The Weighted Averages Algorithm Revisited

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***Abstract*—The classic weighted averages (WA) algorithm for the evaluation of Sommerfeld-like integrals is reviewed and reappraised. As a result, a new version of the WA algorithm, called generalized WA, is introduced. The new version can be considered as a generalization of the well established Hölder and Cèsaro means, used to sum divergent series. Generalized WA exhibits a more compact formulation, devoid of iterative and recursive steps, and a wider range of applications. It is more robust, as it provides a unique formulation, valid for monotonic and oscillating functions. The implementation of the new version is easier and more economical in terms of basic operations. Preliminary numerical examples show that generalized WA also outperforms in terms of accuracy the classic WA algorithm, which is currently recognized as the most competitive algorithm to evaluate Sommerfeld integral tails.**

***Index Terms*—Computational electromagnetics, convergence accelerators, extrapolation techniques, Green's functions, multilayered substrates, sommerfeld integrals, stratified media, weighted averages.**

I. Introduction

THE weighted averages (WA) algorithm was introduced within the microwave and antenna communities in the early eighties [1], [2], as a technique to evaluate the tail of the Sommerfeld integrals arising in the formulation of Green's functions for planar multilayered problems [3], [4]. The WA algorithm transforms these infinite integrals into an infinite sequence of partial finite integrals and acts upon this sequence as a convergence accelerator. Its original formulation was essentially heuristic and based on intuitive considerations. No attempt was made to connect it to the existing mathematical knowledge and its use was solely justified a posteriori by the excellent results it usually yielded.

Then, in 1998, a seminal paper by Michalski [5] provided a rigorous frame for the WA, classifying it in the family of "integration-then-summation" procedures and identifying it mathematically as an extension of the Euler transformation [6, p. 230]. The paper also stressed the fact, already mentioned in the original publications [1], [2], that WA could also be efficiently applied to improper integrals, exhibiting oscillating divergent integrands and not being defined in the usual Riemann's sense [6, p.7]. After an intensive and thorough comparison with other algorithms, Michalski concluded that *"the W-transformation and the weighted averages method emerge as the most versatile and efficient currently known convergence accelerators for Sommerfeld integral tails"*.

Although prior to 1998, WA had been already known in specialized circles and had been successfully used [7]–[9], it was the weight of Michalski's work that gave it a serious push and launched WA as the most popular technique for the numerical evaluation of Sommerfeld and related integrals. Nowadays, the main application of WA is the computation of Green's functions (or of Method-of-Moments matrices entries) in the integral equation formulation of multilayered problems. In this application, WA results in robust, efficient and easy to implement numerical quadrature algorithms. These algorithms are customarily included in existing software tools, either for direct use or as a benchmark to evaluate the accuracy of faster but approximate methods like the complex image formulations [10]–[15]. Today, the WA-based algorithm remains as popular as ever, as witnessed by several recently published papers [16]–[18].

It is also worth mentioning that WA has successfully spread and disseminated outside our microwave & antenna community, since printed multilayered antennas and circuits are not the only possible application of Sommerfeld integrals and stratified media theory. Indeed, WA is nowadays well known and used in domains like Optics [19], Plasma Physics [20], Geology [21], Geophysics [22], Ground Penetrating Radar (GPR, [23]) and Lightning and related EMC problems [24].

The WA algorithm got his respectability letters within the mathematical world in 2000, when H.H.H. Homeier dissociated it from the narrow realm of Sommerfeld integrals and included it, under the name "Mosig-Michalski algorithm", in his exhaustive study of scalar Levin-type sequence transformations [25]. From Homeier's vantage point, WA is seen as a pure algebraic convergence accelerator for series, able to transform a given sequence into a faster convergent one.

This paper revisits the weighted averages algorithm and offers some new insights into it. It is no longer possible to ignore the efforts accomplished in the last decades to provide a rigorous mathematical framework for the WA algorithm. However, a successful practical implementation can still be essentially based on the intuitive principles developed in the earlier papers. This contribution attempts to maintain this fresh approach, reducing the mathematical developments and justifying them, whenever possible, by physical arguments. A new, original version of WA is developed, based on a more general integral and hence exhibiting a wider range of applications. The resulting algorithm successfully compares with existing versions in terms of simplicity, computational speed and accuracy.

II. The Basic Integral

The WA procedure can be rigorously developed and explained in the mathematical context of discrete sequences,

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assimilating it to an integration-then-summation-procedure combined with partition-extrapolation strategies [5], [25].

