Note on noncrossing path in colored convex sets *

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Abstract

Consider a $2n$ element colored point set, n points red and n points blue, in convex position in the plane. Erdős asked to estimate the number of points in the longest noncrossing path such that edges join points of different color and are straight line segments. Kynčl, Pach and Tóth in 2008 gave a construction proving the upper bound $\frac{4}{3}n + O(\sqrt{n})$. This bound is conjectured to be tight. For an arbitrary coloring they gave a lower bound $n + \Omega(\sqrt{\frac{n}{\log n}})$.

In this paper we improve the previous lower bound to $n + \Omega(\sqrt{n})$. We also present a class of configurations that shows the $\frac{4}{3}n + O(\sqrt{n})$ upper bound.

1 Introduction

In this paper edges of a graph will be always considered to be straight line segments. If we have a point set in general position, it is not so hard to see that there is a noncrossing Hamiltonian path on that set. In the case when our point set is colored we get to new interesting problems. We will restrict the edges to connect points of different color. Consider *balanced* colorings of a $2n$ element point set, that is let n points be red and n points be blue. If the color classes are separated by a line, then there is a noncrossing Hamiltonian path on the point set [1]. If the color classes are not separated by a line,

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then the previous statement does not hold for $n \geq 8$, even if the points are in convex position. By the existence of halving lines the result in [1] gives at least n points on the longest noncrossing path for any point set on n points.

Before turning our attention to the main topic of the paper (the convex case) we mention a related, interesting class of point sets. A *convex* or a *concave chain* is a finite set of points in the plane lying on the graph of a strictly convex or a strictly concave function, respectively. A *double-chain* consists of a convex chain and a concave chain such that any line determined by any of the chains does not intersect the other chain.

Cibulka, Kynčl, Mészáros, Stolař, Valtr [3] in 2009 proved if both chains contain at least one fifth of all the points, then there exists a Hamiltonian, noncrossing path. On the other hand, they showed that the above property does not hold for double-chains in which one of the chains contains at most $\approx 1/29$ of all the points.

Now we state of a problem of Erdős, that is the root of our discussion. He [4] asked what happens if we restrict the points to be in convex position.

$$\ell(\mathcal{P}) = \max_{U \textit{ is a noncrossing path}} \ell(U),$$

where $\ell(U)$ is the number of points on U.

$$\ell(n) = \min_{\mathcal{P} \textit{ is balanced}} \ell(\mathcal{P}),$$

where $\mathcal{P}$ is any colored planar $2n$ element convex point set.

Without loss of generality we may assume that the points are on a circle. Erdős conjectured that the following configuration was asymptotically extremal. If n is divisible by four, divide the circle into four intervals that consist of $\frac{n}{2}$ red, $\frac{n}{4}$ blue, $\frac{n}{2}$ red and $\frac{3n}{4}$ blue points, respectively. In this configuration there are $\frac{3n}{2} + 2$ points on the longest noncrossing path.

Jan Kynčl, János Pach and Géza Tóth [7] disproved the above conjecture in 2008 and gave the $\frac{4}{3}n + O(\sqrt{n})$ upper and $n + \Omega\left(\sqrt{\frac{n}{\log n}}\right)$ lower bound.

Abellanas et al. had a very similar construction for the same upper bound [2] independently at the same time. It is conjectured that this upper bound is asymptotically tight.

In this paper we improve the above lower bound. We consider a $2n$ element point set in convex position with a balanced coloring. We show that there are at least $n + \Omega(\sqrt{n})$ points on the longest noncrossing path. Regarding the upper bound previously there were two very similar constructions in [7] and in [2]. Here we present a class of configurations for the $\frac{4}{3}n + O(\sqrt{n})$ upper bound. This class was also found by Jan Kynčl [6] using computer search.

