On the Area Requirements of Euclidean Minimum Spanning Trees *

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Abstract

In their seminal paper on Euclidean minimum spanning trees *[Discrete & Computational Geometry, 1992]*, Monma and Suri proved that any tree of maximum degree 5 admits a planar embedding as a Euclidean minimum spanning tree. The algorithm they presented constructs embeddings with exponential area; however, the authors conjectured that $c^n \times c^n$ area is sometimes required to embed an n-vertex tree of maximum degree 5 as a Euclidean minimum spanning tree, for some constant $c > 1$. In this paper, we prove the first exponential lower bound on the area requirements for embedding trees as Euclidean minimum spanning trees.

*This work is partially supported by the Italian Ministry of Research, Grant number RBIP06BZW8, FIRB project "Advanced tracking system in intermodal freight transportation", by the Swiss National Science Foundation, Grant No. 200021-125287/1, and by the Centre Interfacultaire Bernoulli (CIB) of EPFL.

1 Introduction

A *Euclidean minimum spanning tree* (MST) of a set P of points in the plane is a tree with a vertex in each point of P and with minimum total edge length. Euclidean minimum spanning trees have several applications in computer science and hence they have been deeply investigated from a theoretical point of view. To cite a few major results, optimal $\Theta(n \log n)$-time algorithms are known to compute an MST of a set of points and it is $\mathcal{NP}$-hard to compute an MST with maximum degree bounded by 2, 3, or 4 [6, 13, 4], while polynomial-time algorithms exist [1, 11, 2, 8] to compute MST with maximum degree bounded by 2, 3, or 4 and total edge length within a constant factor from the optimal one.

An *MST embedding* of a tree T is a plane embedding of T such that the MST of the points where the vertices of T are drawn coincides with T. In this paper we consider the problem of constructing MST embeddings of trees. Several results are known related to such a problem. No tree having a vertex of degree at least 7 admits an MST embedding. Further, deciding whether a tree with degree 6 admits an MST embedding is $\mathcal{NP}$-hard [3]. However, restricting the attention to trees of degree 5 is not a limitation since: (i) every planar point set has an MST with maximum degree 5 [12], and (ii) every tree of maximum degree 5 admits an MST embedding in the plane [12].

Monma and Suri's proof [12] that every tree of maximum degree 5 admits an MST embedding in the plane is a strong combinatorial result; on the other hand, their algorithm for constructing MST embeddings seems to be useless in practice, since the constructed embeddings have $2^{\Theta(k^2)}$ area for trees of height k (hence, in the worst case the area requirement of such drawings is $2^{\Theta(n^2)}$). However, Monma and Suri conjectured that there exist trees of maximum degree 5 that require $c^n \times c^n$ area in *any* MST embedding, for some constant $c > 1$. The problem of determining whether or not the area upper bound for MST embeddings of trees can be improved to polynomial is reported also in [3, 10, 7]. Recently, MST embeddings in polynomial area have been proved to exist for trees with maximum degree 4 [9, 5].

In this paper, we prove that there exist n-vertex trees of maximum degree 5 requiring $2^{\Omega(n)}$ area in any MST embedding. Our lower bound is achieved by considering an n-vertex tree T^*, shown in Fig. 1, composed of a degree-5 *complete tree* T_c with a constant number of vertices and of a set of degree-5 *caterpillars*, each one attached to a distinct leaf of T_c. The complete tree T_c forces the angles incident to an end-vertex of the backbone of at least one of the caterpillars to be very small, that is, between 60° and 61°. Using this as a starting point, we prove that each angle incident to a vertex of the caterpillar is either very small, that is, between 60° and 61°, or is very large, that is, between 89.5° and 90.5°. As a consequence, we show that the lengths of the edges of the backbone of the caterpillar decrease exponentially along the caterpillar, thus obtaining the claimed area bound.

The paper is organized as follows. In Sect. 2 we give some definitions and preliminaries; in Sect. 3 we give some geometric lemmata; in Sect. 4 we argue about the angles and the edge lengths of the MST embeddings of T^*; in Sect. 5 we prove the area lower bound; finally, in Sect. 6 we give remarks and conclusions. Some proofs have been omitted for space limitations and can be found in the Appendix.

2 Preliminaries

A *rooted tree* is a tree with one distinguished vertex, called *root*. The *depth* of a vertex in a rooted tree is its distance from the root, that is, the number of edges in the path from the root to the vertex. The *height* of a rooted tree is the maximum depth of one of its vertices. A *complete tree* is such that every path from the root to a leaf has the same number of vertices and every vertex has the same degree. A *caterpillar* is a tree such that removing the leaves yields a path, called the *backbone* of the caterpillar.

A *minimum spanning tree* MST of a set of n points in the plane is a tree spanning the n points and having minimum total edge length. Given a tree T, the *MST embedding problem* asks for a mapping of the vertices of T to points in the plane such that the MST of such points is isomorphic to T. Such a mapping provides a straight-line drawing of T, that is called an *MST embedding* of T.

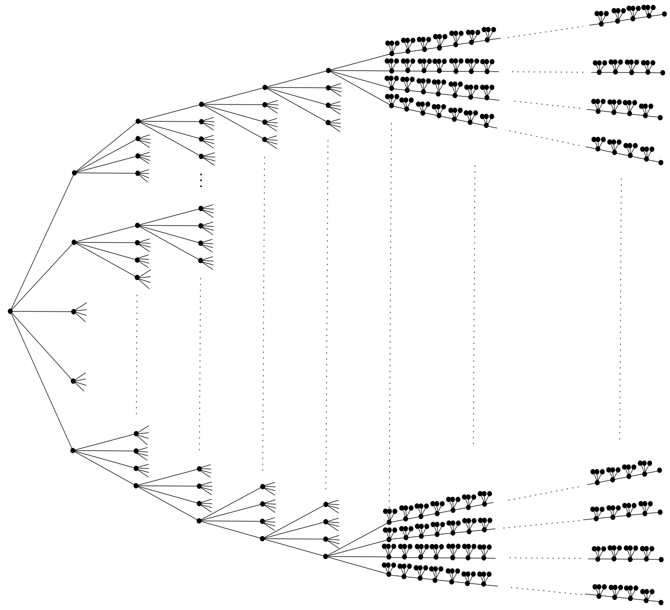


Figure 1: A tree T^* requiring $2^{\Omega(n)}$ area in any MST embedding.

The *area* of an MST embedding is the area of a rectangle enclosing such an embedding. The concept of area of an MST embedding only makes sense once a *resolution rule* is fixed, i.e., a rule that does not allow vertices to be arbitrarily close (*vertex resolution rule*), or edges to be arbitrarily short (*edge resolution rule*). In fact, without any of such rules, one could just construct MST embeddings with arbitrarily small area. In the following we will hence suppose that any two vertices have distance at least one unit. With such an assumption, in order to prove that an n-vertex tree T requires $f(n)$ area in any MST embedding, it suffices to prove that the ratio between the longest and the shortest edge of any MST embedding is $f(n)$, and that both dimensions have at least constant size.

Consider any MST embedding of a tree T rooted at a node r. The *clockwise path* $Cl(u)$ of a vertex $u \neq r$ of T is the path $v_0, v_1, \ldots, v_k$ such that $v_0 = u$, (v_i, v_{i+1}) is the edge following the edge from v_i to its parent in the clockwise order of the edges incident to v_i, for $i = 0, \ldots, k-1$, and v_k is a leaf. The *counterclockwise path* $Ccl(u)$ of a vertex $u \neq r$ of T is defined analogously. Denote by $d(a, b)$ the Euclidean distance between two vertices a and b (or between two points a and b) and denote by $|e|$ the length of an edge e. Further, $k(c, r)$ denotes the circle centered at a point c and having radius r.

Next, we define an n-vertex tree T^* that requires $\Omega(2^n)$ area in any MST embedding. Let T_c be a complete tree of height six and degree five. Let r be the root of T_c. Augment T_c by inserting a degree-five caterpillar at each leaf of T_c. That is, for each leaf l of T_c, insert a caterpillar C_l whose every non-leaf vertex has degree five, such that l is an end-vertex of the backbone of C_l, the parent of l in T_c is a leaf of C_l, and C_l and T_c do not share any other vertex. Denote by T^* the resulting tree.

3 Geometric Lemmata

In this section we give some properties for MST embeddings. The first four lemmata are well-known.

Lemma 1 *A straight-line drawing of a tree T is an MST embedding of T if and only if, for each pair of vertices u and v of T, $d(u, v) \geq |e|$, for each edge e in the path connecting u and v in T.*

Lemma 2 *In any MST embedding of a tree, any angle between two adjacent segments is at least* $60°$.

Lemma 3 *Consider any MST embedding Γ of a tree T. Consider any subtree T' of T. Then, Γ restricted to the vertices and edges of T' is an MST embedding of T'.*

Lemma 4 *Any MST embedding of a tree T is planar.*

The next lemma bounds the length of an edge in an MST embedding in terms of the length of an adjacent edge and of the size of the angle between them.

Lemma 5 *Let e_1 and e_2 be two edges consecutively incident to the same vertex and let $\alpha \leq 90°$ be the angle they form. Then, $2|e_1|\cos(\alpha) \leq |e_2| \leq \frac{|e_1|}{2\cos(\alpha)}$.*

Proof: Refer to Fig. 2(a). Let $e_1 = (u, v)$ and $e_2 = (u, z)$. If $|e_1| < 2|e_2|\cos\alpha$, then $|(v, z)| < |(u, z)|$, thus contradicting Lemma 1. Hence, $|e_1| \geq 2|e_2|\cos\alpha$. Analogously, $|e_2| \geq 2|e_1|\cos\alpha$. □

Consider an edge $e = (u, v)$ in an MST embedding of a tree T. Let $e_1 = (u, p)$ be the edge following e in the counterclockwise order of the edges incident to u and $e'_1 = (v, q)$ be the edge following e in the clockwise order of the edges incident to v. Let α (β) be the angle defined by a counterclockwise (resp. clockwise) rotation of e around u (resp. around v) bringing e to coincide with e_1 (resp. with e'_1). See Fig. 2(b). The next lemma, that establishes a strong lower bound on β provided that α is sufficiently small, is one of our main tools for the remainder of the paper.

Lemma 6 *Suppose that $\alpha \leq 80°$. Then, $\beta \geq 120° - \alpha/2$.*

Proof: First, we determine restrictions on the region where q lies, once the drawings of e and e_1 are fixed. Refer to Figs. 3(a) and 3(b). By Lemma 1, $d(q, u) \geq d(u, v)$ holds. Then, q is outside $k(u, |e|)$. Still by Lemma 1, $d(p, q) \geq d(p, u)$ and $d(p, q) \geq d(u, v)$ hold. Then, q is outside $k(p, m)$, where $m = \max\{|e|, |e_1|\}$. Again by Lemma 1, $d(p, q) \geq d(v, q)$ holds. Denote by $l^{|}_{pv}$ the line orthogonal to $\overline{pv}$ passing through the midpoint of $\overline{pv}$; then, q is in the half-plane delimited by $l^{|}_{pv}$ and not containing p. Suppose, w.l.o.g. up to a rotation, a reflection, and a translation of the drawing, that e is horizontal, that u is at point $(0, 0)$, that v is to the right of u, and that both p and q are above the horizontal line through u and v. We can suppose that q is to the left of the vertical line l_v through v, since otherwise $\beta \geq 90° \geq 120° - \alpha/2$, where the last inequality holds by Lemma 2, and there is nothing to prove.