However, the author believes that the basic steps and properties of a WA procedure are better explained within the frame of the evaluation of some generic improper integrals. This was also the original framework of the algorithm [1], [2] and this is the area where WA consistently produces its best results. Therefore, let us start by considering a generic complex integral:

$$I(\gamma) = \int_a^\infty f(x)\exp(-\gamma x)dx \tag{1}$$

where

— $[a,\infty]$ is a semi-infinite interval on the real axis x,

— the function f will be considered, for the sake of simplicity, as being real-valued and behaving asymptotically as a power function $O(x^q)$; complex functions can be dealt with by considering successively their real and imaginary parts.

— γ is a complex parameter $\gamma = \alpha + j\beta$ satisfying the conditions: $\alpha \geq 0$ and $|\gamma| \neq 0$

It is worth mentioning that in the strict sense this integral is not a Sommerfeld integral. The later could be defined, introducing a notation compatible with (1), as

$$g(\alpha,\beta) = \int_0^\infty J_n(\beta x)\tilde{g}(x)\exp(-\alpha\sqrt{x^2-k^2})x^{n+1}dx \tag{2}$$

However, it is obvious that a Sommerfeld integral can be included in the generic integral (1) by using the asymptotic expansion of Bessel functions and considering a function $\tilde{f}$ given by

$$\tilde{f}(x) \sim \frac{1}{\sqrt{x}}\tilde{g}(x)x^{n+1} \tag{3}$$

Therefore, without loss of generality, the WA algorithm will be constructed based on the generic integral (1) and for any value of the complex parameter γ, satisfying the condition $\mathrm{Re}[\gamma] = \alpha \geq 0$.

If $\alpha > 0$, then the integral I in (1) is defined in the traditional Riemann sense, because the integrand converges exponentially at infinity. However, if $\alpha = 0$ and hence $\gamma = j\beta$, there is no exponential decrease to guarantee convergence. Indeed, if the asymptotic behavior of the function f is given by a power $q \geq 0$, the integrand doesn't converge and the integral is not defined in the Riemann sense. In these cases, a physical meaning can still be assigned to the integral I defining it in the Abel sense [26, p.381] as

$$I(0+j\beta) = \lim_{\alpha\to 0} I(\alpha + j\beta) \tag{4}$$

The Abel definition of convergence for improper integrals is a concept derived from the definition of "Abel's summability" for infinite divergent series and has a fully rigorous mathematical interpretation [27, p.71]. However, any electrical engineer can understand "Abel's summability" in a very intuitive way, relating it to our understanding of electromagnetic phenomena arising in lossless media. A lossless medium is an ideal abstraction, exhibiting a pure imaginary propagation constant $\gamma = j\beta$. Its direct mathematical treatment involves frequently some difficulties, but the lossless situation can be always viewed as the limiting case when the losses vanish $(\alpha \to 0)$ of a physical lossy medium, with a complex propagation constant $\gamma = \alpha + j\beta$.

In stratified media, Sommerfeld integrals usually lose their standard Riemann interpretation when source and observer are at the same level with respect to the stratified medium interfaces. In these situations, the physical counterpart of the Abel definition is just to consider the improper Sommerfeld integral as a limiting situation when the observer's level tends towards the source's level.

Using Abel's definition, it is easy to obtain results for improper integrals like

$$\int_0^\infty x\sin\beta x dx = 0 \quad ; \quad \int_0^\infty x\cos\beta x dx = \frac{-1}{\beta^2} \tag{5a}$$

$$\int_0^\infty xJ_0(\beta x)dx = 0 \quad ; \quad \int_0^\infty xJ_1(\beta x)dx = \frac{1}{\beta^2} \tag{5b}$$

that have no meaning in the Riemann sense. These results can be also easily justified within the frame of distribution theory [28], essentially by using the generalized function result

$$\lim_{x\to\infty}\exp(j\beta x) = 0 \tag{6}$$

These integrals (5) constitute obvious benchmarks for the purposes of this paper. Any worthy numerical quadrature algorithm should be able to confirm the theoretical values of these improper integrals with a minimum number of evaluations of the integrand.

Obviously it is possible to directly discretize the Abel limiting process of (4). Within this strategy, a series of values of the integral $I_i = I(\alpha_i + j\beta)$ are numerically computed for some small values of α and an extrapolation is made to obtain the value of the integral for $\alpha = 0$. However, usually this approach is not very efficient. An extrapolation towards zero is only accurate when made from situations corresponding to very small values of α, but integrals with very small values of α are hard to evaluate numerically.

What is needed here is a numerical method than can deal directly with the case $\alpha = 0$ and obtain Abel's values as the result of a simple arithmetic process. This was the historic goal of the WA algorithm.

