Approximate Ergodic Capacity of a Class of Fading $2 \times 2 \times 2$ Networks

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***Abstract*—We study a 2-user 2-hop network with 2 relays in which channel coefficients are independently drawn from continuous distributions and vary over time. For a broad class of channel distributions, we characterize the ergodic sum capacity within a constant number of bits/sec/Hz, independent of signal-to-noise ratio. Specifically, we characterize the ergodic sum capacity within 4 bits/sec/Hz for independent and identically distributed (i.i.d.) uniform phase fading and approximately 4.7 bits/sec/Hz for i.i.d. Rayleigh fading. For achievability, we propose ergodic interference neutralization in which the relays amplify and forward their received signals with appropriate delays such that interference can be neutralized at each destination.**

I. Introduction

In recent years, there has been great progress on understanding fundamentals of multi-source wireless networks. One of the remarkable achievements is an *approximate capacity*, which characterizes the capacity within a constant number of bits/sec/Hz independent of signal-to-noise ratio (SNR) and channel parameters. It has been proved in [1] that simple Han–Kobayashi type scheme can achieve the capacity of the two-user Gaussian interference channel within one bit/sec/Hz. Approximate capacity has been also characterized for many-to-one or one-to-many interference channel [2] and two-way channel [3].

For more than two-user, *interference alignment* can significantly improve the overall rate of the Gaussian interference channel. It has been originally proved in [4] that interference alignment can achieve the optimal degrees of freedom (DoF) of $K/2$ for the time-varying K-user Gaussian interference channel. The recently proposed *ergodic interference alignment* in [5] makes interference aligned in finite SNR and, as a result, provides significant rate improvement compared to the conventional time-sharing strategy at finite SNR. Similar concept has been independently proposed in [6] for finite-field networks.

In spite of recent achievements on interference channels or multi-source single-hop networks, understanding of *multi-source multi-hop networks* is still in progress. For multi-source multi-hop networks, interference can not only be aligned, but it can be cancelled through multiple paths, which is referred to as *interference neutralization*. The work in [7] has exploited interference alignment to neutralize interference at final destinations, which is referred to as *aligned interference neutralization*, and showed that the optimal 2 DoF is achievable for 2-user 2-hop networks with 2 relays. Similar concept of ergodic interference alignment has been proposed for interference neutralization in [8] showing that *ergodic interference neutralization* achieves the optimal DoF of K-user K-hop networks with K relays in each layer for isotropic fading. The result in [7] has been recently generalized to 2-user multi-hop networks [9], [10].

In this paper, we consider *fading 2-user 2-hop networks with 2 relays*. Our aim is to *characterize the ergodic sum capacity within a constant number of bits/sec/Hz, independent of SNR*. We notice that the best known capacity characterization for these networks is within $o(\log \mathrm{SNR})$ [7], which can be arbitrarily large as SNR increases. For a broad class of channel distributions including independent and identically distributed (i.i.d.) Rayleigh fading, we characterize the ergodic sum capacity within a constant number of bits/sec/Hz. We propose ergodic interference neutralization that able to completely neutralize interference at each destination in the finite SNR regime.

II. System Model

Throughout the paper, we will use $\mathbf{A}$, $\mathbf{a}$, and $\mathcal{A}$ to denote a matrix, vector, and set, respectively. Let $\mathbf{A}^T$ (or $\mathbf{a}^T$) and $\mathbf{A}^\dagger$ (or $\mathbf{a}^\dagger$) denote the transpose and conjugate transpose of $\mathbf{A}$ (or $\mathbf{a}$), respectively. Let $\mathcal{N}_{\mathbb{C}}(\mu, \sigma^2)$ denote the circularly symmetric complex Gaussian distribution with mean μ and variance σ^2.

A. Fading $2 \times 2 \times 2$ Networks

We study a 2-user 2-hop network with 2 relays in which each source wishes to transmit an independent message to its destination. The input–output relation of the first hop at time t is given by

$$\mathbf{y}_R[t] = \mathbf{H}[t]\mathbf{x}[t] + \mathbf{z}_R[t], \tag{1}$$

where $\mathbf{H}[t] = [[h_{11}[t], h_{12}[t]]^T, [h_{21}[t], h_{22}[t]]^T]^T$ is the complex channel matrix of the first hop at time t, $\mathbf{y}_R[t] = [y_{R,1}[t], y_{R,2}[t]]^T$ is the received signal vector of the relays at time t, $\mathbf{x}[t] = [x_1[t], x_2[t]]^T$ is the transmit signal vector of the sources at time t, and $\mathbf{z}_R[t] = [z_{R,1}[t], z_{R,2}[t]]^T$ is the