Second, we discuss about the intersections of $k(p, m)$ with l_v. The distance from p to l_v is less than $|e|$, because p is to the right of the vertical line through u, given that $\alpha \leq 80°$. It follows that $k(p, m)$ has exactly two intersections with l_v, given that $m \geq |e|$. Moreover both of such intersections lie not below v as the distance between p and v is at least m, by Lemma 1, and hence the distance between p and any point of l_v below v is strictly greater than m, while $k(p, m)$ has radius exactly m. Denote by h and b the highest and the lowest of such two intersections, respectively.

Third, we prove the claimed lower bound for β. We distinguish the case in which the intersection of $l^{|}_{pv}$ with l_v is not higher than h (Case 1), as in Fig. 3(a), or is higher than h (Case 2), as in Fig. 3(b).

We discuss Case 1. The region R_1 of the plane in which q can lie is bounded by l_v from the right, by $k(u, |e|)$ from the left, and either by $k(p, m)$ or by $l^{|}_{pv}$ from above (depending on whether the

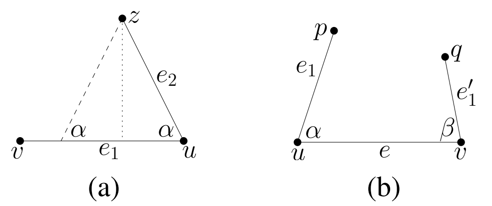
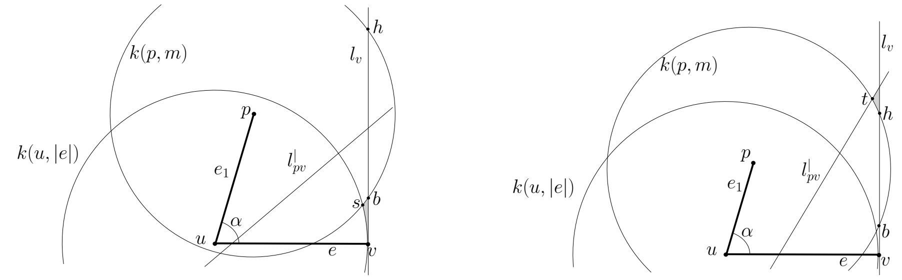


Figure 2: (a) Illustration for the proof of Lemma 5. (b) The setting for Lemma 6.



(a) (b)

Figure 3: Illustration for the proof of Lemma 6. In (a) and (b) the shaded region is R_1 and R_2, resp.

intersection point of $l^{|}_{pv}$ with l_v is higher or lower than b). Hence, such a region is a subset of the region bounded by l_v from the right, by $k(u, |e|)$ from the left, and by $k(p, m)$ from above. Then, denoting by s the intersection point between $k(p, m)$ and $k(u, |e|)$, we have $\beta \geq \widehat{uvs}$. Namely, the line through v and s has R_1 to its right. Hence, we assume that q lies at s. Denote by γ the angle $\widehat{vus}$. Then, we have $s \equiv (|e| \cos\gamma, |e| \sin\gamma)$ and $\beta = \frac{180° - \gamma}{2}$, where the last equality uses the fact that $|\overline{us}| = |\overline{uv}|$. Observe also that $p \equiv (|e_1| \cos\alpha, |e_1| \sin\alpha)$. We further distinguish two cases, namely the one in which $|e| \geq |e_1|$ (Case 1a) and the one in which $|e_1| \geq |e|$ (Case 1b).

Suppose that we are in Case 1a. Then, s is one of the intersection points of $k(u, |e|)$ and of $k(p, |e|)$, that has equation $(x - (|e_1| \cos\alpha))^2 + (y - (|e_1| \sin\alpha))^2 = |e|^2$. From the equation of $k(p, |e|)$ we get $x^2 - 2x|e_1| \cos\alpha + |e_1|^2 \cos^2\alpha + y^2 - 2y|e_1| \sin\alpha + |e_1|^2 \sin^2\alpha = |e|^2$. Then, since the equation of $k(u, |e|)$ is $x^2 + y^2 = |e|^2$ and since $k(u, |e|)$ and $k(p, |e|)$ pass through s, we get $|e|^2 - 2(|e_1||e| \cos\alpha \cos\gamma + |e_1||e| \sin\alpha \sin\gamma) + |e_1|^2 = |e|^2$. Thus, $2|e|(\cos(\alpha - \gamma)) = |e_1|$, hence $\gamma = \alpha - \arccos\left(\frac{|e_1|}{2|e|}\right)$. Since $|e| \geq |e_1|$, we have $\frac{|e_1|}{2|e|} \leq \frac{|e|}{2|e|} = \frac{1}{2}$, hence $\arccos\left(\frac{|e_1|}{2|e|}\right) \geq 60°$ and $\gamma \leq \alpha - 60°$. Using $\beta = (180° - \gamma)/2$, we get $\beta \geq \frac{180° - (\alpha - 60°)}{2} = 120 - \alpha/2$.

Case 1b is analogous to Case 1a. Namely, from the equations $x^2 + y^2 = |e|^2$ and $(x - (|e_1| \cos\alpha))^2 + (y - (|e_1| \sin\alpha))^2 = |e_1|^2$ of $k(u, |e|)$ and $k(p, |e_1|)$ and from the fact that $k(u, |e|)$ and $k(p, |e_1|)$ pass through s, analogously to Case 1a we get $\gamma = \alpha - \arccos\left(\frac{|e|}{2|e_1|}\right)$. Since $|e_1| \geq |e|$, we get $\arccos\left(\frac{|e|}{2|e_1|}\right) \geq 60°$, hence $\gamma \leq \alpha - 60°$, and finally $\beta \geq 120 - \alpha/2$.

We discuss Case 2. In this case, q lies either in region R_1, defined as in Case 1, or in the region R_2 bounded by l_v from the right, by $k(p, m)$ from below, and by $l^{|}_{pv}$ from above. If q is inside R_1, the proof is the same as in Case 1. If q is inside R_2, the minimum value of β is achieved when q is at the intersection point t between $k(p, m)$ and $l^{|}_{pv}$. Namely, the line through v and t has R_2 to its right. We prove that in Case 2 it holds $|e_1| < |e|$. Suppose, for a contradiction, that $|e_1| \geq |e|$. Consider a segment $\overline{vw}$ parallel to e_1 such that $|e_1| = |\overline{vw}|$. Observe that $\overline{pw} = |e|$. Then, $l^{|}_{pv}$ crosses polygon (u, v, w, p) on segments $\overline{up}$ and $\overline{vw}$, and the intersection of $l^{|}_{pv}$ with l_v is inside (u, v, w, p). On the other hand, h is above the line through p and w, thus contradicting the assumptions of Case 2. Moreover, since the slope of $l^{|}_{pv}$ increases while decreasing the length of $|e_1|$, the smaller is $|e_1|$, the smaller is $\widehat{uvt}$. Hence, by Lemma 5, we can assume that $|e_1| = 2|e| \cos\alpha$. Since $|e_1| < |e|$, $k(p, |e|)$ has equation $(x - (|e_1| \cos\alpha))^2 + (y - (|e_1| \sin\alpha))^2 = |e|^2$. Observe that $|\overline{tv}| = |e|$. Namely, the distance of every point of $l^{|}_{pv}$ from p and from v is the same, and the distance of t from p is $|e|$, given that t belongs to $k(p, |e|)$. Then, β can be computed by assuming that q is at one of the intersections of $k(p, |e|)$ and $k(v, |e|)$. Observe that $k(v, |e|)$ has equation $(x - |e|)^2 + y^2 = |e|^2$, that is $x^2 - 2x|e| + y^2 = 0$. Subtracting the last one from the equation of $k(p, |e|)$ we get $-x^2 + 2x|e| - y^2 + x^2 + y^2 - 2x|e_1| \cos\alpha - 2y|e_1| \sin\alpha + |e_1|^2 \cos^2\alpha + |e_1|^2 \sin^2\alpha = |e|^2$. From such a formula we get $2x|e| - 2x|e_1| \cos\alpha - 2y|e_1| \sin\alpha + |e_1|^2 = |e|^2$. Then, using $|e_1| = 2|e| \cos\alpha$ and using $t \equiv$

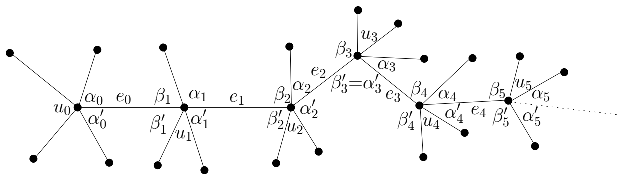


Figure 4: An embedding of C^*.

$(|e| - |e|\cos\beta, |e|\sin\beta)$, where the coordinates of t descend from the fact that $|tv| = |e|$, we get $2(|e|-|e|\cos\beta)|e|-2(|e|-|e|\cos\beta)(2|e|\cos\alpha)\cos\alpha-2(|e|\sin\beta)(2|e|\cos\alpha)\sin\alpha+(2|e|\cos\alpha)^2 = |e|^2$. Hence, $2|e|^2-2|e|^2\cos\beta-4|e|^2\cos^2\alpha+4|e|^2\cos^2\alpha\cos\beta-4|e|^2\cos\alpha\sin\alpha\sin\beta+4|e|^2\cos^2\alpha = |e|^2$. Thus we get $\cos\beta - 2\cos\alpha\,(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = \frac{1}{2}$ and hence $\cos\beta - 2\cos\alpha\cos(\alpha + \beta) = \frac{1}{2}$. Manipulating the last equation we get $\cos\beta - 2\cos\left(\frac{(2\alpha+\beta)-\beta}{2}\right)\cos\left(\frac{(2\alpha+\beta)+\beta}{2}\right) = \frac{1}{2}$. Using $\cos\theta\cos\phi = \frac{\cos(\theta+\phi)+\cos(\theta-\phi)}{2}$, we get $\cos\beta - (\cos(2\alpha+\beta) + \cos(\beta)) = \frac{1}{2}$, hence $\cos(2\alpha+\beta) = \frac{-1}{2}$. Since $\alpha, \beta \geq 60°$ by Lemma 2, we have that $2\alpha+\beta \geq 180°$. By the assumptions on α and β, $2\alpha+\beta \leq 280°$. It follows that $\cos(2\alpha+\beta) = \frac{-1}{2}$ is achieved with $2\alpha+\beta = 240°$. Hence, $\beta = 240° - 2\alpha \geq 120° - \frac{\alpha}{2}$, where the last inequality holds for all $\alpha \leq 80°$. □

4 Angles and Edge Lengths in MST Embeddings

In this section we consider the MST embeddings of T^* and argue about the angles and the edge lengths in each of such embeddings. We start by providing a lemma about the complete tree T_c .

Lemma 7 *In any MST embedding of T^* there exists a vertex u of T_c with depth five such that two angles consecutively incident to u and not adjacent to the edge from u to its parent sum up to at most* $121°$.

Consider any MST embedding of T^*; by Lemma 7, there exists a caterpillar C^* such that one of the end-vertices u_0 of the backbone of C^* is incident to an edge of T_c that is adjacent to two angles α_0 and α'_0 summing up to at most 121°. Denote by $u_0, u_1, u_2, \ldots, u_k$ the vertices of the backbone of C^* and by e_i the backbone edge connecting u_i and u_{i+1}, for $i = 0, \ldots, k-1$. We call *outgoing angles α_i and α'_i* the angles adjacent to e_i and incident to u_i; we call *incoming angles β_{i+1} and β'_{i+1}* the angles adjacent to e_i and incident to u_{i+1}. An edge e incident to u_i that is not the incoming edge of u_i is *in position* $j \in \{1, 2, 3, 4\}$ if e is the j-th edge in the clockwise order of the edges incident to u_i starting at e_{i-1}. Note that, if e_{i+1} is in position 1 (respectively 4), the incoming angle β_{i+1} and the outgoing angle α_{i+1} (respectively the incoming angle β'_{i+1} and the outgoing angle α'_{i+1}) coincide. See Fig. 4.

