = s$ (and all their variants obtained by permutation). These correspond to the cases $K = 2$ and $K = 1$, respectively, mentioned in Section VI-B.

Using the notation of the previous sections, recalling that $h_i = (1-sx_i)e^{-sx_i}$, we can compute the Jacobian of the reduced model, which is:

$$\hat{J} = \begin{pmatrix} -h_1 - \frac{1}{2}h_3 & \frac{1}{2}h_2 - \frac{1}{2}h_3 \\ \frac{1}{2}h_1 - \frac{1}{2}h_3 & -h_2 - \frac{1}{2}h_3 \end{pmatrix}$$

We can then compute its two eigenvalues letting $det(\hat{J} - \lambda \cdot I) = 0$, where I denotes the 2 dimensional identity matrix, and solving for λ. The eigenvalues are

$$\lambda_{1,2} = -\frac{1}{2}(h_1 + h_2 + h_3) \pm \frac{1}{2}\sqrt{(h_1 + h_2 + h_3)^2 - 3(h_1h_2 + h_1h_3 + h_2h_3)}$$

Now, we can specialize this formula for the stationary points of the system. From Proposition 6.1, we know that the symmetric stationary point $x_{sym} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is stable for $s < 3$ and unstable for $s > 3$. Moreover, for $s = 3$ both eigenvalues are 0, so for $s = 3$ we have a phase shift.

Consider now an asymmetric stationary point. All these points have two coordinates equal among them and the other different. W.l.g, we can always assume that the first and the second coordinate are equal, hence we let $h_1 = h_2 = \beta$ and $h_3 = \gamma$. We obtain the following eigenvalues:

$$\lambda_1 = -\frac{1}{2}\beta - \gamma \qquad \lambda_2 = -\frac{3}{2}\beta$$

Assume now the equilibrium is of the form $(\frac{\varphi(\alpha)}{s}, \frac{\varphi(\alpha)}{s}, \frac{\alpha}{s})$, with $\beta = (1 - \varphi(\alpha))e^{-\varphi(\alpha)}$ and $\gamma = (1-\alpha)e^{-\alpha}$. As $\varphi(\alpha) \leq 1$, we have that $\beta > 0$, and so $\lambda_2 \leq 0$ in all such points. As for λ_1, we can observe that the condition $\lambda_1 < 0$, after some algebra, is equivalent to

$$\frac{(1-\alpha)e^{-\alpha}}{(1-\varphi(\alpha))e^{-\varphi(\alpha)}} > -\frac{1}{2}$$

Differentiating both sides of the equality $\alpha e^{-\alpha} = \varphi(\alpha)e^{-\varphi(\alpha)}$, we obtain that the left hand side of the previous condition is equal to $\varphi'(\alpha)$. Now, $\varphi'(\alpha)$ increases monotonically to zero, and it is equal to -1 for $\alpha = 1$. Furthermore, it equals $-\frac{1}{2}$ for $\alpha = 1.605654$, i.e. the point minimising $2\varphi(\alpha) + \alpha$, the equation defining the stationary points under examination. In particular, $\alpha = 1.605654$ corresponds to the value $s = 2.745644$, which is the minimum value of s for which stationary points of the form $(\frac{\varphi(\alpha)}{s}, \frac{\varphi(\alpha)}{s}, \frac{\alpha}{s})$ exist. Now, recall that the equation $2\varphi(\alpha)+\alpha = s$, has one solution for $s = 2.745644$, two asymmetric solutions for $2.745644 < s \leq 3$, and one asymmetric solution for $s > 3$ (see Case $K = 2$ in Section VI-B). For $2.745644 < s \leq 3$, the two solutions for α are one below and one above 1.605654 thus leading to $\lambda_1 > 0$ and $\lambda_1 < 0$, respectively. It follows that one stationary point is unstable (a saddle point) because one eigenvalue has a negative real part and the other a positive one. The other stationary point is stable, because both eigenvalues have a negative real part. In particular, the saddle node is the point closer to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For what concerns the situation for $s > 3$ the only solution for α is always above 1.605654 leading to negative eigenvalues and thus stable points.

As for $s = 2.745644$ the situation is more complicated because in that case λ_1 has a zero real part, which means that we are dealing with a degenerate case which would require further analysis.