2 Notations

Let $\mathbf{P}$ be a planar point set of $2n$ points in convex position. We index our points according to their circular order along the perimeter of their convex hull: $P_1, P_2, \ldots, P_{2n}$ where the arithmetics of the indices is the modulo $2n$ arithmetics. Two elements of $\mathbf{P}$, P_i and P_j, define two arcs (two subsets of $\mathbf{P}$): $a(P_i, P_j) = \{P_i, P_{i+1}, \ldots, P_j\}$ and $a(P_j, P_i) = \{P_j, P_{j+1}, \ldots, P_i\}$. Let A be an arc. The complement of A (in $\mathbf{P}$) will be also an arc: the complement of arc $a(P_i, P_j)$ is the arc $a(P_{j+1}, P_{i-1})$. The closed straight line segment determined by P_i and P_j is denoted by $[P_iP_j]$. Segments $[P_iP_j]$ and $[P_kP_l]$ are *crossing* if and only if the four indices are pairwise different, furthermore P_k and P_l lie in different arcs determined by P_i and P_j. It is easy to see that "to be crossing" is a symmetric relation. A path in P is just an ordered subset of $\mathbf{P}$: $p_1, p_2, \ldots, p_\ell$. We can think about a path as the sequence of segments $[p_i, p_{i+1}]$. A path is noncrossing if it consists of pairwise noncrossing segments. The *length* of a path P is the number of points in P, and we use the notation $length(P)$.

A coloring of $\mathbf{P}$ is a function $c : \mathbf{P} \to \{\text{red, blue}\}$. The coloring c is balanced if $|c^{-1}(\text{red})| = |c^{-1}(\text{blue})| (= |\mathbf{P}|/2 = n)$. We denote $c^{-1}(\text{red})$ by $\mathcal{R}$, respectively $\mathcal{B} = c^{-1}(\text{blue})$. Furthermore $\mathcal{P}$ always denotes a $2n$ element convex planar point set $\mathbf{P}$ with a balanced coloring c.

Observe, as we restricted the edges to connect points of different color for each path $p_1, p_2, \ldots, p_\ell$ the color of p_i and the color of p_j are the same if and only if i and j are of the same parity. In other words, as we walk along the path the red and blue points alternate. When we want to emphasize this property we will say that a path is *alternating*.

We can partition $\mathcal{P}$ into disjoint nonempty arcs in such a way that each arc is monochromatic and the sequence of these monochromatic arcs $R_1, B_1, R_2, B_2, \ldots, R_r, B_r$ along the perimeter is alternating in color. The arc R_i is a red monochromatic arc and B_i is the next blue monochromatic arc in the above sequence. Arcs R_i and B_i are called *runs* for every $i \in \{1, \ldots, r\}$. The common number of red arcs and blue arcs is called the run parameter of the colored point set, $run(\mathcal{P})$.

From the definitions above one can see that the problem is really a combinatorial question. For example we can assume that $\mathbf{P}$ is on a circle or on an ellipse.

In the next section we summarize the previous result. Finally we describe our improvement.

3 Initial observations

Let s be any line that is disjoint from $\mathcal{P}$ and cuts our point set into two nonempty parts. Then s determines two complementary arcs: A and A^c. We call these arcs/point sets the *sides* of s. We call s an *axe*.

The *elements* of a matching on point set $\mathcal{P}$ are edges determined by two elements of $\mathcal{P}$. The endpoints of an edge are called *matched* points. A matching M of $\mathcal{P}$ is a *separated* matching with axe s if the following three properties are satisfied:

(a) any element of M crosses the axe s (that is if we take two matched elements of $\mathcal{P}$, then they belong to different sides of s);

(b) different elements of M do not cross each other;

(c) the two endpoints of any element of M have different colors (that is, our earlier rule that edges connect points of different colors apply here, too)

In other words a matching is separated (with axe s) if and only if it matches pairs of points from different sides of s with different colors in a noncrossing way. The *size* of a separated matching M is the number of points in M.

The elements of a separated matching can be easily joined to form a noncrossing path. Our next claim summarizes this observation.

Observation 1. *If M is a separated matching, then we can find a noncrossing alternating path of length $2 \cdot |M|$.*

This can be improved easily by using edges (if possible) that are not intersecting the line s. First we need a definition.

Let M be a separated matching (with axe s, cutting our point set into two sides: arcs A and A^c). Let $alt(M)$ be the number of alternations between the colors along A following the perimetrical order. Note that arc A contains one endpoint of each element of M. We call a separated matching M with $alt(M) = 0$ an *Erdős matching*.

For any separated matching M there is a path P on the point set of M so that the edges of M give the odd edges of P. Note, this path is not unique. There are exactly two such paths depending on it which color we choose to be the color of the first point of P. If we want to enlarge P, we may incorporate new points into P if possible. The new edges we get may or may not cross the line s. If an edge does not cross s, we call it a *side* edge.

Observation 2. *([7]) There exist side edges and a suitable Erdős matching so that they can be connected into a path of $n + run(\mathcal{P}) - 1$ points.*

The following lemma says that in certain sense every path is a separated matching improved by side edges.