III. The Classic WA Algorithm

Before introducing the proposed new approach to WAs, it is worth to briefly recall the "classic" WA algorithm, as introduced in [1]–[3] and extensively developed in [5]. The basic idea is to consider "partial" or "finite" integrals I_n (where the infinite integration interval $[a,\infty]$ is replaced by $[a,x_n]$) and the corresponding "remainders" R_n (over the interval $[x_n,\infty]$):

$$I_n = \int_a^{x_n} f(x)\exp(-\gamma x)dx \tag{7}$$

$$R_n = I - I_n = \int_{x_n}^\infty f(x)\exp(-\gamma x)dx \tag{8}$$

In the original WA formulation, the tails are approximated integrating by parts the integral (8). This yields an infinite series expansion for the remainder [3]:

$$R_n = I - I_n = \exp(-\gamma x_n) \sum_{k=1}^{\infty} f_n^{(k-1)} \gamma^{-k} \tag{9}$$

where the coefficient $f_n^{(k-1)}$ stands for the $(k-1)$th order derivative:

$$f_n^{(k)} = \left. \frac{\partial^k f(x)}{dx^k} \right|_{x=x_n} \quad ; \quad f_n^{(0)} = f(x_n) \tag{10}$$

Equation (9) can be formally written as

$$I - I_n = O(\gamma^{-1}) \tag{11}$$

which shows that, at least asymptotically, the error incurred when approximating the infinite integral I by any partial integral I_n is of order γ^{-1}.

If (9) is written twice for two different subscripts n and $n+1$ (corresponding to two different upper limits x_n and x_{n+1} for the partial integrals), we can formally eliminate the term γ^{-1} among the two equations and obtain the result:

$$\frac{\exp(\gamma x_n)}{f_n^{(0)}}(I - I_n) - \frac{\exp(\gamma x_{n+1})}{f_{n+1}^{(0)}}(I - I_{n+1}) = O(\gamma^{-2}) \tag{12}$$

The classic WA algorithm exploits this result. If the term in $O(\gamma^{-2})$ is neglected in (12), then we obtain the *first-order WA approximation* $I_n^{(1)}$ for the infinite integral:

$$I \simeq I_n^{(1)} = \frac{\dfrac{\exp(\gamma x_n)}{f(x_n)} I_n - \dfrac{\exp(\gamma x_{n+1})}{f(x_{n+1})} I_{n+1}}{\dfrac{\exp(\gamma x_n)}{f(x_n)} - \dfrac{\exp(\gamma x_{n+1})}{f(x_{n+1})}} \tag{13}$$

Although, essentially, the WA principle will work for an arbitrary choice of the integration limits x_n and x_{n+1}, the original formulation suggested to select integration limits such that the condition $\beta(x_{n+1} - x_n) = \pi$ is fulfilled. This results in the classic WA formula:

$$I_n^{(1)} = \frac{w_n^{(0)} I_n^{(0)} + w_{n+1}^{(0)} I_{n+1}^{(0)}}{w_n^{(0)} + w_{n+1}^{(0)}} \tag{14}$$

where a more comprehensive notation using a "0" superscript has been introduced for the partial integrals $I_n^{(0)} = I_n$ and the "weights" w_n^0 are given by

$$w_n^{(0)} = \frac{\exp(\alpha x_n)}{f(x_n)} \tag{15}$$

The next step is to apply the idea recursively. We consider a sequence of N partial integrals $I_n, n = 1, 2 \ldots N$ and we apply the WA procedure, as formalized by (14), to every couple of consecutive partial integrals. This results in a new sequence of $N-1$ first-order WA estimations $I_n^{(1)}$, $n = 1, 2 \ldots N-1$.

Nothing prevents us from applying again a WA procedure to the first-order WA estimations and to generalize (14) as a multilevel procedure, defined by the expression

$$I_n^{(i+1)} = \frac{w_n^{(i)} I_n^{(i)} + w_{n+1}^{(i)} I_{n+1}^{(i)}}{w_n^{(i)} + w_{n+1}^{(i)}} \tag{16}$$

Obviously, N-1 successive applications of (16) will produce a single last result $I_1^{(N-1)}$ which should be considered as the best estimation of the infinite integral I that can be extracted from the sequence $I_n^{(0)}$.

The critical point here is how to select the weights when using (16) at successive levels $(i - 1, 2 \ldots, N-2)$. Several efficient strategies, blending simple mathematical reasoning with heuristic considerations and leading to quasi-optimal results, were already proposed in the pioneer times of WA [1], [3]. The question was definitely settled by Michalski [5] who gave rigorous and exhaustive developments for the most interesting analytical forms of these weights.