First, we prove that the outgoing and the incoming angles incident to a vertex of the backbone of C^* are either *small angles*, that is, between 60° and 61°, or *large angles*, that is between 89.5° and 90.5°. More precisely, the incoming angles are always large, while the outgoing angles are either both small or one large and one small. Indeed, observe that the outgoing angles of u_0 are both small by Lemma 7.

Suppose that a backbone edge e_i is in position 2 or 3 and that the incoming angles of u_i are at least 89.5°. By Lemma 2, each of the outgoing angles of u_i is at most 61° (recall that e_i is in position 2 or 3). Then, by Lemma 6, the incoming angles of u_{i+1} are at least 89.5°. Hence, if e_i is in position 2 or 3 and the incoming angles of u_i are at least 89.5°, the incoming angles of u_{i+1} are also at least 89.5°.

If e_i is in position 1 or 4, Lemma 6 is not useful to provide lower bounds on the values of both the incoming angles of u_{i+1}. Namely, one of the outgoing angles of u_i, say α_i, coincides with one of the incoming angles of u_i, say β_i. Hence, $\alpha_i{=}\beta_i$ is large and no lower bound for β_{i+1} can obtained by Lemma 6. However, we can prove that even if the outgoing angle α_i of a backbone vertex u_i is large, the incoming angle β_{i+1} of the next backbone vertex u_{i+1} is large, provided that the following condition

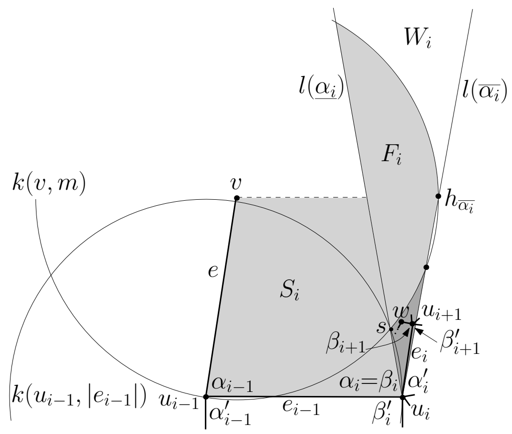


Figure 5: The setting for Lemmata 8–12. The dark-shaded region is R_i. To improve the readability, angles and edge lengths in the illustration do not correspond to actual angles and edge lengths.

is satisfied: The clockwise path $Cl(u_i)$ of u_i is contained in a bounded region R_i that is a subset of a wedge W_i with angle 1° centered at u_i. We will later prove (in Lemma 13) that, if such a condition is satisfied by a node u_i incident to a large outgoing angle α_i, then β_{i+1} is large and moreover $Cl(u_{i+1})$ is contained in a bounded region R_{i+1} that is a subset of a wedge W_{i+1} with angle 1° centered at u_{i+1}. However, before that, we have to prove that such a condition is satisfied by a node u_i if α_{i-1} is small.

Suppose, w.l.o.g. up to a rotation, a reflection, and a translation of the drawing, that e_{i-1} is horizontal, with u_i to the right of u_{i-1}, and that e_i is in position 1. Denote by $e = (u_{i-1}, v)$ (by $e^* = (u_{i+1}, w)$) the edge following e_{i-1} in the counterclockwise (resp. clockwise) order of the edges incident to u_{i-1} (resp to u_{i+1}). Denote by $l(\underline{\alpha_i})$ (by $l(\overline{\alpha_i})$) the half-line with slope 90.5° (resp. with slope 89.5°) starting at u_i. Finally, denote by W_i the closed wedge with angle 1° delimited by $l(\underline{\alpha_i})$ and $l(\overline{\alpha_i})$. See Fig. 5.

We will bound the region in which $Cl(u_i)$ lies from the right, from the left, and from above. Let $m = \max\{|e|, |e_{i-1}|\}$. Concerning the bound from the left, we can prove that the intersection point s of the circles $k(v, m)$ and $k(u_{i-1}, |e_{i-1}|)$ is not to the left of $l(\underline{\alpha_i})$, as stated in the following.

Lemma 8 *Suppose that $\alpha_{i-1} \leq 61^\circ$. Then, s is not to the left of $l(\underline{\alpha_i})$.*

We continue with the bound from the right.

Lemma 9 *Suppose that $\beta'_i \geq 89.5^\circ$. Then vertex u_{i+1} is not to the right of $l(\overline{\alpha_i})$.*

In order to derive the bound from above, we first prove that $k(v, m)$ intersects $l(\overline{\alpha_i})$ twice and we then argue about the distance between u_i and the highest intersection point $h_{\overline{\alpha_i}}$ of $k(v, m)$ with $l(\overline{\alpha_i})$.

Lemma 10 *Suppose that $\alpha_{i-1} \leq 61^\circ$. Then, $k(v, m)$ intersects $l(\overline{\alpha_i})$ twice.*

Lemma 11 *The distance between u_i and $h_{\overline{\alpha_i}}$ is at least* $1.604|e_{i-1}|$.

We are now ready to state the following:

Lemma 12 *Suppose that $\alpha_{i-1} \leq 61^\circ$, that $\beta'_i, \beta'_{i+1} \geq 89.5^\circ$, and that $|e_i| \leq \frac{|e_{i-1}|}{10}$. Then, $Cl(u_i)$ is inside a bounded region R_i that is a subset of W_i.*

Proof: Let R_i be the bounded region delimited by $l(\underline{\alpha_i})$ from the left, by $l(\overline{\alpha_i})$ from the right, and by $k(v, m)$ from above. We prove that $Cl(u_i)$ is inside R_i.

First, we prove that u_{i+1} is in R_i. By the assumption that $\alpha_{i-1} \leq 61^\circ$ and by Lemma 6, u_{i+1} is not to the left of $l(\underline{\alpha_i})$. By the assumption that $\beta'_i \geq 89.5^\circ$ and by Lemma 9, u_{i+1} is not to the right of $l(\overline{\alpha_i})$. Hence, u_{i+1} is in W_i. By the assumption that $\alpha_{i-1} \leq 61^\circ$ and by Lemma 10, $k(v, m)$ intersects $l(\overline{\alpha_i})$. Moreover, v is to the left of $l(\underline{\alpha_i})$. Namely, $v \equiv (|e|\cos\alpha_{i-1}, |e|\sin\alpha_{i-1})$. Further, if $y = |e|\sin\alpha_{i-1}$, then the x-coordinate of $l(\underline{\alpha_i})$ is $x = |e_{i-1}| - (|e|\sin\alpha_{i-1})/\tan 89.5^\circ$. Since $|e_{i-1}| \geq 2|e|\cos\alpha_{i-1}$ (by Lemma 5) and $60^\circ \leq \alpha_{i-1} \leq 61^\circ$ (by assumption and by Lemma 2), we have $|e_{i-1}| - |e|\sin\alpha_{i-1}/\tan 89.5^\circ \geq 2\cos 61^\circ|e| - |e|\sin 61^\circ/\tan 89.5^\circ \geq 0.96|e| > |e|\cos 60^\circ \geq |e|\cos\alpha_{i-1}$. Since v is to the left of $l(\underline{\alpha_i})$ and since $k(v, m)$ intersects $l(\overline{\alpha_i})$, there exists a bounded region F_i of W_i, delimited by $k(v, m)$ from above and from below, by $l(\underline{\alpha_i})$ from the left, and by $l(\overline{\alpha_i})$ from the right, in which u_{i+1} can not lie, as otherwise Lemma 1 would be violated. By Lemma 11, the distance between u_i and every point above F_i is at least $1.604|e_{i-1}|\cos 0.5^\circ > 1.4|e_{i-1}|$. Hence, by the assumption that $|e_i| \leq |e_{i-1}|/10$, u_{i+1} is not above F_i. It follows that u_{i+1} is in R_i.

Next, we prove that w is in R_i. Observe that $\beta_{i+1} \leq 90.5^\circ$, by the assumption that $\beta'_{i+1} \geq 89.5^\circ$ and since the three angles incident to u_{i+1} and different from β_{i+1} and β'_{i+1} sum up to at least 180° (by Lemma 2). Hence, e^* can not cross $l(\overline{\alpha_i})$. Since $\beta_i, \beta_{i+1} \leq 90.5^\circ$, the angle defined by a clockwise rotation bringing a horizontal line to coincide with e^* is at most 1°. Since the x-coordinate of u_{i+1} is at most $|e_{i-1}| + \frac{|e_{i-1}|\sin 0.5}{10}$, the y-coordinate of the line through e^* if $x = |e|\cos\alpha_{i-1}$ is at most $\frac{|e_{i-1}|}{10} + \tan 1^\circ(|e_{i-1}| + \frac{|e_{i-1}|\sin 0.5}{10} - |e|\cos\alpha_{i-1}) \leq \frac{|e|}{20\cos 61^\circ} + \tan 1^\circ(\frac{|e|}{2\cos 61^\circ} + \frac{|e|\sin 0.5}{20\cos 61^\circ} - |e|\cos 61^\circ) < 0.112|e| < |e|\sin\alpha_{i-1}$, since $\alpha_{i-1} \leq 61^\circ$, by assumption, and $2|e_{i-1}|\cos\alpha_{i-1} \leq |e|$, by Lemma 5. Then, the line through e^* crosses the vertical line through v below v. Since the y-coordinate of every point above F_i is at least $1.4|e_{i-1}|$, by Lemma 11, e^* can not cross $k(v, m)$. Further, the region S_i bounded by e from the left, by e_{i-1} from below, by $l(\underline{\alpha_i})$ from the right, and by the horizontal line through v from above entirely belongs to $k(v, m) \cup k(u_{i-1}, |e_{i-1}|)$, by Lemma 8; since the y-coordinate of w is at most $0.112|e| < |e|\sin\alpha_{i-1}$, if e^* crosses $l(\underline{\alpha_i})$, then either w is in S_i, thus violating Lemma 1, or e^* crosses an edge of T^*, thus violating Lemma 4. Hence, w is in R_i.

Finally, consider the rest of $Cl(u_i)$. The angle defined by a clockwise rotation bringing an edge g_1 of $Cl(u_i)$ to overlap with the next edge g_2 of $Cl(u_i)$ is at most 120°, since the four other angles incident to the vertex shared by g_1 and g_2 sum up to at least 240° (by Lemma 2). Hence, no edge of $Cl(u_i)$ crosses $l(\overline{\alpha_i})$ or $k(v, m)$, as otherwise such an edge crosses an edge of T^*, thus violating Lemma 4. Moreover, no edge of $Cl(u_i)$ crosses $l(\underline{\alpha_i})$, as otherwise either one end-vertex of such an edge is in S_i, thus violating Lemma 1, or the edge crosses an edge of T^*, thus violating Lemma 4. □

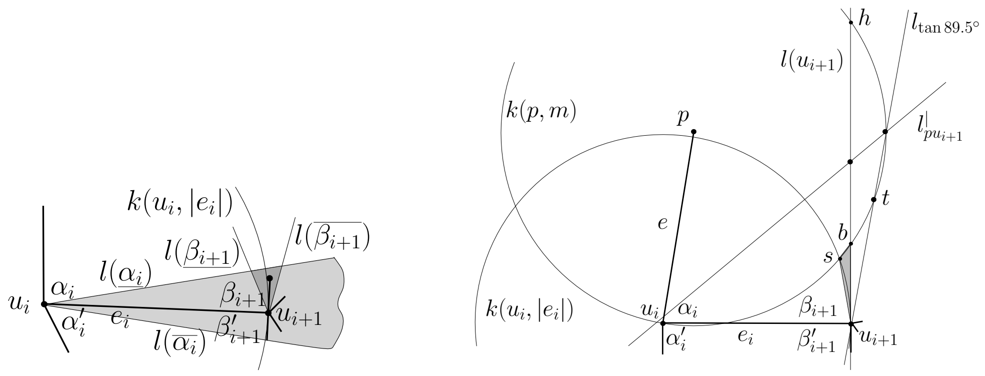
Lemma 12 assumes that $|e_i| \leq \frac{|e_{i-1}|}{10}$. The reason why we can assume such a ratio will be made clear at the end of the section and then exploited in the inductive proof presented in Section 5.