Observation 3. *Let P be an arbitrary noncrossing path in $\mathcal{P}$. Take a line s such that the first and last edge of P are crossed by s. Then we can choose a separated matching M with axe s from the edges of P in such a way, that the number of points in M is at least $length(P) - 2run(\mathcal{P}) + 1$.*

Note that the axe is not uniquely defined.

Proof. Throw away all edges of P that are not crossed by the line s. Thus we obtain subpaths $P_1, \dots, P_l$ of P. From each subpath of odd length delete the first edge. In the remainder of each subpath keep every other edge starting from the first edge. In such a way we get a separated matching.

Let t denote the number of subpaths of odd length among $P_1, \dots, P_l$. When we throw away the first edges of paths of odd length we delete t points from P. Erasing every other edge in a path of even length does not decrease the points of P. Thus what remains is to calculate the number of side edges that we disregarded.

For $i \in \{1, \dots, l-1\}$ the ending point of P_{i-1} and the starting point of P_i are on the same side of the line s. Observe, a side edge between the subpaths P_{i-1} and P_i has its endpoints in different runs. Therefore, when we disregard the side edges we delete at most $2run(\mathcal{P}) - l - 1$ points from P.

Altogether we erased at most $2run(\mathcal{P}) + t - l - 1$ points from P. So the size of the separated matching M is at least $length(P) - 2run(\mathcal{P}) - t + l + 1$. As t is at most l the claim follows.

□

If someone considers the observed examples in the literature, then $length(P)$ is between n and $2n$, while $run(\mathcal{P})$ is $o(n)$. If $run(\mathcal{P})$ is linear in n the longest noncrossing path beats the best known lower bound by Observation 2. Hence, assuming $run(\mathcal{P}) = o(n)$ is reasonable, we should concentrate on separated matchings.

Let $m(\mathcal{P})$ denote the maximum size of the separated matchings on the point set $\mathcal{P}$.

4 Previous methods

The idea of the obvious lower bound is very simple.

Take any line s that cuts the point set $\mathcal{P}$ into two parts. If $|A \cap \mathcal{R}|, |A^c \cap \mathcal{B}| \geq t$, then take t many points from $A \cap \mathcal{R}$ and t many points from $A^c \cap \mathcal{B}$. The two t element point sets are separated by the line s. We can match their elements in a noncrossing way and hence we obtain a separated matching of $2t$ points.

Observation 4. *There is an Erdős matching M of size at least n and hence a noncrossing path of length n.*

All what we described was known to Erdős. He showed in the following way that it was easy to find a separated matching of size n. Take any halving line s of the point set $\mathcal{P}$. Let A be the arc with red majority ($|A \cap \mathcal{R}| \geq |A \cap \mathcal{B}|$). It turns out that A^c must have blue majority. Hence the parameter t in the above argument is at least $n/2$. We obtain a noncrossing alternating path of length at least n.

The main ingredient of the improvement is summarized in the following observation. First we need to introduce a simple notion. Let A be an arc in $\mathcal{P}$ of even size. Then there is a unique partition of A into two arcs of the same size that we call *half-arcs*.

Observation 5. *(Implicit in [7]) If we can find an arc A, with half-arcs: $A = A_r \dot{\cup} A_b$ such that $|A_r \cap \mathcal{R}| - |A_r \cap \mathcal{B}| \geq t$ and $|A_b \cap \mathcal{B}| - |A_b \cap \mathcal{R}| \geq t$, then there is a separated matching M of size at least $n + t$, moreover $alt(M) \leq 1$.*

The essence of the proof of the lower bound in [7] is a clever way to define an arc A, where the red-blue coloring is unbalanced assuming that $run(\mathcal{P})$ is small. We do the same using a completely different idea and obtain a better result in the following section.

5 Improved lower bound

The basic idea of our improvement is a simple visualisation/coding of the colored $\mathcal{P}$. The code-diagram will be part of the grid G consisting of the (x, y) points with integer coordinates. We walk along the perimeter making steps from a point to the succeeding one. Depending on the color of the passed point we make a step on the grid G. Each step increases the x-coordinate by 1. The change of the y coordinate will code the color of the passed point: if it was red, then the step increases the y-coordinate by 1; if it was blue, then the step decreases the y-coordinate by 1. We show an example for the coloring and coding on Figure 1.