IV. Weighted Averages, Euler and Hölder

The recursive application of WA according to (16) can be viewed as a triangular process, transforming the original sequence $I_n^{(0)}$ into a new sequence $I_1^{(i)}$ according to the scheme:

$$\begin{array}{cccc} I_1^{(0)} & I_2^{(0)} & I_3^{(0)} & \ldots \\ I_1^{(1)} & I_2^{(1)} & \ldots & \\ I_1^{(2)} & \ldots & & \\ \ldots & & & \end{array} \tag{17}$$

In the above scheme, the first horizontal row is the original sequence of partial integrals $I_n^{(0)}$. Successive rows are obtained by recursive application of (16). At the end, the first vertical column contains the transformed sequence $I_1^{(i)}$ and $I_1^{(N-1)}$ is the best possible estimation of the infinite integral I.

In the particular case when all the weights are equal to one (simple arithmetic means instead of weighted averages) the above sequence transformation is equivalent to the Euler transformation, commonly used to accelerate the convergence of series [6]. Hence, WAs can be correctly described as *a much more efficient and generalized version of the Euler method* [5]. In turn, and under its "iterated means" form (17), the Euler transformation belongs to a class of summation techniques geared to divergent infinite series and known as Hölder and Cèsaro means [27, p. 94]. The relevant point here is that it can be demonstrated [27, p. 108] that summability in the sense of Hölder and Cèsaro, whenever possible, produces results in agreement with Abel's summability, for which we have established a physical meaning in Section II. This is the ultimate justification for the success of WA when dealing with divergent improper integrals.

To clarify these rather theoretical concepts, let us consider the integral

$$I = \frac{1}{\pi} \int_0^{\infty} x \sin \beta x dx = 0 \tag{18}$$

whose meaning in the Abel's sense can be easily established (5a). The partial integrals are readily evaluated as

$$I_n^{(0)} = \frac{1}{\pi}\int_0^{n\pi} x \sin \beta x dx = (-1)^{n-1} n \tag{19}$$

And thus we have the typical divergent oscillating sequence

$$I_n^{(0)} = 1, -2, +3, -4 \dots . \tag{20}$$

Here, the Hölder means process produces quickly the true value:

$$\begin{array}{cccc} 1 & -2 & 3 & -4 \\ \frac{-1}{2} & \frac{+1}{2} & \frac{-1}{2} & \\ 0 & 0 & & \\ 0 & & & \end{array} \tag{21}$$

V. The Generalized WA Procedure

Previously proposed WA algorithms are essentially more powerful versions of the Euler/Hölder algorithm, in which simple arithmetic means are replaced by weighted means. But it is well known that the Euler/Hölder procedure can be easily generalized in order to act simultaneously on the N members of a given sequence, rather than on two consecutive elements every time. The same strategy could be applied to WA. Hence, the generalized WA procedure will also start with a sequence of N partial integrals $I_n = I_n^{(0)}$. But now, the best possible evaluation of the infinite integral I will be obtained by performing a unique weighted average applied simultaneously to all the partial integrals.

To develop this generalized WA algorithm, let us start by recalling that the classic WA expression (13) was obtained by writing twice the asymptotic expansion (9) and eliminating the term γ^{-1}.

This procedure can be formally written in a matricial form as

$$\begin{bmatrix} -\exp(\gamma x_n) & f(x_n) \\ -\exp(\gamma x_{n+1}) & f(x_{n+1}) \end{bmatrix} \begin{bmatrix} I \\ \gamma^{-1} \end{bmatrix} = \begin{bmatrix} -\exp(\gamma x_n) I_n \\ -\exp(\gamma x_{n+1}) I_{n+1} \end{bmatrix} + O(\gamma^{-2}) \tag{22}$$

If the terms $O(\gamma^{-2})$ are neglected and the above expression is considered as a linear system, we can formally solve it for the unknown I. Using Cramer's rule we obtain the result as a quotient of two determinants:

$$I \simeq I_n^{(1)} = \frac{\begin{vmatrix} -I_n \exp(\gamma x_n) & f(x_n) \\ -I_{n+1} \exp(\gamma x_{n+1}) & f(x_{n+1}) \end{vmatrix}}{\begin{vmatrix} -\exp(\gamma x_n) & f(x_n) \\ -\exp(\gamma x_{n+1}) & f(x_{n+1}) \end{vmatrix}} \tag{23}$$

and this is exactly the classic WA formula (13).

So now, the obvious strategy is to generalize the above procedure to N terms.

We start writing the asymptotic expansion (13) as

$$-I \exp(\gamma x_n) + \sum_{k=1}^{N-1} f_n^{(k-1)} \gamma^{-k} = -I_n \exp(\gamma x_n) + O(\gamma^{-N}) \tag{24}$$

and apply it to N different integration limits x_n. Then, we can neglect the higher-order terms $O(\gamma^N)$, consider the resulting equations as a linear system, and formally solve it for the unknown I.