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noise vector of the relays at time t. Similarly, the input–output relation of the second hop at time t is given by

$$\mathbf{y}[t] = \mathbf{G}[t]\mathbf{x}_R[t] + \mathbf{z}[t], \tag{2}$$

where $\mathbf{G}[t] = [[g_{11}[t], g_{12}[t]]^T, [g_{21}[t], g_{22}[t]]^T]^T$ is the complex channel matrix of the second hop at time t, $\mathbf{y}[t] = [y_1[t], y_2[t]]^T$ is the received signal vector of the destinations at time t, $\mathbf{x}_R[t] = [x_{R,1}[t], x_{R,2}[t]]^T$ is the transmit signal vector of the relays at time t, and $\mathbf{z}[t] = [z_1[t], z_2[t]]^T$ is the noise vector of the destinations at time t. Each source and relay should satisfy the average power constraint P, i.e., $\mathsf{E}[|x_i[t]|^2] \le P$ and $\mathsf{E}[|x_{R,i}[t]|^2] \le P$ for $i \in \{1,2\}$. The elements of $\mathbf{z}_R[t]$ and $\mathbf{z}[t]$ are i.i.d. drawn from $\mathcal{N}_{\mathbb{C}}(0,1)$.

We assume that *channel coefficients are independent of each other and vary over time*. Throughout the paper, we consider *symmetric channel distribution* such that the direct channel coefficients $h_{ii}[t]$ and $g_{ii}[t]$ are drawn from a continuous function $f_d(x)$, $x \in \mathbb{C}$, and the cross channel coefficients $h_{ij}[t]$ and $g_{ij}[t]$ are drawn from a continuous function $f_c(x)$, $x \in \mathbb{C}$, where $i,j \in \{1,2\}$, $i \ne j$. Without loss of generality, we assume $\mathsf{E}[|h_{ii}[t]|^2] = \mathsf{E}[|g_{ii}[t]|^2] = 1$ and $\mathsf{E}[|h_{ij}[t]|^2] = \mathsf{E}[|g_{ij}[t]|^2] = \sigma_c^2 \in [0,1]$. We further assume that the sources do not know any channel state information (CSI) and the relays and the destinations know global CSI. That is, at time t, each relay and destination know $\mathbf{H}[t]$ and $\mathbf{G}[t]$.

Remark 1: Notice that the considered class of channel distributions includes *symmetric Rayleigh fading*. In this case, $f_d(\cdot)$ and $f_c(\cdot)$ are given by $\mathcal{N}_{\mathbb{C}}(0,1)$ and $\mathcal{N}_{\mathbb{C}}(0,\sigma_c^2)$, respectively.

B. Setup

Based on the network model, we consider a set of length-n block codes. Let W_i be the message of source i uniformly distributed over $\{1,\cdots,2^{nR_i}\}$, where R_i is the rate of source i. A rate pair (R_1,R_2) is said to be *achievable* if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that the probabilities of error for W_1 and W_2 converge to zero as n increases. The sum capacity C_{sum} is defined as the maximum achievable sum rate.

III. Ergodic Interference Neutralization

Interference can be neutralized by transmitting signals using a certain pair of $\mathbf{H}[t_1]$ and $\mathbf{G}[t_2]$ at the sources and the relays respectively such that $\mathbf{G}[t_2]\mathbf{H}[t_1]$ becomes a diagonal matrix with non-zero diagonal elements. Although finding a pair of channel instances having exact prescribed values is impossible, such a pairing can be done approximately by partitioning the channel space of each hop. We first explain channel space partition of each hop and then define the pairing rule of the partitioned channel spaces in the following subsection.

A. Partitioning and Pairing of Channel Space

We first partition the channel space of each hop, i.e, $\mathbb{C}^{2\times 2}$ space. Define $\mathcal{Q} = \Delta(\mathbb{Z}^{2\times 2} + \jmath\mathbb{Z}^{2\times 2})$, where $\Delta > 0$ is the quantization interval. For a quantized channel matrix $\mathbf{Q} \in \mathcal{Q}$, define $\mathcal{A}(\mathbf{Q})$ as the set of all $\mathbf{A} \in \mathbb{C}^{2\times 2}$ whose closest point in $\mathcal{Q}$ is equal to $\mathbf{Q}$. Specifically, $\mathcal{A}(\mathbf{Q}) = \{\mathbf{A} \big| -\frac{\Delta}{2} \le a_{ij} - q_{ij} < \frac{\Delta}{2} \text{ for all } i,j \in \{1,2\}, \mathbf{A} \in \mathbb{C}^{2\times 2}\}$, where a_{ij} and q_{ij} denote the (i,j)th element of $\mathbf{A}$ and $\mathbf{Q}$, respectively. We will use $\mathcal{A}(\mathbf{Q})$ for partitioning the first-hop and the second-hop channel spaces in the next subsection.