We can now prove that the condition that the clockwise path of each vertex is inside a bounded region propagates along the vertices of the backbone. Refer to Fig. 6(a).

Lemma 13 *Suppose that $\alpha_i \geq 89.5^\circ$, that $\beta'_{i+1} \geq 89.5^\circ$, and that $Cl(u_i)$ is in a bounded region R_i that is a subset of a wedge W_i centered at u_i with angle 1°. Then, $\beta_{i+1} \geq 89.5^\circ$. Moreover, $Cl(u_{i+1})$ is in a bounded region R_{i+1} that is a subset of a wedge W_{i+1} centered at u_{i+1} with angle 1°.*

Proof: Since $Cl(u_i)$ is in R_i, it follows that u_{i+1} is in R_i. Then, w is not inside $k(u_i, |e_i|)$, as otherwise Lemma 1 would be violated. Hence, the minimum value of $\widehat{u_i u_{i+1} w} = \beta_{i+1}$ is achieved if w is on $k(u_i, |e_i|)$, inside R_i, and hence inside W_i. If w is on $k(u_i, |e_i|)$, then triangle $\Delta(u_i u_{i+1} w)$ is isosceles. Since $\widehat{u_{i+1} u_i w} \leq 1^\circ$, then $\beta_{i+1} \geq 89.5$, thus proving the first part of the lemma.

Next, let $l(\underline{\beta_{i+1}})$ ($l(\overline{\beta_{i+1}})$) be the half-line starting at u_{i+1} such that a 89.5° (resp. 90.5°) clockwise rotation around u_{i+1} brings e_i to overlap with $l(\underline{\beta_{i+1}})$ (resp. with $l(\overline{\beta_{i+1}})$). Define R_{i+1} as the intersection of R_i and the wedge delimited by $l(\underline{\beta_{i+1}})$ and $l(\overline{\beta_{i+1}})$. Then R_{i+1} is bounded as R_i is; further, R_{i+1} is a subset of a wedge W_{i+1} centered at u_{i+1} with angle 1°. We prove that $Cl(u_{i+1})$ is in R_{i+1}.



(a) (b)

Figure 6: (a) Illustration for Lemma 13. The dark-shaded region is R_{i+1}. (b) Illustration for Lemma 14. The dark-shaded region is R_1. To improve the readability, angles and edge lengths in the illustrations do not correspond to actual angles and edge lengths.

Since $\beta'_{i+1} \geq 89.5^\circ$ and the three angles incident to u_{i+1} and different from β_{i+1} and β'_{i+1} sum up to at least 180°, it holds $\beta_{i+1} \leq 90.5^\circ$. Since $Cl(u_i)$ is in R_i and the angle defined by a clockwise rotation bringing an edge g_1 of $Cl(u_i)$ to overlap with the next edge g_2 of $Cl(u_i)$ is at most 120°, as the four other angles incident to the vertex shared by g_1 and g_2 sum up to at least 240° (by Lemma 2), then every vertex of $Cl(u_{i+1})$ is not to the right of $l(\overline{\beta_{i+1}})$, as otherwise an edge of such a path crosses e_i or (u_{i+1}, w), thus contradicting Lemma 4. The region delimited by e_i from below, by $l(\underline{\beta_{i+1}})$ from the right, and by $l(\underline{\alpha_i})$ from above is a subset of $k(u_i, |e_i|)$ since the line through u_{i+1} and through the intersection point of $k(u_i, |e_i|)$ and $l(\underline{\alpha_i})$ forms with e_i an angle which is at least 89.5°. Hence, if an edge of $Cl(u_{i+1})$ crosses $l(\underline{\beta_{i+1}})$, then either a vertex of $Cl(u_{i+1})$ is in $k(u_i, |e_i|)$, thus violating Lemma 1, or an edge of $Cl(u_{i+1})$ crosses e_i or (u_{i+1}, w), thus violating Lemma 4. It follows that $Cl(u_{i+1})$ is in R_{i+1}. □

We now deal with the edge lengths in any MST embedding of T^*. Consider a backbone edge $e_i = (u_i, u_{i+1})$ such that the outgoing angle α_i is small. Assume w.l.o.g. up to a rotation, a reflection, and a translation of the drawing, that e_i is horizontal with u_{i+1} to the right of u_i. Assume that u_i has coordinates $(0, 0)$. Let $e^* = (u_{i+1}, q)$ ($e = (u_i, p)$) be the edge following e_i in the clockwise (resp. counterclockwise) order of the edges incident to u_{i+1} (resp. to u_i). Let α_i and β_{i+1} be the angles delimited by e_i and e and by e_i and e^*, respectively. Let $m = \max\{|e|, |e_i|\}$. Further, let $l(u_{i+1})$ be the vertical line through u_{i+1} and $l^{|}_{pu_{i+1}}$ the line orthogonal to $\overline{pu_{i+1}}$ through the midpoint of such a segment. Let b and h be the lowest and the highest intersection point of $k(p, m)$ and $l(u_{i+1})$, respectively. Let s be the rightmost intersection point of $k(p, m)$ and $k(u_i, |e_i|)$. Refer to Fig. 6(b). We have the following:

Lemma 14 *Suppose that* $\alpha_i \leq 61^\circ$ *and that* $\beta_{i+1} \leq 90.5^\circ$. *Then, it holds* $\frac{|e^*|}{|e_i|} \leq 0.073$.

Proof: We distinguish two cases, namely the one in which $\beta_{i+1} \leq 90^\circ$ and the one in which $90^\circ < \beta_{i+1} \leq 90.5^\circ$. By assumption, no other values of β_{i+1} have to be considered to prove the lemma.

Suppose that $\beta_{i+1} \leq 90^\circ$. We claim that the maximum value of $|e^*|$ is achieved when q is either at b or at s. Namely, by Lemma 1, we have that: (i) q is outside $k(p, m)$; (ii) q is in the half-plane that is delimited by $l^{|}_{pu_{i+1}}$ and that does not contain p; and (iii) q is outside $k(u_i, |e_i|)$. Further, q is not to the right of $l(u_{i+1})$ since $\beta_{i+1} \leq 90^\circ$. Hence, as long as $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below h, q is in the region R_1 bounded by $l(u_{i+1})$ from the right, by $k(p, m)$ from above, and by $k(u_i, |e_i|)$ from below. Such a region is a subset of triangle $\Delta(u_{i+1}, s, b)$, since $\overline{sb}$ is a chord of $k(p, m)$ and $\overline{u_{i+1}s}$ is a chord of $k(u_i, |e_i|)$. Hence, the farthest point from u_{i+1} inside R_1 is either b or s.

Claim 1 *The intersection of* $l^{|}_{pu_{i+1}}$ *and* $l(u_{i+1})$ *is below* h.

We now further distinguish the two cases in which $|e^*| = |\overline{u_{i+1}b}|$ and $|e^*| = |\overline{u_{i+1}s}|$.

Suppose that the farthest point from u_{i+1} inside R_1 is b. We compute $|\overline{u_{i+1}b}|$. The equation of $k(p,m)$ is $(x - |e|\cos\alpha_i)^2 + (y - |e|\sin\alpha_i)^2 = m^2$. Setting $x = |e_i|$ into such an equation we get the y-coordinate of b, that is $y = |e|\sin\alpha_i - \sqrt{m^2 - |e_i|^2 + 2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} = |\overline{u_{i+1}b}|$.

First, suppose that $|e_i| \geq |e|$. Then, $|\overline{u_{i+1}b}| = |e|\sin\alpha_i - \sqrt{2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e|\sin\alpha_i - \sqrt{2|e|^2\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e_i|(\sin\alpha_i - \sqrt{2\cos\alpha_i - \cos^2\alpha_i})$. Studying the derivative of $2\cos\alpha_i - \cos^2\alpha_i$, we get that such a function is monotonically decreasing with α_i, hence $|\overline{u_{i+1}b}| \leq |e_i|(\sin 61^\circ - \sqrt{2\cos 61^\circ - \cos^2 61^\circ}) < 0.0176$. Second, suppose that $|e| \geq |e_i|$. Then $|\overline{u_{i+1}b}| = |e|\sin\alpha_i - \sqrt{|e|^2 - |e_i|^2 + 2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e|\sin\alpha_i - \sqrt{2|e_i||e|\cos\alpha_i - |e|^2\cos^2\alpha_i} \leq |e|\sin\alpha_i - \sqrt{3|e|^2\cos^2\alpha_i} = |e|(\sin\alpha_i - \sqrt{3}\cos\alpha_i) \leq |e_i|\frac{\sin\alpha_i - \sqrt{3}\cos\alpha_i}{2\cos\alpha_i}$, where we used twice $|e_i| \geq 2|e|\cos\alpha_i$, which holds by Lemma 5. Since $\tan\alpha_i$ is monotonically increasing with α_i between 60° and 61°, we get $\frac{|\overline{u_{i+1}b}|}{|e_i|} \leq \frac{\tan 61^\circ}{2} - \frac{\sqrt{3}}{2} = 0.036$.

Suppose that the farthest point from u_{i+1} inside R_1 is s. We have that $s \equiv (|e_i|\cos\gamma, |e_i|\sin\gamma)$, where $\gamma = \widehat{u_{i+1}u_i s}$. Exactly as in the proof of Lemma 6, we derive $\gamma \leq \alpha_i - 60^\circ \leq 1^\circ$. Hence, $|\overline{u_{i+1}s}| = \sqrt{(|e_i|\sin\gamma)^2 + (|e_i| - |e_i|\cos\gamma)^2} = |e_i|\sqrt{2 - 2\cos\gamma} \leq |e_i|\sqrt{2 - 2\cos 1^\circ} < 0.0175|e_i|$, where we used the fact that $\cos\gamma$ is monotonically decreasing between 0° and 1°.

Suppose that $90^\circ < \beta_{i+1} \leq 90.5^\circ$. We claim that $|e^*|$ is at most $|u_{i+1}t|$, where t is the intersection point of $k(p,m)$ and the line $l_{\tan 89.5^\circ}$ through u_{i+1} with slope $\tan 89.5^\circ$. First, p is to the left of $l(u_{i+1})$, since $|e|\cos\alpha_i < 2|e|\cos\alpha_i \leq |e_i|$, which holds by Lemma 5; further, by Lemma 10 (where α_i, $k(p,m)$, and $l_{\tan 89.5^\circ}$ replace α_{i-1}, $k(v,m)$, and $l(\overline{\alpha_i})$, resp.), $l_{\tan 89.5^\circ}$ intersects $k(p,m)$ twice. Denote by $l^{|}_{pu_{i+1}}$ the line orthogonal to $\overline{pu_{i+1}}$ through the midpoint of $\overline{pu_{i+1}}$. We have the following:

Claim 2 *The distance between u_{i+1} and the intersection point $h^{|}(p, u_{i+1}, \tan 89.5^\circ)$ of $l^{|}_{pu_{i+1}}$ and $l_{\tan 89.5^\circ}$ is at most $0.66|e_{i-1}|$.*

By Lemma 11 (where u_{i+1}, $k(p,m)$, and $l_{\tan 89.5^\circ}$ replace u_i, $k(v,m)$, and $l(\overline{\alpha_i})$) the distance between u_{i+1} and the highest intersection point of $k(p,m)$ and $l_{\tan 89.5^\circ}$ is at least $1.604|e_{i-1}|$. Hence, q is not above $k(p,m)$, as otherwise it is above $l^{|}_{pu_{i+1}}$, thus contradicting Lemma 1, and is not inside $k(p,m)$, again by Lemma 1. Then q is below $k(p,m)$, and hence $|e^*|$ is at most $|u_{i+1}t|$. Then, we have:

Claim 3 *If $|e| \geq |e_i|$, it holds $\frac{|\overline{u_{i+1}t}|}{|e_i|} < 0.056$; if $|e_i| \geq |e|$, it holds $\frac{|\overline{u_{i+1}t}|}{|e_i|} < 0.0723$.*

Such a claim concludes the proof of the lemma. □

Next, we present a lemma asserting that if β_i and β'_i are large enough, then all the edges incident to u_i have about the same length. Denote by e_{i-1}, e^1_i, e^2_i, e^3_i, and e^4_i the clockwise or the counterclockwise order of the edges incident to u_i, where β_i and β'_i are both incident to e_{i-1}.