The determinant of such a linear system is

$$D_N = \begin{vmatrix} -\exp(\gamma x_1) & f_1^{(0)} & f_1^{(1)} & \dots & f_1^{(N-2)} \\ -\exp(\gamma x_2) & f_1^{(0)} & f_2^{(1)} & \dots & f_2^{(N-2)} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ -\exp(\gamma x_N) & f_N^{(0)} & f_N^{(1)} & \dots & f_N^{(N-2)} \end{vmatrix} \tag{25}$$

Hence, applying again Cramer's rule, we can obtain our "best" estimation I_N^* of the infinite integral I as a quotient of determinants, thus generalizing (23). The denominator in this quotient is the determinant D_N and the numerator a determinant obtained replacing in the first column of (25) the elements $-\exp(\gamma x_n)$ by $-\exp(\gamma x_n) I_n$.

Both determinants can be expanded by their first column and the final expression is the *generalized WA formula*:

$$I_N^* = \frac{\sum_{n=1}^{N} w_n I_n}{\sum_{n=1}^{N} w_n} \tag{26}$$

This is the sought-after linear combination of partial integrals allowing an estimation of the infinite integral which is optimal in the sense of the asymptotic expansion (24). In the above expression, the generalized weights are given by

$$w_n = (-1)^{n+1} \exp(\gamma x_n) M_n \tag{27}$$

where M_n are the minors of the determinant D_N, obtained by deleting the first column and the n-th row.

VI. The Generalized Weights

Expression (27) for the generalized weights is of little practical interest, as it involves the computation of determinants whose elements include the values of the function $f(x)$ and its $N-2$ derivatives at N points x_n. Computing determinants is usually a cumbersome and time-consuming task. Fortunately enough, an interesting analytical treatment is possible in some cases of interest.

It was stated at the beginning of the paper that the function $f(x)$ was assumed to behave asymptotically as a power. In other words:

$$\lim_{x \to \infty} [f(x) - C x^q] = 0 \tag{28}$$

with C and q being some real constants.

If we replace in the determinant D_N the function $f(x)$ by its asymptotic approximation (28), we obtain for the minors M_n in (27) the eldritch expression:

$$M_n = C^{N-1} \frac{q!}{q-N+1!} V_n \left(\frac{1}{x_n} \prod_{j=1}^{N} x_j \right)^{q-N+2} \tag{29}$$

where the V_n are Vandermonde's determinants:

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & .. & x_1^{N-2} \\ 1 & x_2 & x_2^2 & .. & x_2^{N-2} \\ : & : & : & : & : \\ 1 & x_{n-1} & x_{n-1}^2 & .. & x_{n-1}^{N-2} \\ 1 & x_{n+1} & x_{n+1}^2 & .. & x_{n+1}^{N-2} \\ : & : & : & : & : \\ 1 & x_N & x_N^2 & .. & x_N^{N-2} \end{vmatrix} \tag{30}$$

These Vandermonde's determinants have well known analytical expressions [29]. When used, the following expression for the weights, devoid of determinants, is obtained:

$$w_n = \frac{(-1)^{n+1} \exp(\gamma x_n) x_n^{N-2-q}}{\prod_{\substack{j=1 \\ (j \neq n)}}^{N} |x_j - x_n|} \tag{31}$$

The above expression for the weights is perhaps not yet very tractable but it has the merit to be quite general. In particular, the abscissas x_n can have arbitrary values.

A. Equidistant Abscissas and Half-Periods

Much simpler expressions for the weights are obtained if some restrictions are applied to the choice of the abscissas. If their values are not conditioned by external circumstances, the obvious choice is to select equidistant abscissas such as $x_{n+1} - x_n = h$. Then, the solution of the Vandermonde's determinant is even easier and a much more compact expression for the weights is obtained:

$$w_n = (-1)^{n+1} \exp(\gamma x_n) \binom{N-1}{n-1} x_n^{N-2-q} \tag{32}$$

A further simplification is obtained if the abscissas are separated by half-periods. This was also the usual choice in the classic WA: $\beta(x_{n+1} - x_n) = \beta h = \pi$.

This choice yields the final, simple expression:

$$w_n = \exp(\alpha x_n) \binom{N-1}{n-1} x_n^{N-2-q} \tag{33}$$

This is a very convenient expression for a practical implementation, since the combinatorial numbers can be easily computed by recursion. Moreover, the weights are always real and positive and this makes of expression (26) a true weighted average of the partial integrals.