Finally, we consider stationary points of the form $(\frac{\alpha}{s}, \frac{\alpha}{s}, \frac{\varphi(\alpha)}{s})$, that exist only for $s > 3$. In this case, $\beta \leq 0$, and so $\lambda_2 \geq 0$: the point is unstable. For completeness, observe that the condition $\lambda_1 < 0$ is equivalent to $\varphi'(\alpha) > -2$, which is always true, as $-1 \leq \varphi'(\alpha) < 0$. It follows that these stationary points are always saddle nodes.

We can summarise the previous discussion in the bifurcation diagram shown in Fig. 5. On the x-axes the value of s is shown. On the y-axes the occupancy measure of x_1 (which happens to be equal to that of x_2 for the stationary points considered). The diagram shows that for $s < 2.7456...$ there is indeed only one stable stationary point at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Interesting behaviour can be observed in the phase transition from a situation with this one single stable stationary point to situations with three locally stable stationary points (and several locally unstable ones). For socialisation factors of between 2.7456 and 3.0 a striking situation can be noted in which multiple distributions of the population can be stable in which the population is neither uniformly distributed, nor gathering mostly in one of the three squares, but always distributed in such a way that two squares have exactly the same occupancy measure. To have a better vi-

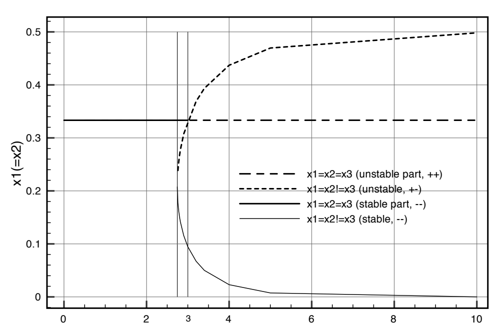
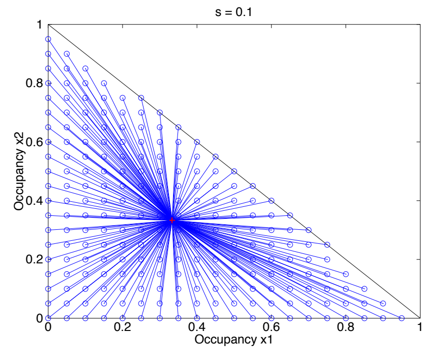
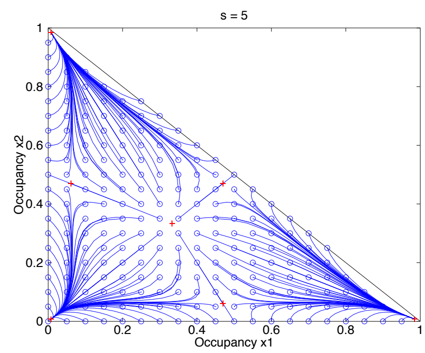


Fig. 5. Bifurcation diagram

sual understanding of the dynamics, we present in Figures 6 and 7 phase plots of the *limit* ODE system for $s = 0.1$ and $s = 5.0$ (Fig. 6) and $s = 3.0$ and $s = 2.7456$ (Fig. 7, see appendix).



(a) $s = 0.1$



(b) $s = 5.0$

Fig. 6. Phase plots of limit ODE

On the x-axes the occupancy measure of square 1 is shown whereas on the y-axes that of square 2 is shown. The third square is not shown because its population can be derived from that in the other squares. Stationary points are indicated by (red) crosses. It can be observed that in Fig 6(a) only one single stationary point at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is present where all trajectories are leading to. From the previous discussion we know that this point is locally stable and the phase plot confirms this. Fig. 6(b) shows 7 stationary points. For the value of $s = 5$ the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ has now become an unstable stationary point, furthermore there are three locally stable stationary points close to the three corners of the diagram, and three unstable stationary points. The three locally stable points reflect three different stable situations that can occur, namely that (almost) all people will eventually end up in one of the three squares. Which square this will be depends clearly on the initial distribution of the people over the squares.