Lemma 15 *Suppose that $\beta_i, \beta'_i \geq 89.5°$. Then* $\max\{e^2_i, e^3_i, e^4_i\} \leq \frac{|e^1_i|}{2\cos(240^\circ - (\beta_i + \beta'_i))} \leq 1.032|e^1_i|$.

Corollary 1 *Suppose that $\alpha_{i-1} \leq 61^\circ$ and that $\beta'_i \geq 89.5^\circ$. Then, all the edges incident to u_i and different from e_{i-1} have length at most $0.1|e_{i-1}|$.*

5 The proof of the area bound

In this section we prove that any MST embedding of T^* is such that, for each backbone vertex u_i of C^*, the outgoing angles of u_i are either both small or one small and one large. As a consequence, we derive a $2^{\Omega(n)}$ lower bound on the area requirements of any MST embedding of T^*. Refer to the same notation as in Section 4. Let k be the number of backbone vertices of C^*.

Lemma 16 *For each $0 \leq i \leq k-2$, one of the following holds: (Condition 1): $\alpha_i, \alpha'_i \leq 61^\circ$; (Condition 2): $\alpha_i \geq 89.5^\circ$, $\alpha'_i \leq 61^\circ$, and $Cl(u_i)$ is in a bounded region R_i that is a subset of a wedge W_i with angle 1° centered at u_i; (Condition 3): $\alpha'_i \geq 89.5^\circ$, $\alpha_i \leq 61^\circ$, and $Ccl(u_i)$ is in a bounded region R_i that is a subset of a wedge W_i with angle 1° centered at u_i.*

Proof: The proof is by induction on i. In the base case $i = 0$ and, by Lemma 7, $\alpha_0, \alpha'_0 \leq 61^\circ$, thus Condition 1 holds. Next we discuss the inductive case.

Suppose that Condition 1 holds for i. By Lemma 6, we have $\beta_{i+1}, \beta'_{i+1} \geq 89.5^\circ$. By Corollary 1, all the edges incident to u_{i+1} and different from e_i have length at most $|e_i|/10$. By Lemma 2, each of the angles incident to u_{i+1} and different from β_{i+1} and β'_{i+1} is at most 61°. Hence, if e_{i+1} is in position 2 or 3, then Condition 1 holds for $i + 1$. If e_{i+1} is in position 1 (that is $\alpha_{i+1} = \beta_{i+1}$), then $\alpha'_{i+1} \leq 61^\circ$. Moreover, by Lemma 6, $\beta'_{i+2} \geq 89.5^\circ$. Then, all the conditions of Lemma 12 are satisfied, namely $\alpha_i \leq 61^\circ$, $\beta'_{i+1}, \beta'_{i+2} \geq 89.5^\circ$, and $|e_{i+1}| \leq |e_i|/10$. Hence, $Cl(u_{i+1})$ is in a bounded region R_{i+1} that is a subset of W_{i+1} and thus Condition 2 holds for $i + 1$. If e_{i+1} is in position 4, then a proof analogous to the one for the case in which e_{i+1} is in position 1 shows that Condition 3 holds for $i + 1$.

Suppose that Condition 2 holds for i (the case in which Condition 3 holds for i can be discussed symmetrically). By Lemma 6, $\beta'_{i+1} \geq 89.5^\circ$. Hence, all the conditions of Lemma 13 are satisfied, namely $\alpha_i \geq 89.5^\circ$, $\beta'_{i+1} \geq 89.5^\circ$, and $Cl(u_i)$ is in a bounded region R_i that is a subset of a wedge W_i with angle 1° centered at u_i. It follows that $\beta_{i+1} \geq 89.5^\circ$ and $Cl(u_{i+1})$ is in a bounded region R_{i+1} that is a subset of a wedge W_{i+1} with angle 1° centered at u_{i+1}. By Lemma 2, each angle incident to u_{i+1} and different from β_{i+1} and β'_{i+1} is at most 61°. Thus, if e_{i+1} is in position 2 or 3, then Condition 1 holds for $i + 1$, and if e_{i+1} is in position 1, then Condition 2 holds for $i + 1$. Suppose that e_{i+1} is in position 4. Since each angle incident to u_{i+1} and different from β_{i+1} and β'_{i+1} is at most 61°, it holds $\alpha_{i+1} \leq 61^\circ$ and then, by Lemma 6, $\beta_{i+2} \geq 89.5^\circ$. Since $\beta_{i+1}, \beta'_{i+1} \geq 89.5^\circ$, by Corollary 1 all the edges incident to u_{i+1} and different from e_i have length at most $|e_i|/10$. Then, all the conditions of the symmetric of Lemma 12 are satisfied, namely $\alpha'_i \leq 61^\circ$, $\beta_{i+1}, \beta_{i+2} \geq 89.5^\circ$, and $|e_{i+1}| \leq |e_i|/10$. Hence, $Ccl(u_{i+1})$ is in a bounded region R_{i+1} that is a subset of W_{i+1} and thus Condition 3 holds for $i + 1$. □

Theorem 1 *Any MST embedding of T^* has $2^{\Omega(n)}$ area.*

Proof: Since the complete tree T_c has constant degree and constant height, then each caterpillar, and in particular C^*, has $k = \Omega(n)$ backbone vertices. By Lemmata 6, 13, and 16, the incoming angles β_i and β'_i are both larger than 89.5°, for each $1 \leq i \leq k - 1$. By Corollary 1, $|e_{i+1}| \leq \frac{|e_i|}{10}$, for each $0 \leq i \leq k - 1$. Hence $\frac{|e_1|}{|e_k|} \geq 10^{k-1} = 2^{\Omega(n)}$. The theorem follows by observing that, in any MST embedding of the root of T_c and of its children, both dimensions have size at least $\sin 30^\circ = 0.5$. □

6 Conclusions

In this paper we have shown trees requiring exponential area in any MST embedding, thus settling a 20-years-old problem proposed by Monma and Suri [12]. The actual conjecture of Monma and Suri states that both coordinate directions of any MST embedding of certain trees have exponential length. However, we believe that some further geometric considerations on the tree T^* we presented in this paper can lead to completely settle the Monma and Suri's conjecture. Observe that the area requirements of the MST embeddings constructed by the algorithm presented by Monma and Suri is $2^{\Omega(n^2)}$, while no $2^{O(n)}$-area MST embeddings are known to exist for all n-vertex degree-5 trees. We believe that such a gap can be closed by further improving our exponential lower bound, as in the following.

Conjecture 1 *Every MST embedding of T^* has $2^{\Omega(n^2)}$ area.*

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Appendix: Omitted Proofs

In this Appendix we present proofs that have been omitted in the main text.

We start with the proof of Lemma 7. In order to do that, we first need the following auxiliary lemma.

Lemma 17 *There exists two consecutive angles τ_1 and τ_2 incident to r such that $\tau_1 + \tau_2 \leq 150^\circ$ and $\tau_1, \tau_2 \leq 80^\circ$.*

Proof: If two among the angles incident to r are greater than 80°, then the other three angles sum up to less than 200°. Hence, by Lemma 2, each of them is at most 80° and any two of them sum up to at most 140°. Since two of such three angles are consecutive, the lemma follows.

If at most one among the angles incident to r is greater than 80°, then the other four angles are each at most 80° and, by Lemma 2, they sum up to at most 300°. Such four angles can be subdivided into two pairs of consecutive angles; since one of such pairs has angles summing up to at most 150°, the lemma follows. □

Lemma 7. *There exists a vertex u of T_c with depth five such that two angles consecutively incident to u and not adjacent to the edge from u to its parent sum up to at most* 121°.

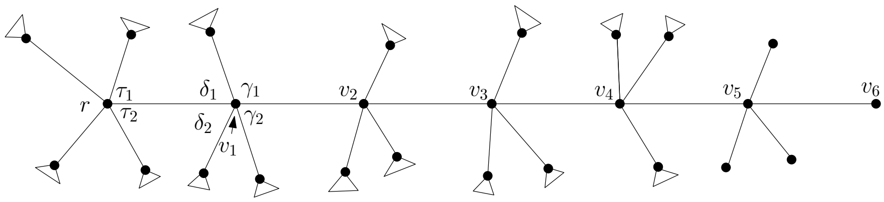

Figure 7: Tree T_c.

Proof: Refer to Fig. 7. Given an edge (u, v), where both u and v are not leaves of T_c, consider the edge (u, u_1) that immediately precedes (u, v) in the clockwise (counterclockwise) order of the edges incident to u. Consider the edge (v, v_1) that immediately precedes (v, u) in the counterclockwise (clockwise, resp.) order of the edges incident to v. Then, $\widehat{u_1uv}$ is *opposite to* $\widehat{v_1vu}$ with respect to (u, v). By Lemma 17, there exists two consecutive angles τ_1 and τ_2 incident to r such that $\tau_1 + \tau_2 \leq 150^\circ$ and $\tau_1, \tau_2 \leq 80^\circ$. Denote by v_1 the neighbor of r such that edge (r, v_1) is adjacent to τ_1 and τ_2. By Lemma 6, the angles opposite to τ_1 and τ_2 with respect to (r, v_1), say δ_1 and δ_2, satisfy $\delta_1 \geq 120^\circ - \tau_1/2$ and $\delta_2 \geq 120^\circ - \tau_2/2$. Hence, $\delta_1 + \delta_2 \geq 240^\circ - (\tau_1 + \tau_2)/2 \geq 240^\circ - 75^\circ = 165^\circ$. Denote by γ_1, γ_2, and γ_3 the angles incident to v_1 different from δ_1 and δ_2 in this clockwise order. Then, we have $\gamma_1 + \gamma_2 \leq 135^\circ$, since $\gamma_1 + \gamma_2 + \gamma_3 \leq 195^\circ$ and $\gamma_3 \geq 60^\circ$. Observe that, since $\gamma_1, \gamma_2 \geq 60^\circ$, we have $\gamma_1, \gamma_2 \leq 75^\circ$. Next, consider the edge (v_1, v_2) adjacent to γ_1 and γ_2. The two angles incident to v_2 and opposite to γ_1 and γ_2 sum up to at least $240^\circ - 135^\circ/2 = 172.5^\circ$. Hence, any two angles consecutively incident to v_2 and not adjacent to (v_1, v_2) sum up to at most 127.5°. Such an argument propagates along any path from v_1 to a leaf. Thus, there exists a path $(r, v_1, v_2, v_3, v_4, v_5, v_6)$ such that the two angles incident to v_1, v_2, v_3, v_4, and v_5 adjacent to edge (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), and (v_5, v_6), resp., sum up to at most 135°, 127.5°, 123.75°, 121.875°, and 120.93875°, respectively. The lemma follows with $u = v_5$. □

Next, we prove the auxiliary lemmata for Lemma 12, that is, we prove Lemmata 8–11.