Now consider the channel space pairing. For $\mathbf{A} \in \mathbb{C}^{2\times 2}$, define

$$F(\mathbf{A}) = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix},$$

where a_{ij} denotes the (i,j)th element of $\mathbf{A}$. Note that $F(\mathbf{Q}) \in \mathcal{Q}$ for any $\mathbf{Q} \in \mathcal{Q}$. The relays will choose a certain time t_2 and amplify and forward $\mathbf{y}_R[t_1]$ such that $\mathbf{H}[t_1] \in \mathcal{A}(\mathbf{Q})$ and $\mathbf{G}[t_2] \in \mathcal{A}(F(\mathbf{Q}))$. Hence the channel subspace $\mathcal{A}(\mathbf{Q})$ of the first-hop will be paired with the channel subspace $\mathcal{A}(F(\mathbf{Q}))$ of the second-hop.

B. Ergodic Interference Neutralization

We first divide a length n block into B sub-blocks having length $n_B = \frac{n}{B}$ each. We assume block transmission. At the first sub-block, the sources transmit their first messages to the relays (the relays do not transmit). At the bth sub-block, $b \in \{2,\cdots,B-1\}$, the sources transmit their bth messages to the relays and the relays amplify and forward the received signals of the $(b-1)$th sub-block to the destinations. At the last sub-block, the relays amplify and forward the received signals of the $(B-1)$th sub-block to the destinations (the sources do not transmit). Hence, the number of effective sub-blocks is equal to $B-1$. Since we can set both n_B and B as large as possible as n increases, the fractional rate loss $1 - \frac{B-1}{B}$ becomes negligible in this case. For simplicity, we describe the proposed scheme based on the first message transmission and omit the sub-block index.

For $M \in \mathbb{Z}_+$, define $\mathcal{Q}' = \{\mathbf{Q} \big| |\text{re}(q_{ij})| \le \Delta M, |\text{im}(q_{ij})| \le \Delta M \text{ for all } i,j \in \{1,2\}, \mathbf{Q} \in \mathcal{Q}\}$, where q_{ij} denote the (i,j)th element of $\mathbf{Q}$ and re$(\cdot)$ and im$(\cdot)$ denote the real and imaginary part of a complex number, respectively. For $\mathbf{Q} \in \mathcal{Q}'$, let $\mathcal{T}_1(\mathbf{Q}) = \{t \big| \mathbf{H}[t] \in \mathcal{A}(\mathbf{Q}), t \in \{1,\cdots,n_B\}\}$, which is the set of time indices of the first hop whose channel instances belong to $\mathcal{A}(\mathbf{Q})$. Similarly, let $\mathcal{T}_2(\mathbf{Q}) = \{t \big| \mathbf{G}[t] \in \mathcal{A}(\mathbf{Q}), t \in \{1,\cdots,n_B\}\}$. The encoding, relaying, and decoding are as follows.

- (Encoding) The sources transmit their messages using Gaussian codebook with power P. Specifically, the transmit signal vector of the sources is given by $\mathbf{x}[t]$ with $\mathsf{E}\left[\mathbf{x}[t]\mathbf{x}[t]^\dagger\right] = P\mathbf{I}$ for $t \in \{1,\cdots,n_B\}$, where $\mathbf{I}$ denotes the identity matrix.[1]
- (Relaying) For all $\mathbf{Q} \in \mathcal{Q}'$, the relays amplify and forward their received signals that were received during $\mathcal{T}_1(\mathbf{Q})$ using the time indices in $\mathcal{T}_2(F(\mathbf{Q}))$. Specifically, the transmit signal vector of the relays is given by

[1]Since we assume Gaussian codebook, one can achieve a rate of $\log(1 + \text{SINR})$ with arbitrarily small probability of error as n_B increases.

$\mathbf{x}_R[t_2] = \gamma\mathbf{\Lambda}\mathbf{y}_R[t_1]$, where $t_1 \in \mathcal{T}_1(\mathbf{Q})$, and $t_2 \in \mathcal{T}_2(F(\mathbf{Q}))$. Here $\mathbf{\Lambda} = [[1,0]^T[0,-1]^T]^T$ and the power amplification factor γ is given by

$$\gamma = \sqrt{\frac{P}{1+(1+\sigma_c^2)P}},$$

which satisfies the average power constraint.

- (Decoding) The destinations decode their messages based on their received signals for $t \in \{1, \cdots, n_B\}$.