VII. Some Numerical Examples

We start by discussing an improper Sommerfeld integral arising in classical EM theory. It is well known [30], that the EM fields of a hertzian dipole located in free space can be derived from a Hertz or a vector potential. For instance, for a z-directed dipole, it is enough to consider a vector potential $\vec{\mathbf{A}} = \hat{e}_z A_z$ colinear with it. The vector potential is then given by the classic electrodynamic free space Green's function, which possesses an integral representation given by the Sommerfeld identity [30]. In spherical/cylindrical coordinates r/ρ, z (here x is the spectral integration variable), we have

$$\begin{aligned} \frac{4\pi}{\mu_0} A_z &= \frac{\exp(-jk_0 r)}{r} \\ &= \int_0^\infty J_0(\rho x) \frac{x}{\sqrt{x^2 - k_0^2}} \exp(-z\sqrt{x^2 - k_0^2}) dx \end{aligned} \tag{34}$$

where k_0 is the free space wavenumber and $r = \sqrt{\rho^2 + z^2}$.

Now, the fields are easily derived from the potentials. For instance, the magnetic field is just linked to the curl of the vector potential, $\mu_0 \vec{\mathbf{H}} = \nabla \times \vec{\mathbf{A}}$. Performing the curl operation in cylindrical coordinates, we reach the conclusion that the magnetic field has only one azimuthal component given by

$$\begin{aligned} 4\pi H_\varphi &= \rho(1 + jk_0 r) \frac{\exp(-jk_0 r)}{r^2} \\ &= \int_0^\infty J_1(\rho x) \frac{x^2}{\sqrt{x^2 - k_0^2}} \\ &\quad \times \exp\left(-z\sqrt{x^2 - k_0^2}\right) dx \end{aligned} \tag{35}$$

In the $z = 0$ plane, the exponential term is annihilated and the infinite integral in (35) is no longer defined in the Riemann' sense, since it exhibits an oscillatory divergent integrand. However, this improper integral represents forcefully a most physical and well defined electrodynamic quantity, namely the magnetic field of a hertzian dipole. If the dipole is embedded in a stratified medium, the integrand in (35) is given by a much more complicated expression, but its essential mathematical properties remain untouched.

The WA algorithms developed in this paper can be applied to the tail part of the integral (35). However, a special tailored numerical technique must be used in the first part of the integration interval, especially around the singular point $x = k_0$. A simple strategy to test for accuracy of an isolated WA algorithm is to consider the static case. It is well known that the tail behaviors of static and full-wave cases are strictly identical. This is due to the fact that when $x \to \infty$, the square root $\sqrt{x^2 - k_0^2}$ behaves asymptotically like x, no matter which finite value takes k_0. On the other hand, no singularities arise at finite points of the integration interval in the static case. Hence, the full static integral can be considered as a tail and evaluated with only WA.

To further simplify the integral expression (35), let us consider a particular case with the numerical values: $k_0 = 0; z = 0; \rho = 1$. This "bare bones" situation results in the expression:

$$4\pi H_\varphi(k_0 = 0; z = 0; \rho = 1) = 1 = \int_0^\infty x J_1(x) dx \tag{36}$$

The improper nature of the integral in (36) is obvious. Yet, we are dealing here with the practical situation where the magnetic

TABLE I

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$I_n^{(0)}$	2.3033	-0.6249	2.9014	-1.1452	3.3650
$I_1^{(n)}$ (Hölder)	2.3033	0.8932	0.9888	0.9985	0.9998
$I_1^{(n)}$ [5]	2.3033	1.3273	1.0124	1.0007	1.0000
I_n^* (this paper)	2.3033	1.0904	1.0002	0.9998	1.0000

TABLE II

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$I_n^{(0)}$	0.8941	-1.3344	1.6656	-1.9419	2.1838
$I_1^{(n)}$ (Hölder)	0.8941	-0.2202	-0.0273	-0.0068	-0.0021
$I_1^{(n)}$ [5]	0.8941	0.1513	0.0084	0.0005	0.0000
I_n^* (this paper)	0.8941	-0.0290	0.0008	0.0000	0.0000

field of a Hertzian dipole must be computed in a very specific location.

In this case, the partial integrals

$$I_n^{(0)} = \int_0^{n\pi} x J_1(x) dx \tag{37}$$

must be evaluated with a numerical quadrature.

Table I compares the different estimations of the integral (36). The first line is formed by the partial integrals $I_n^{(0)}$ themselves (37). They constitute the sequence of a wildly divergent series. The second line gives the Euler/Hölder means $I_1^{(i)}$, which clearly improves the situation. Next, we consider a classic WA approach. As discussed at the end of Section III, several choices for the weights are possible and have been discussed in the literature. Here, we use the so-called "Michalski asymptotic weights [5, eqn.39], considered to be one of the most efficient choices. Results for this classic implementation are given in the third line of Table I. They represent a sizeable improvement on Hölder. Finally, the new generalized WA, with weights given by (33), results in the sequence given in the fourth line of Table I, which provides the fast converging sequence for the infinite integral (36).