VII. Comparison of the ODE for population level N and the limit ODE.

We have studied the stationary points of the limiting ODE, and we have also observed a close correspondence between the phase plots (obtained numerically) of the limit ODE (see Figs. 6 and 7) and the stationary behavior of the stochastic system when N is large (see Figs. 1 (a) and (b)). In particular we see that the stochastic systems tends to concentrate on one of the stationary points. We now give a theoretical result that shows that this is expected in theory.

First note that the stochastic model, where N is finite, is a continuous Markov chain on a finite state space; the transition is fully connected, therefore, by standard Markov chain theory, it has a unique stationary probability, say ϖ^N, which is also its unique invariant probability and is the limiting probability distribution for any initial condition. The question is what becomes ϖ^N when N is large. We do not have a complete answer to this question, but we can give some partial answer. It is natural to assume that the mass of ϖ^N must tend to be concentrated on the stationary points of the ODE, but this is a misconception [4], as the stationary behaviour of the ODE may not be concentrated on stationary points (there may be limit cycles or even chaotic behavior). In our case, though, we can show the following.

Theorem 7.1: Let ϖ^* be a probability distribution on the N-simplex that is a limit of some subsequence of ϖ^N when $N \to \infty$; the limit is for the usual Prokhorov metric. Then the support of ϖ^* is included in the set of stationary points of the limit ODE. Furthermore, if the limit ODE has a unique stationary point then ϖ^N converges to a unique distribution, which is the Dirac mass at the unique stationary point.

The proof of this result relies on the fact that the stochastic model is reversible, which itself is due to the fact that the topology matrix Q is symmetric [11]. We conjecture that the mass of any limiting probability ϖ^* must in fact be concentrated on the *stable* stationary points, but a proof of this is for further study. Note that if we focus on the ODE defined by the Bio-PEPA model at level N, we can observe that the symmetric point $x_{sym} = (\frac{1}{n}, .., \frac{1}{n})$ is also a stationary state, whose stability can be analysed as for the limit case. In particular, x_{sym} is stable for $s < N(1 - e^{-n/N})$, which returns the condition $s < n$ in the limit $N \to \infty$.

VIII. Conclusions

We have considered a variant of the crowd dynamics model by Rowe and Gomez [1] in which the chat-probability has been defined as the level of socialisation. For this variant we have analysed the mean field limit and provided a stability analysis. We have compared the results of the limit ODE with the ODE for population level N which forms the underlying semantic model of a Bio-PEPA specification of the crowd dynamics. The conjecture is that the ODE for a sufficiently high population level N provides a good approximation of the limit ODE. This conjecture has been partially proven and partially supported by numerical results. Our results show that the stationary behavior can be very different from what was obtained with previous models; in particular, we find that there may emerge non-symmetric stationary points in fully symmetric configurations.

Future work is developing along a few main directions. Although the simple topology addressed in this paper, chosen for reasons of validation of the approach, can be analysed analytically, more complex topologies and models, addressing issues such as the relative attractiveness of squares, may easily turn out to be too complex to be studied analytically. Fluid flow approximation with Bio-PEPA and the underlying ODE for population level N may be an alternative to obtain rather quickly an approximate insight in the dynamic behaviour of such more complicated crowd models.

Acknowledgments: This research has been partially funded by the EU-IP project ASCENS (nr. 257414).

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APPENDIX

A. Bio-PEPA and Fluid Flow Analysis

We briefly recall Bio-PEPA [3], a language that has recently been developed for the modelling and analysis of biochemical systems. The main components of a Bio-PEPA system are the *"species" components*, describing the behaviour of individual entities, and the *model component*, describing the interactions between the various species. The initial amounts of each type of entity or species are given in the model component.

The syntax of the Bio-PEPA components is defined as:

$$S ::= (\alpha, \kappa) \ \mathtt{op} \ S \mid S + S \mid C$$
$$\text{with } \mathtt{op} = \downarrow \mid \uparrow \mid \oplus \mid \ominus \mid \odot$$
$$\text{and } P ::= P \underset{\mathcal{L}}{\bowtie} P \mid S(x)$$