Suppose that the sources transmit at time $t_1 \in \mathcal{T}_1(\mathbf{Q})$ and the relays amplify and forward their received signals at time $t_2 \in \mathcal{T}_2(F(\mathbf{Q}))$. For this case, from (1) and (2), the received signal vector of the destinations is given by

$$\mathbf{y}[t_2] = \gamma\mathbf{G}[t_2]\mathbf{\Lambda}\mathbf{H}[t_1]\mathbf{x}[t_1] + \gamma\mathbf{G}[t_2]\mathbf{\Lambda}\mathbf{z}_R[t_1] + \mathbf{z}[t_2], \quad (3)$$

where we use $\mathbf{x}_R[t_2] = \gamma\mathbf{\Lambda}\mathbf{y}_R[t_1]$. Notice that if $\mathbf{H}[t] \notin \mathcal{A}(\mathbf{Q})$ for all $\mathbf{Q} \in \mathcal{Q}'$, then the corresponding source signals will not be delivered to the destinations. Also if $\text{card}(\mathcal{T}_1(\mathbf{Q})) > \text{card}(\mathcal{T}_2(F(\mathbf{Q})))$, then a subset of source signals will not be delivered to the destinations, where $\text{card}(\cdot)$ denotes the cardinality of a set. Similarly, the relays will not utilize a subset of time indices if $\mathbf{G}[t] \notin \mathcal{A}(\mathbf{Q})$ for all $\mathbf{Q} \in \mathcal{Q}'$ or $\text{card}(\mathcal{T}_1(\mathbf{Q})) < \text{card}(\mathcal{T}_2(F(\mathbf{Q})))$. As a consequence, among the n_B received signals at each destination, only a subset of the received signals contains information about the desired message. We will show that the fraction of the received signals that contain no information converges to zero as n_B increases in the next subsection.

C. Achievable Rate

The following theorem shows an achievable rate of ergodic interference neutralization.

Theorem 1: For the fading $2 \times 2 \times 2$ network,

$$R_i = \mathsf{E}\left[\log\left(1 + \frac{P^2|\det(\mathbf{H})|^2}{1+P(|h_{jj}|^2+|h_{ij}|^2+1+\sigma_c^2)}\right)\right]$$

is achievable for $i,j \in \{1,2\}$ and $i \neq j$, where the expectation is over the channel coefficients.[2]

For convenience, let us denote R_{IN} by the achievable sum rate of ergodic interference neutralization in Theorem 1.

Example 1 (Achievable Sum Rate): Fig. 1 plots R_{IN} for i.i.d. uniform phase fading and i.i.d. Rayleigh fading. Since channel coefficients are i.i.d, C_{sum} is upper bounded by the ergodic capacity of the multiple-input and multiple-output (MIMO) channel from the sources to the relays, which is given by

$$R_{\text{MIMO}} = \mathsf{E}\left[\log\det(\mathbf{I} + P\mathbf{H}\mathbf{H}^\dagger)\right]. \quad (4)$$

[2]Since the channel coefficients are drawn i.i.d. over time, we omit the time index for notational simplicity.

Simulation results show that $R_{\text{MIMO}} - R_{\text{IN}}$ is approximately 4 for i.i.d. uniform phase fading and 4.1 for i.i.d. Rayleigh fading at high SNR.

The rest of this subsection is the proof of Theorem 1. We first introduce the following two lemmas.

Lemma 1: For any $\mathbf{Q} \in \mathcal{Q}$,

$$\mathsf{P}\left[\mathbf{H}[t_1] \in \mathcal{A}(\mathbf{Q})\right] = \mathsf{P}\left[\mathbf{G}[t_2] \in \mathcal{A}(F(\mathbf{Q}))\right].$$

Proof: We refer to the full paper [11] in preparation. ■

Lemma 2: The probability that

$$\left|\frac{\text{card}(\mathcal{T}_1(\mathbf{Q}))}{n_B} - \mathsf{P}[\mathbf{H}[t] \in \mathcal{A}(\mathbf{Q})]\right| \leq \delta$$

and

$$\left|\frac{\text{card}(\mathcal{T}_2(\mathbf{Q}))}{n_B} - \mathsf{P}[\mathbf{G}[t] \in \mathcal{A}(\mathbf{Q})]\right| \leq \delta$$

for all $\mathbf{Q} \in \mathcal{Q}'$ is greater than $1 - \text{card}(\mathcal{Q}')/(2n_B\delta^2)$.