Lemma 8. *Suppose that $\alpha_{i-1} \leq 61^\circ$. Then, s is not to the left of $l(\underline{\alpha_i})$.*

Proof: The statement can be proved using exactly the same considerations as in the proof of

Lemma 6. Namely, a lower bound of $120^{\circ} - \frac{\alpha_{i-1}}{2}$ for the slope of the line through u_i and s can be computed exactly as in Lemma 6. Since $\alpha_{i-1} \leq 61^{\circ}$, the statement follows. □

Lemma 9. *Suppose that $\beta'_i \geq 89.5^{\circ}$. Then vertex u_{i+1} is not to the right of $l(\overline{\alpha_i})$.*

Proof: By Lemma 2 the three angles incident to u_i and different from β_i and β'_i sum up to at least 180°. The lemma follows by the assumption that $\beta'_i \geq 89.5^{\circ}$. □

Lemma 10. *Suppose that $\alpha_{i-1} \leq 61^{\circ}$. Then, $k(v,m)$ intersects $l(\overline{\alpha_i})$ twice.*

Proof: We prove that $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice. Suppose, w.l.o.g. up to a translation of the coordinate system that u_{i-1} has coordinates $(0,0)$. Then $k(v,m)$ has equation $(y - |e|\sin\alpha_{i-1})^2 + (x - |e|\cos\alpha_{i-1})^2 = m^2$ and $l(\overline{\alpha_i})$ has equation $y = \tan 89.5^{\circ}(x - |e_{i-1}|)$. Substituting the second equation into the first one, we get that the x-coordinates of the intersections of $k(v,m)$ and $l(\overline{\alpha_i})$ satisfy $x^2\tan^2 89.5^{\circ} + |e_{i-1}|^2\tan^2 89.5^{\circ} + |e|^2\sin^2\alpha_{i-1} - 2|e_{i-1}|x\tan^2 89.5^{\circ} - 2|e|x\tan 89.5^{\circ}\sin\alpha_{i-1} + 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1} + x^2 + |e|^2\cos^2\alpha_{i-1} - 2|e|x\cos\alpha_{i-1} = m^2$. Simplifying the previous equation we get $(\tan^2 89.5^{\circ} + 1)x^2 - 2(|e_{i-1}|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|\cos\alpha_{i-1})x + |e_{i-1}|^2\tan^2 89.5^{\circ} + 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|^2 - m^2 = 0$. Thus $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice if and only if $(|e_{i-1}|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|\cos\alpha_{i-1})^2 - (\tan^2 89.5^{\circ} + 1)(|e_{i-1}|^2\tan^2 89.5^{\circ} + 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|^2 - m^2) \geq 0$. To prove that the last inequality holds, we distinguish two cases, namely the one in which $|e| \geq |e_{i-1}|$ and the one in which $|e_{i-1}| \geq |e|$.

First, suppose that $|e| \geq |e_{i-1}|$, that is, $m = |e|$. Then, we have to prove that $(|e_{i-1}|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|\cos\alpha_{i-1})^2 - (\tan^2 89.5^{\circ} + 1)(|e_{i-1}|^2\tan^2 89.5^{\circ} + 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1}) \geq 0$, that is, $|e_{i-1}|^2\tan^4 89.5^{\circ} + |e|^2\tan^2 89.5^{\circ}\sin^2\alpha_{i-1} + |e|^2\cos^2\alpha_{i-1} + 2|e_{i-1}||e|\tan^3 89.5^{\circ}\sin\alpha_{i-1} + 2|e_{i-1}||e|\tan^2 89.5^{\circ}\cos\alpha_{i-1} + 2|e|^2\tan 89.5^{\circ}\sin\alpha_{i-1}\cos\alpha_{i-1} - |e_{i-1}|^2\tan^4 89.5^{\circ} - |e_{i-1}|^2\tan^2 89.5^{\circ} - 2|e_{i-1}||e|\tan^3 89.5^{\circ}\sin\alpha_{i-1} - 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1} \geq 0$. Simplifying the previous one and using $|e| \geq |e_{i-1}|$ and $2|e|\cos\alpha_{i-1} \leq |e_{i-1}|$ (by Lemma 5), we get that, in order to prove the previous inequality, it suffices to prove that $|e_{i-1}|^2\tan^2 89.5^{\circ}\sin^2\alpha_{i-1} + |e_{i-1}|^2\cos^2\alpha_{i-1} + 2|e_{i-1}|^2\tan^2 89.5^{\circ}\cos\alpha_{i-1} + 2|e_{i-1}|^2\tan 89.5^{\circ}\sin\alpha_{i-1}\cos\alpha_{i-1} - |e_{i-1}|^2\tan^2 89.5^{\circ} - 4|e_{i-1}|^2\tan 89.5^{\circ}\sin\alpha_{i-1}\cos\alpha_{i-1} \geq 0$. Moreover, since $\sin 60^{\circ} \leq \sin\alpha_{i-1} \leq \sin 61^{\circ}$ and $\cos 61^{\circ} \leq \cos\alpha_{i-1} \leq \cos 60^{\circ}$ (by hypothesis and by Lemma 2), we get that the previous inequality is implied by $|e_{i-1}|^2(\tan^2 89.5^{\circ}\sin^2 60^{\circ} + \cos^2 61^{\circ} + 2\tan^2 89.5^{\circ}\cos 61^{\circ} + 2\tan 89.5^{\circ}\sin 60^{\circ}\cos 61^{\circ} - \tan^2 89.5^{\circ} - 4\tan 89.5^{\circ}\sin 61^{\circ}\cos 60^{\circ}) > 9345|e_{i-1}|^2 > 0$. Thus, if $|e| \geq |e_{i-1}|$ then $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice.

Second, suppose that $|e_{i-1}| \geq |e|$, that is, $m = |e_{i-1}|$. Then, we have to prove that $(|e_{i-1}|\tan^2 89.5^{\circ} + |e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|\cos\alpha_{i-1})^2 - (\tan^2 89.5^{\circ} + 1)(|e_{i-1}|^2\tan^2 89.5^{\circ} + 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1} + |e|^2 - |e_{i-1}|^2) \geq 0$, that is, $|e_{i-1}|^2\tan^4 89.5^{\circ} + |e|^2\tan^2 89.5^{\circ}\sin^2\alpha_{i-1} + |e|^2\cos^2\alpha_{i-1} + 2|e_{i-1}||e|\tan^3 89.5^{\circ}\sin\alpha_{i-1} + 2|e_{i-1}||e|\tan^2 89.5^{\circ}\cos\alpha_{i-1} + 2|e|^2\tan 89.5^{\circ}\sin\alpha_{i-1}\cos\alpha_{i-1} - |e_{i-1}|^2\tan^4 89.5^{\circ} - |e_{i-1}|^2\tan^2 89.5^{\circ} - 2|e_{i-1}||e|\tan^3 89.5^{\circ}\sin\alpha_{i-1} - 2|e_{i-1}||e|\tan 89.5^{\circ}\sin\alpha_{i-1} - |e|^2\tan^2 89.5^{\circ} - |e|^2 + |e_{i-1}|^2\tan^2 89.5^{\circ} + |e_{i-1}|^2 \geq 0$. Simplifying the previous one and using $|e| \leq |e_1|$ and $|e| \geq 2|e_1|\cos\alpha_{i-1}$ (by Lemma 5), we get that, in order to prove the previous inequality, it suffices to prove that $4|e_{i-1}|^2\tan^2 89.5^{\circ}\sin^2\alpha_{i-1}\cos^2\alpha_{i-1} + 4|e_{i-1}|^2\cos^4\alpha_{i-1} + 4|e_{i-1}|^2\tan^2 89.5^{\circ}\cos^2\alpha_{i-1} + 8|e_{i-1}|^2\tan 89.5^{\circ}\sin\alpha_{i-1}\cos^3\alpha_{i-1} - |e_{i-1}|^2\tan^2 89.5^{\circ} - 2|e_{i-1}|^2\tan 89.5^{\circ}\sin\alpha_{i-1} - |e_{i-1}|^2\tan^2 89.5^{\circ} - |e_{i-1}|^2 + |e_{i-1}|^2\tan^2 89.5^{\circ} + |e_{i-1}|^2 \geq 0$. Moreover, since $\sin 60^{\circ} \leq \sin\alpha_{i-1} \leq \sin 61^{\circ}$ and $\cos 61^{\circ} \leq \cos\alpha_{i-1} \leq \cos 60^{\circ}$ (by hypothesis and by Lemma 2), we get that the previous inequality is implied by $|e_{i-1}|^2(4\tan^2 89.5^{\circ}\sin^2 60^{\circ}\cos^2 61^{\circ} + 4\cos^4 61^{\circ} + 4\tan^2 89.5^{\circ}\cos^2 61^{\circ} + 8\tan 89.5^{\circ}\sin 60^{\circ}\cos^3 61^{\circ} - \tan^2 89.5^{\circ} - 2\tan 89.5^{\circ}\sin 61^{\circ}) \geq 8363|e_{i-1}|^2 > 0$. Thus, even if $|e_{i-1}| \geq |e|$ then $l(\overline{\alpha_i})$ intersects $k(v,m)$ twice. □

Lemma 11. *The distance between u_i and $h_{\overline{\alpha_i}}$ is at least $1.604|e_{i-1}|$.*

Proof: By the proof of Lemma 10, we have that the intersection points of $k(v,m)$ with $l(\overline{\alpha_i})$ satisfy $(\tan^2 89.5^\circ+1)x^2-2(|e_{i-1}|\tan^2 89.5^\circ+|e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|\cos\alpha_{i-1})x+|e_{i-1}|^2\tan^2 89.5^\circ+2|e_{i-1}||e|\tan 89.5^\circ\sin\alpha_{i-1}+|e|^2-m^2=0$. To lower bound the distance between u_i and $h_{\overline{\alpha_i}}$ we distinguish two cases, namely the one in which $|e|\geq|e_{i-1}|$ and the one in which $|e_{i-1}|\geq|e|$.

First, suppose that $|e|\geq|e_{i-1}|$. By the computation in the proof of Lemma 10, the discriminant of the equation describing the x-coordinates of the intersections of $k(v,m)$ with $l(\overline{\alpha_i})$ is at least $9345|e_{i-1}|^2$. Hence, since $\sin 60^\circ\leq\sin\alpha_{i-1}\leq\sin 61^\circ$ and $\cos 61^\circ\leq\cos\alpha_{i-1}\leq\cos 60^\circ$ (by hypothesis and by Lemma 2) and since $|e|\geq|e_{i-1}|$ and $2|e|\cos\alpha_{i-1}\leq|e_{i-1}|$ (by Lemma 5), we get that $h_{\overline{\alpha_i}}$ has x-coordinate which is at least $\frac{|e_{i-1}|\tan^2 89.5^\circ+|e_{i-1}|\tan 89.5^\circ\sin 60^\circ+|e_{i-1}|\cos 61^\circ+|e_{i-1}|\sqrt{9345}}{\tan^2 89.5^\circ+1}>1.014|e_{i-1}|$. Plugging such a lower bound into the equation $y=\tan 89.5^\circ(x-|e_{i-1}|)$ of $l(\overline{\alpha_i})$ we get that the y-coordinate of $h_{\overline{\alpha_i}}$ is at least $1.604|e_{i-1}|$. Hence, the distance between $h_{\overline{\alpha_i}}$ and u_i is at least $|e_{i-1}|\sqrt{(1.604)^2+(0.014)^2}>1.604|e_{i-1}|$.