For the sake of completeness, we have also tested the companion integral:

$$I = \int_0^{\infty} x J_0(x) dx = 0 \tag{38}$$

Its physical meaning is less evident, but on the other hand the partial integrals have a simple value:

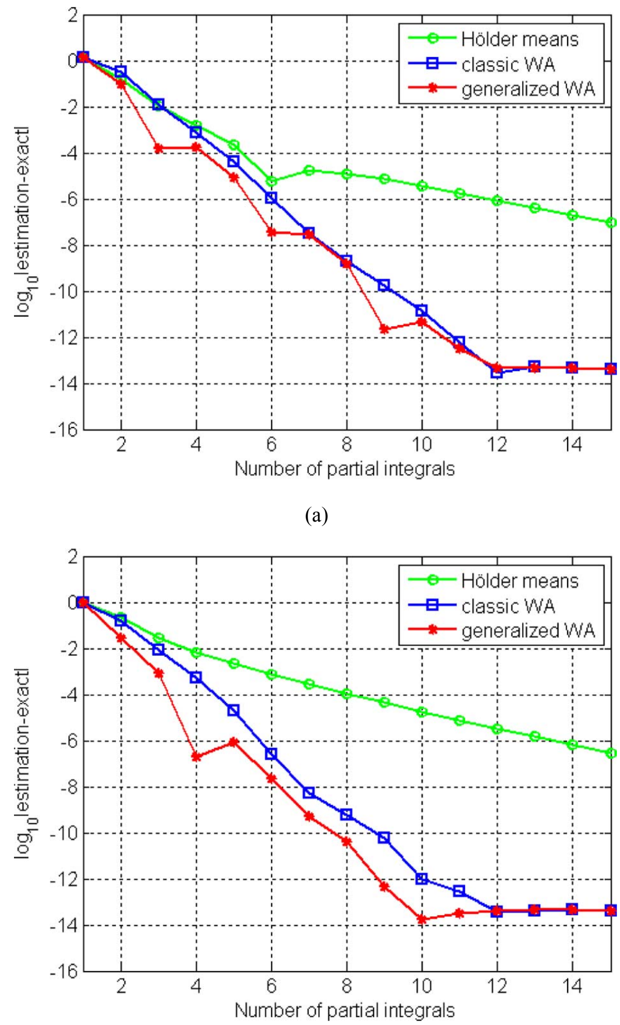
$$I_n^{(0)} = \int_0^{n\pi} x J_0(x) dx = n\pi J_1(n\pi) \tag{39}$$

and we can safely say that the WA algorithm is the sole responsible factor for the accuracy of the final results.

Table II is the exact counterpart of Table I for this integral (38), with Hölder, the classic and the generalized WA showing a behavior very similar to the previous example.

In order to provide a more precise comparison between these different approaches, Figs. 1(a) and (b) depict the logarithms of the absolute errors associated with Tables I and II.

It can be easily seen that the generalized WA always outperforms the classic WA by a factor roughly between 1 and 10, and both are much better than Euler/Hölder.



(a)

(b)

Fig. 1. (a) Logarithm of the absolute error associated with different estimations of the infinite improper integral (36). (b) Logarithm of the absolute error associated with different estimations of the infinite improper integral (38).

As a final numerical example, we consider a true Sommerfeld integral, corresponding to the static term in the scalar potential associated with a point source in free space [1]:

$$I = \frac{1}{\pi} \int_0^{\infty} J_0(\beta x) \exp(-\alpha x) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \tag{40}$$

Here x would be the spectral variable, β the radial cylindrical coordinate and α the vertical cylindrical coordinate. Also, the

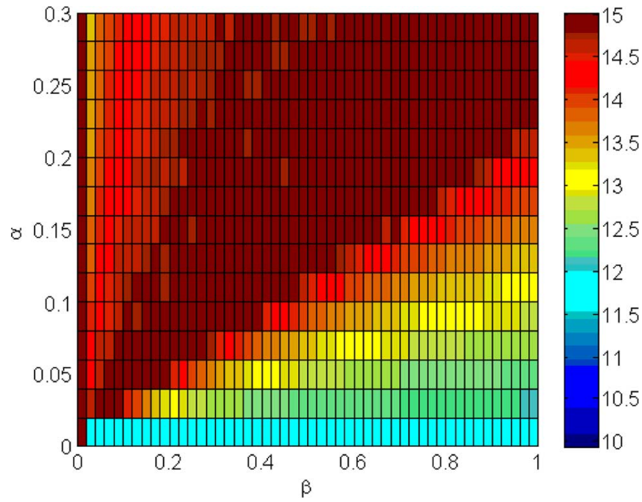


Fig. 2. Number of exact digits provided by the application of the generalized WA algorithm (26) and (33) to 10 partial estimations of the Sommerfeld integral (40).