So the height of the walk reflects how the colors are changing. Since we code a balanced coloring the walk ends at the level of starting. We can fold our diagram to the surface of a cylinder to obtain a closed walk that reflects the circular behaviour of our geometric point set. We call the horizontal line through the lowest point of our diagram the 0-*level*. We choose our coordinate

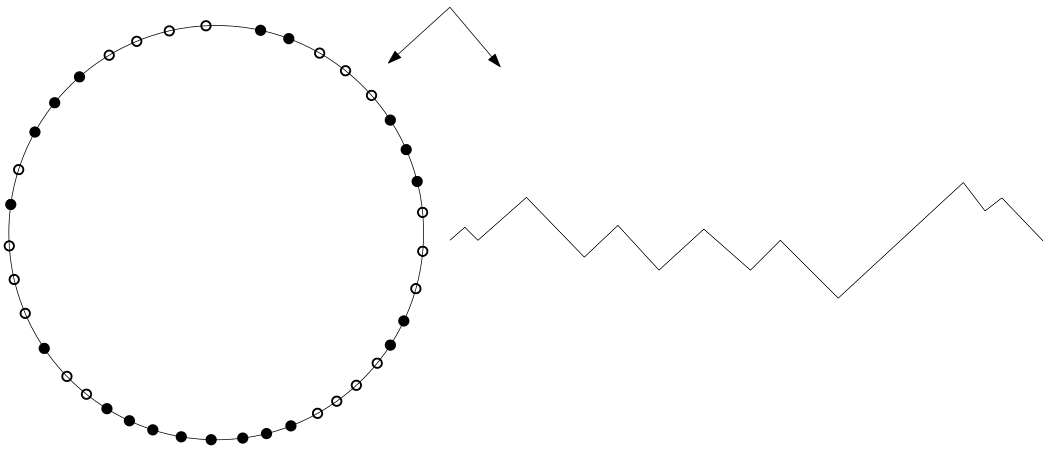

Figure 1: How to code $\mathcal{P}$ as a Dyck path

system such a way that the 0-level is the x-axis. For a non-negative integer h the line described by equation $y = h$ and it is called h-level. Note, the h-level and h'-level are neighboring levels if and only if $|h - h'| = 1$.

We cut the closed walk at any point that belongs to the 0-level. This way we obtain a Dyck path (for example see [10]) coding our colored point set. Actually, our code contains all the combinatorial information we need to consider the problem.

Our Dyck path has $2n$ steps. Each step starts on a level and ends at a neighboring level. Walking through a run (monochromatic arc) our code changes its height monotonicly. Hence for any h there are at most $run(\mathcal{P})$ many steps stepping up to the h-level and there are at most $run(\mathcal{P})$ many steps stepping down to the h-level. We choose t so that more than n steps are above the t-level and t is maximal among the levels with this property. The 0-level has all $2n$ many steps above it. As we lift the level one by one we decrease the number of "steps above" by at most $2 \cdot run(\mathcal{P})$. From this it is straightforward to derive that

$$t \geq \left\lfloor \frac{n-1}{2 \cdot run(\mathcal{P})} \right\rfloor .$$

Let σ be the i-th step of our Dyck path. Let σ' be the $2n + 1 - i$-th step, the *symmetric pair* of σ. Note, t is chosen in such a way that we can find a step σ above t-level with its symmetric pair also above t-level. Indeed: If we consider each step below t-level and its symmetric pair, we cannot obtain all the steps. Any remaining step is suitable for σ.

Theorem 6. *Let $\mathcal{P}$ be a $2n$ element point set in convex position with balanced* *coloring. There is a separated matching M with $alt(M) = 2$ and of size at least*

$$n + \left\lfloor \frac{n-1}{2 \cdot run(\mathcal{P})} \right\rfloor.$$

Proof. Let t be the level as above and let σ and σ' be two symmetric steps of the coded $\mathcal{P}$. Steps σ and σ' correspond to two points S and S' of $\mathcal{P}$. Then σ and σ' define two complementary arcs A and A^c of $\mathcal{P}$. Let F and L be the points corresponding to the first and the last step, respectively of the Dyck path. One of the two arcs say A contains the point F and hence it contains the point L, a neighbor of F on the Dyck path. According to the symmetricity of σ and σ' F and L are the two middle points of A. $A = a(S', L)\dot{\cup}a(F, S)$ is the partition of A into two half-arcs. As we walk from F to S the coding Dyck path raises from 0-level to above t-level. This color coding implies that $|a(F, S) \cap \mathcal{R}| - |a(F, S) \cap \mathcal{B}| > t$. At the same time $|a(S', L) \cap \mathcal{B}| - |a(S', L) \cap \mathcal{R}| > t$.