Second, suppose that $|e_{i-1}|\geq|e|$. By the computation in the proof of Lemma 10, the discriminant of the equation describing the x-coordinates of the intersections of $k(v,m)$ with $l(\overline{\alpha_i})$ is at least $8363|e_{i-1}|^2$. Hence, since $\sin 60^\circ\leq\sin\alpha_{i-1}\leq\sin 61^\circ$ and $\cos 61^\circ\leq\cos\alpha_{i-1}\leq\cos 60^\circ$ (by hypothesis and by Lemma 2) and since $|e|\leq|e_{i-1}|$ and $|e|\geq 2|e_{i-1}|\cos\alpha_{i-1}$ (by Lemma 5), we get that $h_{\overline{\alpha_i}}$ has x-coordinate which is at least $\frac{|e_{i-1}|\tan^2 89.5^\circ+2|e_{i-1}|\tan 89.5^\circ\sin 60^\circ\cos 61^\circ+2|e_{i-1}|\cos^2 61^\circ+|e_{i-1}|\sqrt{8363}}{\tan^2 89.5^\circ+1}>1.014|e_{i-1}|$. Again, this yields a $1.604|e_{i-1}|$ lower bound for the y-coordinate of $h_{\overline{\alpha_i}}$ and to a $1.604|e_{i-1}|$ lower bound for the the distance between $h_{\overline{\alpha_i}}$ and u_i. □

Next, we prove the claims formulated in the proof of Lemma 14.

Claim 1. *The intersection of $l^{|}_{pu_{i+1}}$ and $l(u_{i+1})$ is below h.*

Proof: As computed in the proof of Claim 2, $l^{|}_{pu_{i+1}}$ has equation $y-\frac{|e|\sin\alpha_i}{2}=\frac{|e_i|-|e|\cos\alpha_i}{|e|\sin\alpha_i}(x-\frac{|e|\cos\alpha_i+|e_i|}{2})$. Intersecting such a line with $l(u_{i+1})$, that has equation $x=|e_i|$, we get $y=\frac{|e|\sin\alpha_i}{2}+\frac{|e_i|^2}{|e|\sin\alpha_i}-\frac{|e_i||e|\cos\alpha_i}{|e|\sin\alpha_i}-\frac{|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}-\frac{|e_i|^2}{2|e|\sin\alpha_i}+\frac{|e|^2\cos^2\alpha_i}{2|e|\sin\alpha_i}+\frac{|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}$. Simplifying the previous formula, the y-coordinate of the intersection of $l^{|}_{pu_{i+1}}$ with $l(u_{i+1})$ is $y=\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}$. Next, we compute the intersection of $k(p,m)$ with $l(u_{i+1})$. The equation of $k(p,m)$ is $(x-(|e|\cos\alpha_i))^2+(y-(|e|\sin\alpha_i))^2=m^2$. Intersecting such a curve with $x=|e_i|$ we get $y^2-2|e|y\sin\alpha_i+|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i=m^2$, that is, the y-coordinate of h is $y=|e|\sin\alpha_i+\sqrt{|e|^2\sin^2\alpha_i-|e|^2-|e_i|^2+m^2+2|e_i||e|\cos\alpha_i}$. Suppose that $|e|\geq|e_i|$, that is, $m=|e|$. Then, in order to prove that $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below h, we have to show that $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}<|e|\sin\alpha_i+\sqrt{|e|^2\sin^2\alpha_i-|e_i|^2+2|e_i||e|\cos\alpha_i}$. Since $|e|\geq|e_i|$ and $2|e|\cos\alpha_i\leq|e_i|$ (by Lemma 5) and since $\sin\alpha_i\geq\sin 60^\circ$ and $\cos\alpha_i\geq\cos 61^\circ$ (by hypothesis and by Lemma 2), we get $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}\leq\frac{|e|^2+|e^2|-4|e|^2\cos^2\alpha_i}{2|e|\sin\alpha_i}=\frac{|e|-2|e|\cos^2\alpha_i}{\sin\alpha_i}\leq|e|\frac{1-2\cos^2 61^\circ}{\sin 60^\circ}<0.61189|e|$. On the other hand, $|e|\sin\alpha_i+\sqrt{|e|^2\sin^2\alpha_i-|e_i|^2+2|e_i||e|\cos\alpha_i}\geq|e|\sin\alpha_i+\sqrt{|e|^2\sin^2\alpha_i-|e|^2+4|e|^2\cos^2\alpha_i}\geq|e|\sin 60^\circ+\sqrt{|e|^2\sin^2 60^\circ-|e|^2+4|e|^2\cos^2 61^\circ}=|e|(\sin 60^\circ+\sqrt{\sin^2 60^\circ-1+4\cos^2 61^\circ}>1.6967|e|$. Thus, if $|e|\geq|e_i|$ then $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below h. Next, suppose that $|e_i|\geq|e|$, that is, $m=|e_i|$. Then, in order to prove that $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below h, we have to show that $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}<|e|\sin\alpha_i+\sqrt{|e|^2\sin^2\alpha_i-|e|^2+2|e_i||e|\cos\alpha_i}$. Since $|e_i|\geq|e|$ and $2|e_i|\cos\alpha_i\leq|e|$ (by Lemma 5) and since $\sin\alpha_i\geq\sin 60^\circ$ and $\cos\alpha_i\geq\cos 61^\circ$ (by hypothesis and by Lemma 2), we get $\frac{|e|^2+|e_i|^2-2|e_i||e|\cos\alpha_i}{2|e|\sin\alpha_i}\leq\frac{|e_i|^2+|e_i|^2-4|e_i|^2\cos^2\alpha_i}{4|e_i|\cos\alpha_i\sin\alpha_i}=\frac{|e_i|-2|e_i|\cos^2\alpha_i}{2\cos\alpha_i\sin\alpha_i}\leq|e_i|\frac{1-2\cos^2 61^\circ}{2\cos 61^\circ\sin 60^\circ}<0.6311|e_i|$. On the other hand, $|e|\sin\alpha_i+\sqrt{|e|^2\sin^2\alpha_i-|e|^2+2|e_i||e|\cos\alpha_i}\geq 2|e_i|\sin\alpha_i\cos\alpha_i+\sqrt{4|e_i|^2\cos^2\alpha_i\sin^2\alpha_i-|e_i|^2+4|e_i|^2\cos^2\alpha_i}\geq 2|e_i|\sin 60^\circ\cos 61^\circ+\sqrt{4|e_i|^2\sin^2 60^\circ\cos^2 61^\circ-|e_i|^2+4|e_i|^2\cos^2 61^\circ}=|e_i|(2\sin 60^\circ\cos 61^\circ+$

$\sqrt{4\sin^2 60^\circ \cos^2 61^\circ - 1 + 4\cos^2 61^\circ}) > 1.643|e_i|$. Thus, even if $|e_i| \geq |e|$ then $l^{|}_{pu_{i+1}}$ intersects $l(u_{i+1})$ below h. □

Claim 2. *The distance between* u_{i+1} *and the intersection point* $h^{|}(p, u_{i+1}, \tan 89.5^\circ)$ *of* $l^{|}_{pu_{i+1}}$ *and* $l_{\tan 89.5^\circ}$ *is at most* $0.66|e_{i-1}|$.

Proof: First, we derive the equation of $l^{|}_{pu_{i+1}}$. Such a line passes through the midpoint of $\overline{pu_{i+1}}$, that has coordinates $(\frac{|e|\cos\alpha_i + |e_i|}{2}, \frac{|e|\sin\alpha_i}{2})$. Moreover, $l^{|}_{pu_{i+1}}$ is orthogonal to the line through v and u_{i+1}, that has equation $y = \frac{x|e|\sin\alpha_i - |e_i||e|\sin\alpha_i}{|e|\cos\alpha_i - |e_i|}$. Hence, the slope of $l^{|}_{pu_{i+1}}$ is $\frac{|e_i| - |e|\cos\alpha_i}{|e|\sin\alpha_i}$. Then, $l^{|}_{pu_{i+1}}$ has equation $y - \frac{|e|\sin\alpha_i}{2} = \frac{|e_i| - |e|\cos\alpha_i}{|e|\sin\alpha_i}(x - \frac{|e|\cos\alpha_i + |e_i|}{2})$. Second, the equation of $l(\overline{\alpha_i})$ is $y = \tan 89.5^\circ (x - |e_i|)$. Intersecting such two lines we get $\tan 89.5^\circ (x - |e_i|) = \frac{|e|\sin\alpha_i}{2} + \frac{|e_i| - |e|\cos\alpha_i}{|e|\sin\alpha_i}(x - \frac{|e|\cos\alpha_i + |e_i|}{2})$, that is $x = \frac{\tan 89.5^\circ |e_i| + \frac{|e|\sin\alpha_i}{2} + \frac{(|e_i| - |e|\cos\alpha_i)(-|e|\cos\alpha_i - |e_i|)}{2|e|\sin\alpha_i}}{\tan 89.5^\circ + \frac{|e|\cos\alpha_i - |e_i|}{|e|\sin\alpha_i}} = \frac{\tan 89.5^\circ |e_i| + \frac{|e|\sin\alpha_i}{2} + \frac{|e|^2\cos^2\alpha_i - |e_i|^2}{2|e|\sin\alpha_i}}{\tan 89.5^\circ + \frac{|e|\cos\alpha_i - |e_i|}{|e|\sin\alpha_i}}$.

Suppose that $|e| \geq |e_i|$. Then, by Lemma 5, $e \leq \frac{|e_i|}{2\cos\alpha_i}$. Using the last two inequalities we get $x \leq \frac{\tan 89.5^\circ |e_i| + \frac{|e_i|\sin\alpha_i}{4\cos\alpha_i} + \frac{\frac{|e_i|^2}{4} - |e_i|^2}{\frac{|e_i|\sin\alpha_i}{\cos\alpha_i}}}{\tan 89.5^\circ + \frac{|e_i|\cos\alpha_i - |e_i|}{|e_i|\sin\alpha_i}} = \frac{\tan 89.5^\circ + \frac{\tan\alpha_i}{4} - \frac{3}{4\tan\alpha_i}}{\tan 89.5^\circ - \frac{1-\cos\alpha_i}{\sin\alpha_i}}|e_i|$. Next, exploiting $\sin 60^\circ \leq \sin\alpha_i \leq \sin 61^\circ$, $\tan 60^\circ \leq \tan\alpha_i \leq \tan 61^\circ$, and $\cos 61^\circ \leq \cos\alpha_i \leq \cos 60^\circ$ (which hold by assumption and by Lemma 2), we get $x \leq \frac{\tan 89.5^\circ + \frac{\tan 61^\circ}{4} - \frac{3}{4\tan 61^\circ}}{\tan 89.5^\circ - \frac{1-\cos 61^\circ}{\sin 60^\circ}}|e_i| < 1.0056|e_i|$. Hence, the y-coordinate of $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ is $y \leq \tan 89.5^\circ (1.0056|e_i| - |e_i|) < 0.642|e_i|$. Finally, the distance between $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ and u_{i+1} is at most $\sqrt{(0.642)^2 + (0.0056)^2}|e_i| < 0.6421|e_i|$, thus proving the claim in the case in which $|e| \geq |e_i|$.