where S is a *species component* and P is a *model component*. In the prefix term $(\alpha, \kappa) \ \mathtt{op} \ S$, κ is the *stoichiometry coefficient* of species S in action α. This arises from the original formulation of the process algebra for modelling biochemical reactions, where the stoichiometric coefficient captures how many molecules of a species are required for a reaction. However it may be interpreted more generally as the multiples of an entity involved in an occurring action. The default value of κ is 1 in which case we simply write α instead of (α, κ). The *prefix combinator* "op" represents the role of S in the action, or conversely the impact that the action has on the species. Specifically, $\downarrow$ indicates a *reactant* which will be consumed in the action, $\uparrow$ a *product* which is produced as a result of the action, $\oplus$ an *activator*, $\ominus$ an *inhibitor* and $\odot$ a generic *modifier*, all of which play a role in an action without being produced or consumed and have a defined meaning in the biochemical context. The operator "+" expresses the choice between possible actions, and the constant C is defined by an equation $C{=}S$. The process $P \underset{\mathcal{L}}{\bowtie} Q$ denotes synchronisation between components P and Q, the set $\mathcal{L}$ determines those actions on which the components P and Q are forced to synchronise, with $\underset{*}{\bowtie}$ denoting a synchronisation on all common actions. In $S(x)$, the parameter $x \in I\!R$ represents the initial amount of the species.

A Bio-PEPA *system* with *locations* consists of a set of species components, also called sequential processes, a model component, and a context (locations, functional/kinetics rates, parameters, etc.). The prefix term $(\alpha, \kappa) \ \mathtt{op} \ S@l$ is used to specify that the action is performed by S in location l. The notation $\alpha[I \rightarrow J] \odot S$ is a shorthand for the pair of reactions $(\alpha, 1){\downarrow}S@I$ and $(\alpha, 1){\uparrow}S@J$ that synchronise on action α [4]. This shorthand is very convenient when modelling agents migrating from one location to another as we will see in the next section. Bio-PEPA is given an operational semantics [3] which is based on Continuous Time Markov Chains (CTMCs).

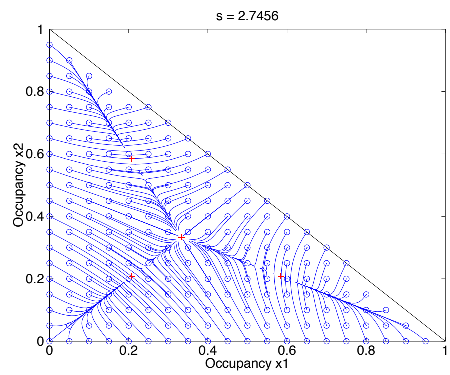
The Bio-PEPA language is supported by a suite of software tools which automatically process Bio-PEPA models and generate internal representations suitable for different types of analysis [3], [8]. These tools include mappings from Bio-PEPA to differential equations (supporting a fluid flow approximation), stochastic simulation models [6], CTMCs with levelsand PRISM models.

A Bio-PEPA model describes a number of sequential components each of which represents a number of entities in a distinct state. The result of an action is to increase the number of some entities and decrease the number of others. Thus the total state of the system at any time can be represented as a vector with entries capturing the counts of each species component (i.e. an aggregated CTMC). This gives rise to a discrete state system which undergoes discrete events. The idea of fluid flow analysis is to approximate these discrete jumps by continuous flows between the states of the system.

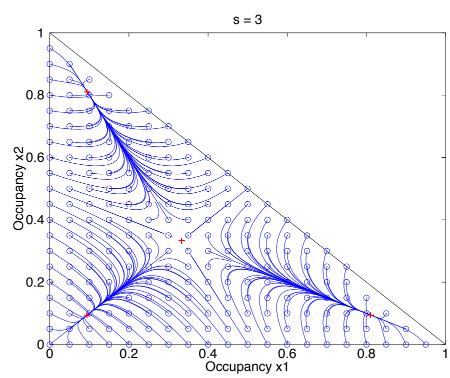
B. Additional phase plots of limit ODE

For $s = 2.7456$ the phase plot in Fig 7(a) predicts 4 stationary points. The one at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is stable, the other three are unstable. In fact, the stability analysis of these points indicates that they are saddle points: they have one positive and one negative eigenvalue.

In Fig. 7(b) another situation is presented. The point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is about to change from stable to unstable. The other three stationary points are locally stable.



(a) $s = 2.7456$



(b) $s = 3.0$

Fig. 7. Phase plots of limit ODE

[4]The concrete syntax for writing this in the Bio-PEPA tool set differs somewhat.