Proof: We refer to Lemma 2.12 in [12] for the proof. ■

For notational simplicity, we use $[\mathbf{A}]_{ij}$ to denote the (i,j)th element of $\mathbf{A}$. Suppose that the sources transmit their messages at $t_1 \in \mathcal{T}_1(\mathbf{Q})$ and the relays amplify and forward their received signals at $t_2 \in \mathcal{T}_2(F(\mathbf{Q}))$, where $\mathbf{Q} \in \mathcal{Q}'$. Denote $\mathbf{H}[t_1] = \mathbf{H}$ and $\mathbf{G}[t_2] = F(\mathbf{H}) + \mathbf{\Delta}$, where $\mathbf{\Delta}$ is the quantization error matrix with respect to $F(\mathbf{H})$. From (3),

$$\begin{aligned}\mathbf{y}[t_2] = \gamma(\det(\mathbf{H})\mathbf{\Lambda} + \mathbf{\Delta}\mathbf{\Lambda}\mathbf{H})\mathbf{x}[t_1] \\ + \gamma(F(\mathbf{H}) + \mathbf{\Delta})\mathbf{\Lambda}\mathbf{z}_R[t_1] + \mathbf{z}[t_2],\end{aligned}$$

where we use $F(\mathbf{H})\mathbf{\Lambda}\mathbf{H} = \det(\mathbf{H})\mathbf{\Lambda}$. Thus, the received signal-to-interference-and-noise ratio (SINR) of destination i is given by (5), where $i,j \in \{1,2\}$, $i \neq j$. Define $R_i(\mathbf{Q}) = \min_{\mathbf{A}\in\mathcal{A}(\mathbf{Q})} \log(1 + \text{SINR}_i)$. Then an achievable rate of destination i is lower bounded by

$$R_i \geq \frac{1}{n_B}\sum_{\mathbf{Q}\in\mathcal{Q}'} R_i(\mathbf{Q})\min\{\text{card}(\mathcal{T}_1(\mathbf{Q})), \text{card}(\mathcal{T}_2(F(\mathbf{Q})))\}.$$

From Lemmas 1 and 2,

$$\text{card}(\mathcal{T}_1(\mathbf{Q})) \geq n_B(\mathsf{P}[\mathbf{H}[t] \in \mathcal{A}(\mathbf{Q})] - \delta)$$

and

$$\begin{aligned}\text{card}(\mathcal{T}_2(F(\mathbf{Q}))) &\geq n_B(\mathsf{P}[\mathbf{G}[t] \in \mathcal{A}(F(\mathbf{Q}))] - \delta) \\ &= n_B(\mathsf{P}[\mathbf{H}[t] \in \mathcal{A}(\mathbf{Q})] - \delta)\end{aligned}$$

for all $\mathbf{Q} \in \mathcal{Q}'$ with probability greater than $1 - \frac{(2M+1)^8}{2n_B\delta^2}$, where we use $\text{card}(\mathcal{Q}') = (2M+1)^8$. Then

$$R_i \geq \sum_{\mathbf{Q}\in\mathcal{Q}'} R_i(\mathbf{Q})\,\mathsf{P}[\mathbf{H}[t] \in \mathcal{A}(\mathbf{Q})] - \delta(2M+1)^8 \max_{\mathbf{Q}\in\mathcal{Q}'} R_i(\mathbf{Q})$$

is achievable with probability greater than $1 - \frac{(2M+1)^8}{2n_B\delta^2}$. Notice that it is possible to set Δ, M, and δ as functions of n_B such that $\Delta \to 0$, $\Delta M \to \infty$, $\delta(2M+1)^8 \max_{\mathbf{Q}\in\mathcal{Q}'} R_i(\mathbf{Q}) \to 0$,

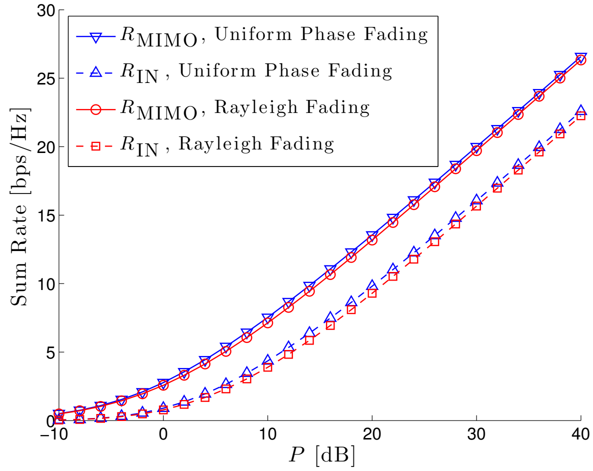

Fig. 1. Achievable sum rates of ergodic interference neutralization.