By Observation 5 our claim is true. □

The following corollary is immediate.

Corollary 7.

$$\ell(n) \geq n - 1 + \sqrt{n-1} = n + \Omega(\sqrt{n}).$$

Proof. We know that $\ell(\mathcal{P}) \geq n + run(\mathcal{P}) - 1$ and $\ell(\mathcal{P}) \geq n + \left\lfloor \frac{n-1}{2 \cdot run(\mathcal{P})} \right\rfloor$. Hence for arbitrary $\mathcal{P}$ we know that the average of the two lower bounds above is also a lower bound. The average of the two bounds is the promised bound by simple arithmetics. □

6 Limits of the known methods

Let us assume that an adversary can fix the initial point of our path in a given $\mathcal{P}$.

Erdős' observation works in this case (starting point given by an adversary): we are guaranteed to find a noncrossing path of length n.

But in the adversary version of the problem we cannot beat the trivial bound. To see this, devide our points into two complementary arcs of equal length. Points in one of the arcs obtain color red, the others will be blue. If the initial point is the middle point of the red arc then the longest noncrossing path has length at most $n + 1$.

Our method works more carefully. We code the coloring by a circular Dyck path and choose an arbitrary step starting at 0-level of this code as an initial point. So we narrow the set of possible initial points to the set of minimal points D. Assume that an adversary picks one element of D and we are forced to start our path from there. Our lower bound is exhibited by a path starting at the point given by the adversary of length at least $n + \Omega(\sqrt{n})$. With this generous setting we cannot improve the order of our lower bound. Consider the following coloring: Let $n = 2k$. Take k red, k blue and then $\sqrt{k}$ red and $\sqrt{k}$ blue points alternating $\sqrt{k}$ many times on the circle. If the adversary party marks the first point of the red run of length k (this point is neighboring to a blue run of $\sqrt{k}$ points), then the longest noncrossing path has the promised length.

The number of the alternations in the matching is fixed in the lower bounds. In the case of Erdős' bound $alt(M) = 0$. In [7] and in our approach for the constructed matching M $alt(M) = 1$. If we insist to come up with a matching part with constant alternation parameter, then we cannot beat the obvious bound n by more than a constant: take the red-blue completely alternating coloring (red, blue, red, blue,...). If we do the alternation in blocks of length $\sqrt{n}$ than even the side edges cannot help, we cannot improve our lower bound of $n + \Omega(\sqrt{n})$ points.

So the moral of the above remarks is that we must choose the initial point of our path carefully and use a lot of alternations when we consider the matching part of the path. The present techniques are not fulfilling these requirements.

7 New constructions

In [7] the upper bound was proved by a single construction and by its analysis. There was even a conjecture that this construction and the construction of Abellanas (that is very much alike) are isolated constructions showing the upper bound of order $\frac{4n}{3}$. We show a different — although — related way to construct a rich family of colored point sets exhibiting the [7] upper bound. We think that these constructions strengthen the belief that the [7] upper bound has the right order of magnitude and might guide the research towards a proof of that.

To describe a colored point set we use the following notation. Let $M_{r\times\ell}$ denote r many consecutive runs altenating in color of length ℓ. We call a building block of this type $M_{r\times\ell}$ a *mixed run*. Regarding the notion of run that we introduced before, in the following we usually say *homogeneous run* to stress that a run is monochromatic. The notations B_L and R_L denote a blue and a red run of length L, respectively. Let $\alpha \in [-1, 1]$, and let $\mathcal{P}_{\alpha,\ell}$ be

$$B_{2L}, R_{(1+\alpha)L}, M_{r\times\ell}, R_{(1+\alpha)L}, B_{2L}, R_{(1-\alpha)L}, M_{r'\times\ell}, R_{(1-\alpha)L},$$