Suppose that $|e| \leq |e_i|$. Then, by Lemma 5, $e \geq 2|e_i|\cos\alpha_i$. Using the last two inequalities we get $x \leq \frac{\tan 89.5^\circ |e_i| + \frac{|e_i|\sin\alpha_i}{2} + \frac{|e_i|^2\cos^2\alpha_i - |e_i|^2}{2|e_i|\sin\alpha_i}}{\tan 89.5^\circ + \frac{2|e_i|\cos^2\alpha_i - |e_i|}{2|e_i|\sin\alpha_i\cos\alpha_i}}$. Next, exploiting $\sin 60^\circ \leq \sin\alpha_i \leq \sin 61^\circ$, $\tan 60^\circ \leq \tan\alpha_i \leq \tan 61^\circ$, and $\cos 61^\circ \leq \cos\alpha_i \leq \cos 60^\circ$ (which hold by assumption and by Lemma 2), we get $x \leq \frac{\tan 89.5^\circ + \frac{\sin 61^\circ}{2} - \frac{1-\cos^2 60^\circ}{2\sin 61^\circ}}{\tan 89.5^\circ - \frac{1-2\cos^2 61^\circ}{2\sin 60^\circ \cos 61^\circ}}|e_i| < 1.0057|e_i|$. Hence, the y-coordinate of $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ is $y \leq \tan 89.5^\circ (1.0057|e_i| - |e_i|) < 0.654|e_i|$. Finally, the distance between $h^{|}(v, u_{i+1}, \overline{\alpha_i})$ and u_{i+1} is at most $\sqrt{(0.654)^2 + (0.0057)^2}|e_i| < 0.655|e_i|$, thus proving the claim in the case in which $|e| \leq |e_i|$. □

Claim 3. *If* $|e| \geq |e_i|$, *it holds* $\frac{|\overline{u_{i+1}t}|}{|e_i|} < 0.056$; *if* $|e_i| \geq |e|$, *it holds* $\frac{|\overline{u_{i+1}t}|}{|e_i|} < 0.0723$.

Proof: Suppose that $|e| \geq |e_i|$. Then we have $m = |e|$. The x-coordinate of t satisfies $(\tan^2 89.5^\circ + 1)x^2 - 2(|e_i|\tan^2 89.5^\circ + |e|\tan 89.5^\circ \sin\alpha_i + |e|\cos\alpha_i)x + |e_i|^2\tan^2 89.5^\circ + 2|e_i||e|\tan 89.5^\circ \sin\alpha_i + |e|^2 - |e|^2 = 0$ (see the proof of Lemma 10), that yields $x = \frac{|e_i|\tan^2 89.5^\circ + |e|\tan 89.5^\circ \sin\alpha_i + |e|\cos\alpha_i}{\tan^2 89.5^\circ + 1} \pm \frac{\sqrt{(|e_i|\tan^2 89.5^\circ + |e|\tan 89.5^\circ \sin\alpha_i + |e|\cos\alpha_i)^2 - (\tan^2 89.5^\circ + 1)(|e_i|^2\tan^2 89.5^\circ + 2|e_i||e|\tan 89.5^\circ \sin\alpha_i)}}{\tan^2 89.5^\circ + 1}$. Simplifying the last equation and observing that the x-coordinate of t is the smallest of the two x-coordinates solving such an equation, we get $x = \frac{|e_i|\tan^2 89.5^\circ + |e|\tan 89.5^\circ \sin\alpha_i + |e|\cos\alpha_i}{\tan^2 89.5^\circ + 1} - \frac{\sqrt{|e|^2\tan^2 89.5^\circ \sin^2\alpha_i + |e|^2\cos^2\alpha_i + 2|e_i||e|\tan^2 89.5^\circ \cos\alpha_i + 2|e|^2\tan 89.5^\circ \sin\alpha_i\cos\alpha_i - |e_i|^2\tan^2 89.5^\circ - 2|e_i||e|\tan 89.5^\circ \sin\alpha_i}}{\tan^2 89.5^\circ + 1}$. Using $|e_i| \leq |e| \leq \frac{|e_i|}{2\cos\alpha_i}$, $\cos 61^\circ \leq \cos\alpha_i \leq \cos 60^\circ$, $\sin 60^\circ \leq \sin\alpha_i \leq \sin 61^\circ$, and $\tan 60^\circ \leq$

$\tan\alpha_i \leq \tan 61^\circ$ we get $x \leq \frac{|e_i|\tan^2 89.5^\circ + \frac{|e_i|\tan 89.5^\circ \tan 61^\circ}{2} + \frac{|e_i|}{2}}{\tan^2 89.5^\circ + 1} - \frac{|e_i|\sqrt{\tan^2 89.5^\circ \sin^2 60^\circ + \cos^2 61^\circ + 2\tan^2 89.5^\circ \cos 61^\circ + 2\tan 89.5^\circ \sin 60^\circ \cos 61^\circ - \tan^2 89.5^\circ - \tan 89.5^\circ \tan 61^\circ}}{\tan^2 89.5^\circ + 1} < \frac{13234.420437 - 96.637136}{13131.5587}|e_i| < 1.00048|e_i|$. Hence, the y-coordinate of t is at most $\tan 89.5^\circ(1.00048|e_i| - |e_i|) < 0.055|e_i|$. It follows that $\frac{|u_{i+1}t|}{|e_i|} \leq \sqrt{0.00048^2 + 0.055^2} < 0.056$.

Suppose that $|e_i| \geq |e|$. Then we have $m = |e_i|$. The x-coordinate of t satisfies $(\tan^2 89.5^\circ + 1)x^2 - 2(|e_i|\tan^2 89.5^\circ + |e|\tan 89.5^\circ \sin\alpha_i + |e|\cos\alpha_i)x + |e_i|^2\tan^2 89.5^\circ + 2|e_i||e|\tan 89.5^\circ \sin\alpha_i + |e|^2 - |e_i|^2 = 0$ (see the proof of Lemma 10). Solving with respect to x, observing that the x-coordinate of t is the smallest of the two x-coordinates solving the previous equation, and using $|e| \leq |e_i| \leq \frac{|e|}{2\cos\alpha_i}$, $\cos 61^\circ \leq \cos\alpha_i \leq \cos 60^\circ$, $\sin 60^\circ \leq \sin\alpha_i \leq \sin 61^\circ$, and $\tan 60^\circ \leq \tan\alpha_i \leq \tan 61^\circ$, analogously to the case in which $|e| \geq |e_i|$ we get $x < 1.00063|e_i|$. Hence, the y-coordinate of t is at most $\tan 89.5^\circ(1.00063|e_i| - |e_i|) < 0.0722|e_i|$. It follows that $\frac{|u_{i+1}t|}{|e_i|} \leq \sqrt{0.00063^2 + 0.0722^2} < 0.0723$. □

Finally, we prove Lemma 15.

Lemma 15. *Suppose that* $\beta_i, \beta'_i \geq 89.5°$. *Then* $\max\{e_i^2, e_i^3, e_i^4\} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))} \leq 1.032|e_i^1|$.

Proof: Denote by γ_1, γ_2, and γ_3 the angles delimited by edges e_i^1 and e_i^2, by edges e_i^2 and e_i^3, and by edges e_i^3 and e_i^4, respectively. Observe that $\beta_i + \beta'_i \geq 179°$, by the lemma's hypotheses, hence $\gamma_1 + \gamma_2 + \gamma_3 \leq 181°$. By Lemma 2, $\gamma_1, \gamma_2, \gamma_3 \geq 60°$, hence we have $\beta_i + \beta'_i \leq 180°$, $\gamma_i \leq 240^\circ - (\beta_i + \beta'_i)$, with $i \in \{1,2,3\}$, and $\gamma_i + \gamma_j \leq 300^\circ - (\beta_i + \beta'_i)$, with $i, j \in \{1,2,3\}$ and $i \neq j$. Further, by Lemma 5, we have $|e_i^2| \leq \frac{|e_i^1|}{2\cos\gamma_1}$, $|e_i^3| \leq \frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2}$, and $|e_i^4| \leq \frac{|e_i^1|}{8\cos\gamma_1\cos\gamma_2\cos\gamma_3}$.

The second inequality directly comes from the fact that $\cos(240° - (\beta_i + \beta'_i)) \geq \cos 61° > 0.484$, hence $\frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))} \leq 1.032|e_i^1|$.

We prove the first inequality. First, $|e_i^2| \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$ directly comes from $|e_i^2| \leq \frac{|e_i^1|}{2\cos\gamma_1}$ and from $\gamma_1 \leq 240^\circ - (\beta_i + \beta'_i)$.

Second, to prove $|e_i^3| \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$, we use $|e_i^3| \leq \frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2}$ and we argue that $\frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$. Observe that $\frac{|e_i^1|}{4\cos\gamma_1\cos\gamma_2} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$ is equivalent to $2\cos\gamma_1\cos\gamma_2 \geq \cos(240^\circ - (\beta_i + \beta'_i))$. Hence, we study the minimum value of $\cos\gamma_1\cos\gamma_2$. Observe that $\cos\gamma_i$ is a function decreasing with γ_i when $0 \leq \gamma_i \leq 90^\circ$, hence, in order to minimize $\cos\gamma_1\cos\gamma_2$, we can assume that $\gamma_3 = 60^\circ$ and thus $\gamma_2 = (300 - \beta_i - \beta'_i) - \gamma_1$. The derivative of $\cos\gamma_1\cos((300 - \beta_i - \beta'_i) - \gamma_1)$ with respect to γ_1 is $-\sin\gamma_1\cos((300 - \beta_i - \beta'_i) - \gamma_1) + \cos\gamma_1\sin((300 - \beta_i - \beta'_i) - \gamma_1) = \sin((300 - \beta_i - \beta'_i) - 2\gamma_1)$. Hence, such a derivative is positive when $60^\circ \leq \gamma_1 < \frac{300 - \beta_i - \beta'_i}{2}$, is null when $\gamma_1 = \frac{300 - \beta_i - \beta'_i}{2}$, and is negative when $\frac{300 - \beta_i - \beta'_i}{2} < \gamma_1 \leq (240 - \beta_i - \beta'_i)$. Thus, the minimum of $\cos\gamma_1\cos\gamma_2$ is achieved either when $\gamma_1 = 60^\circ$ and $\gamma_2 = 240 - \beta_i - \beta'_i$ or when $\gamma_1 = 240 - \beta_i - \beta'_i$ and $\gamma_2 = 60^\circ$. Thus, we get $2\cos\gamma_1\cos\gamma_2 \geq \cos(240^\circ - (\beta_i + \beta'_i))$.

Third, to prove that $|e_i^4| \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$, we use $|e_i^4| \leq \frac{|e_i^1|}{8\cos\gamma_1\cos\gamma_2\cos\gamma_3}$ and we argue that $\frac{|e_i^1|}{8\cos\gamma_1\cos\gamma_2\cos\gamma_3} \leq \frac{|e_i^1|}{2\cos(240^\circ - (\beta_i + \beta'_i))}$. Similarly to the previous proof, it suffices to show that $4\cos\gamma_1\cos\gamma_2\cos\gamma_3 \geq \cos(240^\circ - (\beta_i + \beta'_i))$. Hence, we study the minimum value of $\cos\gamma_1\cos\gamma_2\cos\gamma_3$. Suppose that γ_3 is fixed to be any angle such that $60^\circ \leq \gamma_3 \leq 240^\circ - (\beta_i + \beta'_i)$. Then, analogously to the previous proof, it can be shown that the minimum value of $\cos\gamma_1\cos\gamma_2$ is achieved when one between γ_1 and γ_2, say γ_1, is $60°$, while the other one, say γ_2, is $300 - \beta_i - \beta'_i - \gamma_3$. Hence, $\cos\gamma_1\cos\gamma_2\cos\gamma_3$ is minimized when $\cos\gamma_2\cos\gamma_3$ is minimized. Then, analogously to the previous proof, it can be shown that the minimum value of $\cos\gamma_2 \cos\gamma_3$ is achieved when one between γ_2 and γ_3, say γ_2, is $60°$, while the other one, say γ_3, is $240 - \beta_i - \beta_i'$. Thus, we get $4\cos\gamma_1 \cos\gamma_2 \cos\gamma_3 \geq \cos(240^{\circ} - (\beta_i + \beta_i'))$. $\square$