and $\frac{(2M+1)^8}{2n_B\delta^2} \to 0$ as $n_B \to \infty$. For example, setting $\Delta = n_B^{-1/(3\times 2^5)}$, $M = n_B^{1/(3\times 2^4)}$, and $\delta = n_B^{-1/3}$ satisfies the above conditions. Hence,

$$\begin{aligned}
&\lim_{n_B\to\infty} R_i \\
&\geq \mathsf{E}_{\mathbf{H}}\left[\log\left(1+\frac{\gamma^2 P|\det(\mathbf{H})|^2}{1+\gamma^2(|[\mathbf{H}]_{jj}|^2+|[\mathbf{H}]_{ij}|^2)}\right)\right] \\
&= \mathsf{E}_{\mathbf{H}}\left[\log\left(1+\frac{P^2|\det(\mathbf{H})|^2}{1+P(|h_{jj}|^2+|h_{ij}|^2+1+\sigma_c^2)}\right)\right]
\end{aligned}$$

is achievable for $i,j \in \{1,2\}$ and $i \neq j$ with probability approaching one. Here we use the fact that

$$\lim_{\Delta\to 0} \mathrm{SINR}_i = \frac{\gamma^2 P|\det(\mathbf{H})|^2}{1+\gamma^2(|[\mathbf{H}]_{jj}|^2+|[\mathbf{H}]_{ij}|^2)}.$$

Therefore Theorem 1 holds.

IV. Approximate Capacity Characterization

In this section, we show that the proposed ergodic interference neutralization achieves the ergodic sum capacity within a constant number of bits/sec/Hz for a broad class of channel distributions including i.i.d. Rayleigh fading.

A. Uniform Phase Fading

We first consider i.i.d. uniform phase fading in which $h_{ij}[t] = \exp(\jmath\theta_{ij}[t])$, $g_{ij}[t] = \exp(\jmath\varphi_{ij}[t])$, and $\theta_{ij}[t]$ and $\varphi_{ij}[t]$ are uniformly distributed over $[0, 2\pi)$. We show that the proposed ergodic interference neutralization achieves $C_{\rm sum}$ within 4 bits/sec/Hz for any $P \geq 0$. Although uniform phase fading violates the channel assumption in Section II-A, $f_d(x)$ and $f_c(x)$ are continuous functions over $x \in \mathbb{C}$, we can modify channel space partition based on phases and show that Theorem 1 still holds.

Before characterizing an approximate capacity, we first introduce the following lemma. This lemma shows the exact solution of $\mathsf{E}_\phi\left[\log\left(1 - x\cos\phi\right)\right]$ for $|x| \leq 1$, which is useful for dealing with uniform phase fading throughout the paper.

Lemma 3: Let ϕ be a random variable uniformly distributed over $[0, 2\pi)$. For $|x| \leq 1$,

$$\mathsf{E}_\phi\left[\log\left(1 - x\cos\phi\right)\right] = \log\left(1+\sqrt{1-x^2}\right) - 1.$$

Proof: We refer to the equation (4.224 12) in [13]. ■

Theorem 2: Consider the fading $2\times 2\times 2$ network. If $h_{ij}[t] = \exp(\jmath\theta_{ij}[t])$, $g_{ij}[t] = \exp(\jmath\varphi_{ij}[t])$, and $\theta_{ij}[t]$ and $\varphi_{ij}[t]$ are uniformly distributed over $[0, 2\pi)$, then

$$R_{\rm IN} = 2\log\left(1+\frac{2P^2}{1+4P}\right)+2\log\left(1+\sqrt{1-(C(P))^2}\right)-2$$

and

$$R_{\rm MIMO} = \log(1+4P+2P^2)+\log\left(1+\sqrt{1-(C(P))^2}\right)-1,$$

where $C(P) = 2P^2/(1+4P+2P^2)$. Furthermore, $C_{\rm sum} - R_{\rm IN} \leq 4$ for any $P \geq 0$.

Proof: Let $\mathbf{H} = [[h_{11}, h_{12}]^T, [h_{21}, h_{22}]^T]^T$ and the distribution of $\mathbf{H}$ is equal to that of $\mathbf{H}[t]$. Then h_{ij} can be represented as $\exp(\jmath\theta_{ij})$, where θ_{ij} is uniformly distributed over $[0, 2\pi)$.