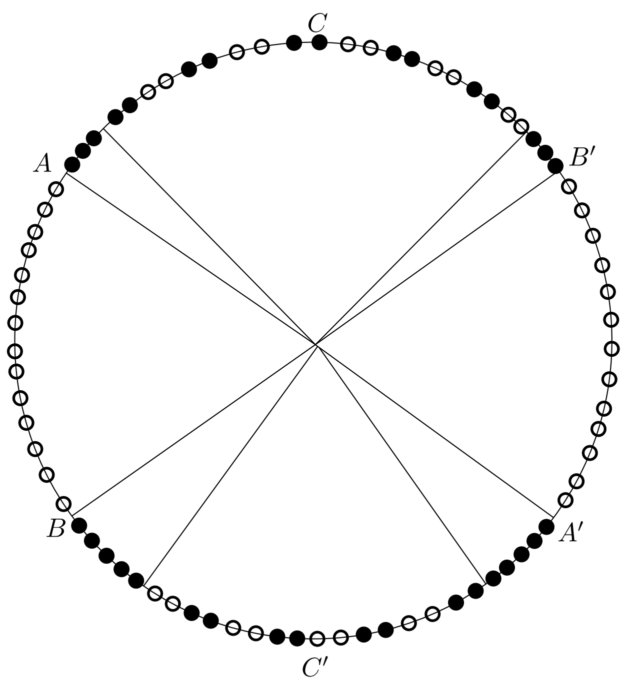

Figure 2: The coloring $\mathcal{P}_{\alpha,\ell}$

where ℓ is arbitrary and r, r' satisfy the following equalities $r\ell = (2 - 2\alpha)L$ and $r'\ell = (2 + 2\alpha)L$. Hence, $\mathcal{P}_{\alpha,\ell}$ is a balanced colored point set of $2n = 12L$ points. We assume that αL is an integer. In the case of $\alpha = -1, 1$ we get the [7] construction if we set $\ell = \Theta(\sqrt{L})$ (they considered this size in order to have $o(n)$ many runs).

On Figure 2 the number of points is proportional to the central angle of the arc. The figure also visualizes six natural cut points of our point set: A, B, C, A', B' and C'. Points A, B, A' and B' seperate two runs, C and C' are at the middle of the mixed runs.

We claim that the size of the largest separated matching in $\mathcal{P}_{\alpha,\ell}$ has the same order of magnitude as the construction of [7].

Theorem 8. *$m(\mathcal{P}_{\alpha,\ell}) \leq \frac{4n}{3} + O(\ell)$.*

Proof. We assume that the axe of an optimal matching divides the point set into upper and lower side and the matched upper and lower points are ordered from left to right. We can partition the edges of the matching into classes in such a way that on each side (upper, lower) the endpoints belong to one run or one mixed run. On each side we complete the points of a class to an arc by adding the possible intermediate points. We call these pairs of arcs *blocks*. We have constant number of blocks.

Consider the case when in a block one arc is a mixed subrun containing an equal number of homogeneous runs of the two colors and the other arc is a homogeneous subrun. Notice that at most half of the mixed subrun is going to be matched in this block. We will throw away some edges to shape mixed subruns as above in every block. Deleting $O(\ell)$ edges from the matching we can guarantee that in each block mixed subruns consist of complete homogeneous runs and also of an equal number of runs of the two colors. We denote the resulting separated matching by M.

Our goal is to take any seperated matching and deform its axe and its edges (if necessary) without significant size decrease in such a way that the axe will cut the point set at two of the six special points (see Figure 2).

We consider a few different cases based on the original position of the axe. If an end of the axe cuts a mixed run or it is placed between a mixed and a homogeneous run, we say that the end of the axe is *mixed.*

1st case: *Both ends of the axe are mixed.* We will change the position of the axe which will modify M. The new axe will be CC' that halves both mixed runs. In the new matching M' the two mixed runs will be completely matched. Around C and C' each mixed run is matched with itself. Observe, that the edges of M that do not involve points from any mixed run will intersect CC' but cannot cross any edge we introduced in M'. Therefore we add these edges to M'.

We will show that the size of M and M' are the same.

If there is an edge going between the two mixed runs in M, then all edges of M are of the following two types. The edges that connect the two mixed runs fall into the first type. Note, only one of the mixed runs can be also connected to homogeneous runs. These edges fall into the second type. There are no more possible edges in M (see the axe). Hence, the size of M is at most $2(2-2\alpha)L + 4\alpha L = 4L = \frac{2n}{3}$. Consequently, we get that $|M| = |M'|$.

So we may assume edges of M that have an endpoint in a mixed run have their other endpoint in a homogeneous run or in the same mixed run. Observe, that the number of all edges that have an endpoint in a mixed run equals to the number of edges if all points of a mixed run are matched within the same mixed run.