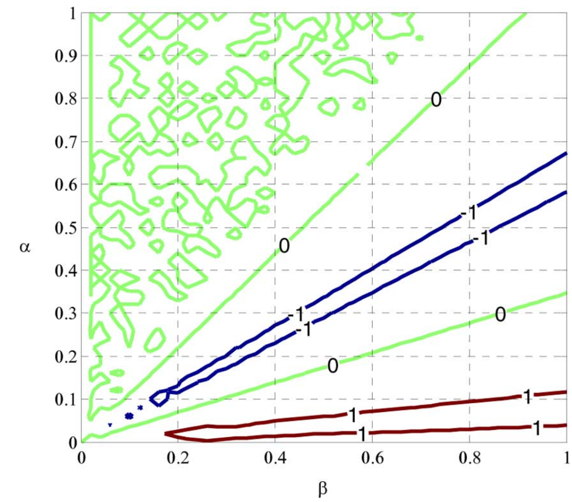


Fig. 3. Difference in digits accuracy between generalized and classic WAs when estimating the Sommerfeld integral (40). Positive values correspond to situations where the new generalized WA outperforms the classic version.

modulus of the WA parameter $\gamma = \alpha + j\beta$ would correspond here to the spherical radial coordinate.

The integral has been evaluated for a full range of parameter values of practical interest: $\mathrm{Re}[\gamma] = \alpha \in [0; 0.3]$ and $\mathrm{Im}[\gamma] = \beta \in [0; 1]$.

Fig. 2 shows the precision achieved for this range of parameters. The generalized WA provided an excellent precision to 12 digits for the most difficult case $\alpha = 0$ and even better results (up to 15 digits) for other parameter values.

A similar figure could be drawn for the classic WA algorithm. Rather, we have compared directly both algorithms by plotting in Fig. 3 the difference between the logarithms of the relative errors produced by generalized and classic WA when estimating the Sommerfeld integral (40). Here, we have considered a larger range for the parameters α, β: $\mathrm{Re}[\gamma] = \alpha \in [0; 1]$ and $\mathrm{Im}[\gamma] = \beta \in [0; 1]$ and only 5 partial integrals have been used.

For lossless and low losses cases ($\alpha \simeq 0$, the region close to the horizontal axis in Fig. 3), the new generalized version has an edge, being one and even two orders of magnitude more accurate. As the losses increase, there is a region where the classic algorithm remains more accurate. Finally, for higher losses (the region above the diagonal $\alpha = \beta$) the differences between both algorithms are negligible and quite randomly distributed.

VIII. Conclusion

The results obtained with the generalized version of the WA algorithm demonstrate the benefits of using this new technique. Generalized WA converges faster in most situations and should be preferred if only a small number of partial integrals is available. In lossless or low losses situations, it outperforms by at least order of magnitude the classic WA, which is currently recognized as the most competitive algorithm to evaluate Sommerfeld integral tails. In addition, regardless of potential improvements in accuracy, the new algorithm has several qualities that must be pointed out. Firstly, its theoretical construction is straightforward, well defined and fully supported by a rigorous mathematical background, as it can be considered as a generalization of the well established Hölder and Cèsaro means used to sum divergent series. Secondly, the iterative and recursive nature of classic WA is eliminated and replaced by a unique weighted means, where the weights are defined in a univocal way. This results in a reduction of the number of operations. For N partial integrals, classic WA will perform $N(N-1)/2$ weighted means, each one involving two partial integrals, so a total of $N(N-1)$ basic operations involving the basic integrals is needed. The new generalized WA applies a weighted means involving the N partial integrals but only once. So the number of basic operations involving the basic integrals is only N. In addition, it is expected that for the above reasons, the new WA should be less sensitive to the propagation of round-off errors.

Due to the simplicity of its formulation, the implemented algorithm is necessarily simpler than a classic WA implementation. Moreover, due to the nature of the weights, involving combinatorial numbers, it should be possible to obtain the estimation I^*_{n+1} as a simple modification of I^*_n, without the need of starting again from scratch.

Finally, the generalized weighted averages algorithm provides a robust approach since the same weights can be applied to both monotonic and oscillating sequences, like those resulting when the complex parameter $\gamma = \alpha + j\beta$, is either purely real or purely imaginary. There is still work to be done, as the main conclusions of this paper should be further confirmed and developed for more involved Sommerfeld integrals, corresponding to real multilayered problems. Other missing items in the exploitation of this new simple algorithm are its application to other infinite integrals and its consideration as a pure convergence accelerator for general sequences.

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