First consider $R_{\rm IN}$. We have

$$\begin{aligned}
R_{\rm IN} &\overset{(a)}{=} 2\,\mathsf{E}_\theta\left[\log\left(1+\frac{2P^2(1-\cos\theta)}{1+4P}\right)\right] \\
&\overset{(b)}{=} 2\,\mathsf{E}_\phi\left[\log\left(1+\frac{2P^2(1-\cos\phi)}{1+4P}\right)\right] \\
&= 2\log\left(1+\frac{2P^2}{1+4P}\right) + 2\,\mathsf{E}_\phi\left[\log\left(1-C(P)\cos\phi\right)\right] \\
&\overset{(c)}{=} 2\log\left(1+\frac{2P^2}{1+4P}\right) \\
&\quad + 2\log\left(1+\sqrt{1-(C(P))^2}\right) - 2, \qquad (6)
\end{aligned}$$

where $\theta = \theta_{11}+\theta_{22}-\theta_{12}-\theta_{21}$ and ϕ is a random variable uniformly distributed over $[0, 2\pi)$. Here, (a) follows since $\sigma_c^2 = 1$ and $|\det(\mathbf{H})|^2 = 2(1-\cos\theta)$, (b) follows since θ mod $[2\pi]$ is uniformly distributed over $[0, 2\pi)$, and (c) follows from Lemma 3 and $|C(P)| \leq 1$. Similarly,

$$\begin{aligned}
R_{\rm MIMO} &= \mathsf{E}_\phi\left[\log((1+2P)^2 - 2P^2(1+\cos\phi))\right] \\
&= \log(1+4P+2P^2) + \mathsf{E}_\phi\left[\log\left(1-C(P)\cos\phi\right)\right] \\
&= \log(1+4P+2P^2) \\
&\quad + \log\left(1+\sqrt{1-(C(P))^2}\right) - 1, \qquad (7)
\end{aligned}$$

$$\mathrm{SINR}_i = \frac{P|(-1)^{i-1}\gamma\det(\mathbf{H}) + \gamma[\mathbf{\Delta\Lambda H}]_{ii}|^2}{1+\gamma^2(|[\mathbf{H}]_{jj}+[\mathbf{\Delta}]_{ii}|^2 + |[\mathbf{H}]_{ij}+[\mathbf{\Delta}]_{ij}|^2) + P|\gamma[\mathbf{\Delta\Lambda H}]_{ij}|^2}, \qquad (5)$$

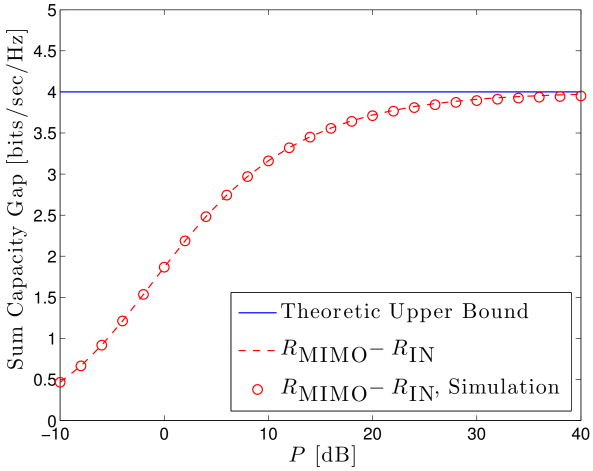

Fig. 2. Gap from the sum capacity for i.i.d. uniform phase fading.

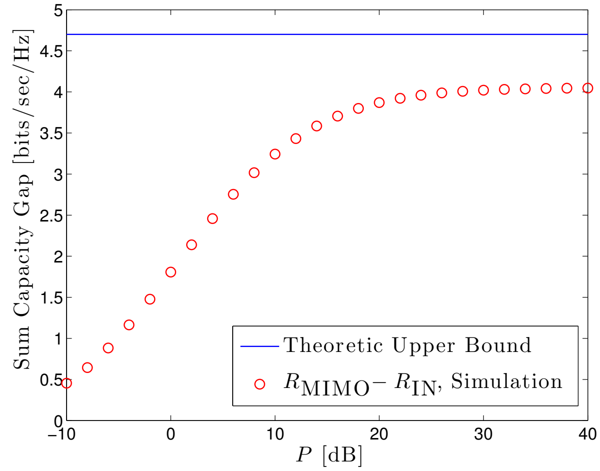

Fig. 3. Gap from the sum capacity for i.i.d. Rayleigh fading.

where ϕ is a random variable uniformly distributed over $[0, 2\pi)$. Lastly, from (6) and (7),

$$\begin{aligned} &R_{\text{MIMO}} - R_{\text{IN}} \\ &= \log\left(\frac{(1+4P)^2}{1+4P+2P^2}\right) - \log\left(1+\sqrt{1-(C(P))^2}\right) + 1 \\ &\overset{(a)}{\le} \log\left(\frac{(1+4P)^2}{1+4P+2P^2}\right) + 1 \\ &\overset{(b)}{\le} 4, \end{aligned}$$

where (a) follows since $0 \le C(P) \le 1$ for any $P \ge 0$ and (b) follows since

$$\begin{aligned} \log\left(\frac{(1+4P)^2}{1+4P+2P^2}\right) &\le \log\left(\frac{(1+4P)^2}{1+2\sqrt{2}P+2P^2}\right) \\ &= 2\log\left(\frac{1+4P}{1+\sqrt{2}P}\right) \\ &\le 3. \end{aligned}$$