After the previous surgery the bound on the number of matched points is straightforward. There are at most $4L$ further points in M'. Hence, we get $8L = \frac{4n}{3}$ points altogether in M' as desired.

2nd case: *None of the ends is mixed.* Observe, that the ends of the axe come from the set of $\{A, A', B, B'\}$. If the axe cuts a homogeneous run to two parts P_1 and P_2, then one of P_1 and P_2 will not contain any endpoints of edges from M. So we can shift the axe to be between two runs in a way that M does not change.

If there are at most $\frac{2n}{3}$ points on one side of the axe, then we are done. Hence, by a symmetry argument we can assume that the axe is AA'. We may assume that at each end of the axe the shorter run is fully matched in the most economical way (to the other neighboring run to the axe). This gives $4L$ point to M.

On the upper side of the axe remain a mixed run of $(2+2\alpha)L$, a red run of $(1-\alpha)L$ and a blue subrun of $(1-\alpha)L)$ points. On the lower side of the axe remain a blue subrun of $(1+\alpha)L$, a red run of $(1+\alpha)L$ and a mixed run of $(2-2\alpha)L$ points. Suppose there is an edge in M with edpoints in different mixed runs. Let t be the number of points in the upper mixed run which are matched in M to the lower mixed run. Therefore, we get at most another $2t+(2+2\alpha)L-t+(2-2\alpha)L-t=4L$ points to M. If there are no edges between the two mixed runs, we may assume that both mixed runs are matched in the most economical way to the available homogeneous runs. Simple calculation gives that there are no more edges that we could add to M. Again we increased M by $4L$ new edges.

We showed that M contains at most $8L=\frac{4n}{3}$ points.

3rd case: *One of the ends is mixed, the other is not.* By a previous observation if the axe cuts a homogeneous run, we shift the axe to be between two runs so that M is not modified. If this end of the axe gets beside a mixed run, we use the argument in the *1st case*.

If there are at most $\frac{2n}{3}$ points on one side of the axe, we are done. Therefore, we claim by a symmetry argument that if the axe is in the upper (lower) mixed run, then the other end of the axe is at B (at A). If a point from the upper mixed run is matched to a point of the lower mixed run, then by simple calculation the size of M is at most $2(1+\alpha)L+(2-2\alpha)L+(2+2\alpha)L\leq 8L=\frac{4n}{3}$.

If there is no edge in M with endpoints from different mixed runs, then the surgery as in *1st case* can be performed for one of the mixed runs. The end of the axe in the mixed run will get to the middle of the mixed run to C or C'. Notice, that one side of the axe contains at most $4L=\frac{2n}{3}$ many points. Hence, the statement follows. □

If we want to restrict the length of the longest noncrossing path we need to restrict the number of runs, too.

Observation 9. *If $\ell=\Theta(\sqrt{n})$, then $\ell(\mathcal{P}_{\alpha,\ell})=\frac{4n}{3}+O(\sqrt{n})$.*

Proof. In any path we can distinguish a separated matching and side edges connected in a suitable way. The size of the maximum separated matching is $\frac{4n}{3} + O(\sqrt{n})$ by Theorem 8. As the number of side edges is restricted to $O(\sqrt{n})$, we showed the desired bound. □

The construction $\mathcal{P}_{\alpha,\ell}$ was found independently by Jan Kynčl [6] using computer search.

8 Further questions

Several directions of research remain open in the area. Our results underline the importance of the following conjecture.

Conjecture. ([7]) For any fixed k and large n, every balanced two-coloring of $2n$ points admits a separated matching of size at least $2n\frac{2k-1}{3k-2} + o(n)$ where k is the run parameter of the point set.

The class of constructions we gave in the previous section, also shows the order of magnitude claimed above. So far it is the strongest evidence by this conjecture.

We underline a more appealing consequence of the Kynčl-Pach-Tóth conjecture

Conjecture. Every balanced two-coloring of $2n$ points admits a separated matching of size $\frac{4}{3}n + O(\sqrt{n})$.

The *discrepancy* is d if on any arc on the circle the difference among the color classes is at most d. Small discrepancy implies a large separated matching [9]. However, already in such small cases as three it is technical to prove $\frac{4n}{3}$. Intuitions suggest that the lower bound for a separated matching when the discrepancy is small $(3, 4, \dots)$ is much closer to $2n$.

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