

# Algebraic Techniques for Linear Deterministic Networks

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**Abstract**—We here summarize some recent advances in the study of linear deterministic networks, recently proposed as approximations for wireless channels. This work started by extending the algebraic framework developed for multicasting over graphs in [1] to include operations over matrices and to admit both graphs and linear deterministic networks as special cases. Our algorithms build on this generalized framework, and provide as special cases unicast and multicast algorithms for deterministic networks, as well as network code designs using structured matrices.

## I. INTRODUCTION

Deterministic network models have recently attracted a significant interest in the literature, in particular in the context of serving as approximate characterizations for wireless networks [9]–[11]. The work in [9], [10] established an information theoretical min-cut max-flow theorem for deterministic networks, showing that the unicast capacity equals the minimum rank of the transfer matrices of cuts separating the source from the destination. In the same work, the authors also extended to deterministic networks the multicasting theorem in network coding [7], and showed that in the presence of  $N$  receivers, if the mincut to each receiver equals at least  $h$ , it is possible to multicast information to all receivers at this rate. The proof uses information theoretical arguments and does not directly lead to polynomial time designs.

For deterministic networks, a polynomial time algorithm that allows unicast information transmission using block length of one over arbitrary networks was first developed in [24], for an extended version see [25]. This work was followed up by three other alternative designs for unicast algorithms [26]–[28]. Additional traffic scenaria over deterministic networks were investigated for example in [31], [32], but for specific network configurations, typically consisting from a small number of nodes.

Network coding solved the problem of multicasting at optimal rate the same information from a single source to multiple receivers [7], [8]. Several algorithms were subsequently developed that offer polynomial time designs for multicasting over graphs [1]–[3]. In particular, Koetter and Medard [1] translated the problem of network code design to an algebraic problem over graphs. This algebraic framework provided the theoretical foundation for the development of most of the finite length coding results and practical algorithms for network coding today.

In our work [29], [30] we have extended the algebraic framework developed for multicasting over graphs in [1] in

two ways: (i) to include operations over matrices and (ii) to accept both graphs and linear deterministic networks as special cases. Independently from our work, the framework in [1] was also extended over deterministic networks in [16]. Building on this extended framework, we propose new algebraic code designs for scalar and vector network coding, that apply both over networks represented as graphs, as well as arbitrary linear deterministic networks. A key ingredient of our designs is the use of vector network coding.

In scalar network coding, when multicasting to  $N$  receivers at rate  $h$ , the source transmits  $h$  scalar values over some finite field. The size of the employed finite field is a design parameter, that for complexity considerations we typically desire to minimize. Intermediate network nodes linearly combine their received symbols by multiplying them with scalar coefficients, called coding coefficients in the literature. The network code design consists of selecting the coding coefficients, and the finite field of operation, so that each receiver has a full rank set of equations to solve and can thus retrieve the source symbols [1].

In vector network coding, the source transmits  $h$  vectors of length  $L$ , where the elements of the vectors are over a fixed finite field  $\mathbb{F}_q$ , for example, the binary field  $\mathbb{F}_2$ . Intermediate network nodes perform coding operations over vectors, namely, multiply their incoming vectors with  $L \times L$  coding matrices and then add them to create the new vectors that they propagate towards the destinations. That is, intermediate nodes linearly combine their incoming vectors using coding matrices, where these matrices play a similar role as scalar coding coefficients in traditional algebraic network coding. The code design consists in selecting the length  $L$  and the  $L \times L$  coding matrices so that each receiver receives information at rate  $h$ . Scalar network coding over a field of size  $\mathbb{F}_q$  can be viewed as a special case of vector network coding with  $L = 1$ .

The contributions in our work [29], [30] include the following:

- 1) We provide a polynomial time algorithm for the design of coding matrices of vector network coding when multicasting to  $N$  receivers. Our metric of optimization is the smallest size  $L$  such that there exist  $L \times L$  coding matrices that allow all  $N$  destinations to successfully decode the source information. The size of these matrices plays the same role as the size of the finite field in traditional network coding. Our algorithm reduces the problem of finding a small size  $L$  to the problem of finding a small

degree coprime factor of an algebraic polynomial, and leads to solutions not possible with using scalar network coding, as illustrated through examples in [29], [30].

- 2) Both for the case of graphs and deterministic networks, we show that  $L \cong \log(N(\log N - 1)h\Lambda)$  is always sufficient, where  $\Lambda$  is a network parameter, and we can find such matrices in polynomial time. We also provide probabilistic guarantees, and show for example that in a fraction  $\frac{1023}{1024}$  of polynomials derived from transfer functions, we will be able to find in polynomial time binary coding matrices of size at most  $3 \times 3$  that lead to a valid code.
- 3) For the case of deterministic networks, our approach offers the first code design for multicasting over arbitrary linear deterministic networks. Moreover, as a special case, it gives an alternative algorithm for a unicast connection over a deterministic network, adding to the existing algorithms [24]–[28].
- 4) Our approach also gives a new algorithm for scalar network code design, that operates in polynomial time. This new algorithm jointly minimizes the employed field size while selecting the coding coefficients. In contrast, existing algorithms [1]–[5] first select a fixed finite field and then proceed to design the network codes over this predetermined field. As a consequence, these algorithms would operate over a field of size  $N$ , i.e., the worst case guarantee. However, our algorithm, due to jointly optimizing the effective field size as well as the coefficients, can result in a much smaller field while still being a polynomial time algorithm. A theoretical side-result of our work is establishing a connection between the problem of identifying the minimum field size required for network coding and finding the smallest coprime factor of algebraic polynomials.

We believe that vector network coding is a very promising research direction, as it offers a natural generalization of network coding, and thus offers a larger space of choices for optimizing cost parameters, such as the operational complexity, or the communication block length.

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