Here we use the fact that $\log\left(\frac{1+4P}{1+\sqrt{2}P}\right)$ is an increasing function of $P \ge 0$ and $\lim_{P\to\infty} \log\left(\frac{1+4P}{1+\sqrt{2}P}\right) = \frac{3}{2}$. Therefore Theorem 2 holds. ■

Example 2 (Capacity Gap: i.i.d. Uniform Phase Fading): Fig. 2 plots $R_{\text{MIMO}} - R_{\text{IN}}$ in Theorem 2 with respect to P for i.i.d. uniform phase fading, which coincides with the simulation result. As proved by Theorem 2, the proposed ergodic interference neutralization achieves C_{sum} within 4 bits/sec/Hz for uniform phase fading. Notice that this theoretic gap is indeed the sum rate gap at high SNR, i.e., $\lim_{P\to\infty}\{R_{\text{MIMO}} - R_{\text{IN}}\}$, which can be seen in Fig. 2.

B. Uniform Phase Fading with Varying Channel Amplitudes

In this subsection, we consider a class of channel distributions in which $f_d(x) = f_c(x)$ is only a function of $|x|$. That is, for given channel amplitudes, their phases are uniformly distributed over $[0, 2\pi)$. We present the main theorem here and refer to the full paper [11] for the proof.

Theorem 3: Consider the fading $2 \times 2 \times 2$ network. If $f_c(x) = f_d(x)$ is only a function of $|x|$, then

$$C_{\text{sum}} - R_{\text{IN}} \le 2\,\mathsf{E}\left[\log\left(\frac{\sqrt{A}(A+B^2)}{B(A+\sqrt{A^2-G^2})}\right)\right] + 2$$

for any $P \ge 0$, where the expectation is over the channel coefficients. Here

$$\begin{aligned} A &= |h_{11}|^2|h_{22}|^2 + |h_{12}|^2|h_{21}|^2, \\ B &= |h_{11}|^2 + |h_{21}|^2 + 2, \\ G &= 2|h_{11}||h_{12}||h_{21}||h_{22}|. \end{aligned}$$

Example 3 (Capacity Gap: i.i.d. Rayleigh Fading): Fig. 3 plots $R_{\text{MIMO}} - R_{\text{IN}}$ with respect to P and also plot its upper bound in Theorem 3 for i.i.d. Rayleigh fading. Since there is no closed form for the upper bound, we numerically evaluate the bound, approximately given by 4.7 for i.i.d. Rayleigh fading. Therefore the proposed ergodic interference neutralization characterizes C_{sum} within approximately 4.7 bits/sec/Hz.

V. Discussions

In this paper, we proposed ergodic interference neutralization for fading $2 \times 2 \times 2$ networks and showed that it characterizes the ergodic sum capacity within 4 bits/sec/Hz for i.i.d. uniform phase fading and approximately 4.7 bits/sec/Hz for i.i.d. Rayleigh fading. Although we mainly considered i.i.d. fading in this paper, the proposed ergodic interference neutralization can characterize the ergodic sum capacity within a constant number of bits/sec/Hz for more general class of channel distributions. We refer to the full paper [11] in preparation for more general results.

Similar analysis used in this paper can be also applied to ergodic interference alignment in [5]. Then we can derive an upper bound on the gap from the ergodic sum capacity of the fading K-user interference channel assuming no power control at the sources. We briefly introduce the following theorem.

Theorem 4: Consider the fading K-user interference channel. Let $R_{\rm IA}$ denote the achievable sum rate of ergodic interference alignment in [5] and $C_{\rm sum}$ denote the sum capacity assuming no power control at the sources. If channel coefficients are i.i.d., then

$$\frac{C_{\rm sum}-R_{\rm IA}}{K}\le\frac{1}{2}\log\left(\frac{3}{2}\right)+\frac{1}{2}\,\mathsf{E}\left[\left|\log\left(\frac{|h_{11}|^2}{|h_{12}|^2}\right)\right|\right]$$

for any $P\ge 0$.

Notice that for i.i.d. Rayleigh fading, the upper bound in Theorem 4 is given by $\frac{1}{2}\log 6$.

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